Introduction to Geometric Measure Theory - Lecture Notes
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Contents

General Notations and Conventions

We list below some basic notation used for objects which are not defined in the text. A more detailed list, including the notation used for objects defined in the text (with references to the pages where each object was defined) may be found at the *list of symbols* at the end.

> **General convention for function spaces:** From chapter 4 on, all function spaces refer to spaces of real-valued functions, unless otherwise specified.

2^X	power set of X 283
$\cup A$	$\{y \mid \exists x \in A, y \in x\}$ 283
$\cap A$	$\{y \mid \forall x \in A, y \in x\}$ 283
$\cup_{\alpha \in A} X_\alpha$	the same as $\cup\{X_\alpha \mid \alpha \in A\}$ 283
$\cup_{\alpha \in A} X_\alpha$	the same as $\cap\{X_\alpha \mid \alpha \in A\}$ 283
$\overline{\mathbb{R}}$	$[-\infty, \infty]$ (extended real numbers) 283
$\mathbb{U}(x, r)$	open ball of center x and radius r 283
$\mathbb{B}(x, r)$	closed ball of center x and radius r 283
$\mathbb{U}$ or $\mathbb{U}^n$	open unit ball in $\mathbb{R}^n$ 283
$\mathbb{B}$ or $\mathbb{B}^n$	closed unit ball in $\mathbb{R}^n$ 283
$\mathbb{S}^n$	unit sphere in $\mathbb{R}^{n+1}$ 283
$\mathbb{C}(x, r, h)$	$\mathbb{U}(p \cdot x, r) \times \mathbb{U}(q \cdot x, h) \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, for $x = (p \cdot x, q \cdot x) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ 283
$\mathbb{C}(x, r)$	$\mathbb{C}(x, r, r)$ 283
$\overline{\mathbb{C}}(x, r, h)$	$\overline{\mathbb{C}(x, r, h)} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ 283
$\overline{\mathbb{C}}(x, r)$	$\overline{\mathbb{C}}(x, r, r)$ 283
$\overline{A}$	closure of A 283
A°	interior of A 283
A^c	complement of A 283
χ_A	characteristic function of A 283

$A \,\Delta\, B$	symmetric difference of A and B, i.e. $(A \setminus B) \dot\cup (B \setminus A)$ 283
$A \Subset X$	$\overline{A}$ is a compact subset of X 283
$\lVert x \rVert$	norm of x (in $\mathbb{R}^n$, the euclidean norm, unless otherwise specified) 283
$\lVert\cdot\rVert_u$	norm of uniform convergence. 283
sgn x	$\frac{x}{\lVert x\rVert}$ if $x \neq 0$ and 0 otherwise 283
$(e_1, \ldots, e_n)$	standard basis of $\mathbb{R}^n$ 283
$\alpha(m)$	$\dfrac{\pi^{m/2}}{\Gamma(m/2+1)}$ (euclidean volume of $\mathbb{B}^m$ if m integer) 283
$\limsup\limits_{x\to x_0} f(x)$	$\inf\limits_{\delta>0} \sup\limits_{x\in\mathbb{U}(x_0,r)} f(x) = \lim\limits_{\delta\to 0} \sup\limits_{x\in\mathbb{U}(x_0,r)} f(x)$ 283
$\liminf\limits_{x\to x_0} f(x)$	$\sup\limits_{\delta>0} \inf\limits_{x\in\mathbb{U}(x_0,r)} f(x) = \lim\limits_{\delta\to 0} \inf\limits_{x\in\mathbb{U}(x_0,r)} f(x)$ 283
LCH	locally compact Hausdorff space 283
LCS	locally compact separable metric space 283
$\mathsf{C_b}$	bounded continuous functions 283
$\mathsf{C^k_b}$	$\mathsf{C^k}$ functions with bounded derivatives up to order k 283
$\mathsf{C_c}$	continuous functions with compact support 283
$\mathsf{C^k_c}$	$\mathsf{C^k}$ functions with compact support 283
$\mathsf{C_0}$	continuous functions which vanish at infinity 283
$\mathsf{C^k_0}$	$\mathsf{C^k}$ functions whose derivatives up to order k vanish at infinity 283
$\mathsf{C^k}(\overline{U})$	$\mathsf{C^k}$ functions on U whose derivatives up to order k extend continuously to $\overline{U}$ 283
gr f	graph of f, i.e. $\{(x,y) \mid y = f(x)\}$ 283
epi f	epigraph of f, i.e. $\{(x,y) \in \text{dom } f \times \mathbb{R} \mid y \geq f(x)\}$ 283
epi_S f	strict epigraph of f, i.e. $\{(x,y) \in \text{dom } f \times \mathbb{R} \mid y > f(x)\}$ 283
hyp f	hypograph of f, i.e. $\{(x,y) \in \text{dom } f \times \mathbb{R} \mid y \leq f(x)\}$ 283
hyp_S f	strict hypograph of f, i.e. $\{(x,y) \in \text{dom } f \times \mathbb{R} \mid y < f(x)\}$ 283
$f \prec U$	$f \in \mathsf{C_c}(U)$ and $0 \leq f \leq 1$ 283
δ_a	Dirac measure centered at a. 283
$\check{f}$	$x \mapsto f(-x)$ 283
$\tau_y f$	$x \mapsto f(x-y)$ 283
Lip f	Lipschitz constant of f 283
$\partial^\alpha f$	For a multi-index $\alpha \in \mathbb{Z}^n_+$, $\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f$ 283
$\lvert\alpha\rvert$	For a multi-index $\alpha \in \mathbb{Z}^n$, $\alpha_1 + \cdots + \alpha_n$. 283
$\mathsf{D}f$	Fréchet derivative of f 283
$\Lambda(n,m)$	set of strictly increasing functions $\{1,\ldots,m\} \to \{1,\ldots,n\}$ 283

