Geometric Measure Theory

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The Carathéodory Construction

1. Let X be a metric space, $\mathcal{F} \subset 2^X$ and $\zeta : \mathcal{F} \to [0, \infty]$.

The idea is to "measure" the elements of $\mathcal{F}$ by means of the *method* or *gauge* ζ and use that to define a Borel measure on X, abstracting the geometric idea underlying the construction of the Lebesgue measure.

2. For $0 < \delta \leq \infty$ we define $\forall A \subset X$,

$$\psi_\delta(A) := \inf\{\sum_{S \in \mathcal{G}} \zeta(S) \mid \mathcal{G} \subset \mathcal{F} \cap \{S \mid \text{diam } S \leq \delta\},$$
$$\mathcal{G} \text{ countable }, A \subset \cup\mathcal{G}\}.$$

3. Define, for each $A \subset X$, $\psi(A) := \sup\{\psi_\delta(A) \mid 0 < \delta \leq \infty\} \in [0, \infty]$.

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Definition (2.1)

With the notation above, we call ψ the *result of Carathéodory's construction from the gauge* ζ *on* $\mathcal{F}$, and we call ψ_δ the *size* δ *approximating measure*.

Proposition (2.2)

Let X be a metric space and ψ be the result of Carathéodory's construction from the gauge ζ on $\mathcal{F} \subset 2^X$. Then ψ is a Borel measure. Besides, if $\mathcal{F} \subset \mathscr{B}_X$, ψ is Borel regular.

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The Carathéodory Construction

Definition (Hausdorff measures; 2.3)

Let X be a metric space and m a nonnegative real number. Take $\mathcal{F} = 2^X$ and $\zeta : 2^X \to [0, \infty]$ given by

$$\zeta(S) := \alpha(m) \frac{(\mathrm{diam}\ S)^m}{2^m},$$

where $\alpha(m) = \frac{\pi^{m/2}}{\Gamma(m/2+1)}$ (i.e. the euclidean volume of $\mathbb{B}^m$ if m integer). The result of Carathéodory's construction from the gauge ζ on 2^X is called *Hausdorff m-dimensional measure* on X, denoted by $\mathcal{H}^m$. We use the notation $\mathcal{H}^m_\delta$ for the size δ approximation of $\mathcal{H}^m$.

Hausdorff measures

Proposition (immediate properties of Hausdorff measure; 2.4)

Let X be a metric space and m a nonnegative real number. The following properties hold for $\mathcal{H}^m$:

1. *The Hausdorff measure is compatible with the operation of taking traces. That is, if X is a metric space and $A \subset X$, the trace of $\mathcal{H}^m$ on A coincides with the m-dimensional Hausdorff measure on A (as a metric subspace of X).*
2. *The Hausdorff measure is invariant by isometries. That is, if Y is another metric space and $f : X \to Y$ is an isometry onto Y, then the pushforward $f_{\#}\mathcal{H}^m$ coincides with the Hausdorff m-dimensional measure on Y.*
3. *If Y is another metric space and $f : X \to Y$ has Lipschitz constant $\operatorname{Lip} f < \infty$, then $\forall A \subset X$, $\mathcal{H}^m(f(A)) \leq (\operatorname{Lip} f)^m \mathcal{H}^m(A)$.*

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4. $\mathcal{H}^m$ also coincides with the result of Carathéodory's construction from ζ (same gauge) on $\mathcal{F}' = \{\text{closed subsets of } X\}$ or $\mathcal{F}'' = \{\text{open subsets of } X\}$. If X is a normed vector space, we may also take $\mathcal{F}''' = \{\text{closed convex subsets of } X\}$.
5. $\mathcal{H}^m$ is a Borel regular measure on X.
6. $\mathcal{H}^0$ coincides with the counting measure on X.

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Corollary (2.5)

If X, Y are metric spaces, m a nonnegative real number and $f : X \to Y$ is an isometry into Y, then $\forall A \subset X$, $\mathcal{H}^m\big(f(A)\big) = \mathcal{H}^m(A)$.

Characterization of $\mathcal{H}^m$–null sets

Exercise ($\mathcal{H}^m$–null sets; 2.6)

Let X be a metric space, $A \subset X$ and $0 < m < \infty$. The following statements are equivalent:

1. $\mathcal{H}^m(A) = 0$.
2. $\exists \delta \in (0, \infty]$ *such that* $\mathcal{H}^m_\delta(A) = 0$.
3. $\forall \epsilon > 0$, $\exists (E_n)_{n\in\mathbb{N}}$ *cover of A such that* $\sum_{n\in\mathbb{N}} (\mathrm{diam}\ E_n)^m < \epsilon$.

Hausdorff dimension

Proposition (2.7)

Let X be a metric space, $A \subset X$ and $0 \leq s < t < \infty$. If $\mathcal{H}^s(A) < \infty$ then $\mathcal{H}^t(A) = 0$.

As a corollary, if $0 \leq s < t < \infty$ and $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = \infty$. It then follows that $\inf\{m \in [0, \infty) \mid \mathcal{H}^m(A) = 0\} = \sup\{m \in [0, \infty) \mid \mathcal{H}^m(A) = \infty\} \in [0, \infty]$.

Definition (Hausdorff dimension; 2.8)

Let X be a metric space and $A \subset X$. The extended real number $\inf\{m \in [0, \infty) \mid \mathcal{H}^m(A) = 0\} = \sup\{m \in [0, \infty) \mid \mathcal{H}^m(A) = \infty\} \in [0, \infty]$ is called *Hausdorff dimension* of A, denoted by $\mathcal{H}\text{-dim } A$.

