Geometric Measure Theory

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Weak derivatives

Definition (weak derivatives and gradients; 5.3)

Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$. We say that:

1. For $1 \leq i \leq n$, u has *weak i-th partial derivative* $v_i \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$ if $\forall \varphi \in \mathrm{C}^\infty_c(\Omega)$,
$$\int_\Omega v_i \varphi \, \mathrm{d}\mathcal{L}^n = -\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}\mathcal{L}^n.$$
2. u has *weak gradient* $v \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$ if $\forall \varphi \in \mathrm{C}^\infty_c(\Omega, \mathbb{R}^n)$,
$$\int_\Omega \langle v, \varphi \rangle \, \mathrm{d}\mathcal{L}^n = -\int_\Omega u \, \mathrm{div}\, \varphi \, \mathrm{d}\mathcal{L}^n. \tag{1}$$

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Sobolev spaces and functions

Definition (5.8)

Let Ω be an open subset of $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}$ and $1 \leq p \leq \infty$. We say that

1. u is a $(1, p)$-*Sobolev function* if $u \in \mathrm{L}^\mathrm{p}(\mathcal{L}^n|_\Omega)$ and, $\forall 1 \leq i \leq n$, u has weak partial derivatives $\frac{\partial u}{\partial x_i} \in \mathrm{L}^\mathrm{p}(\mathcal{L}^n|_\Omega)$. Notation: $\mathrm{W}^{1,\mathrm{p}}(\Omega)$.
2. u is a *local* $(1, p)$-*Sobolev function* if $u \in \mathrm{L}^\mathrm{p}_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$ and, $\forall 1 \leq i \leq n$, u has weak partial derivatives $\frac{\partial u}{\partial x_i} \in \mathrm{L}^\mathrm{p}_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$. Notation: $\mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}(\Omega)$.

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Weak derivatives, bis

Definition (weak derivatives and gradients, bis; 7.1)

Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$. We say that:

1. For $1 \leq i \leq n$, u has *weak i-th partial derivative* $\mu_i \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}) \equiv \mathrm{C}_\mathrm{c}(\Omega, \mathbb{R})^*$ if $\forall \varphi \in \mathrm{C}^\infty_\mathrm{c}(\Omega)$,

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Weak derivatives, bis

Remark (7.2)

1. For $1 \leq i \leq n$, $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$ admits weak i-th partial derivative if, for each compact $K \subset \Omega$, there exists $C_K < \infty$ such that
$$\sup\{\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, \mathrm{d}\mathcal{L}^n \mid \varphi \in \mathrm{C}^\infty_\mathrm{c}(\Omega), \mathrm{spt}\ \varphi \subset K, \|\varphi\|_u \leq 1\} \leq C_K.$$

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3. Weak partial derivatives or weak gradients, if exist, are unique.

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Weak derivatives, bis

Remark

4. $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$ has weak gradient $\mu = (\mu_1, \ldots, \mu_n) \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ iff it has weak partial derivatives of first order $\mu_i \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R})$ for $1 \leq i \leq n$.
5. If $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$ has weak i-th partial derivative $v_i \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$ in the sense of definition 1, then it has weak i-th partial derivative $\mathcal{L}^n \llcorner v_i \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R})$ in the sense of definition 3. Thus, considering the injection $\mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega) \subset \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R})$ given by $v \mapsto \mathcal{L}^n \llcorner v$, we see that definition 1 may be considered a particular case of definition 3.

Weak derivatives, bis

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4. $u \in L^1_{\text{loc}}(\mathcal{L}^n|_\Omega)$ has weak gradient $\mu = (\mu_1, \ldots, \mu_n) \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^n)$ iff it has weak partial derivatives of first order $\mu_i \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R})$ for $1 \leq i \leq n$.
5. If $u \in L^1_{\text{loc}}(\mathcal{L}^n|_\Omega)$ has weak i-th partial derivative $v_i \in L^1_{\text{loc}}(\mathcal{L}^n|_\Omega)$ in the sense of definition 1, then it has weak i-th partial derivative $\mathcal{L}^n \llcorner v_i \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R})$ in the sense of definition 3. Thus, considering the injection $L^1_{\text{loc}}(\mathcal{L}^n|_\Omega) \subset \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R})$ given by $v \mapsto \mathcal{L}^n \llcorner v$, we see that definition 1 may be considered a particular case of definition 3.

Weak derivatives, bis

Exercise (7.3)

Weak gradients may be also characterized by means of Gauss-Green identity in gradient form. That is, let Ω *be an open subset of* $\mathbb{R}^n$ *and* $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$*; then* u *admits weak gradient* $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ iff $\forall \varphi \in \mathrm{C}^\infty_{\mathrm{c}}(\Omega)$,

$$\int_\Omega u \nabla\varphi \, \mathrm{d}\mathcal{L}^n = -\int_\Omega \varphi \, \mathrm{d}\mu. \tag{3}$$

Exercise (7.4)

Let Ω *be an open subset of* $\mathbb{R}^n$, $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$ *and* $1 \leq i \leq n$*. If there exists* $\mu_i = \frac{\partial^w u}{\partial x_i} \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R})$*, then* $\forall \varphi \in \mathrm{C}^1_{\mathrm{c}}(\Omega)$,

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}\mathcal{L}^n = -\int_\Omega \varphi \, \mathrm{d}\mu_i$$

BV functions

Definition (7.5)

Let Ω be an open subset of $\mathbb{R}^n$.

i) We denote by $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ the set of functions $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$ which admit weak partial gradient $\nabla^{\mathrm{w}} u \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.

ii) We say that $u \in \mathrm{BV}(\Omega)$ if $u \in \mathrm{L}^1(\mathcal{L}^n|_\Omega)$ and u admits weak gradient $\nabla^{\mathrm{w}} u \in \mathcal{M}(\Omega, \mathbb{R}^n)$.

iii) We say that $E \subset \Omega$ is a *set of locally finite perimeter* in Ω if $\chi_E \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. We say that E is a *Caccioppoli set* or a *set of finite perimeter* in Ω if $\chi_E \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $\nabla^{\mathrm{w}} \chi_E \in \mathcal{M}(\Omega, \mathbb{R}^n)$.

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$W^{1,1} \subset BV$

Example (7.6)

Let $\Omega \subset \mathbb{R}^n$ open and $f \in W^{1,1}_{loc}(\Omega)$. Then $f \in BV_{loc}(\Omega)$ and its measure-weak gradient is given by $\mathcal{L}^n \llcorner \nabla^w f \in \mathcal{M}_{loc}(\Omega, \mathbb{R}^n)$.
The inclusion $W^{1,1}_{loc}(\Omega) \subset BV_{loc}(\Omega)$ is strict; for instance, if $u = \chi_{(0,\infty)}$ on $\Omega = \mathbb{R}$, $\nabla^w u$ coincides with the Dirac measure $\delta_0 \in \mathcal{M}(\mathbb{R}, \mathbb{R})$, so that $\nabla^w u \perp \mathcal{L}^n$, hence $u \in BV(\mathbb{R}) \setminus W^{1,1}_{loc}(\mathbb{R})$.

