Geometric Measure Theory

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5–times covering lemma

Definition (2.12)

Let X be a metric space, $\mathcal{F}$ a collection of balls in X and $A \subset X$. We say that $\mathcal{F}$ is a *fine cover A*, or that $\mathcal{F}$ *covers A finely*, if $\mathcal{F}$ is a cover of A such that, $\forall x \in A$, $\inf\{\operatorname{diam} B \mid x \in B \in \mathcal{F}\} = 0$.

Corollary (2.13)

Let X be a metric space, $A \subset X$, $\mathcal{F} \subset 2^X$ a family of nondegenerate closed balls of X which covers A finely. Then there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that, for all $F \subset \mathcal{F}$ finite, $A \setminus \cup_{B \in F} B \subset \cup_{B \in \mathcal{G} \setminus F} 5B$.

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Proof.

- Since the cover $\mathcal{F}$ is fine, we may assume that $\sup\{\operatorname{diam} B \mid B \in \mathcal{F}\} \leq 1$; otherwise, discard the balls in $\mathcal{F}$ with diameter > 1, so that the remaining balls still cover A finely.
- Take $\mathcal{G} \subset \mathcal{F}$ as in the previous remark. Let $x \in A \setminus \cup_{B \in F} B$. Since F is finite, $\cup_{B \in F} B$ is closed, hence there exists $r > 0$ such that $\mathbb{U}(x, r) \cap \cup_{B \in F} B = \emptyset$.
- Since $\mathcal{F}$ covers A finely, there exists $B \in \mathcal{F}$ such that $x \in B$ and $\operatorname{diam} B < r$, so that $B \subset \mathbb{U}(x, r)$, thus $B \cap \cup_{B \in F} B = \emptyset$. Take $B' \in \mathcal{G}$ such that $B' \cap B \neq \emptyset$ (hence $B' \notin F$) and $\operatorname{diam} B < 2 \operatorname{diam} B'$, so that $x \in B \subset 5B'$. Then $A \setminus \cup_{B \in F} B \subset \cup_{B \in \mathcal{G} \setminus F} 5B$, as asserted.

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Vitali's covering theorem

Corollary (Vitali's covering theorem for the Lebesgue measure;2.14)

Let $A \subset \mathbb{R}^n$ and $\mathcal{F}$ a collection of nondegenerate closed balls in $\mathbb{R}^n$ which covers A finely. Then, for every $\epsilon > 0$, there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{L}^n(\cup\mathcal{G}) \leq \mathcal{L}^n(A) + \epsilon$ and $\mathcal{L}^n(A \setminus \cup\mathcal{G}) = 0$.

Corollary (filling open sets with balls with respect to Lebesgue measure; 2.15)

Let $U \subset \mathbb{R}^n$ be an open set and $\mathcal{F}$ a family of nondegenerate closed balls contained in U which covers U finely (for instance, if $\mathcal{F}$ is the family of all nondegenerate closed balls contained in U, or the family of all such balls with diameters bounded by a fixed $\delta > 0$). Then there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{L}^n(U \setminus \cup\mathcal{G}) = 0$.

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Notation

Notation for sections

For $E \subset X \times Y$ and $(x_0, y_0) \in X \times Y$,

- $E_{x_0} := \{y \in Y \mid (x_0, y) \in E\}$ (the x_0-*section* of E);
- $E_{y_0} := \{x \in X \mid (x, y_0) \in E\}$ (the y_0-*section* of E).

Steiner Symmetrization

Definition (2.16)

Let $(e_1, \ldots, e_n)$ be the standard basis of $\mathbb{R}^n$ and identify $\mathbb{R}^{n-1} \equiv \langle e_1, \ldots, e_{n-1} \rangle$, $\mathbb{R} \equiv \langle e_n \rangle$, so that $\mathbb{R}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R}$. We define the *Steiner symmetrization* with respect to $\mathbb{R}^{n-1}$ to be the map $\mathrm{S_{e_n}} : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$ defined by (see figure 1):

$$\mathrm{S_{e_n}}(A) := \bigcup_{\{x' \in \mathbb{R}^{n-1} \mid A_{x'} \neq \emptyset\}} \{(x', x_n) \mid |x_n| \leq \frac{1}{2} \mathcal{L}^1(A_{x'})\}.$$