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Exercise (properties of Hausdorff dimension; 2.9)

Let X be a metric space.

a) *If $Y \subset X$ is a metric subspace of X and $A \subset Y$, the Hausdorff dimension of A as a subset of the metric space Y is the same for A as a subset of the metric space X.*

b) *The Hausdorff dimension is invariant by isometries, i.e. if Y is a metric space, $f : X \to Y$ an isometry into Y and $A \subset X$, then $\mathcal{H}\text{-dim } A = \mathcal{H}\text{-dim } f(A)$.*

c) *Let X, Y be metric spaces and $f : X \to Y$ be a Lipschitz map. For all $A \subset X$, $\mathcal{H}\text{-dim } f(A) \leq \mathcal{H}\text{-dim } A$. In particular, if f is bi-Lipschitz onto its image (i.e. f is Lipschitz and has a Lipschitz inverse $f^{-1} : \operatorname{Im} f \to X$), then $\forall A \subset X$, $\mathcal{H}\text{-dim } f(A) = \mathcal{H}\text{-dim } A$.*

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Hausdorff dimension

d) (monotonicity) If $A \subset B \subset X$, $\mathcal{H}$-dim $A \leq \mathcal{H}$-dim B.

e) (stability with respect to countable unions) If $A = \cup_{n \in \mathbb{N}} A_n \subset X$, then $\mathcal{H}$-dim $A = \sup\{\mathcal{H}\text{-dim } A_n \mid n \in \mathbb{N}\}$.

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Próximo objetivo:

$$\mathcal{H}^n = \mathcal{L}^n$$

in $\mathbb{R}^n$.

Notation for Balls

For a metric space (X, d):

- $\mathbb{B}(x, r) := \{y \in X \mid d(y, x) \leq r\}$;
- $\mathbb{U}(x, r) := \{y \in X \mid d(y, x) < r\}$;
- if B is a closed ball and $t \geq 1$,

$$tB := \cup\{B' \subset X \text{ closed ball} \mid B' \cap B \neq \emptyset, \operatorname{diam} B' \leq \frac{t-1}{2} \operatorname{diam} B\}.$$

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5–times covering lemma

Theorem (5–times covering lemma; 2.10)

Let X be a metric space and $\mathcal{F} \subset 2^X$ a family of nondegenerate closed balls in X such that $\sup\{\operatorname{diam} B \mid B \in \mathcal{F}\} < \infty$. Then there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\cup_{B\in\mathcal{F}} B \subset \cup_{B\in\mathcal{G}} 5B$.

5–times covering lemma

Proof:

Let $R := \sup\{\text{diam } B \mid B \in \mathcal{F}\} < \infty$. For any $B \in \mathcal{F}$, diam $B \in (0, R] = \sqcup_{j\in\mathbb{N}}(\frac{R}{2^j}, \frac{R}{2^{j-1}}]$. Thus, putting $\forall j \in \mathbb{N}, \mathcal{F}_j := \{B \in \mathcal{F} \mid \text{diam } B \in (\frac{R}{2^j}, \frac{R}{2^{j-1}}]\}$, we have $\sqcup_{j\in\mathbb{N}} \mathcal{F}_j = \mathcal{F}$.

Define inductively $(\mathcal{G}_j)_{j\in\mathbb{N}}$ by:

1. $\mathcal{G}_1$ is a maximal disjoint subfamily of $\mathcal{F}_1$, obtained by an application of Zorn's lemma to the set of all disjoint subfamilies of $\mathcal{F}_1$ partially ordered by inclusion;
2. Once defined $\mathcal{G}_1 \subset \mathcal{F}_1, \ldots, \mathcal{G}_{j-1} \subset \mathcal{F}_{j-1}$, we take a maximal disjoint subfamily $\mathcal{G}_j$ of $\mathcal{F}'_j := \{B \in \mathcal{F}_j \mid \forall B' \in \cup_{i=1}^{j-1}\mathcal{G}_i, B \cap B' = \emptyset\}$, obtained by an application of Zorn's lemma to the set of all disjoint subfamilies of $\mathcal{F}'_j \subset \mathcal{F}_j$ partially ordered by inclusion.

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5–times covering lemma

Proof(cont.)

We contend that $\mathcal{G} := \cup_{j\in\mathbb{N}}\mathcal{G}_j \subset \mathcal{F}$ satisfies the thesis of the theorem. Indeed, it is clear, by construction, that $\mathcal{G}$ is a disjoint subfamily of $\mathcal{F}$. On the other hand, for any $B \in \mathcal{F}_j$, there exists $B' \in \cup_{i=1}^{j}\mathcal{G}_i$ such that $B \cap B' \neq \emptyset$, otherwise $\mathcal{G}_j \sqcup \{B\} \supsetneq \mathcal{G}_j$ would be a disjoint subfamily of $\mathcal{F}_j'$, violating the maximality of $\mathcal{G}_j$. Since diam $B \leq \frac{R}{2^{j-1}} = 2\frac{R}{2^j}$ and $\frac{R}{2^j} <$ diam B', it follows that diam $B < 2\,$diam B', so that $B \subset 5B'$.

5–times covering lemma

Remark (2.11)

1. Note that, if X is separable, then $\mathcal{G}$ is countable (since any disjoint family of sets with nonempty interiors in X is countable).
2. We have actually proved a stronger statement than the thesis: there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that, for any $B \in \mathcal{F}$, $\exists B' \in \mathcal{G}$ with $B \cap B' \neq \emptyset$ and $\operatorname{diam} B < 2 \operatorname{diam} B'$ (thus $B \subset 5B'$).