Locality of the weak derivative

Theorem (7.7)

Let $\Omega \subset \mathbb{R}^n$, $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$ and $\mathcal{F} \subset 2^\Omega$ an open cover of Ω. Then f admits weak partial derivatives of first order on Ω iff $\forall U \in \mathcal{F}$, $f|_U$ admits weak partial derivatives of first order on U. Moreover, weak derivatives commute with restrictions (for a Radon measure, "restriction" here means "trace").

Corollary (6.15)

Let $\Omega \subset \mathbb{R}^n$ open, $1 \leq p \leq \infty$ and $f : \Omega \to \mathbb{R}$ Lebesgue measurable. Then $f \in \mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}(\Omega)$ iff for all open $V \Subset \Omega$, $f|_V \in \mathrm{W}^{1,\mathrm{p}}(V)$.

Corollary (7.9)

Let $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}$ Lebesgue measurable. Then $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ iff for all open $V \Subset \Omega$, $f|_V \in \mathrm{BV}(V)$.

$W^{1,p}$ norm

Definition (6.1)

Let $\Omega \subset \mathbb{R}^n$ open and $f \in W^{1,1}_{loc}(\Omega)$, i.e. $f \in L^1_{loc}(\mathcal{L}^n|_\Omega)$ admits weak partial derivatives of first order. We define

- for $1 \leq p < \infty$, $\|f\|_{W^{1,p}(\Omega)} := (\int_\Omega |f|^p + \|\nabla f\|^p \, d\mathcal{L}^n)^{1/p} \in [0, \infty]$;
- for $p = \infty$, $\|f\|_{W^{1,\infty}(\Omega)} := \big\| |f| + \|\nabla f\| \big\|_{L^\infty(\Omega)} \in [0, \infty]$.

Proposition (6.2)

Let $\Omega \subset \mathbb{R}^n$ open. For $1 \leq p \leq \infty$, $W^{1,p}(\Omega)$ is a Banach space (for $p = 2$, it is a Hilbert space). It is reflexive for $1 < p < \infty$ and it is separable for $1 \leq p < \infty$.

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BV norm

Proposition (7.10)

Let Ω *be an open subset of* $\mathbb{R}^n$*. Then* $\mathrm{BV}(\Omega)$ *is a Banach space with the norm*

$$\|f\|_{\mathrm{BV}(\Omega)} := \|f\|_{\mathrm{L}^1(\Omega)} + |\nabla^{\mathrm{w}} f|(\Omega). \tag{4}$$

Gauss-Green measure, exterior normal and perimeter measure

Definition (7.12)

Let Ω be an open subset of $\mathbb{R}^n$ and $E \subset \Omega$ such that $\chi_E \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$.

- $\mu_E := -\nabla^{\mathrm{w}} \chi_E \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ (attention to the minus sign) is called the *Gauss-Green measure* of E.
- Let $(\nu_E, |\mu_E|)$ be the polar decomposition of μ_E. We call the positive Radon measure $\mathrm{P}(E, \cdot) := |\mu_E|$ on Ω the *perimeter measure of E* and ν_E the *exterior normal* to E.

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Remark (7.13)

Let $E \subset \Omega$ such that $\chi_E \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $\partial^\Omega E = \Omega \cap \partial E$ be the topological boundary of E in Ω. Then

1) spt $\mu_E \subset \partial^\Omega E$. Since ν_E is determined up to $|\mu_E|$-null sets, we may and do assume henceforth that $\nu_E = 0$ on $\Omega \setminus \partial^\Omega E$ and we identify ν_E with a Borelian map $\partial^\Omega E \to \mathbb{R}^n$;
2) for all $\varphi \in \mathrm{C}^1_\mathrm{c}(\Omega, \mathbb{R}^n)$,

$$\int_E \operatorname{div} \varphi \, \mathrm{d}\mathcal{L}^n = \int_{\partial^\Omega E} \varphi \cdot \nu_E \, \mathrm{d}|\mu_E|. \tag{5}$$

We call the above equality the *generalized Gauss-Green theorem*.

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Lipschitz epigraphs have locally finite perimeter

Proposition (7.14)

Let $n \geq 2$, $f : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz and $\Omega := \mathrm{epi_S}\ f$. Then Ω is a set of locally finite perimeter in $\mathbb{R}^n$, $|\mu_\Omega| = \mathcal{H}^{n-1} \llcorner \partial\Omega$ and ν_Ω coincides with the unit outer normal to $\partial\Omega$, i.e.

$$\nu(x) = \frac{\big(\nabla f(x'), -1\big)}{\sqrt{1 + \|\nabla f(x')\|^2}}$$

on each point point $x = \big(x', f(x')\big)$ in $\partial\Omega = \mathrm{gr}\ f$ whose abscissa x' is a differentiability point of f.

Lipschitz domains

Notation for cylinders

Let $\mathbb{R}^n \equiv \mathbb{R}^k \times \mathbb{R}^{n-k}$, $x \in \mathbb{R}^n$, $0 < r, h \leq \infty$, $p : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ and $q : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be the projections on the first and second factors, respectively.

- $\mathbb{C}(x, r, h) := \mathbb{U}(p \cdot x, r) \times \mathbb{U}(q \cdot x, h) \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$.
- $\overline{\mathbb{C}}(x, r, h) := \mathbb{B}(p \cdot x, r) \times \mathbb{B}(q \cdot x, h) = \overline{\mathbb{C}(x, r, h)}$.

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Lipschitz domains

Definition (6.33)

Let $n \geq 2$, $U \subset \mathbb{R}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R}$ open and $\Omega \subset U$ an open subset of U. We say that Ω is a *Lipschitz domain* if for all $x \in \partial^U \Omega = \partial\Omega \cap U$, there exist:

1. a rigid motion $\Phi \in \mathrm{SE}(n)$ with $\Phi(0) = x$;
2. $f : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz with $f(0) = 0$;
3. $\mathbb{C}(0, r, h) \subset \mathbb{R}^{n-1} \times \mathbb{R}$ open cylinder

satisfying the following conditions:

- $C := \Phi\big(\mathbb{C}(0, r, h)\big) \subset U$;
- $\Phi\big(\mathrm{gr}\ f \cap \mathbb{C}(0, r, h)\big) = C \cap \partial\Omega$
- $\Phi\big(\mathrm{epi}_S\ f \cap \mathbb{C}(0, r, h)\big) = C \cap \Omega$,

where $\mathrm{epi}_S\ f = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y > f(x)\}$.

Lipschitz domains

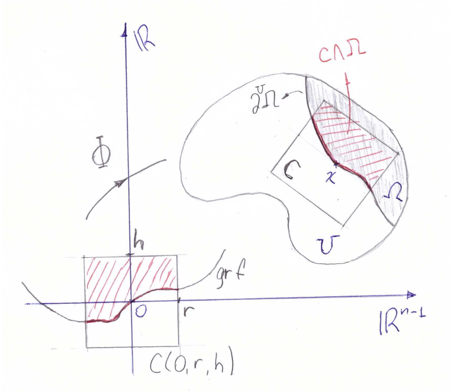

Figure: Lipschitz Domain

Smooth partitions of unity on open sets of $\mathbb{R}^n$

Definition (6.6)

Let $\Omega \subset \mathbb{R}^n$ open. A *smooth partition of unity* of Ω is a family $(\xi_\alpha)_{\alpha\in A}$ such that:

PU1) $\forall \alpha \in A$, $\xi_\alpha \in C^\infty(\Omega, [0,1])$ and $(\operatorname{spt} \xi_\alpha)_{\alpha\in A}$ is a locally finite family of subsets of Ω;

PU2) $\forall x \in \Omega$, $\sum_{\alpha\in A} \xi_\alpha(x) = 1$.