Given $a \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$, we define similarly the Steiner symmetrization $\mathrm{S_a}$ with respect to the $(n-1)$-dimensional subspace $\langle a \rangle^\perp$: take any orthogonal map $\phi \in \mathrm{O}(n)$ such that $\phi(a) = e_n$ (hence $\phi(\langle a \rangle^\perp) = \mathbb{R}^{n-1}$) and put $\mathrm{S_a} := \phi^{-1} \circ \mathrm{S_{e_n}} \circ \phi$.

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Steiner Symmetrization

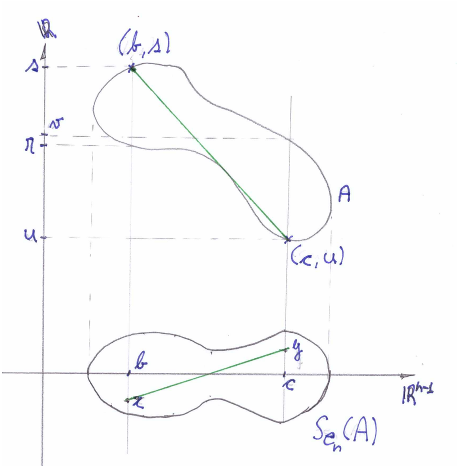

Figure: Steiner Symmetrization

Steiner Symmetrization

Proposition (properties of Steiner symmetrization; 2.17)

Let $a \in \mathbb{S}^{n-1}$.

i) $\forall A \subset \mathbb{R}^n$, $\operatorname{diam} \mathrm{S_a}(A) \leq \operatorname{diam} A$.

ii) *If $A \subset \mathbb{R}^n$ is $\mathcal{L}^n$-measurable, then so is $\mathrm{S_a}(A)$ and* $\mathcal{L}^n(A) = \mathcal{L}^n\big(\mathrm{S_a}(A)\big)$.

Steiner Symmetrization

Lemma (2.18)

Let $f : \mathbb{R}^n \to [0, \infty]$ *be* $\mathcal{L}^n$*-measurable. Then* $\operatorname{hyp} f := \{(x, t) \in \mathbb{R}^n \times [0, \infty) \mid t \leq f(x)\} \subset \mathbb{R}^{n+1}$ *is* $\mathcal{L}^{n+1}$*-measurable.*

The isodiametric inequality

Theorem (isodiametric inequality; 2.19)

The Lebesgue measure of any subset of $\mathbb{R}^n$ is at most the measure of an euclidean ball with the same diameter. That is, for all $A \subset \mathbb{R}^n$,

$$\mathcal{L}^n(A) \leq \alpha(n)\left(\frac{\operatorname{diam} A}{2}\right)^n.$$

Exercise (2.20)

Show an example of a set $A \subset \mathbb{R}^n$ which is not contained in any ball with diameter $\operatorname{diam} A$.

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$\mathcal{L}^n = \mathcal{H}^n$

Theorem (2.21)

For all $\delta \in (0, \infty]$ *and* $n \in \mathbb{N}$, $\mathcal{H}^n = \mathcal{H}^n_\delta = \mathcal{L}^n$ *in* $\mathbb{R}^n$.

Corollary (2.22)

$\mathcal{H}$-dim $\mathbb{R}^n = n$.

Proof.

Apply the stability with respect to countable unions of the Hausdorff dimension to $\mathbb{R}^n = \cup_{k \in \mathbb{N}} C_k$, where each C_k is a nondegenerate cube with finite Lebesgue measure, i.e. $0 < \mathcal{H}^n(C_k) < \infty$, so that $\forall k \in \mathbb{N}$, $\mathcal{H}$-dim $C_k = n$. □

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$\mathcal{L}^n = \mathcal{H}^n$

Exercise (2.23)

If E is a k-dimensional subspace of a normed space X, then $\mathcal{H}\text{-dim } E = k$.