Geometric Measure Theory

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Measures

Definition (outer measures; def. 1.1)

A *measure* on a set X is a set function $\mu : 2^X \to [0, \infty]$ such that:

- (M1) $\mu(\emptyset) = 0$,
- (M2) (*monotonicity*) $\mu(A) \le \mu(B)$ whenever $A \subset B$,
- (M3) (*countable subadditivity*) $\mu(\cup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu(A_n)$.

Measures

Definition (Carathéodory measurability; 1.2)

Given a measure μ on a set X, a subset $A \subset X$ is called *measurable with respect to* μ (or μ*-measurable*, or simply *measurable*) if it satisfies the *Carathéodory condition*:

$$\forall T \subset X, \mu(T) = \mu(T \cap A) + \mu(T \setminus A).$$

We denote by $\sigma(\mu)$ the set of measurable subsets of X with respect to μ.

Measures

Definition (algebras and σ-algebras; 1.6)

Given a set X, $\mathcal{M} \subset 2^X$ is called an *algebra* of subsets of X if:

- $\emptyset \in \mathcal{M}$;
- $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$;
- $A, B \in \mathcal{M} \Rightarrow A \cup B \in \mathcal{M}$.

$\mathcal{M}$ is called a *σ-algebra* if it is an algebra closed under countable unions. The sets in $\mathcal{M}$ are called *measurable with respect to* $\mathcal{M}$, or *$\mathcal{M}$-measurable*.

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Measures

Definition (measures on σ-algebras; 1.6)

Given a σ-algebra $\mathcal{M} \subset 2^X$, we call a set function $\mu : \mathcal{M} \to [0, \infty]$ a *measure on* $\mathcal{M}$ if it satisfies:

- (M1) $\mu(\emptyset) = 0$,
- (M2) (*countable additivity*) $\mu(\dot\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

Measures

Theorem (Carathéodory; 1.7)

If μ is a measure on a set X, then $\sigma(\mu)$ is a σ-algebra and the restriction of μ to $\sigma(\mu)$ is a complete measure on $\sigma(\mu)$.

Theorem (1.8)

If $\mathcal{M}$ is a σ-algebra of subsets of X and $\mu : \mathcal{M} \to [0, \infty]$ is a measure on $\mathcal{M}$, then the set function:

$$\begin{array}{rccc} \mu^* : & 2^X & \longrightarrow & [0, \infty] \\ & A & \longmapsto & \inf\{\mu(E) \mid A \subset E \in \mathcal{M}\} \end{array}$$

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Measures

Definition (1.9)

A measure $\mu : 2^X \to [0, \infty]$ is called:

- *regular*, if $\forall A \subset X$, $\exists E \in \sigma(\mu)$ such that $A \subset E$ and $\mu(A) = \mu(E)$.
- *finite* (respectively, σ-finite) if so is its restriction to $\sigma(\mu)$.

Proposition (Continuity properties of measures; 1.11)

For a measure μ on X, the following properties hold:

1. *(continuity from below) if $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\sigma(\mu)$, then $\mu(\cup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$,*
2. *(continuity from above) if $(E_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $\sigma(\mu)$ and $\mu(E_1) < \infty$, then $\mu(\cap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$.*

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Restrictions, traces and pushforwards

Definition (Restrictions and traces of measures; 1.13)

Let μ be a measure on a set X and $A \subset X$. We define the:

- *restriction* of μ to A, denoted by $\mu \llcorner A$, as the measure $2^X \to [0, \infty]$ given by $E \mapsto \mu(A \cap E)$.
- *trace* of μ on A, denoted by $\mu|_A$, as the measure $2^A \to [0, \infty]$ given by $E \mapsto \mu(E)$.

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Restrictions, traces and pushforwards

Definition (Pushforward of measures; 1.14)

Let μ be a measure on the set X and $f : X \to Y$ a map into the set Y. We define a measure $2^Y \to [0, \infty]$ on Y by:

$$A \subset Y \mapsto \mu\big(f^{-1}(A)\big),$$

called *pushforward of μ by f* and denoted by $f_{\#}\mu$.