If $\mathcal{F}$ is an open cover of Ω, we say that a smooth partition of unity $(\xi_\alpha)_{\alpha\in A}$ of Ω is *subordinate* to $\mathcal{F}$ if $\forall \alpha \in A$, there exists $U \in \mathcal{F}$ such that $\operatorname{spt} \xi_\alpha \subset U$.

Existence of partitions of unity on open sets of $\mathbb{R}^n$

Theorem (6.8)

Let $\Omega \subset \mathbb{R}^n$ open and $(U_\alpha)_{\alpha \in A}$ a locally finite open cover of Ω with $\forall \alpha \in A$, $U_\alpha \Subset \Omega$. Then there exists a smooth partition of unity $(\xi_\alpha)_{\alpha \in A}$ of Ω such that, $\forall \alpha \in A$, $\operatorname{spt} \xi_\alpha \Subset U_\alpha$.

Corollary

Let $\Omega \subset \mathbb{R}^n$ be open and $\mathcal{F}$ an open cover of Ω. Then there exists a partition of unity $(\xi_V)_{V \in \mathcal{F}}$ of Ω strictly subordinate to $\mathcal{F}$, i.e. such that for all $V \in \mathcal{F}$, $\operatorname{spt} \xi_V \subset V$.

Existence of partitions of unity on open sets of $\mathbb{R}^n$

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Let $\Omega \subset \mathbb{R}^n$ be open and $\mathcal{F}$ an open cover of Ω. Then there exists a partition of unity $(\xi_V)_{V \in \mathcal{F}}$ of Ω strictly subordinate to $\mathcal{F}$, i.e. such that for all $V \in \mathcal{F}$, $\operatorname{spt} \xi_V \subset V$.

Gaus-Green theorem for Lipschitz domains

Theorem (7.16)

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Then Ω is a set of locally finite perimeter in $\mathbb{R}^n$ and $|\mu_\Omega| = \mathcal{H}^{n-1} \llcorner \partial\Omega$.

Gaus-Green theorem for Lipschitz domains

Lemma (7.15)

Let $n \geq 2$, $f : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz, U' an open subset of $\mathbb{R}^n$, $E' := U' \cap \operatorname{epis}_S f$ and $\nu' : \partial \operatorname{epis}_S f \to \mathbb{R}^n$ the unit outer normal to $\partial \operatorname{epis}_S f$. Let $\Phi \in \mathrm{SE}(n)$ be a rigid motion, $U := \Phi(U')$, $E := \Phi(E')$ and $\nu := \Phi_* \nu'$, i.e. $\nu : \partial\Phi(\operatorname{epis}_S f) \to \mathbb{R}^n$ is given by $x \mapsto \mathrm{D}\Phi\big(\Phi^{-1}(x)\big) \cdot \nu'\big(\Phi^{-1}(x)\big)$. Then:

1. $\Phi_{\#}\big(\mathcal{H}^{n-1} \llcorner \partial \operatorname{epis}_S f\big) = \mathcal{H}^{n-1} \llcorner \partial\Phi(\operatorname{epis}_S f)$.
2. *E is a set of locally finite perimeter in U, $|\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^U E$ and its exterior normal is given by $\nu_E = \nu|_{\partial^U E}$.*

Gaus-Green theorem for Lipschitz domains

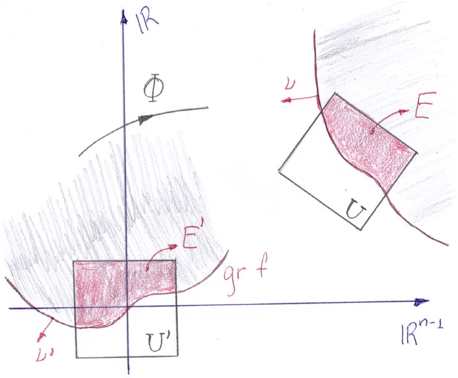

Figure: Gauss-Green measure of a Lipschitz Domain

Gaus-Green theorem for Lipschitz domains

Corollary

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Then $\mathcal{H}^{n-1} \llcorner \partial\Omega$ is a Radon measure.

Remark (outer normal to a Lipschitz domain)

With the notation from the proof of the previous theorem, for each $i \geq 1$, the exterior normal to Ω coincides $\mathcal{H}^{n-1} \llcorner \partial\Omega$-a.e. with ν_i on $\partial\Omega \cap U_i = \partial^{U_i}\Omega$. In particular, if $\partial\Omega$ is a C^1 hypersurface on a neighborhood of $p \in \partial\Omega$, we may choose ν_Ω on this neighborhood as the usual outer unit normal from Differential Geometry.

Exercises

Exercise (Complements of sets of locally finite perimeter)

Let Ω be an open subset of $\mathbb{R}^n$ and $E \subset \Omega$ be a set of locally finite perimeter in Ω. Then $\Omega \setminus E$ has locally finite perimeter in Ω and

$$\mu_{\Omega \setminus E} = -\mu_E.$$

Exercise (Sets of finite perimeter under scaling and translation)

Let E be a set of locally finite perimeter in $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and $\lambda > 0$. Then $x + \lambda E$ is a set of locally finite perimeter in $\mathbb{R}^n$ and

$$\mu_{x+\lambda E} = \Phi_{\#}\mu_E,$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is given by $y \mapsto x + \lambda y$. In particular, if E has finite perimeter, so does $x + \lambda E$ and $P(x + \lambda E, \mathbb{R}^n) = \lambda^{n-1} P(E, \mathbb{R}^n)$.

Standard mollifier in $\mathbb{R}^n$

Definition (1.112)

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be the smooth function given by

$$\phi(x) := \begin{cases} c \exp\left(\frac{1}{\|x\|^2 - 1}\right) & \text{if } \|x\| < 1 \\ 0 & \text{if } \|x\| \geq 1, \end{cases}$$

where c is chosen so that $\int_{\mathbb{R}^n} \phi(x)\,\mathrm{d}x = 1$. The family $(\phi_t)_{>0}$ in $\mathrm{C}_c^\infty(\mathbb{R}^n)$ defined by we define $\phi_t : \mathbb{R}^n \to \mathbb{C}$ by

$$\phi_t(x) := t^{-n}\phi(t^{-1}x). \tag{6}$$

is called *standard mollifier* in $\mathbb{R}^n$.
Note that spt $\phi = \mathbb{B}(0, 1)$, so that $\forall t > 0$, spt $\phi_t = \mathbb{B}(0, t)$.

