Geometric Measure Theory

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General hypothesis

We fix a locally compact Hausdorff space X, which will be assumed σ-compact, unless otherwise specified.

Notation

We denote by

- $C_c(X, \mathbb{R}^n)$ the space of continuous functions $f : X \to \mathbb{R}^n$ with spt f compact;
- $C_0(X, \mathbb{R}^n)$ the space of continuous functions $f : X \to \mathbb{R}^n$ which vanish at infinity, i.e. such that $\forall \epsilon > 0$, $\exists K \subset X$ compact such that $\|f\| < \epsilon$ on $X \setminus K$.
- $C_b(X, \mathbb{R}^n)$ the space of bounded continuous functions $f : X \to \mathbb{R}^n$.

$\mathbb{R}^n$-valued Radon measures

Definition (4.1)

We say that a linear functional $\mu : \mathrm{C_c}(X, \mathbb{R}^n) \to \mathbb{R}$ is an $\mathbb{R}^n$*-valued Radon measure* on X if, for each compact $K \subset X$, the restriction of μ to $\mathrm{C_c^K}(X, \mathbb{R}^n) := \{f \in \mathrm{C_c}(X, \mathbb{R}^n) \mid \operatorname{spt} f \subset K\}$, endowed with $\|\cdot\|_u$, is linear continuous; that is, if $\exists C_K \geq 0$ such that

$$\sup\{\mu \cdot f \mid f \in \mathrm{C_c^K}(X, \mathbb{R}^n), \|f\|_u \leq 1\} \leq C_K. \tag{LF cont}$$

If the condition above holds with a constant $C \geq 0$ which does not depend on K, i.e. if μ is linear continuous on $\mathrm{C_c}(X, \mathbb{R}^n)$ endowed with $\|\cdot\|_u$, we call μ a *finite* $\mathbb{R}^n$*-valued Radon measure* on X.

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Remark (4.2)

1. The definition adopted for an $\mathbb{R}^n$-valued Radon measure on X is equivalent to saying that $\mu : \mathrm{C_c}(X, \mathbb{R}^n) \to \mathbb{R}$ is linear continuous with respect to the natural topological vector space topology on $\mathrm{C_c}(X, \mathbb{R}^n)$, which is an inductive limit of Fréchet spaces (an *LF space* for short).
2. For those fluent in locally convex spaces: if X is an open set in some Euclidean space, $\mathrm{C_c^\infty}(X, \mathbb{R})^n$ has a continuous dense inclusion in $\mathrm{C_c}(X, \mathbb{R}^n) \equiv \mathrm{C_c}(X, \mathbb{R})^n$. That means that the dual of $\mathrm{C_c}(X, \mathbb{R})^n$ may be identified with a linear subspace of the dual of $\mathrm{C_c^\infty}(X, \mathbb{R})^n$, i.e. every $\mathbb{R}^n$-valued Radon measure on X is an $\mathbb{R}^n$-valued Schwartz distribution on X.

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$\mathbb{R}^n$-valued Radon measures on open sets of Euclidean spaces

Exercise (4.3)

Let X be an open subset of $\mathbb{R}^m$ and $(U_k)_{k\in\mathbb{N}}$ be an increasing sequence of relatively compact open subsets of X such that $\cup_{k\in\mathbb{N}} U_k = X$. Let $\mu : \mathrm{C}_\mathrm{c}^\infty(X, \mathbb{R}^n) \to \mathbb{R}$ be a linear map such that $\forall k \in \mathbb{N}$, $\mu|_{\left(\mathrm{C}_\mathrm{c}^\infty(U_k,\mathbb{R}^n),\|\cdot\|_u\right)}$ is continuous. Then μ may be uniquely extended to a continuous linear map $\mathrm{C}_\mathrm{c}(X, \mathbb{R}^n) \to \mathbb{R}$.

Recall

Notation

Let X be a locally compact Hausdorff space, $U \subset X$ open and f a function on X.
$f \prec U$ means that $0 \leq f \leq 1$, $f \in \mathrm{C_c}(X, \mathbb{R})$ and $\operatorname{spt} f \subset U$.

Lemma (Urysohn's lemma for LCH; 4.5)

If X is a locally compact Hausdorff space, $U \subset X$ open and $K \subset U$ compact, then there exists $f \in \mathrm{C_c}(X, \mathbb{R})$ such that $\chi_K \leq f \prec U$.

Theorem (Tietze's extension theorem for LCH; 4.6)

If X is a locally compact space, $K \subset X$ compact and $f : K \to \mathbb{R}$ continuous, then f admits a continuous extension $\tilde{f} : X \to \mathbb{R}$. Moreover, we may take $\tilde{f}$ with compact support and, if f is bounded, we may also take $\tilde{f}$ such that $\|\tilde{f}\|_u = \|f\|_u$.

