Geometric Measure Theory

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August 18, 2019

Up to the end of this section we fix a metric space (X, d).

Upper and lower n-dimensional densities

Definition (3.1)

Let $A \subset X$, $x \in X$, $n > 0$ real and μ a measure on X. We define:

1) the *n-dimensional upper density of A at x with respect to* μ:

$$\Theta^{*n}(\mu, A, x) := \limsup_{r \to 0} \frac{\mu(A \cap \mathbb{B}(x, r))}{\alpha(n) r^n} \in [0, \infty].$$

2) the *n-dimensional lower density of A at x with respect to* μ:

$$\Theta_*^{n}(\mu, A, x) := \liminf_{r \to 0} \frac{\mu(A \cap \mathbb{B}(x, r))}{\alpha(n) r^n} \in [0, \infty].$$

If $\Theta^{*n}(\mu, A, x) = \Theta_*^n(\mu, A, x)$, we denote their common value by $\Theta^n(\mu, A, x)$ and call it *density of A at x with respect to* μ.
For $A = X$, we use the notations $\Theta^{*n}(\mu, x)$, $\Theta_*^n(\mu, x)$ and $\Theta^n(\mu, x)$.

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If $\Theta^{*n}(\mu, A, x) = \Theta_*^n(\mu, A, x)$, we denote their common value by $\Theta^n(\mu, A, x)$ and call it *density of A at x with respect to μ*.
For $A = X$, we use the notations $\Theta^{*n}(\mu, x)$, $\Theta_*^n(\mu, x)$ and $\Theta^n(\mu, x)$.

Remark (3.2)

With the notation above:

1) Note that $\Theta^{*n}(\mu, A, x) = \Theta^{*n}(\mu \llcorner A, x)$ and $\Theta^{n}_{*}(\mu, A, x) = \Theta^{n}_{*}(\mu \llcorner A, x)$.
2) If $U \subset X$ is an open set and $x \in U$, $\Theta^{*n}(\mu, A, x) = \Theta^{*n}(\mu \llcorner U, A, x)$ and $\Theta^{n}_{*}(\mu, A, x) = \Theta^{n}_{*}(\mu \llcorner U, A, x)$.

Lemma (3.3)

If μ is a locally finite Borel measure on X, $A \subset X$, $x \in X$ and $n > 0$ real, then, the definitions of $\Theta^{*n}(\mu, A, x)$ or $\Theta_*^n(\mu, A, x)$ do not change if we use open balls instead of closed balls.

Proposition (3.4)

If μ is a locally finite Borel measure on X, $A \subset X$ and $n > 0$ real, then the functions $X \to [0, \infty]$ given by $x \in X \mapsto \Theta^{*n}(\mu, A, x)$ and $x \in X \mapsto \Theta_*^n(\mu, A, x)$ are Borelian.

Corollary (3.5)

If μ is a locally finite Borel measure on X, $A \subset X$ and $n > 0$ real, then the set $Y := \{x \in X \mid \Theta^{*n}(\mu, A, x) = \Theta_*^n(\mu, A, x)\}$ is Borel measurable and $\Theta^n(\mu, A, \cdot) : Y \to [0, \infty]$ is Borelian.

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Comparison density theorem

Theorem (3.6)

Let μ be a Borel measure on a metric space X, $n > 0$ real, $t \geq 0$ and $A \subset A_1 \subset X$. If $\forall x \in A$, $\Theta^{*n}(\mu, A_1, x) \geq t$ then $t\mathcal{H}^n(A) \leq \mu(A_1)$.

Upper density theorem

Theorem (3.7)

Let μ be a Borel regular measure on a metric space X, $n > 0$ real and $B \in \sigma(\mu)$ with $\mu(B) < \infty$. Then $\Theta^{*n}(\mu, B, x) = 0$ for $\mathcal{H}^n$-a.e. $x \in X \setminus B$.

Exercise (3.8)

If μ is an open σ-finite Borel regular measure on a metric space X, the thesis in the previous theorem holds for all $B \in \sigma(\mu)$, i.e. the hypothesis of $\mu(B)$ being finite may be dropped.

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If μ is an open σ-finite Borel regular measure on a metric space X, the thesis in the previous theorem holds for all $B \in \sigma(\mu)$, i.e. the hypothesis of $\mu(B)$ being finite may be dropped.