Approximation by mollifiers on $\mathbb{R}^n$

Theorem (1.111)

Let $\phi \in \mathrm{L}^1(\mathcal{L}^n)$ with $\int \phi \, \mathrm{d}\mathcal{L}^n = a$ and $f : \mathbb{R}^n \to \mathbb{C}$.

i) *If $1 \leq p < \infty$ and $f \in \mathrm{L}^p(\mathcal{L}^n)$, then $\phi_t * f \overset{t \to 0}{\to} af$ in $\mathrm{L}^p(\mathcal{L}^n)$.*

ii) *If f is uniformly continuous and either (1) f is bounded or (2) spt ϕ is compact, then $\phi_t * f \overset{t \to 0}{\to} af$ uniformly in $\mathbb{R}^n$.*

iii) *If f is continuous on an open set $U \subset \mathbb{R}^n$ and either (1) $f \in \mathrm{L}^\infty(\mathcal{L}^n)$ or (2) $f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathcal{L}^n)$ and spt ϕ is compact, then $\phi_t * f \overset{t \to 0}{\to} af$ uniformly on compact subsets of U.*

Approximation by mollifiers on open subsets of $\mathbb{R}^n$

Definition (6.17)

For each $t > 0$, let

$$\Omega_t := \{x \in \mathbb{R}^n \mid \mathbb{B}(x,t) \subset \Omega\} = \{x \in \mathbb{R}^n \mid d(x, \Omega^c) > t\},$$

so that $(\Omega_t)_{t>0}$ is a family of open subsets of Ω which increases to Ω as $t \downarrow 0$.

Let $(\phi_t)_{t>0}$ be the standard mollifier in $\mathbb{R}^n$. For each $t > 0$ and $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$, we define $f_t : \Omega_t \to \mathbb{R}$ by, $\forall x \in \Omega_t$,

$$f_t(x) := (\phi_t * f)(x) = \int_{\mathbb{B}(x,t)} f(y)\phi_t(x-y)\,\mathrm{d}\mathcal{L}^n(y).$$

We call f_t the *t-approximation or t-regularization of f*.

Approximation by mollifiers on open subsets of $\mathbb{R}^n$

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Remark

1. $f_t(x)$ is well-defined since, for $x \in \Omega_t$, $\mathbb{B}(x, t) \subset \Omega$;
2. if $\Omega = \mathbb{R}^n$, then $\Omega_t = \mathbb{R}^n$ for all $t > 0$.

Approximation by mollifiers on open subsets of $\mathbb{R}^n$

Definition (convergence in the sense of $\mathrm{L}^{\mathrm{p}}_{\mathrm{loc}}$ and $\mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}$; 6.18)

Let $1 \leq p \leq \infty$, $f : \Omega \to \mathbb{R}$ $\mathcal{L}^n$-measurable and, for each $k \in \mathbb{N}$, let $f_k : \operatorname{dom} f_k \subset \Omega \to \mathbb{R}$ be $\mathcal{L}^n$-measurable.

- We say that $(f_k)_{k\in\mathbb{N}}$ *converges to f in the sense of* $\mathrm{L}^{\mathrm{p}}_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$ (notation: "$f_k \to f$ in $\mathrm{L}^{\mathrm{p}}_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$") if, for all open $V \Subset \Omega$, there exists $k_0 \in \mathbb{N}$ (possibly depending on V) such that $\forall k \geq k_0$, $V \subset \operatorname{dom} f_k$ and $\|f_k - f\|_{\mathrm{L}^{\mathrm{p}}(\mathcal{L}^n|_V)} \to 0$.
- If $\forall k \in \mathbb{N}$, $\operatorname{dom} f_k$ is open, f and f_k belong to $\mathrm{L}^1_{\mathrm{loc}}$ on their domains and admit weak partial derivatives of first order, we say that $(f_k)_{k\in\mathbb{N}}$ *converges to f in the sense of* $\mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}(\Omega)$ (notation: "$f_k \to f$ in $\mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}(\Omega)$") if, for all open $V \Subset \Omega$, there exists $k_0 \in \mathbb{N}$ (possibly depending on V) such that $\forall k \geq k_0$, $V \subset \operatorname{dom} f_k$ and $\|f_k - f\|_{\mathrm{W}^{1,\mathrm{p}}(V)} \to 0$.

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Approximation by mollifiers on open subsets of $\mathbb{R}^n$

Remark (6.19)

1. We make similar definitions of convergence in the sense of L^p_{loc} or in the sense of $W^{1,p}_{loc}$ for a family $(f_\epsilon)_{\epsilon>0}$ in place of $(f_k)_{k\in\mathbb{N}}$.
2. What we have in mind is the family $(f_t)_{t>0}$ of the regularized functions of some $f \in L^1_{loc}(\mathcal{L}^n|_\Omega)$, cf. definition 26.
3. For a sequence $(f_k)_{k\in\mathbb{N}}$ in $L^p_{loc}(\mathcal{L}^n|_\Omega)$ and $f \in L^p_{loc}(\mathcal{L}^n|_\Omega)$, the convergence defined above coincides with the convergence in the natural topology of $L^p_{loc}(\mathcal{L}^n|_\Omega)$, which is a Fréchet space topology induced by the family of seminorms $\{\|\cdot\|_{L^p(\mathcal{L}^n|_V)} \mid V \Subset \Omega \text{ open}\}$.
4. Similarly, for a sequence $(f_k)_{k\in\mathbb{N}}$ in $W^{1,p}_{loc}(\Omega)$ and $f \in W^{1,p}_{loc}(\Omega)$, the convergence defined above coincides with the convergence in the natural topology of $W^{1,p}_{loc}(\Omega)$, which is a Fréchet space topology induced by the family of seminorms $\{\|\cdot\|_{W^{1,p}(V)} \mid V \Subset \Omega \text{ open}\}$.

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Approximation by mollifiers on open subsets of $\mathbb{R}^n$

Theorem (mollifiers, part II)

Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$, $(\phi_\epsilon)_{\epsilon>0}$ the standard mollifier and $f_\epsilon = \phi_\epsilon * f : \Omega_\epsilon \to \mathbb{R}$ the ϵ-approximation of f, cf. definition 24.

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iii) $\lim_{\epsilon\to 0} f_\epsilon(x) = f(x)$ *if* $x \in \Omega$ *is a Lebesgue point of* f*; in particular,* $f_\epsilon \to f$ $\mathcal{L}^n$*-a.e. on* Ω.

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Theorem (mollifiers, part II)

Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$, $(\phi_\epsilon)_{\epsilon>0}$ *the standard mollifier and* $f_\epsilon = \phi_\epsilon * f : \Omega_\epsilon \to \mathbb{R}$ *the* ϵ*-approximation of* f*, cf. definition 24.*

i) $\forall \epsilon > 0$, $f_\epsilon \in \mathrm{C}^\infty(\Omega_\epsilon)$.

ii) $\forall \epsilon > 0$, $\forall \varphi \in \mathrm{C}^0_c(\Omega_\epsilon)$, $\int f_\epsilon \varphi \, \mathrm{d}\mathcal{L}^n = \int f \varphi_\epsilon \, \mathrm{d}\mathcal{L}^n$.

iii) $\lim_{\epsilon\to 0} f_\epsilon(x) = f(x)$ *if* $x \in \Omega$ *is a Lebesgue point of* f*; in particular,* $f_\epsilon \to f$ $\mathcal{L}^n$*-a.e. on* Ω.

iv) *If* $f \in \mathrm{C}(\Omega)$, $f_\epsilon \to f$ *uniformly on compact subsets of* Ω.

v) *If* $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$ *for some* $1 \le p < \infty$*, then* $f_\epsilon \to f$ *in the sense of* $\mathrm{L}^p_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$.

vi) *If* $f \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ *for some* $1 \le p \le \infty$*, then* $\forall \epsilon > 0, \forall 1 \le i \le n$,

$$\frac{\partial f_\epsilon}{\partial x_i} = \phi_\epsilon * \frac{\partial^w f}{\partial x_i} = \left(\frac{\partial^w f}{\partial x_i}\right)_\epsilon$$

Approximation by mollifiers on open subsets of $\mathbb{R}^n$

Theorem (mollifiers, part II)

Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$, $(\phi_\epsilon)_{\epsilon>0}$ *the standard mollifier and* $f_\epsilon = \phi_\epsilon * f : \Omega_\epsilon \to \mathbb{R}$ *the* ϵ*-approximation of* f*, cf. definition 24.*

i) $\forall \epsilon > 0$, $f_\epsilon \in \mathrm{C}^\infty(\Omega_\epsilon)$.