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Recall

Theorem (Riesz representation theorem for positive linear functionals; 4.7)

Let X be a locally compact Hausdorff space and $L : \mathrm{C_c}(X, \mathbb{R}) \to \mathbb{R}$ a positive linear functional, i.e. L is linear and $L \cdot f \geq 0$ whenever $f \geq 0$. Then there exists a unique Radon measure η on X which represents L, i.e. $\forall f \in \mathrm{C_c}(X, \mathbb{R})$, $L \cdot f = \int f \, \mathrm{d}\eta$. Moreover, on open sets η is given by

$$\eta(U) = \sup\{L \cdot f \mid f \prec U\}.$$

Remark (4.8)

Every positive linear functional on $\mathrm{C_c}(X,\mathbb{R})$ is an $\mathbb{R}$-valued Radon measure on X, i.e. positivity implies continuity on $\mathrm{C_c}(X,\mathbb{R})$.

Proof.

Given $K \subset X$ compact, take $\Phi \in \mathrm{C_c}(X,\mathbb{R})$ given by lemma 3 such that $\chi_K \leq \Phi \prec X$. For all $f \in \mathrm{C_c^K}(X,\mathbb{R})$ with $f \neq 0$, we have $\frac{|f|}{\|f\|_u} \leq \Phi$, so that $\Phi \pm \frac{f}{\|f\|_u} \geq 0$ and $\Phi \pm \frac{f}{\|f\|_u} \in \mathrm{C_c}(X,\mathbb{R})$. Hence $0 \leq L(\Phi \pm \frac{f}{\|f\|_u}) = L(\Phi) \pm \frac{L(f)}{\|f\|_u}$, which implies $|L(f)| \leq L(\Phi)\|f\|_u$. The continuity condition (LF cont) is then satisfied with $C_K := L(\Phi)$. □

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Riesz representation theorem for Radon measures

Theorem (4.9)

Let X be a σ-compact locally compact Hausdorff space and $\mu : \mathrm{C_c}(X, \mathbb{R}^n) \to \mathbb{R}$ an $\mathbb{R}^n$-valued Radon measure on X. Then there exists a unique Radon measure λ on X and a Borel measurable map $\nu : X \to \mathbb{R}^n$ unique up to λ-null sets such that $\|\nu\| = 1$ λ-a.e. on X and $\forall f \in \mathrm{C_c}(X, \mathbb{R}^n)$,

$$\mu \cdot f = \int \langle f, \nu \rangle \, \mathrm{d}\lambda, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$. Moreover,

i) *$\forall U \subset X$ open,*

$$\lambda(U) = \sup\{\mu \cdot f \mid f \in \mathrm{C_c}(X, \mathbb{R}^n), \|f\| \prec U\}. \tag{2}$$

ii) *μ is a finite $\mathbb{R}^n$-valued Radon measure* iff *λ is a finite Radon measure; if that is the case, $\|\mu\|_{\mathrm{C_0}(X,\mathbb{R}^n)^*} = \lambda(X)$.*

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Riesz representation theorem for Radon measures

Remark (4.10)

Note that, in (2), $\sup\{\mu \cdot f \mid f \in \mathrm{C_c}(X, \mathbb{R}^n), \|f\| \prec U\} = \sup\{|\mu \cdot f| \mid f \in \mathrm{C_c}(X, \mathbb{R}^n), \|f\| \prec U\}$. Indeed, if $f \in \mathrm{C_c}(X, \mathbb{R}^n)$ and $\|f\| \prec U$, so does $-f$, and $\mu \cdot (-f) = -\mu \cdot f$, hence either $\mu \cdot f$ or $\mu \cdot (-f)$ coincides with $|\mu \cdot f|$.

Lemma (4.11)

Let X be a locally compact Hausdorff space, $f : X \to [0, \infty)$ bounded Borelian and μ a σ-finite Radon measure on $\mathscr{B}_X$. Then $\lambda := f\mu : \mathscr{B}_X \to [0, \infty]$ given by $A \mapsto \int_A f \, \mathrm{d}\mu$ is a Radon measure on $\mathscr{B}_X$.

Total variation and polar decomposition

Definition (4.13)

Let μ be an $\mathbb{R}^n$-valued Radon measure on a σ-compact locally compact Hausdorff space X. With the same notation of theorem 7, λ is called the *total variation of* μ, and the pair (ν, λ) is called the *polar decomposition of* μ. Henceforth, we will use the notation $|\mu| := \lambda$ to denote the total variation of μ, and

$$\mu = \nu|\mu|$$

with the meaning that $(\nu, |\mu|)$ is the polar decomposition of μ.