Restrictions, traces and pushforwards

Proposition (1.15)

Let μ be a measure on the set X, $A \subset X$ and $f : X \to Y$. The following properties hold:

i) $\sigma(\mu) \subset \sigma(\mu \llcorner A)$.

ii) *If $E \in \sigma(\mu)$, then $E \cap A \in \sigma(\mu|_A)$. Besides, if $A \in \sigma(\mu)$, then $\sigma(\mu|_A) = \sigma(\mu) \cap 2^A = \{E \in \sigma(\mu) \mid B \subset A\}$.*

iii) *For $B \subset Y$, $f^{-1}(B)$ is μ-measurable* iff *$\forall A \subset X$, B is $f_\#(\mu \llcorner A)$-measurable.*

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Borel measures

Recall

Given a subset $S \subset 2^X$, there exists a smallest σ-algebra of subsets of X which contains S, that is, the intersection of the family of σ-algebras that contain S, which we denote by $\sigma(S)$.

Definition (Borel measures; 1.16)

For a topological space (X, τ), we define its *Borel* σ-algebra as the σ-algebra generated by τ, i.e. $\sigma(\tau)$. We denote it by $\mathscr{B}_X$.

- We say that a measure μ on X is a *Borel measure* if each Borel set is μ-measurable, i.e. if $\mathscr{B}_X \subset \sigma(\mu)$.
- A *Borel regular* measure on X is a Borel measure on X which satisfies: $\forall A \subset X$, $\exists E \in \mathscr{B}_X$ such that $A \subset E$ and $\mu(A) = \mu(E)$.

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Borel measures

Theorem (Carathéodory's criterion)

A measure μ *on a metric space* (X, d) *is Borel* iff

$$\mu(A \cup B) = \mu(A) + \mu(B) \tag{Ca}$$

whenever $A, B \subset X$ *satisfy* $d(A, B) := \inf\{d(a, b) \mid a \in A, b \in B\} > 0$.

Regularity properties of Borel measures

Definition (1.22)

A Borel measure μ on a topological space (X, τ) is called:

- *open σ-finite* if there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of X such that $X = \cup_{n\in\mathbb{N}} U_n$ and $\forall n \in \mathbb{N}$, $\mu(U_n) < \infty$.
- *locally finite* if, for each $x \in X$, there exists an open neighborhood U of x such that $\mu(U) < \infty$.

Example

A locally finite Borel measure on a second countable topological space is open σ-finite.

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Nice measures

There are two main classes of measures which interact nicely with the topology:

- locally finite Borel regular measures on separable metric spaces;
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Regularity properties of Borel measures

Theorem (approximation by open and closed sets; 1.23)

Let μ be an open σ-finite Borel regular measure on a topological space (X, τ) for which each closed set is a G_δ. The following approximation properties hold:

i) *(approximation by open sets from the outside)* $\forall A \subset X$, $\mu(A) = \inf\{\mu(U) \mid A \subset U \in \tau\}$,

ii) *(approximation by closed sets from the inside)* $\forall A \in \sigma(\mu)$, $\mu(A) = \sup\{\mu(C) \mid C \subset A, C \text{ closed}\}$.

Regularity properties of Borel measures

Lemma (1.25)

Let X be a set, $S \subset 2^X$ and $\mathcal{F} \subset 2^X$ such that:

- *$\mathcal{F}$ is closed under countable intersections and countable unions.*
- *If $A \in S$, both A and its complement A^c belong to $\mathcal{F}$.*

Then $\mathcal{F} \supset \sigma(S)$.

Proof.

Let $\mathcal{G} := \{A \in \mathcal{F} \mid A^c \in \mathcal{F}\}$. Then:

1) $S \subset \mathcal{G}$.
2) $\mathcal{G}$ is closed under complementation.
3) $\mathcal{G}$ is closed under countable unions.