Density theorem for the Lebesgue measure

Corollary (3.9)

If $B \subset \mathbb{R}^n$ is $\mathcal{L}^n$-measurable, then $\Theta^n(\mathcal{L}^n, B, x)$ exists for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$, $\Theta^n(\mathcal{L}^n, B, x) = 1$ for $\mathcal{L}^n$-a.e. $x \in B$ and $\Theta^n(\mathcal{L}^n, B, x) = 0$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n \setminus B$.

Upper and lower densities of a measure relative another

Definition (3.12)

Let X be a metric space, μ and ν measures on X, and $x \in X$. We define the *upper* and *lower density of μ relative to ν at x* by, respectively:

$$\Theta^{*\nu}(\mu, x) := \limsup_{r \to 0} \frac{\mu\big(\mathbb{B}(x, r)\big)}{\nu\big(\mathbb{B}(x, r)\big)} \in [0, \infty],$$

$$\Theta_*^{\nu}(\mu, x) := \liminf_{r \to 0} \frac{\mu\big(\mathbb{B}(x, r)\big)}{\nu\big(\mathbb{B}(x, r)\big)} \in [0, \infty],$$

where we adopt the extended arithmetic rules $\frac{0}{0} := 0$, $\frac{\infty}{\infty} := 0$.

If $\Theta^{*\nu}(\mu, x) = \Theta_*^{\nu}(\mu, x)$, we say that the *density of μ relative to ν at x* exists and denote it by $\Theta^{\nu}(\mu, x) := \Theta^{*\nu}(\mu, x) = \Theta_*^{\nu}(\mu, x)$.

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Remark (3.13)

If $X = \mathbb{R}^n$, $A \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$ and μ a measure on $\mathbb{R}^n$, the n-dimensional upper and lower densities of A at x with respect to μ, defined in 1, are special cases of the previous definition: $\Theta^{*n}(\mu, A, x) = \Theta^{*\mathcal{L}^n}(\mu \llcorner A, x)$ and $\Theta^{n}_{*}(\mu, A, x) = \Theta^{\mathcal{L}^n}_{*}(\mu \llcorner A, x)$.

Lemma (3.14)

If μ and ν are locally finite Borel measures on a metric space X, and $x \in X$, then the definitions of $\Theta^{*\nu}(\mu, x)$ or $\Theta^{\nu}_{*}(\mu, x)$ do not change if we use open balls instead of closed balls.

Proposition (3.15 and 3.16)

Let μ and ν be locally finite Borel measures on a metric space X. Suppose that X is separable or that ν is finite on all closed balls of X. Then the functions $X \to [0, \infty]$ given by $x \in X \mapsto \Theta^{*\nu}(\mu, x)$ and $x \in X \mapsto \Theta^{\nu}_{*}(\mu, x)$ are Borelian.

Corollary (3.17)

Let μ and ν be locally finite Borel measures on a metric space X, with X separable or ν finite on all closed balls of X. Then the set $Y := \{x \in X \mid \Theta^{*\nu}(\mu, x) = \Theta^{\nu}_{*}(\mu, x)\}$ is Borel measurable and $\Theta^{\nu}(\mu, \cdot) : Y \to [0, \infty]$ is Borelian.

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Recall

Definition (2.12)

Let X be a metric space, $\mathcal{F}$ a collection of balls in X and $A \subset X$. We say that $\mathcal{F}$ is a *fine cover A*, or that $\mathcal{F}$ *covers A finely*, if $\mathcal{F}$ is a cover of A such that, $\forall x \in A$, $\inf\{\operatorname{diam} B \mid x \in B \in \mathcal{F}\} = 0$.

Corollary (Vitali's covering theorem for the Lebesgue measure;2.14)

Let $A \subset \mathbb{R}^n$ and $\mathcal{F}$ a collection of nondegenerate closed balls in $\mathbb{R}^n$ which covers A finely. Then, for every $\epsilon > 0$, there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{L}^n(\cup\mathcal{G}) \leq \mathcal{L}^n(A) + \epsilon$ and $\mathcal{L}^n(A \setminus \cup\mathcal{G}) = 0$.

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Symmetric Vitali property (SVP)

Definition (3.18)

Let X be a metric space, $\mathcal{F}$ a collection of balls in X and $A \subset X$. We say that $\mathcal{F}$ is a *strongly fine cover A*, or that $\mathcal{F}$ *covers A finely in the strong sense*, if $\mathcal{F}$ is a cover of A such that, $\forall x \in A$, $\inf\{r > 0 \mid \mathbb{B}(x, r) \in \mathcal{F}\} = 0$.

It is clear that every strongly fine cover of A is a fine cover of A in the sense of definition 12, but the converse does not hold.