Total variation and polar decomposition

Example (4.14)

1. Let μ be a locally finite Borel measure on X. Then μ induces a positive linear functional $\hat{\mu}$ on $C_c(X, \mathbb{R})$, given by $\hat{\mu} \cdot f := \int f \, d\mu$. If μ is a Radon measure, then $\hat{\mu} = 1 \cdot \mu$ is the polar decomposition of $\hat{\mu}$.
2. Similarly, let ν be a signed measure on $\mathscr{B}_X$ whose total variation $|\nu|$ is locally finite. Then ν induces a continuous linear functional $\hat{\nu}$ on $C_c(X, \mathbb{R})$ given by $\hat{\nu} \cdot f := \int f \, d\nu$.
3. Let $X = \mathbb{R}$ and I be the positive linear functional defined on $C_c(X, \mathbb{R})$ by the Riemann integral, i.e. $I \cdot f := \int_a^b f(x) \, dx$ for $a < b$ such that spt $f \subset [a, b]$. The polar decomposition of I is $I = 1 \cdot \mathcal{L}^n$. In particular, that could have been taken as the definition of the Lebesgue measure, i.e. it is the total variation of the positive linear functional induced by the Riemann integral.

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Properties of the total variation, part I

Proposition (4.15)

Let μ and ν be $\mathbb{R}^n$-valued Radon measures on a σ-compact locally compact Hausdorff space X and $c \in \mathbb{R}$. Then:

i) $|\mu + \nu| \leq |\mu| + |\nu|$, *with equality if* $|\mu| \perp |\nu|$.

ii) $|c\mu| = |c||\mu|$.

Integration with respect to $\mathbb{R}^n$-valued Radon measures

Definition (4.16)

Let μ be an $\mathbb{R}^n$-valued Radon measure on a σ-compact locally compact Hausdorff space X, with polar decomposition $\mu = \nu|\mu|$.

1. A vector Borelian map $f : X \to \mathbb{R}^n$ is called *summable with respect to* μ if $f \in \mathrm{L}^1(|\mu|, \mathbb{R}^n) \equiv \mathrm{L}^1(|\mu|)^n$. For such f, we define

$$\int f \cdot \mathrm{d}\mu := \int \langle f, \nu \rangle \, \mathrm{d}|\mu| \in \mathbb{R}.$$

2. An scalar Borelian map $f : X \to \mathbb{R}$ is called *summable with respect to* μ if $f \in \mathrm{L}^1(|\mu|)$. For such f, we define

$$\int f \, \mathrm{d}\mu := \int f\nu \, \mathrm{d}|\mu| = \left(\int f\nu_1 \, \mathrm{d}|\mu|, \ldots, \int f\nu_n \, \mathrm{d}|\mu|\right) \in \mathbb{R}^n.$$

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Integration with respect to $\mathbb{R}^n$-valued Radon measures

Remark (4.17)

1. Note that $\mathrm{C_c}(X, \mathbb{R}^n) \subset \mathrm{L}^1(|\mu|, \mathbb{R}^n)$ and the integral defined above extends $\mu : \mathrm{C_c}(X, \mathbb{R}^n) \to \mathbb{R}$, i.e. $\forall f \in \mathrm{C_c}(X, \mathbb{R}^n)$,

$$\int f \cdot \mathrm{d}\mu = \mu \cdot f.$$

2. The integrals defined above satisfy the usual linearity and convergence properties and the following versions of the triangle inequality:

$$|\int f \cdot \mathrm{d}\mu| \leq \int \|f\| \,\mathrm{d}|\mu| \quad \text{and} \quad \|\int f \,\mathrm{d}\mu\| \leq \int |f| \,\mathrm{d}|\mu|,$$

for $f \in \mathrm{L}^1(|\mu|, \mathbb{R}^n)$ or $f \in \mathrm{L}^1(|\mu|)$, respectively.

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$\mathbb{R}^n$-valued measure on a σ-algebra

Definition (4.18)

Let X be a set and $\mathcal{M}$ a σ-algebra of subsets of X. We say that a map $\mu : \mathcal{M} \to \mathbb{R}^n$ is an $\mathbb{R}^n$-valued measure on $\mathcal{M}$ if

VM1) $\mu(\emptyset) = 0$;

VM2) μ is σ-additive, i.e. for all countable disjoint family $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$,

$$\mu(\cup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

Notation

We denote by $\mathscr{B}_X^c$ the set of Borel subsets of X which are relatively compact.

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$\mathbb{R}^n$-valued Radon measures as set functions

Definition (4.19)

1. a *finite $\mathbb{R}^n$-valued Radon measure set function* on X is an $\mathbb{R}^n$-valued measure on $\mathscr{B}_X$.
2. an *$\mathbb{R}^n$-valued Radon measure set function* on X is a set function $\mu : \mathscr{B}_X^c \to \mathbb{R}^n$ such that, for all $K \subset X$ compact, $\mu|_{\mathscr{B}_K} : \mathscr{B}_K \to \mathbb{R}^n$ is an $\mathbb{R}^n$-valued measure on $\mathscr{B}_K$.