Therefore, $\mathcal{G}$ is a σ-algebra which contains S, i.e. $\sigma(S) \subset \mathcal{G} \subset \mathcal{F}$. □

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Regularity properties of Borel measures

Lemma (1.27)

Let μ be a Borel measure on a topological space (X, τ) for which each closed set is a G_δ. If $B \in \mathscr{B}_X$ and $\mu(B) < \infty$, for all $\epsilon > 0$ there exists a closed set $C \subset B$ such that $\mu(B \setminus C) < \epsilon$.

Radon measures

Definition (Radon measures; 1.28)

A *Radon measure* on a locally compact Hausdorff topological space (X, τ) is a Borel measure μ on X such that:

- R1) (*finiteness on compact sets*) if K is a compact subset of X, then $\mu(K) < \infty$,
- R2) (*interior regularity for open sets*) for all $U \subset X$ open, $\mu(U) = \sup\{\mu(K) \mid K \subset U, K \text{ compact}\}$,
- R3) (*exterior regularity*) for all $A \subset X$, $\mu(A) = \inf\{\mu(U) \mid A \subset U, U \text{ open}\}$.

Radon measures

Remark (1.29)

1. Note that, by condition R2 in the definition above, every Radon measure is Borel regular.
2. A measure $\mu : \mathscr{B}_X \to [0, \infty]$ is called a *Radon measure on* $\mathscr{B}_X$ if its canonical extension $\mu^* : 2^X \to [0, \infty]$ is an exterior Radon measure as defined above. That is equivalent to saying that μ satisfies R1, R2 and R3 for any Borel set $A \subset X$.

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Radon measures

Exercise (1.31)

If μ is a Radon measure on a locally compact Hausdorff space (X, τ), then μ is inner regular on all σ-finite μ-measurable sets. In particular, if μ is σ-finite, then it is inner regular on all μ-measurable sets.

Exercise (1.32)

Let X be a locally compact separable metric space. Then μ is a Radon measure on X iff *μ is a locally finite Borel regular measure on X. Moreover, if μ is such a measure, then μ is σ-finite, hence it is inner regular on all μ-measurable sets by the previous exercise.*

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Radon measures

Corollary (1.33)

It follows from the previous exercise that, if X is a locally compact separable metric space and $\mu : \mathscr{B}_X \to [0, \infty]$ is a measure which is finite on compact subsets of X, then the canonical extension of μ to a measure on X is a Radon measure.

Support of a measure on a topological space

Definition (1.34)

Let μ be a measure on a topological space X.

- We say that μ is *concentrated* on a set $A \subset X$ if $\mu(X \setminus A) = 0$.
- The *support* of μ, denoted by $\operatorname{spt} \mu$, is the complement of the union of all open sets $V \subset X$ such that $\mu(V) = 0$.

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Support of a measure on a topological space

Proposition (1.35)

If μ is a measure on a second countable topological space or if μ is a Radon measure on a locally compact Hausdorff topological space, then μ is concentrated on its support. Actually, its support is the smallest closed set on which μ is concentrated.

Support of a measure on a topological space

Proof.

- If μ is a measure on a second countable topological space X, by Lindelöf's theorem we may cover $X \setminus \text{spt } \mu$ by countably many open sets of measure zero, thus $\mu(X \setminus \text{spt } \mu) = 0$.
- If μ is a Radon measure on a locally compact Hausdorff topological space, for each compact $K \subset X \setminus \text{spt } \mu$, we may cover K with finitely many open sets of measure zero, hence $\mu(K) = 0$. By interior regularity, it follows that $\mu(X \setminus \text{spt } \mu) = \sup\{\mu(K) \mid K \subset X \setminus \text{spt } \mu, K \text{ compact}\} = 0$. In any of the two cases, its clear that spt μ the smallest closed set on which μ is concentrated.