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vi) *If* $f \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ *for some* $1 \le p \le \infty$*, then* $\forall \epsilon > 0, \forall 1 \le i \le n$,

$$\frac{\partial f_\epsilon}{\partial x_i} = \phi_\epsilon * \frac{\partial^{\mathrm{w}} f}{\partial x_i} = \left(\frac{\partial^{\mathrm{w}} f}{\partial x_i}\right)_\epsilon$$

Approximation by mollifiers on open subsets of $\mathbb{R}^n$

Corollary (6.21)

Let $1 \leq p < \infty$, $(\phi_t)_{t>0}$ *the standard mollifier and* $f \in \mathrm{W}^{1,p}(\mathbb{R}^n)$. *Then:*

i) $\forall \epsilon > 0$, $f_\epsilon = \phi_\epsilon * f \in \mathrm{C}^\infty(\mathbb{R}^n) \cap \mathrm{W}^{1,p}(\mathbb{R}^n)$ *and* $f_\epsilon \to f$ *in* $\mathrm{W}^{1,p}(\mathbb{R}^n)$ *as* $\epsilon \to 0$.

ii) *There exists a sequence* $(f_k)_{k\in\mathbb{N}}$ *in* $\mathrm{C}_c^\infty(\mathbb{R}^n)$ *such that* $f_k \to f$ *in* $\mathrm{W}^{1,p}(\mathbb{R}^n)$.

Regularization of Radon measures and BV functions

Proposition (7.19)

Let $(\phi_t)_{t>0}$ be the standard mollifier on $\mathbb{R}^m$. Then, for each $\epsilon > 0$, the convolution with ϕ_ϵ defines a continuous linear map $\phi_\epsilon * : \mathrm{C_c}(\mathbb{R}^m, \mathbb{R}^n) \to \mathrm{C_c}(\mathbb{R}^m, \mathbb{R}^n)$.

Remark

Similarly, given an open subset $\Omega \subset \mathbb{R}^m$, the convolution with ϕ_ϵ defines a continuous linear map $\phi_\epsilon * : \mathrm{C_c}(\Omega_\epsilon, \mathbb{R}^n) \to \mathrm{C_c}(\Omega, \mathbb{R}^n)$. It then follows that $(\phi_\epsilon *)^{\mathrm{t}} : \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n) \to \mathcal{M}_{\mathrm{loc}}(\Omega_\epsilon, \mathbb{R}^n)$ is a well defined linear map. We shall omit the "t" in the notation of this transpose, i.e. we denote it with the same notation "$\phi_\epsilon *$".

Regularization of Radon measures and BV functions

Definition (7.20)

Let Ω be an open subset of $\mathbb{R}^m$, $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $(\phi_t)_{t>0}$ the standard mollifier on $\mathbb{R}^m$. We define the *t-approximation or t-regularization of μ* by $\mu_t := \phi_t * \mu \in \mathcal{M}_{\mathrm{loc}}(\Omega_t, \mathbb{R}^n)$.

Remark (7.21)

The definition above extends definition 24 for $\mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$. That is, considering the embedding $\mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega, \mathbb{R}^n) \subset \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ given by $f \mapsto \mathcal{L}^n|_\Omega \llcorner f$, we have

$$(\mathcal{L}^n|_\Omega \llcorner f)_\epsilon = \mathcal{L}^n|_\Omega \llcorner (f_\epsilon) \in \mathcal{M}_{\mathrm{loc}}(\Omega_\epsilon, \mathbb{R}).$$

Regularization of Radon measures and BV functions

Proposition (7.22)

With the notation from the previous definition, let $\Omega \subset \mathbb{R}^m$ open and $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Define $\mu^\epsilon : \Omega_\epsilon \to \mathbb{R}^n$ by

$$\mu^\epsilon(x) := \int_\Omega \phi_\epsilon(x - y)\, \mathrm{d}\mu(y).$$

Then $\mu^\epsilon \in C^\infty(\Omega_\epsilon, \mathbb{R}^n)$ and

$$\mu_\epsilon = \mathcal{L}^n|_{\Omega_\epsilon} \llcorner \mu^\epsilon.$$

In particular, $\mu_\epsilon \ll \mathcal{L}^n|_{\Omega_\epsilon}$ and $|\mu_\epsilon| \leq |\mu|_\epsilon$.

Regularization of Radon measures and BV functions

Theorem (Weak-star convergence of regularized Radon measures; 7.23)

Let Ω be an open subset of $\mathbb{R}^m$ and $\mu \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^n)$. Then, as $\epsilon \downarrow 0$,

$$\mu_\epsilon \overset{*}{\rightharpoonup} \mu \text{ and } |\mu_\epsilon| \overset{*}{\rightharpoonup} |\mu|,$$

in the sense that, for all $\varphi \in \mathrm{C_c}(\Omega, \mathbb{R}^n)$, $\mu_\epsilon \cdot \varphi \to \mu \cdot \varphi$ and similarly for the total variations. Moreover, for all $\epsilon > 0$ and $E \in \mathscr{B}_{\Omega_\epsilon}$,

$$|\mu_\epsilon|(E) \leq |\mu|(E_\epsilon),$$

where $E_\epsilon := E + \mathbb{U}(0, \epsilon)$ is the ϵ-neighborhood of E.

Regularization of Radon measures and BV functions

Proposition (regularization of BV functions; 7.24)

Let Ω *be an open subset of* $\mathbb{R}^n$, $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, $(\phi_\epsilon)_{\epsilon>0}$ *the standard mollifier on* $\mathbb{R}^n$, $f_\epsilon := \phi_\epsilon * f \in \mathrm{C}^\infty(\Omega_\epsilon)$ *and* $(\nabla^{\mathrm{w}} f)_\epsilon := \phi_\epsilon * \nabla^{\mathrm{w}} f \in \mathcal{M}_{\mathrm{loc}}(\Omega_\epsilon, \mathbb{R}^n)$. *Then:*

i) $(\nabla^{\mathrm{w}} f)_\epsilon = \mathcal{L}^n|_{\Omega_\epsilon} \llcorner \nabla(f_\epsilon)$.

ii) $f_\epsilon \to f$ *in the sense of* $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$.

iii) *For each open* $V \Subset \Omega$,

$$\big(\mathcal{L}^n|_{\Omega_\epsilon} \llcorner \nabla(f_\epsilon)\big)|_V \overset{*\mathrm{f}}{\rightharpoonup} (\nabla^{\mathrm{w}} f)|_V \text{ and } \big(\mathcal{L}^n|_{\Omega_\epsilon} \llcorner \|\nabla(f_\epsilon)\|\big)|_V \overset{*\mathrm{f}}{\rightharpoonup} |\nabla^{\mathrm{w}} f|\big|_V$$

as $\epsilon \downarrow 0$.