Notation

- $\mathcal{M}(X)^n$ or $\mathcal{M}(X, \mathbb{R}^n)$ for finite $\mathbb{R}^n$-valued Radon measures on X
- $\mathcal{M}_{\text{loc}}(X)^n$ or $\mathcal{M}_{\text{loc}}(X, \mathbb{R}^n)$ for $\mathbb{R}^n$-valued Radon measures on X

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Remark (4.20)

Each $\mu \in \mathcal{M}(X, \mathbb{R}^n)$ determines an element of $\mathcal{M}_{\text{loc}}(X, \mathbb{R}^n)$ by restriction of $\mu : \mathscr{B}_X \to \mathbb{R}^n$ to $\mathscr{B}_X^c$.
Since X is σ-compact, μ is uniquely determined by its restriction to $\mathscr{B}_X^c$, i.e. the association $\mu \in \mathcal{M}(X, \mathbb{R}^n) \mapsto \mu|_{\mathscr{B}_X^c} \in \mathcal{M}_{\text{loc}}(X, \mathbb{R}^n)$ is linear 1-1 and allows us to identify $\mathcal{M}(X, \mathbb{R}^n)$ with a linear subspace of $\mathcal{M}_{\text{loc}}(X, \mathbb{R}^n)$.

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Induced $\mathbb{R}^n$-valued Radon measure set functions

Definition (4.21)

Let μ be an $\mathbb{R}^n$-valued Radon measure on a σ-compact locally compact Hausdorff space X. The $\mathbb{R}^n$-valued Radon measure set function induced by μ is the set function $\hat{\mu} : \mathscr{B}_X^c \to \mathbb{R}^n$ defined, for all $A \in \mathscr{B}_X^c$, by

$$\hat{\mu}(A) := \int \chi_A \, \mathrm{d}\mu \in \mathbb{R}^n.$$

If μ is finite, we define $\hat{\mu} : \mathscr{B}_X \to \mathbb{R}^n$ by the same formula.

Induced $\mathbb{R}^n$-valued Radon measure set functions

Proposition (4.22)

With the notation from the previous definition:

i) *$\hat{\mu}$ is a (finite) $\mathbb{R}^n$-valued Radon measure set function on X if μ is a (finite) $\mathbb{R}^n$-valued Radon measure on X.*

ii) *The maps $I : C_c(X, \mathbb{R}^n)^* \to \mathcal{M}_{loc}(X, \mathbb{R}^n)$ and $I : C_0(X, \mathbb{R}^n)^* \to \mathcal{M}(X, \mathbb{R}^n)$ defined by $\mu \mapsto \hat{\mu}$ are linear 1-1 and commute with the inclusions, i.e. the following diagram is commutative:*

$$\begin{array}{ccc} C_c(X, \mathbb{R}^n)^* & \xrightarrow{\ I\ } & \mathcal{M}_{loc}(X, \mathbb{R}^n) \\ \uparrow & & \uparrow \\ C_0(X, \mathbb{R}^n)^* & \xrightarrow{\ I\ } & \mathcal{M}(X, \mathbb{R}^n) \end{array}$$

Induced $\mathbb{R}^n$-valued Radon measure set functions

Remark

If X is a locally compact separable metric space, $I : C_c(X, \mathbb{R}^n)^* \to \mathcal{M}_{loc}(X, \mathbb{R}^n)$ and $I : C_0(X, \mathbb{R}^n)^* \to \mathcal{M}(X, \mathbb{R}^n)$ are surjective, i.e.

$$C_c(X, \mathbb{R}^n)^* \equiv \mathcal{M}_{loc}(X, \mathbb{R}^n)$$
$$C_0(X, \mathbb{R}^n)^* \equiv \mathcal{M}(X, \mathbb{R}^n)$$

Restrictions of $\mathbb{R}^n$-valued Radon measures

Definition (4.31)

Let X be a locally compact separable metric space, $\mu \in \mathrm{C_c}(X, \mathbb{R}^n)^*$ an $\mathbb{R}^n$-valued Radon measure and $g \in \mathrm{L}^1_{\mathrm{loc}}(|\mu|)$ (in particular, if $g : X \to \mathbb{R}$ a bounded Borelian function on X). We define the *restriction of* μ *to* g, denoted by $\mu \llcorner g$, as the continuous linear functional on $\mathrm{C_c}(X, \mathbb{R}^n)$ given by

$$\mu \llcorner g \cdot f := \int \langle fg, \nu \rangle \, \mathrm{d}|\mu|$$

if $(\nu, |\mu|)$ is the polar decomposition of μ.

Notation

If λ is a positive measure on X and $h \in \mathrm{L}^+(\lambda)$,

$$\lambda \llcorner h := h\lambda$$

Restrictions of $\mathbb{R}^n$-valued Radon measures

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If λ is a positive measure on X and $h \in \mathrm{L}^+(\lambda)$,

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Restrictions of $\mathbb{R}^n$-valued Radon measures

Remark (4.32)