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□

Regularity properties of restrictions and pushforwards

Proposition (Regularity properties of restrictions; 1.36)

Let μ be a measure on the set X and $A \subset X$. The following properties hold:

1. *If X is a metric space, μ is a Borel regular measure on X and either 1) $A \in \mathscr{B}_X$ or 2) $A \in \sigma(\mu)$ and $\mu(A) < \infty$, then $\mu \llcorner A$ is Borel regular.*
2. *If X is a locally compact separable metric space, μ a Radon measure on X and either 1) $A \in \mathscr{B}_X$ or 2) $A \in \sigma(\mu)$ and $\mu(A) < \infty$, then $\mu \llcorner A$ is a Radon measure.*

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If both X and Y are separable locally compact metric spaces, f a continuous proper map and μ a Radon measure on X, then $f_{\#}\mu$ is a Radon measure on Y, and $\operatorname{spt} f_{\#}\mu = f(\operatorname{spt} \mu)$.

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Measurable maps

Definition (Measurable spaces and measurable maps; 1.39)

A *measurable space* is a pair $(X, \mathcal{M})$ where X is a set and $\mathcal{M}$ is a σ-algebra of subsets of X. The elements of $\mathcal{M}$ are called $\mathcal{M}$-*measurable* subsets of X.

Given measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, a map $f : X \to Y$ is called *measurable with respect to* $\mathcal{M}$ *and* $\mathcal{N}$ if, $\forall A \in \mathcal{N}$, $f^{-1}(A) \in \mathcal{M}$.

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If X (or Y) is a topological space, we shall tacitly assume that the σ-algebra $\mathcal{M}$ is the Borel σ-algebra $\mathscr{B}_X$. Thus, for instance:

- For X and Y topological spaces, a map $f : X \to Y$ is called *Borelian* or *Borel measurable* if it is measurable with respect to $\mathscr{B}_X$ and $\mathscr{B}_Y$.
- For $X = \mathbb{R}$ and Y a topological space (in particular, for $Y = \mathbb{R}$ or $\mathbb{C}$), a map $f : X \to Y$ is called *Lebesgue measurable* if it is measurable with respect to $\mathscr{L}$ and $\mathscr{B}_Y$, where $\mathscr{L}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$.

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Definition (μ-measurable maps)

Let μ be a measure on the set X and Y a topological space. A function $f : \text{dom } f \subset X \to Y$ is called *measurable with respect to* μ if the following conditions hold:

1. its domain covers almost all of X, i.e. $\mu(X \setminus \text{dom } f) = 0$,
2. for all $B \in \mathscr{B}_Y$, $f^{-1}(B)$ is μ-measurable.

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$f : \text{dom } f \subset X \to Y$ is measurable with respect to μ in the sense of the definition above iff any extension of f to a a map $X \to Y$ is measurable with respect to $\sigma(\mu)$ and $\mathscr{B}_Y$.

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Properties of measurable maps

Theorem (Properties of measurable maps; 1.41)

Let $(X,\mathcal{M})$, $(Y,\mathcal{N})$, $(Z,\mathcal{O})$ be measurable spaces. The following properties hold:

1. *$f: X \to Y$ is measurable* iff *given $S \subset 2^Y$ such that $\sigma(S) = \mathcal{N}$, for all $B \in S$, $f^{-1}(B) \in \mathcal{M}$.*
2. *If $f: X \to Y$ and $g: Y \to Z$ are both measurable maps, so is $g \circ f$.*
3. *If X and Y are topological spaces and $f: X \to Y$ is continuous, then it is Borelian.*
4. *For $Y = \overline{\mathbb{R}}$, if $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable maps $X \to \overline{\mathbb{R}}$, the following maps $X \to \overline{\mathbb{R}}$ are measurable: $\inf_{n\in\mathbb{N}} f_n$, $\sup_{n\in\mathbb{N}} f_n$, $\liminf f_n$, $\limsup f_n$. In particular, if $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent, the limit function is measurable. More generally, if Y is a metric space and $(f_n)_{n\in\mathbb{N}}$ is a pointwise convergent sequence of measurable maps $X \to Y$, the limit function is measurable.*