Definition (3.19)

We say that a measure μ on a metric space X satisfies the *symmetric Vitali property (SVP)* if, for all $A \subset X$ with $\mu(A) < \infty$ and for all $\mathcal{F}$ strongly fine cover of A by nondegenerate closed balls, there exists a countable disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\mu(A \setminus \cup\mathcal{G}) = 0$.

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Remark (3.20)

1) It is clear that, if a measure μ on a metric space X has SVP, so does any restriction of μ, i.e. $\forall Y \subset X$, $\mu \llcorner Y$ has SVP.

2) If a measure μ on a metric space X is σ-finite and has SVP, then μ is concentrated on its support, i.e. $\mu(X \setminus \operatorname{spt} \mu) = 0$.
Proof: Let $X = \cup_{k\in\mathbb{N}} A_k$, with $\forall k \in \mathbb{N}$, $A_k \in \sigma(\mu)$ and $\mu(A_k) < \infty$. For each $k \in \mathbb{N}$, the family of nondegenerate closed balls $\mathcal{F} = \{\mathbb{B}(x,r) \mid x \in X \setminus \operatorname{spt} \mu, r > 0, \mu(\mathbb{B}(x,r)) = 0\}$ covers $A_k \setminus \operatorname{spt} \mu$ finely in the strong sense. Hence, there exists a countable disjoint subfamily $\mathcal{G}_k \subset \mathcal{F}$ such that $\mu((A_k \setminus \operatorname{spt} \mu) \setminus \cup\mathcal{G}_k) = 0$; since $\mu(\cup\mathcal{G}_k) = 0$, we conclude that $\mu(A_k \setminus \operatorname{spt} \mu) = 0$. Therefore $X \setminus \operatorname{spt} \mu = \cup_{k\in\mathbb{N}}(A_k \setminus \operatorname{spt} \mu)$ has μ-measure zero.

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   Proof: Let $X = \cup_{k\in\mathbb{N}} A_k$, with $\forall k \in \mathbb{N}$, $A_k \in \sigma(\mu)$ and $\mu(A_k) < \infty$. For each $k \in \mathbb{N}$, the family of nondegenerate closed balls $\mathcal{F} = \{\mathbb{B}(x, r) \mid x \in X \setminus \text{spt } \mu, r > 0, \mu(\mathbb{B}(x, r)) = 0\}$ covers $A_k \setminus \text{spt } \mu$ finely in the strong sense. Hence, there exists a countable disjoint subfamily $\mathcal{G}_k \subset \mathcal{F}$ such that $\mu((A_k \setminus \text{spt } \mu) \setminus \cup\mathcal{G}_k) = 0$; since $\mu(\cup\mathcal{G}_k) = 0$, we conclude that $\mu(A_k \setminus \text{spt } \mu) = 0$. Therefore $X \setminus \text{spt } \mu = \cup_{k\in\mathbb{N}}(A_k \setminus \text{spt } \mu)$ has μ-measure zero.

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Doubling property implies SVP

Proposition (3.21)

Let X be a separable metric space and μ a finite Borel regular measure on X. Assume that μ satisfies the doubling property:

$$\exists C > 0,\ \forall B \subset X \textit{ nondegenerate closed ball, } \mu(5B) \leq C\mu(B).$$

Then μ has the symmetric Vitali property.

Besicovitch covering theorem

Theorem (3.24)

For each $n \in \mathbb{N}$, there exists a natural constant $N = N(n)$, depending only on n, which satisfies the following property: if $\mathcal{F}$ is any family of nondegenerate closed balls in $\mathbb{R}^n$ with $\sup\{\operatorname{diam} B \mid B \in \mathcal{F}\} < \infty$ and A is the set of centers of the balls in $\mathcal{F}$, then exist $\mathcal{G}_1, \dots, \mathcal{G}_N$ such that, for $1 \leq i \leq N$, $\mathcal{G}_i$ is a disjoint subfamily of $\mathcal{F}$ and $\cup_{i=1}^N \mathcal{G}_i$ covers A.

Corollary (3.25)

Let μ be a Borel measure in $\mathbb{R}^n$, $A \subset \mathbb{R}^n$ with $\mu(A) < \infty$ and $\mathcal{F}$ a family of nondegenerate closed balls which covers A finely in the strong sense. Then, for any open set $U \supset A$, there exists a countable disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\cup \mathcal{G} \subset U$ and $\mu(A \setminus \cup \mathcal{G}) = 0$.

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Proposition (Borel measures on subsets of $\mathbb{R}^n$ satisfy SVP; 3.23)

Let X be a metric subspace of $\mathbb{R}^n$ and μ a Borel measure on X. Then μ satisfies the symmetric Vitali property.