1. The polar decomposition of $\mu \llcorner g$ is $(\frac{g\nu}{|g|}, |g||\mu|)$, where we define $\frac{g\nu}{|g|} := 0$ on the Borel set $\{g = 0\}$. In particular,

$$|\mu \llcorner g| = |\mu| \llcorner |g|.$$

2. If μ is a positive Radon measure on X (which we identify with the element of $C_c(X, \mathbb{R})^*$ whose polar decomposition is $(1, \mu)$) and $A \in \mathscr{B}_X$, then $\mu \llcorner \chi_A$ coincides with the positive Radon measure $\mu \llcorner A$. We extend this notation for an arbitrary $\mu \in C_c(X, \mathbb{R}^n)^*$, i.e. we use the notation $\mu \llcorner A$ in place of $\mu \llcorner \chi_A$. It then follows from the previous item that

$$|\mu \llcorner A| = |\mu| \llcorner A.$$

Restrictions of $\mathbb{R}^n$-valued Radon measures

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Restrictions of $\mathbb{R}^n$-valued Radon measures

Remark (4.32)

3. We may similarly define $\mu \llcorner g \in C_c(X, \mathbb{R}^n)^*$ for $\mu \in C_c(X, \mathbb{R})^*$ and $g \in L^1_{loc}(|\mu|, \mathbb{R}^n)$:

$$\mu \llcorner g : f \in C_c(X, \mathbb{R}^n) \mapsto \int \langle f, g \rangle \nu \, \mathrm{d}|\mu|$$

where $(\nu, |\mu|)$ is the polar decomposition of μ. Then $(\frac{g\nu}{\|g\|}, \|g\||\mu|)$ is the polar decomposition of $\mu \llcorner g$. In particular,

$$|\mu \llcorner g| = |\mu| \llcorner \|g\|.$$

4. As a final generalization of the restriction operation, we may define $\mu \llcorner T \in C_c(X, \mathbb{R}^m)^*$ for $\mu \in C_c(X, \mathbb{R}^n)^*$ and $T \in L^1_{loc}(|\mu|, L(\mathbb{R}^m, \mathbb{R}^n))$ by $f \in C_c(X, \mathbb{R}^m) \mapsto \int \langle T \cdot f, \nu \rangle \, \mathrm{d}|\mu|$, where $(\nu, |\mu|)$ is the polar decomposition of μ.

Restrictions of $\mathbb{R}^n$-valued Radon measures

Remark (4.32)

3. We may similarly define $\mu \llcorner g \in \mathrm{C_c}(X,\mathbb{R}^n)^*$ for $\mu \in \mathrm{C_c}(X,\mathbb{R})^*$ and $g \in \mathrm{L}^1_{\mathrm{loc}}(|\mu|,\mathbb{R}^n)$:

$$\mu \llcorner g : f \in \mathrm{C_c}(X,\mathbb{R}^n) \mapsto \int \langle f, g \rangle \nu \,\mathrm{d}|\mu|$$

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4. As a final generalization of the restriction operation, we may define $\mu \llcorner T \in \mathrm{C_c}(X,\mathbb{R}^m)^*$ for $\mu \in \mathrm{C_c}(X,\mathbb{R}^n)^*$ and $T \in \mathrm{L}^1_{\mathrm{loc}}\big(|\mu|, \mathrm{L}(\mathbb{R}^m,\mathbb{R}^n)\big)$ by $f \in \mathrm{C_c}(X,\mathbb{R}^m) \mapsto \int \langle T \cdot f, \nu \rangle \,\mathrm{d}|\mu|$, where $(\nu, |\mu|)$ is the polar decomposition of μ.

Restrictions of $\mathbb{R}^n$-valued Radon measures

Remark (4.32)

Note that, defining $T^* : X \to \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ by $x \mapsto T(x)^*$, we have, $\forall f \in \mathrm{C_c}(X, \mathbb{R}^m)$:

$$\mu \llcorner T \cdot f = \int \langle T \cdot f, \nu \rangle \, \mathrm{d}|\mu| = \int \langle f, \frac{T^* \cdot \nu}{\|T^* \cdot \nu\|} \rangle \|T^* \cdot \nu\| \, \mathrm{d}|\mu|.$$

Fundamental lemma of the Calculus of Variations

Exercise (4.34)

Let X be an open set in $\mathbb{R}^m$. If $\mu : \mathrm{C_c}(X, \mathbb{R}^n) \to \mathbb{R}$ is an $\mathbb{R}^n$-valued Radon measure on X such that $\mu \cdot f = 0$ for all $f \in \mathrm{C_c^\infty}(X, \mathbb{R}^n)$, then $\mu = 0$. In particular, if $g \in \mathrm{L^1_{loc}}(\mathcal{L}^m|_X, \mathbb{R}^n)$ and

$$\int_X \langle f, g \rangle \, \mathrm{d}\mathcal{L}^m = 0$$

for all $f \in \mathrm{C_c^\infty}(X, \mathbb{R}^n)$, then $g = 0$ $\mathcal{L}^m$-a.e. on X.

Trace of $\mathbb{R}^n$-valued Radon measures

Definition (4.35)

Let X be a locally compact separable metric space and $A \subset X$ a locally compact subspace of X (i.e the intersection of an open with a closed subset of X). If μ is an $\mathbb{R}^n$-valued Radon measure on X with polar decomposition $(\nu, |\mu|)$, we define an $\mathbb{R}^n$-valued Radon measure $\mu|_A$ on A by

$$f \in \mathrm{C_c}(A, \mathbb{R}^n) \mapsto \int \langle \widetilde{f}, \nu \rangle \, \mathrm{d}|\mu|,$$

where $\widetilde{f} : X \to \mathbb{R}^n$ is the extension of f by 0 in the complement of A.