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Properties of measurable maps

Corollary (1.42)

If $f, g : X \to \overline{\mathbb{R}}$ *are both measurable, so are* $\max\{f, g\}$ *and* $\min\{f, g\}$*. In particular, both* $f^+ := \max\{f, 0\}$ *and* $f^- := \max\{-f, 0\}$ *are measurable.*

Induced σ-algebras

Definition (σ-algebra induced by a family of maps; 1.43)

Let X be a set, $(Y_\alpha, \mathcal{N}_\alpha)_{\alpha\in A}$ a family of measurable spaces and $(X \xrightarrow{f_\alpha} Y_\alpha)_{\alpha\in A}$ a family of maps defined on X. The smallest σ-algebra on X for which $\forall\alpha \in A$, f_α is measurable (i.e. the intersection of the family of σ-algebras which make all f_α's measurable maps) is called *σ-algebra induced by* $(f_\alpha)_{\alpha\in A}$, denoted by $\sigma\big((f_\alpha)_{\alpha\in A}\big)$.

Proposition (1.44)

With the notation from the previous definition, let $\mathcal{M} = \sigma\big((f_\alpha)_{\alpha\in A}\big)$.

i) *If $\forall\alpha \in A$, $\mathcal{N}_\alpha = \sigma(S_\alpha)$, then $\mathcal{M} = \sigma\big(\{V \subset X \mid \exists\alpha \in A, \exists D \in S_\alpha, V = f_\alpha^{-1}(D)\}\big)$.*

ii) *If $(Z, \mathcal{O})$ is a measurable space, then a map $g : Z \to X$ is measurable with respect to $\mathcal{O}$ and $\mathcal{M}$* iff *$\forall\alpha \in A$, $f_\alpha \circ g$ is measurable.*

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Induced σ-algebras

Examples of induced σ-algebras

Product σ-algebra Let $(X_\alpha, \mathcal{M}_\alpha)_{\alpha \in A}$ be a family of measurable spaces. On the product $X = \prod_{\alpha \in A} X_\alpha$, the σ-algebra induced by the family of projections $(\mathrm{pr}_\alpha : X \to X_\alpha)_{\alpha \in A}$ is called *product σ-algebra*, denoted by $\otimes_{\alpha \in A} \mathcal{M}_\alpha$.

Pullback Let $(Y, \mathcal{N})$ be a measurable space and $f : X \to Y$. The σ-agebra on X induced by $\{f\}$ is called *pullback* of $\mathcal{N}$, denoted by $f^*\mathcal{N}$.

Note that $f^*\mathcal{N} = \{f^{-1}(V) : V \in \mathcal{N}\}$. In particular, if $X \subset Y$ e $f = i$ is the inclusion $X \to Y$, the *pullback* $i^*\mathcal{N}$ coincides with $\{B \cap X : B \in \mathcal{N}\}$, called *restriction* or *trace* of $\mathcal{N}$ on X, usually denoted by $\mathcal{N}|_X$. In this situation, if $X \in \mathcal{N}$, then $\mathcal{N}|_X = \{B \in \mathcal{N} \mid B \subset X\}$.

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Products, pullbacks and traces

Products, pullbacks and traces

- a map taking values on a product of measurable spaces endowed with the product σ-algebra is measurable iff each of its components is measurable.
- if a map f takes values on a measurable space $(Y, \mathcal{N})$ and has its image contained in a subset X, than f is measurable iff it is measurable as a map taking values on X endowed with the trace σ-algebra.
- If X is a topological space and $A \subset X$, $\mathscr{B}_X|_A = \mathscr{B}_A$, i.e. the trace σ-algebra of $\mathscr{B}_X$ on A coincides with the Borel σ-algebra of A endowed with the relative topology.