General comparison density theorem

Theorem (3.26)

Let μ and ν be open σ-finite Borel regular measures on a metric space X such that ν has the symmetric Vitali property, $t \geq 0$ and $A \subset X$. If $\forall x \in A$, $\Theta^{*\nu}(\mu, x) \geq t$ then $t\nu(A) \leq \mu(A)$.

Corollary (3.27)

Let μ and ν be open σ-finite Borel regular measures on a metric space X such that ν has the symmetric Vitali property. Then $\Theta^{*\nu}(\mu, x) < \infty$ for ν-a.e. $x \in X$.

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General upper density theorem

Theorem (3.28)

Let μ be a Borel regular measure on a metric space X, ν an open σ-finite Borel regular measure on X with the symmetric Vitali property, and $A \in \sigma(\mu)$ with $\mu(A) < \infty$. Then $\Theta^{*\nu}(\mu \llcorner A, x) = 0$ for ν-a.e. $x \in X \setminus A$.

Theorem (general density theorem; 3.29)

Let μ be an open σ-finite Borel regular measure on a metric space X with symmetric Vitali property and $A \in \sigma(\mu)$. Then the density $\Theta^{\mu}(\mu \llcorner A, \cdot)$ coincides μ-a.e. on X with χ_A, i.e.

$$\Theta^{\mu}(\mu \llcorner A, x) = \lim_{r \to 0} \frac{\mu(A \cap \mathbb{B}(x,r))}{\mu(\mathbb{B}(x,r))} = \begin{cases} 1 & \text{for } \mu\text{-a.e. } x \in A, \\ 0 & \text{for } \mu\text{-a.e. } x \in X \setminus A. \end{cases}$$

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Lusin's theorem

Theorem (1.112)

Let μ be a Borel regular measure on a metric space X (respectively, a Radon measure on a locally compact Hausdorff space X), Y a separable metric space, $f : \operatorname{dom} f \subset X \to Y$ a μ-measurable map. Then, for each $A \in \sigma(\mu)$ with $\mu(A) < \infty$ and for each $\epsilon > 0$, there exists a closed (respectively, compact) set $C \subset A$ such that $\mu(A \setminus C) < \epsilon$ and $f|_C$ is continuous.

General Lebesgue differentiation theorem

Corollary (3.30)

Let μ be an open σ-finite Borel regular measure on a metric space X with symmetric Vitali property and $f : X \to \mathbb{C}$ a μ-measurable function satisfying one of the following conditions:

1. *$f \in \mathrm{L}^1(\mu)$ or*
2. *X is separable and $f \in \mathrm{L}^1_{\mathrm{loc}}(\mu)$, i.e. $\forall x \in X$, $\exists r > 0$, $\int_{\mathbb{B}(x,r)} |f| \,\mathrm{d}\mu < \infty$.*

Then, for μ-a.e. $x \in X$:

$$\lim_{r \to 0} \frac{1}{\mu\big(\mathbb{B}(x,r)\big)} \int_{\mathbb{B}(x,r)} f \,\mathrm{d}\mu = f(x).$$

Lebesgue Points

Corollary (3.31)

Let X be a separable metric space, μ an open σ-finite Borel regular measure on X with symmetric Vitali property, $1 \leq p < \infty$ and $f \in \mathrm{L}^{\mathrm{p}}_{\mathrm{loc}}(\mu)$, i.e. $\forall x \in X, \exists r > 0, \int_{\mathbb{B}(x,r)} |f|^p \,\mathrm{d}\mu < \infty$. Then, for μ-a.e. $x \in X$,

$$\lim_{r \to 0} \frac{1}{\mu\big(\mathbb{B}(x,r)\big)} \int_{\mathbb{B}(x,r)} |f(y) - f(x)|^p \,\mathrm{d}\mu(y) = 0. \tag{1}$$

Definition (Lebesgue Points; 3.32)

With the same notation from the previous corollary, a point $x \in X$ for which (1) holds is called *Lebesgue point of f with respect to μ*.