Trace of $\mathbb{R}^n$-valued Radon measures

Proposition (4.36)

With the notation above, $\mu|_A$ is a well-defined $\mathbb{R}^n$-valued Radon measure on A and it is finite if so is μ. Moreover, the polar decomposition of $\mu|_A$ is $(\nu|_A, |\mu|\big|_A)$.

Continuity of linear maps on $C_c(X, \mathbb{R}^n)$

Definition (4.37)

Let X and Y be locally compact separable metric spaces.

i) We say that $A \subset C_c(X, \mathbb{R}^n)$ is *bounded* it there exists $K \subset X$ compact such that $A \subset C_c^K(X, \mathbb{R}^n)$ and A is bounded in the latter space (i.e. it bounded as a subset of the Banach space $C_c^K(X, \mathbb{R}^n)$).

ii) We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in $C_c(X, \mathbb{R}^n)$ converges to $x \in C_c(X, \mathbb{R}^n)$ if there exists $K \subset X$ compact such that the image of the sequence is contained in $C_c^K(X, \mathbb{R}^n)$, $x \in C_c^K(X, \mathbb{R}^n)$ and $x_n \to x$ in $C_c^K(X, \mathbb{R}^n)$.

iii) We say that a linear map $T : C_c(X, \mathbb{R}^n) \to C_c(Y, \mathbb{R}^m)$ is continuous if one of the following equivalent conditions hold:

- $T(A)$ is bounded whenever $A \subset C_c(X, \mathbb{R}^n)$ is bounded.
- $T(x_n) \to 0$ whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $C_c(X, \mathbb{R}^n)$ such that $x_n \to 0$.

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   - $T(A)$ is bounded whenever $A \subset C_c(X, \mathbb{R}^n)$ is bounded.
   - $T(x_n) \to 0$ whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $C_c(X, \mathbb{R}^n)$ such that $x_n \to 0$.

Transposition

Proposition (4.39)

Let X and Y be locally compact separable metric spaces and $T : \mathrm{C_c}(X, \mathbb{R}^n) \to \mathrm{C_c}(Y, \mathbb{R}^m)$ a linear map.

i) *If T is continuous and μ is an $\mathbb{R}^m$-valued Radon measure on Y, then $\mu \circ T$ is an $\mathbb{R}^n$-valued Radon measure on X.*

ii) *If T is continuous with respect to the $\mathrm{C_0}$ topology (i.e. the topology induced by $\|\cdot\|_u$) on both domain and codomain, and μ is a finite $\mathbb{R}^m$-valued Radon measure on Y, then $\mu \circ T$ is a finite $\mathbb{R}^n$-valued Radon measure on X.*

Transposition

Definition (4.40)

With the notation from the previous proposition, we define the *transpose* of T, $T^{\mathrm{t}} : \mathrm{C_c}(Y, \mathbb{R}^m)^* \to \mathrm{C_c}(X, \mathbb{R}^n)^*$ in case (i) or $T^{\mathrm{t}} : \mathrm{C_0}(Y, \mathbb{R}^m)^* \to \mathrm{C_0}(X, \mathbb{R}^n)^*$ in case (ii), by $T^{\mathrm{t}} \cdot \mu := \mu \circ T$.

Transposition

Example (4.41)

Let X be a locally compact separable metric space.

1) Let $T : X \to \mathrm{L}(\mathbb{R}^m, \mathbb{R}^n)$ be a continuous map. We define $\hat{T} : \mathrm{C_c}(X, \mathbb{R}^m) \to \mathrm{C_c}(X, \mathbb{R}^n)$ by $(\hat{T} \cdot f)(x) := T(x) \cdot f(x)$. Then $\hat{T}$ is linear continuous and its transpose is given by $\mu \mapsto \mu \llcorner T$.
2) Let $U \subset X$ open. The inclusion $\mathrm{C_c}(U, \mathbb{R}^n) \subset \mathrm{C_c}(X, \mathbb{R}^n)$ (which maps $f \in \mathrm{C_c}(U, \mathbb{R}^n)$ to its extension by 0 on the complement of U) is clearly continuous; its transpose coincides with $\mu \mapsto \mu|_U$.

Pushforward

Proposition

Let X and Y be locally compact separable metric spaces and $f : X \to Y$ a continuous proper map. Then both $(\circ f) : C_c(Y, \mathbb{R}^n) \to C_c(X, \mathbb{R}^n)$ and $(\circ f) : C_0(Y, \mathbb{R}^n) \to C_0(X, \mathbb{R}^n)$ given by $g \mapsto g \circ f$ are well-defined and linear continuous.