Proposition (7.25)

Let Ω be an open subset of $\mathbb{R}^n$ and $(f_k)_{k\in\mathbb{N}}$ a sequence in $\mathrm{BV}_{\mathrm{loc}}(\Omega)$.

i) *If $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $f_k \to f$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$, then $\nabla^{\mathrm{w}} f_k \overset{*}{\rightharpoonup} \nabla^{\mathrm{w}} f$.*

ii) *If $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$, $f_k \to f$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$ and there exists $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ such that $\nabla^{\mathrm{w}} f_k \overset{*}{\rightharpoonup} \mu$, then $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $\nabla^{\mathrm{w}} f = \mu$.*

Proposition (Product rule for BV; 7.27)

Let Ω be an open subset of $\mathbb{R}^n$, $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $g : \Omega \to \mathbb{R}$ locally Lipschitz. Then $fg \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $\nabla^{\mathrm{w}}(fg) = \nabla^{\mathrm{w}} f \llcorner g + \mathcal{L}^n \llcorner f \, \nabla^{\mathrm{w}} g$.

Variation of a function in L^1_{loc}

Definition (7.28)

Let $\Omega \subset \mathbb{R}^n$ open and $f \in L^1_{loc}(\Omega)$. We define, for each open $V \subset \Omega$,

$$\mathrm{Var}(f, V) := \sup\{\int f \operatorname{div} \varphi \, d\mathcal{L}^n \mid \varphi \in C^\infty_c(V, \mathbb{R}^n), \|\varphi\|_u \leq 1\}.$$

Exercise (variation of a function in L^1_{loc}; 7.29)

Let $\Omega \subset \mathbb{R}^n$ open and $f \in L^1_{loc}(\Omega)$. Define, for each $B \subset \Omega$,

$$\mathrm{Var}_f(B) := \inf\{\mathrm{Var}(f, U) \mid U \text{ open}, B \subset U\}.$$

Then Var_f is a Borel regular measure on U which extends the variation $\mathrm{Var}(f, \cdot)$. Moreover, $f \in BV_{loc}(\Omega)$ if, and only if, Var_f is a positive Radon measure on Ω, in which case it coincides with $|\nabla^w f|$.
We call Var_f the variation measure of f.

Variation of a function in L^1_{loc}

Definition (7.28)

Let $\Omega \subset \mathbb{R}^n$ open and $f \in L^1_{loc}(\Omega)$. We define, for each open $V \subset \Omega$,

$$\operatorname{Var}(f, V) := \sup\{\int f \operatorname{div} \varphi \, \mathrm{d}\mathcal{L}^n \mid \varphi \in C^\infty_c(V, \mathbb{R}^n), \|\varphi\|_u \leq 1\}.$$

Exercise (variation of a function in L^1_{loc}; 7.29)

Let $\Omega \subset \mathbb{R}^n$ open and $f \in L^1_{loc}(\Omega)$. Define, for each $B \subset \Omega$,

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Then Var_f is a Borel regular measure on U which extends the variation $\operatorname{Var}(f, \cdot)$. Moreover, $f \in BV_{loc}(\Omega)$ if, and only if, Var_f is a positive Radon measure on Ω, in which case it coincides with $|\nabla^w f|$.
We call Var_f the variation measure of f.

Lower semicontinuity of the variation

Proposition (7.30)

Let $\Omega \subset \mathbb{R}^n$ open, $(f_i)_{i\in\mathbb{N}}$ a sequence in $\mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$ and $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$ such that $f_i \to f$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathcal{L}^n|_\Omega)$. Then, for all $V \subset \Omega$ open,

$$\mathrm{Var}(f, V) \leq \liminf \mathrm{Var}(f_i, V).$$

In particular, if $f_i \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ for all $i \in \mathbb{N}$ and the second member of the equality above is finite for each open $V \Subset \Omega$, then $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$.

Approximation of Sobolev functions by smooth functions

Theorem (Meyers-Serrin's theorem;6.24)

Let $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$ *open and* $u \in \mathrm{W}^{1,p}(\Omega)$. *There exists a sequence* $(u_k)_{k \in \mathbb{N}}$ *in* $\mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{1,p}(\Omega)$ *such that* $u_k \to u$ *in* $\mathrm{W}^{1,p}(\Omega)$.

Approximation of Sobolev functions by smooth functions

Theorem (6.34)

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. If $1 \leq p < \infty$ and $f \in \mathrm{W}^{1,p}(\Omega)$, there exists $(f_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,p}(\Omega) \cap \mathrm{C}^\infty(\overline{\Omega})$ such that $f_k \to f$ in $\mathrm{W}^{1,p}(\Omega)$. Moreover, if $f \in \mathrm{W}^{1,p}(\Omega) \cap \mathrm{C}(\overline{\Omega})$, the sequence $(f_k)_{k\in\mathbb{N}}$ may be chosen so that it also converges to f uniformly on $\overline{\Omega}$.

Corollary (6.43)

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. If $1 \leq p < \infty$ and $f \in \mathrm{W}^{1,p}(\Omega)$, there exists $(f_k)_{k\in\mathbb{N}}$ in $\mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n)$ such that $f_k|_\Omega \to f$ in $\mathrm{W}^{1,p}(\Omega)$. Moreover, if $f \in \mathrm{W}^{1,p}(\Omega) \cap \mathrm{C}(\overline{\Omega})$, the sequence $(f_k)_{k\in\mathbb{N}}$ may be chosen so that it also converges to f uniformly on compact subsets of $\overline{\Omega}$.

Approximation of BV functions by smooth functions

Theorem (7.33; Almgren)

Let Ω be an open subset of $\mathbb{R}^n$ and $f \in \mathrm{BV}(\Omega)$. There exists a sequence $(f_i)_{i\in\mathbb{N}} \in \mathrm{BV}(\Omega) \cap \mathrm{C}^\infty(\Omega)$ such that $f_i \to f$ in $\mathrm{L}^1(\mathcal{L}^n|_\Omega)$ and $|\nabla^{\mathrm{w}} f_i|(\Omega) \to |\nabla^{\mathrm{w}} f|(\Omega)$.

Corollary (7.35)

Let $\Omega = \mathbb{R}^n$ or Ω be a Lipschitz domain in $\mathbb{R}^n$, and $f \in \mathrm{BV}(\Omega)$. There exists a sequence $(f_i)_{i\in\mathbb{N}} \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n)$ such that $f_i|_\Omega \to f$ in $\mathrm{L}^1(\mathcal{L}^n|_\Omega)$ and $|\nabla^{\mathrm{w}} f_i|(\Omega) \to |\nabla^{\mathrm{w}} f|(\Omega)$.