Lebesgue Points

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Lebesgue points with noncentered balls

Corollary (3.33)

Let $1 \leq p < \infty$ *and* $f \in \mathrm{L}^{\mathrm{p}}_{\mathrm{loc}}(\mathcal{L}^n)$*. Then, for each Lebesgue point* x *of* f *with respect to* $\mathcal{L}^n$ *(in particular, for* $\mathcal{L}^n$*-a.e.* $x \in \mathbb{R}^n$*),*

$$\lim_{B \downarrow \{x\}} \frac{1}{\mathcal{L}^n(B)} \int_B |f(y) - f(x)|^p \, \mathrm{d}\mathcal{L}^n(y) = 0,$$

where the limit is taken over all closed balls B *containing* x *with* $\operatorname{diam} B \to 0$*.*

Absolute continuity and mutual singularity

Definition (3.34)

Let μ and ν be Borel measures on a topological space X. We say that:

1. μ is *absolutely continuous* with respect to ν (notation: $\mu \ll \nu$) if $\forall A \subset X$, $\nu(A) = 0$ implies $\mu(A) = 0$.
2. μ and ν are *mutually singular* (notation: $\mu \perp \nu$) if there exists $A \in \mathscr{B}_X$ such that μ is concentrated on A and ν is concentrated on $X \setminus A$.

Lebesgue decomposition theorem

Lemma (3.36)

Let μ be a σ-finite Borel measure and ν a Borel regular measure on a metric space X. Then there exists $B \in \mathscr{B}_X$ such that ν is concentrated on B^c and $\mu \llcorner B^c \ll \nu$, so that

$$\mu = \mu \llcorner B + \mu \llcorner B^c, \quad \mu \llcorner B \perp \nu, \mu \llcorner B^c \ll \nu. \tag{LD}$$

Moreover:

1. *$B \in \mathscr{B}_X$ satisfying (LD) is unique up to μ-null sets, i.e. if $B' \in \mathscr{B}_X$ also satisfies (LD), then $B \bigtriangleup B'$ is μ-null.*
2. *the decomposition (LD) is unique in the sense that, if $\nu = \mu_s + \mu_a$ with $\mu_s \perp \nu$ and $\mu_a \ll \nu$, then $\mu_s = \mu \llcorner B$ and $\mu_a = \mu \llcorner B^c$.*

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Comparison theorem for lower densities

Theorem (3.38)

Let μ and ν be open σ-finite Borel regular measures on a metric space X, $t \geq 0$ and $A \subset X$ with $\forall x \in A$, $\Theta_*^{\nu}(\mu, x) \leq t$.

i) *If μ has SVP, then $\mu(A) \leq t\,\nu(A)$.*
ii) *If ν has SVP and B is given by the previous lemma, so that (LD) holds, then $\mu(A \setminus B) \leq t\,\nu(A)$.*

Differentiation theorem for Borel measures on metric spaces

Theorem (3.39)

Let μ and ν be open σ-finite Borel regular measures on a metric space X. Suppose that X is separable or that ν is finite on closed balls of X.

1. *The set $Y := \{x \in X \mid \Theta^{*\nu}(\mu, x) = \Theta_*^\nu(\mu, x)\}$ is Borel measurable and $\Theta^\nu(\mu, \cdot) : Y \to [0, \infty]$ is Borelian.*
2. *If ν has SVP, $Y_f := \{x \in Y \mid \Theta^\nu(\mu, x) < \infty\}$ is a Borel measurable subset of X whose complement is ν-null.*
3. *If both μ and ν have SVP, $\mu(Y^c) = \nu(Y^c) = 0$.*

Lebesgue-Besicovitch-Radon-Nikodym differentiation theorem

Theorem (3.40)

Let μ and ν be open σ-finite Borel regular measures on a metric space X. Suppose that X is separable or that ν is finite on closed balls of X, and that ν has SVP.

1. *Let $\mu = \mu_s + \mu_a$ be the Lebesgue decomposition of μ with respect to ν, i.e. $\mu_s = \mu \llcorner B$ and $\mu_a = \mu \llcorner B^c$, where $B \in \mathscr{B}_X$ is given by lemma 29. Then, for all $A \in \mathscr{B}_X$,*

$$\mu_a(A) = \int_A \Theta^\nu(\mu, x)\,\mathrm{d}\nu(x),$$

   so that, for all $A \in \mathscr{B}_X$, $\mu(A) = \int_A \Theta^\nu(\mu, x)\,\mathrm{d}\nu(x) + \mu_s(A)$.

2. *If μ also has SVP, in lemma 29 we can take $B' = \{x \in X \mid \Theta^\nu(\mu, x) = \infty\}$ in place of B.*

Corollary (3.41)

With the same hypothesis from the previous theorem, $\Theta^{\nu}(\mu,\cdot)$ *coincides* ν*-a.e. with the Radon-Nikodym derivative* $\frac{d(\mu_a|_{\mathscr{B}_X})}{d(\nu|_{\mathscr{B}_X})}$.