Definition

With the notation from the previous definition, the transposes $(\circ f)^t : C_c(X, \mathbb{R}^n)^* \to C_c(Y, \mathbb{R}^n)^*$ and $(\circ f)^t : C_0(X, \mathbb{R}^n)^* \to C_0(Y, \mathbb{R}^n)^*$ are called *pushforward by f* and denoted by $f_\# : \mu \mapsto f_\# \mu$.

Pushforward

Proposition

Let X and Y be locally compact separable metric spaces and $f : X \to Y$ a continuous proper map. Then both $(\circ f) : \mathrm{C_c}(Y, \mathbb{R}^n) \to \mathrm{C_c}(X, \mathbb{R}^n)$ and $(\circ f) : \mathrm{C_0}(Y, \mathbb{R}^n) \to \mathrm{C_0}(X, \mathbb{R}^n)$ given by $g \mapsto g \circ f$ are well-defined and linear continuous.

Definition

With the notation from the previous definition, the transposes $(\circ f)^{\mathrm{t}} : \mathrm{C_c}(X, \mathbb{R}^n)^* \to \mathrm{C_c}(Y, \mathbb{R}^n)^*$ and $(\circ f)^{\mathrm{t}} : \mathrm{C_0}(X, \mathbb{R}^n)^* \to \mathrm{C_0}(Y, \mathbb{R}^n)^*$ are called *pushforward by f* and denoted by $f_\# : \mu \mapsto f_\# \mu$.

Pushforward

Proposition

Let X and Y be locally compact separable metric spaces, $f : X \to Y$ a continuous proper map and $\mu \in \mathrm{C_c}(X, \mathbb{R}^n)^*$ with polar decomposition $(\nu_X, |\mu|)$. Suppose that there exists a Borelian map $\nu_Y : Y \to \mathbb{R}^n$ such that $\nu_Y \circ f = \nu_X$. Then the polar decomposition of $f_\# \mu$ is $(\nu_Y, f_\# |\mu|)$. In particular, if μ is a positive Radon measure on X, the pushforward of μ by f in the sense of definition above coincides with the pushforward in the sense of positive measures.

Weak-star convergence

Definition (4.47)

Let X be a locally compact separable metric space. We say that

i) a sequence $(\mu_k)_{k\in\mathbb{N}}$ in $\mathrm{C_c}(X,\mathbb{R}^n)^*$ is *weakly-star convergent* to $\mu \in \mathrm{C_c}(X,\mathbb{R}^n)^*$ (notation: $\mu_k \overset{*}{\rightharpoonup} \mu$) if, for all $f \in \mathrm{C_c}(X,\mathbb{R}^n)$, $\int f \cdot \mathrm{d}\mu_k \to \int f \cdot \mathrm{d}\mu$;

ii) a sequence $(\mu_k)_{k\in\mathbb{N}}$ in $\mathrm{C_0}(X,\mathbb{R}^n)^*$ is *weakly-star convergent in the sense of finite measures* to $\mu \in \mathrm{C_0}(X,\mathbb{R}^n)^*$ (notation: $\mu_k \overset{*\mathrm{f}}{\rightharpoonup} \mu$) if, for all $f \in \mathrm{C_0}(X,\mathbb{R}^n)$, $\int f \cdot \mathrm{d}\mu_k \to \int f \cdot \mathrm{d}\mu$.

Weak-star convergence

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ii) a sequence $(\mu_k)_{k\in\mathbb{N}}$ in $\mathrm{C_0}(X,\mathbb{R}^n)^*$ is *weakly-star convergent in the sense of finite measures* to $\mu \in \mathrm{C_0}(X,\mathbb{R}^n)^*$ (notation: $\mu_k \overset{*\,\mathrm{f}}{\rightharpoonup} \mu$) if, for all $f \in \mathrm{C_0}(X,\mathbb{R}^n)$, $\int f \cdot \mathrm{d}\mu_k \to \int f \cdot \mathrm{d}\mu$.

Weak-star convergence

Remark (4.48)

Both types of convergence above are actually the same notion, i.e. convergence of sequences with respect to weak star topologies: the first type in the weak-star dual of $C_c(X, \mathbb{R}^n)$ and the second in the weak-star dual of $C_0(X, \mathbb{R}^n)$.

Weak-star convergence

Proposition (relation between weak-star convergence and weak-star convergence in the sense of finite measures; 4.49)

Let X be a locally compact separable metric space, $(\mu_k)_{k\in\mathbb{N}}$ a sequence in $\mathrm{C_c}(X,\mathbb{R}^n)^*$ and $\mu \in \mathrm{C_c}(X,\mathbb{R}^n)^*$. The following conditions are equivalent:

i) $\mu_k \overset{*}{\rightharpoonup} \mu$ *and* $\sup_{k\in\mathbb{N}}|\mu_k|(X) < \infty$.

ii) $(\mu_k)_{k\in\mathbb{N}}$ *is a sequence in* $\mathrm{C_0}(X,\mathbb{R}^n)^*$, $\mu \in \mathrm{C_0}(X,\mathbb{R}^n)^*$ *and* $\mu_k \overset{*\,\mathrm{f}}{\rightharpoonup} \mu$.