Approximation of BV functions by smooth functions

Remark (7.34)

With the same hypothesis from theorem 38, if $f \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\mathcal{L}^n|_\Omega)$, there exists a sequence $(f_i)_{i \in \mathbb{N}} \in \mathrm{BV}(\Omega) \cap \mathrm{C}^\infty(\Omega)$ such that $f_i \to f$ in $\mathrm{L}^1(\mathcal{L}^n|_\Omega)$, $|\nabla^{\mathrm{w}} f_i|(\Omega) \to |\nabla^{\mathrm{w}} f|(\Omega)$ and, for all $i \in \mathbb{N}$, $\|f_i\|_{\mathrm{L}^\infty(\mathcal{L}^n|_\Omega)} \leq 3\|f\|_{\mathrm{L}^\infty(\mathcal{L}^n|_\Omega)}$.

Product rule for BV, part II

Proposition (Product rule for BV, part II; 7.36)

Let Ω be an open subset of $\mathbb{R}^n$. If $f, g \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\mathcal{L}^n|_\Omega)$, then $fg \in \mathrm{BV}(\Omega)$.

Trace theorem for Sobolev functions on Lipschitz domains

Theorem (6.48, 6.51)

Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ a Lipschitz epigraph or a Lipschitz domain with $\partial\Omega$ bounded, and $1 \leq p < \infty$. Then there exists a unique bounded linear operator $T : \mathrm{W}^{1,\mathrm{p}}(\Omega) \to \mathrm{L}^{\mathrm{p}}(\mathcal{H}^{n-1}|_{\partial\Omega})$ such that, for all $f \in \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^n)$, $T \cdot (f|_\Omega) = f|_{\partial\Omega}$.

Moreover, the Gauss-Green formula holds for all $f \in \mathrm{W}^{1,1}(\Omega)$, i.e. denoting by ν the unit outer normal to $\partial\Omega$,

$$\int_\Omega \nabla^{\mathrm{w}} f \, \mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} T \cdot f \, \nu \, \mathrm{d}\mathcal{H}^{n-1},$$

with a similar equality in divergence form.

Trace theorem for Sobolev functions on Lipschitz domains

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Moreover, the Gauss-Green formula holds for all $f \in \mathrm{W}^{1,1}(\Omega)$, i.e. denoting by ν the unit outer normal to $\partial\Omega$,

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with a similar equality in divergence form.

Trace theorem for BV functions on Lipschitz domains

Theorem (7.36, 7.40)

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ a Lipschitz epigraph or a Lipschitz domain with $\partial\Omega$ bounded. Then there exists a unique bounded linear operator $T : \mathrm{BV}(\Omega) \to \mathrm{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$ such that, for all $f \in \mathrm{BV}(\Omega)$ and all $\varphi \in \mathrm{C}^1_c(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int_\Omega f \operatorname{div} \varphi \, \mathrm{d}\mathcal{L}^n = -\int_\Omega \varphi \cdot \mathrm{d}\,\nabla^{\mathrm{w}} f + \int_{\partial\Omega} Tf \, \varphi \cdot \nu \, \mathrm{d}\mathcal{H}^{n-1}, \tag{7}$$

where ν the unit outer normal to $\partial\Omega$.

Moreover, for all $f \in \mathrm{BV}(\Omega)$ and for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$,

$$\lim_{r \to 0} \fint_{\mathbb{B}(x,r)\cap\Omega} \left| f(y) - Tf(x) \right| \mathrm{d}\mathcal{L}^n(y) = 0. \tag{8}$$

Trace theorem for BV functions on Lipschitz domains

Theorem (7.36, 7.40)

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ a Lipschitz epigraph or a Lipschitz domain with $\partial\Omega$ bounded. Then there exists a unique bounded linear operator $T : \mathrm{BV}(\Omega) \to \mathrm{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$ such that, for all $f \in \mathrm{BV}(\Omega)$ and all $\varphi \in \mathrm{C}^1_\mathrm{c}(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int_\Omega f \operatorname{div} \varphi \, \mathrm{d}\mathcal{L}^n = -\int_\Omega \varphi \cdot \, \mathrm{d}\,\nabla^{\mathrm{w}} f + \int_{\partial\Omega} Tf \, \varphi \cdot \nu \, \mathrm{d}\mathcal{H}^{n-1}, \tag{7}$$

where ν the unit outer normal to $\partial\Omega$.
Moreover, for all $f \in \mathrm{BV}(\Omega)$ and for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$,

$$\lim_{r\to 0} \fint_{\mathbb{B}(x,r)\cap\Omega} \big|f(y) - Tf(x)\big| \, \mathrm{d}\mathcal{L}^n(y) = 0. \tag{8}$$

Trace theorem for BV functions on Lipschitz domains

Corollary (7.41)

With the same hypothesis of the previous theorem, if $f \in \mathrm{BV}(\Omega) \cap \mathrm{C}(\overline{\Omega})$, *then* $Tf = f|_{\partial\Omega}$.

Extension of BV functions on Lipschitz domains

Theorem (7.43)

Let $n \geq 2$ and Ω an open subset of $\mathbb{R}^n$ which is a Lipschitz epigraph or a Lipschitz domain with $\partial\Omega$ bounded. Given $f \in \mathrm{BV}(\Omega)$ and $g \in \mathrm{BV}(\mathbb{R}^n \setminus \overline{\Omega})$, let F be $\mathcal{L}^n$-measurable function defined by $F|_\Omega = f$ and $F|_{\overline{\Omega}^c} = g$.
Then $F \in \mathrm{BV}(\mathbb{R}^n)$ and

$$\nabla^{\mathrm{w}} F = i_\# \nabla^{\mathrm{w}} f + i_\# \nabla^{\mathrm{w}} g - \mathcal{H}^{n-1} \llcorner \partial\Omega \llcorner (Tf - Tg)\nu, \tag{9}$$

where $i_\# \nabla^{\mathrm{w}} f$ and $i_\# \nabla^{\mathrm{w}} g$ are the pushforwards of $\nabla^{\mathrm{w}} f \in \mathcal{M}(\Omega, \mathbb{R}^n)$ and $\nabla^{\mathrm{w}} g \in \mathcal{M}(\overline{\Omega}^c, \mathbb{R}^n)$ by the respective inclusions, ν is the unit outer normal of Ω and T denotes both trace operators $\mathrm{BV}(\Omega), \mathrm{BV}(\overline{\Omega}^c) \to \mathrm{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$.

Extension of BV functions on Lipschitz domains

Corollary (7.44)

Let $n \geq 2$ and Ω an open subset of $\mathbb{R}^n$ which is a Lipschitz epigraph or a Lipschitz domain with $\partial\Omega$ bounded. The extension by 0 *defines a bounded linear operator* $\mathrm{BV}(\Omega) \to \mathrm{BV}(\mathbb{R}^n)$.

