

Polar actions on compact rank one symmetric spaces are taut

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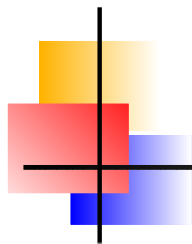
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- A polar representation of a compact Lie group in an Euclidean space is orbit equivalent to the isotropy representation of a symmetric space (Dadok, 1985).



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- If L_q is Morse, the weak Morse inequalities say that $\mu_k(L_q|_{M^c}) \geq \beta_k(M^c; F)$ for all k and $c > 0$; L_q is called **F -perfect** if these are equalities.



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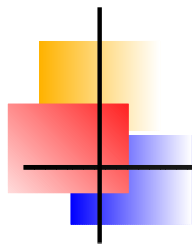
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- E_q is a Morse function iff q is not a focal point of M , γ is a critical point of E_q iff it is a geodesic perpendicular to M at $\gamma(0)$, and the Morse index thm holds.
- E_q is bounded below and satisfies the Palais-Smale condition, so for a Morse function E_q , $\mu_k(E_q|\mathcal{M}^c)$ is finite and the weak Morse inequalities hold; E_q is **F -perfect** if $\mu_k(E_q|\mathcal{M}^c) = \beta_k(\mathcal{M}^c; F)$ for all k and $c > 0$.



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- There are nonpolar, taut actions on Euclidean space (G.-Thorbergsson, 2000) and on compact rank one symmetric spaces (Biliotti-G., 2005).
- A hyperpolar action of a compact Lie group on a complete Riemannian manifold is \mathbf{Z}_2 -taut (Bott-Samelson and Conlon).



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- We will only use that a polar action on $\mathbf{Ca}P^2 = \mathbf{F}_4/\mathbf{Spin}(9)$ of cohomogeneity greater than one is given by one of the following subgroups of \mathbf{F}_4 :

$$\mathbf{Spin}(8), \quad S^1 \cdot \mathbf{Spin}(7), \quad \mathbf{SU}(2) \cdot \mathbf{SU}(4), \quad \mathbf{SU}(3) \cdot \mathbf{SU}(3).$$



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- **Proposition** Let $\pi : \hat{X} \rightarrow X$ be a Riemannian submersion and a principal G -bundle. Let M be a properly embedded submanifold of X and $\hat{M} = \pi^{-1}(M)$. Then \hat{M} is taut in \hat{X} iff M is taut in X .



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- **Lemma** (Heintze, Liu, Olmos, 2000) Let $\pi : \hat{X} \rightarrow X$ be a Riemannian submersion, M a properly embedded submanifold of X and $\hat{M} = \pi^{-1}(M)$. Then a normal vector $\hat{\xi}$ is a multiplicity- m focal direction of \hat{M} iff $\pi_*\hat{\xi}$ is a multiplicity- m focal direction of M .



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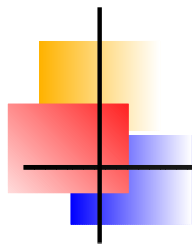
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The map

$$\Phi : \mathcal{P}(X, M \times q) \times \mathcal{P}(G, G \times 1) \rightarrow \mathcal{P}(\hat{X}, \hat{M} \times \hat{q}),$$

given by $\Phi(x, g)(t) = \hat{x}(t)g(t)$, where $\hat{x}(t)$ is the horizontal lift of $x(t)$ with $\hat{x}(1) = \hat{q}$, is a diffeomorphism.

$\mathcal{P}(G, G \times 1)$ is contractible.



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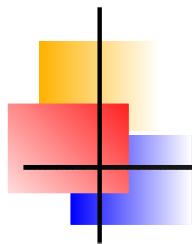
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Then there are homeomorphisms

$${}_cX/\bar{N} \rightarrow X/G \quad \text{and} \quad \bar{X}/\bar{G} \rightarrow X/G.$$



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Floyd:

$$\beta(\Omega^L) \leq \beta(\Omega^{H_{n-1}}) \leq \dots \leq \beta(\Omega^{H_0}).$$



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