Weak-star convergence

Proposition (4.50)

Let X and Y be locally compact separable metric spaces and $T : \mathrm{C_c}(X, \mathbb{R}^n) \to \mathrm{C_c}(Y, \mathbb{R}^m)$ linear continuous. Then $T^{\mathrm{t}} : \mathrm{C_c}(Y, \mathbb{R}^m)^* \to \mathrm{C_c}(X, \mathbb{R}^n)^*$ preserves weak-star convergence of sequences. The same holds for weak-star convergence in the sense of finite measures if T is continuous with respect to the $\mathrm{C_0}$ topologies.

Foliations by Borel sets for positive Radon measures

Proposition (4.53)

Let X be a locally compact separable metric space, μ a positive Radon measure on X and $(E_\alpha)_{\alpha \in A}$ a disjoint family of Borel sets in X. Then $\{\alpha \in A \mid \mu(E_\alpha) > 0\}$ is countable.

Characterization of weak-star convergence for positive Radon measures

Theorem (4.54)

Let X be a locally compact separable metric space, $(\mu_k)_{k\in\mathbb{N}}$ a sequence of positive Radon measures in X and μ a positive Radon measure in X. The following conditions are equivalent:

i) $\mu_k \overset{*}{\rightharpoonup} \mu$.

ii) *For all $K \subset X$ compact and for all $U \subset X$ open,*

$$\mu(K) \geq \limsup \mu_k(K) \quad \textit{and} \quad \mu(U) \leq \liminf \mu_k(U).$$

iii) *For all $E \in \mathscr{B}_X^c$ such that $\mu(\partial E) = 0$, $\mu_k(E) \to \mu(E)$.*

Weak convergence and total variation

Proposition (4.57)

Let X be a locally compact separable metric space and $(\mu_k)_{k\in\mathbb{N}}$ a sequence in $\mathrm{C_c}(X,\mathbb{R}^n)^*$ weakly-star convergent to $\mu \in \mathrm{C_c}(X,\mathbb{R}^n)^*$. Then, for every $A \subset X$ open, $|\mu|(A) \leq \liminf |\mu_k|(A)$.

Proposition (4.58)

Let X be a locally compact separable metric space and $(\mu_k)_{k\in\mathbb{N}}$ a sequence in $\mathrm{C_c}(X,\mathbb{R}^n)^*$ weakly-star convergent to $\mu \in \mathrm{C_c}(X,\mathbb{R}^n)^*$. If $|\mu_k|(X) \to |\mu|(X) < \infty$, then $|\mu_k| \xrightarrow{*\mathrm{f}} |\mu|$.

Weak convergence and total variation

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Let X be a locally compact separable metric space and $(\mu_k)_{k\in\mathbb{N}}$ a sequence in $\mathrm{C_c}(X,\mathbb{R}^n)^*$ weakly-star convergent to $\mu \in \mathrm{C_c}(X,\mathbb{R}^n)^*$. Then, for every $A \subset X$ open, $|\mu|(A) \leq \liminf|\mu_k|(A)$.

Proposition (4.58)

Let X be a locally compact separable metric space and $(\mu_k)_{k\in\mathbb{N}}$ a sequence in $\mathrm{C_c}(X,\mathbb{R}^n)^*$ weakly-star convergent to $\mu \in \mathrm{C_c}(X,\mathbb{R}^n)^*$. If $|\mu_k|(X) \to |\mu|(X) < \infty$, then $|\mu_k| \xrightarrow{*\,\mathrm{f}} |\mu|$.

De La Vallée Poussin Theorem

Theorem (4.61)

Let X be a locally compact separable metric space and $(\mu_k)_{k\in\mathbb{N}}$ be a sequence of finite $\mathbb{R}^n$-valued Radon measures on X such that $\sup\{|\mu_k|(X) \mid k \in \mathbb{N}\} < \infty$. *Then there exists a finite $\mathbb{R}^n$-valued Radon measure μ on X and a subsequence $(\mu_{k_j})_{j\in\mathbb{N}}$ of $(\mu_k)_{k\in\mathbb{N}}$ such that $\mu_{k_j} \overset{*\,\mathrm{f}}{\rightharpoonup} \mu$. Moreover,* $|\mu|(X) \leq \liminf|\mu_{k_j}|(X)$.

De La Vallée Poussin Theorem

Corollary (4.63)

Let X be a locally compact separable metric space and $(\mu_k)_{k\in\mathbb{N}}$ be a sequence of $\mathbb{R}^n$-valued Radon measures on X such that, for any $K \subset X$ compact, $\sup\{|\mu_k|(K) \mid k \in \mathbb{N}\} < \infty$*. Then there exists an $\mathbb{R}^n$-valued Radon measure μ on X and a subsequence $(\mu_{k_j})_{j\in\mathbb{N}}$ of $(\mu_k)_{k\in\mathbb{N}}$ such that $\mu_{k_j} \stackrel{*}{\rightharpoonup} \mu$.*