Rellich-Kondrachov

Theorem (6.77)

Let Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$, $1 \leq p < n$ *and* $1 \leq q < p^*$, *where* p^* *is the Sobolev conjugate of* p. *Then*

$$\mathrm{W}^{1,\mathrm{p}}(\Omega) \Subset \mathrm{L}^{\mathrm{q}}(\mathcal{L}^n|_\Omega),$$

i.e. $\mathrm{W}^{1,\mathrm{p}}(\Omega) \subset \mathrm{L}^{\mathrm{q}}(\mathcal{L}^n|_\Omega)$ *with compact inclusion.*

Compactness theorem for BV

Theorem (7.46)

Let $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain and* $(f_i)_{i\in\mathbb{N}}$ *a sequence in* $\mathrm{BV}(\Omega)$ *such that*

$$\sup\{\|f_i\|_{\mathrm{BV}(\Omega)} \mid i \in \mathbb{N}\} < \infty.$$

Then there exists $f \in \mathrm{BV}(\Omega)$ *and a subsequence* $(f_{i_j})_{j\in\mathbb{N}}$ *of* $(f_i)_i$ *such that* $f_{i_j} \to f$ *in* $\mathrm{L}^1(\mathcal{L}^n|\Omega)$*.*

Support of the Gauss-Green measure

Proposition (7.50)

If $E \subset \Omega$ is a set of locally finite perimeter in the open subset Ω of $\mathbb{R}^n$, then

$$\operatorname{spt} \mu_E = \{x \in \Omega \mid \forall r > 0, 0 < |E \cap \mathbb{U}(x,r)| < \alpha(n) r^n\} \subset \partial^\Omega E.$$

Moreover, there exists a Borel set $F \subset \Omega$ in the same $\mathrm{L}^1_{\mathrm{loc}}$ class of E such that $\mu_F = \partial^\Omega F$.

Operations with Sets of Finite Perimeter, part I

Proposition (7.51)

Let Ω be an open subset of $\mathbb{R}^n$. If E, F are sets of (locally) finite perimeter in Ω, then so are $E \cup F$ and $E \cap F$. Moreover,

$$|\mu_{E\cup F}| + |\mu_{E\cap F}| \leq |\mu_E| + |\mu_F|.$$

Compactness from perimeter bounds

Definition (7.52)

Let $(E_i)_{i\in\mathbb{N}}$ be a sequence of Lebesgue measurable sets in $\mathbb{R}^n$ and E a Lebesgue measurable set in $\mathbb{R}^n$. We say that

$$E_i \rightharpoonup E$$

if $\|\chi_{E_i} - \chi_E\|_{\mathrm{L}^1(\mathcal{L}^n)} = |E_i \triangle E| \to 0$.
We say that $E_i \overset{\text{loc}}{\rightharpoonup} E$ if $\chi_{E_i} \to \chi_E$ in $\mathrm{L}^1_{\text{loc}}(\mathcal{L}^n)$.

Compactness from perimeter bounds

Theorem (7.53)

Let $R > 0$ *and* $(E_i)_{i\in\mathbb{N}}$ *be a sequence of sets of finite perimeter in* $\mathbb{R}^n$ *such that*

$$\sup_{i\in\mathbb{N}} \mathrm{P}(E_i) < \infty,$$

$$E_i \subset \mathbb{U}(0, R) \quad \forall i \in \mathbb{N}.$$

Then there exists a set $E \subset \mathbb{U}(0, R)$ *of finite perimeter in* $\mathbb{R}^n$ *and a subsequence* $(E_{i_j})_{j\in\mathbb{N}}$ *of* $(E_i)_{i\in\mathbb{N}}$ *such that*

$$E_{i_j} \rightharpoonup E \quad \text{and} \quad \mu_{E_{i_j}} \xrightarrow{*} \mu_E.$$

Compactness from perimeter bounds

Corollary (7.55)

Let $(E_i)_{i\in\mathbb{N}}$ be a sequence of sets of locally finite perimeter in $\mathbb{R}^n$ such that, for all $R > 0$,

$$\sup_{i\in\mathbb{N}} \mathrm{P}\big(E_i, \mathbb{U}(0, R)\big) < \infty.$$

Then there exists a set E of locally finite perimeter in $\mathbb{R}^n$ and a subsequence $(E_{i_j})_{j\in\mathbb{N}}$ of $(E_i)_{i\in\mathbb{N}}$ such that

$$E_{i_j} \xrightarrow{loc} E \quad \text{and} \quad \mu_{E_{i_j}} \xrightarrow{*} \mu_E.$$

Compactness from perimeter bounds

Lemma (7.54)

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $E \subset \mathbb{R}^n$ be a set of locally finite perimeter. Then $E \cap \Omega$ is a set of finite perimeter in $\mathbb{R}^n$ and

$$\mathrm{P}(E \cap \Omega) \leq \mathrm{P}(E, \Omega) + \mathrm{P}(\Omega).$$

Plateau problem in K with boundary data M

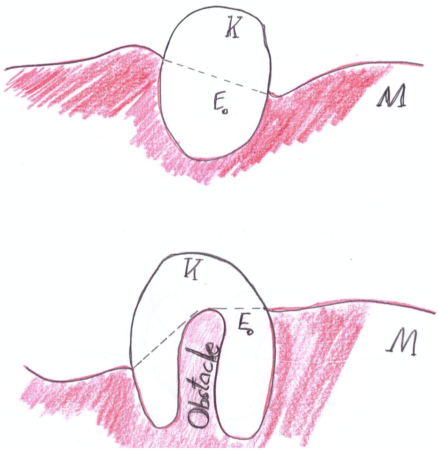

Figure: Plateau problem in K with boundary data M

Plateau problem in K with boundary data M

Proposition (Minimizers for the Plateau problem in K with boundary data M)

Let $K \subset \mathbb{R}^n$ be a compact set and M be a set of locally finite perimeter in $\mathbb{R}^n$. Then there exists $E_0 \subset \mathbb{R}^n$ of locally finite perimeter which minimizes the functional

$$E \mapsto \mathrm{P}(E, K)$$

in the class $\mathcal{E} := \{E \subset \mathbb{R}^n \mid \chi_E \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ and $E \setminus K = M \setminus K\}$.

Relative isoperimetric problem in Ω

Definition

Let $\Omega \subset \mathbb{R}^n$ open, $m \in (0, |\Omega|)$ and

$$\alpha(m, \Omega) := \inf\{\mathrm{P}(E, \Omega) \mid E \subset \Omega, \chi_E \in \mathrm{BV}(\Omega), |E| = m\}.$$

We say that a set $E \subset \Omega$ of finite perimeter in Ω is a *relative isoperimetric set in* Ω if spt $\mu_E = \partial^\Omega E$ and $\mathrm{P}(E, \Omega) = \alpha(|E|, \Omega)$.

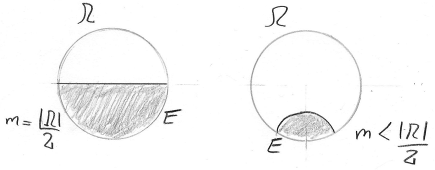

Figure: Relative isoperimetric problem in Ω

Existence of relative isoperimetric sets on bounded Lipschitz domains

Proposition (Existence of relative isoperimetric sets on bounded Lipschitz domains)

Let Ω be a bounded Lipschitz domain and $m \in (0, |\Omega|]$. Then there exists a set $E \subset \Omega$ such that $\chi_E \in \mathrm{BV}(\Omega)$, $|E| = m$ and $\mathrm{P}(E, \Omega) = \alpha(m, \Omega)$.