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Products, pullbacks and traces

Proposition (Product of Borel σ-algebras; 1.47)

Let $(X_\alpha, \tau_\alpha)_{\alpha \in A}$ be a family of topological spaces and $X = \prod_{\alpha \in A} X_\alpha$ endowed with the product topology. Then:

i) $\bigotimes_{\alpha \in A} \mathscr{B}_{X_\alpha} \subset \mathscr{B}_X$.

ii) *Equality holds in the previous item if A is countable and $\forall \alpha \in A, \tau_\alpha$ is second countable.*

Products, pullbacks and traces

Corollary (1.48)

For any $n \in \mathbb{N}$, $\mathscr{B}_{\mathbb{R}^n} = \otimes_1^n \mathscr{B}_{\mathbb{R}}$. In particular, if $(X, \mathcal{M})$ is a measure space, a map $f = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ is measurable iff *each component f_i is measurable, $1 \leq i \leq n$.*

Corollary (1.49)

Let $(X, \mathcal{M})$ be a measurable space, Y a topological space, $f_1, \ldots, f_n : X \to \mathbb{R}$ measurable maps and $\Phi : \mathbb{R}^n \to Y$ Borelian. Then $\Phi(f_1, \ldots, f_n) : X \to Y$ is measurable. In particular, sums, products and differences of measurable maps $X \to \mathbb{R}$ are measurable.

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Measurable partitions

Proposition (1.50)

Let $(X, \mathcal{M})$, $(Y, \mathcal{N})$ be measurable spaces, and $(A_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{M}$ such that $\cup_{n\in\mathbb{N}} A_n = X$. Then a map $f : X \to Y$ is measurable iff $\forall n \in \mathbb{N}$, $f|_{A_n} : A_n \to Y$ is measurable, where each A_n is endowed with the trace σ-algebra.

Example (1.51)

Let $+ : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be arbitrarily defined on $\{(+\infty, -\infty), (-\infty, +\infty)\}$ and in the usual way on the complement of this set. Taking $A_1 = \{(+\infty, -\infty), (-\infty, +\infty)\}$, $A_2 = \mathbb{R} \times \{+\infty\} \cup \{+\infty\} \times \mathbb{R}$, $A_3 = \mathbb{R} \times \{-\infty\} \cup \{-\infty\} \times \mathbb{R}$ and $A_4 = \mathbb{R} \times \mathbb{R}$ in the previous preposition, is clear that $+$ is Borelian. Thus, if $(X, \mathcal{M})$ is a measurable space and $f, g : X \to \overline{\mathbb{R}}$ are measurable maps, so is $f + g$. We can treat similarly the difference, product and quotient of extended real valued measurable maps.

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Let $(X, \mathcal{M})$, $(Y, \mathcal{N})$ *be measurable spaces, and* $(A_n)_{n\in\mathbb{N}}$ *a sequence in* $\mathcal{M}$ *such that* $\cup_{n\in\mathbb{N}} A_n = X$. *Then a map* $f : X \to Y$ *is measurable iff* $\forall n \in \mathbb{N}$, $f|_{A_n} : A_n \to Y$ *is measurable, where each* A_n *is endowed with the trace* σ*-algebra.*

Example (1.51)

Let $+ : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be arbitrarily defined on $\{(+\infty, -\infty), (-\infty, +\infty)\}$ and in the usual way on the complement of this set. Taking $A_1 = \{(+\infty, -\infty), (-\infty, +\infty)\}$, $A_2 = \mathbb{R} \times \{+\infty\} \cup \{+\infty\} \times \mathbb{R}$, $A_3 = \mathbb{R} \times \{-\infty\} \cup \{-\infty\} \times \mathbb{R}$ and $A_4 = \mathbb{R} \times \mathbb{R}$ in the previous preposition, is clear that $+$ is Borelian. Thus, if $(X, \mathcal{M})$ is a measurable space and $f, g : X \to \overline{\mathbb{R}}$ are measurable maps, so is $f + g$. We can treat similarly the difference, product and quotient of extended real valued measurable maps.