CHAPTER 1

Measure and Integration Theory

1.1. Measures

DEFINITION 1.1. A *measure* on a set X is a set function $\mu : 2^X \to [0, \infty]$ such that:

M1) $\mu(\emptyset) = 0$,

M2) (*monotonicity*) $\mu(A) \leq \mu(B)$ whenever $A \subset B$,

M3) (*countable subadditivity*) $\mu(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

WARNING. Our nomenclature is in accordance with the one commonly used in *Geometric Measure Theory*. However, most textbooks on *Real Analysis* (see, for instance, [**Fol99**]) call such a set function an *outer measure*, reserving the name *measure* for a countably additive set function defined on a σ-algebra $\mathcal{M}$ of subsets of X, as defined below in 1.6. We shall use the term "measure" for both types of set functions, if no confusion arises; if the context does not make it clear, we may use, for clarification, "measure on a σ-algebra" or "measure on $\mathcal{M}$" for countably additive set functions on σ-algebras, or "outer measure" for the set functions introduced in the previous definition.

DEFINITION 1.2. Given a measure μ on a set X, a subset $A \subset X$ is called *measurable with respect to* μ (or μ-*measurable*, or simply *measurable*) if it satisfies the *Carathéodory condition*:

$$\forall T \subset X, \mu(T) = \mu(T \cap A) + \mu(T \setminus A).$$

We denote by $\sigma(\mu)$ the set of measurable subsets of X with respect to μ.

EXAMPLE 1.3. The following are examples of measures:

1) Let X be a set and $\mu : 2^X \to [0, \infty]$ be defined by $\mu(A) := \operatorname{card}(A)$ if A is finite and $\mu(A) := \infty$ otherwise. Then μ is a measure on X, called *counting measure*, and it can be readily checked that $\sigma(\mu) = 2^X$.
2) Let X be a set, $a \in X$ and $\mu : 2^X \to [0, \infty]$ be defined by $\mu(A) := 1$ if $a \in A$ and $\mu(A) := 0$ otherwise. Then μ is a measure on X, called *Dirac measure centered at* a, denoted by δ_a (or simply δ if $a = 0$). It can be readily checked that $\sigma(\delta_a) = 2^X$.

3) Let $X = \mathbb{R}^n$ and $\mu : 2^X \to [0, \infty]$ be defined by $\mu(A) := \inf\{\sum_{Q \in \mathcal{A}} \mathrm{vol}(Q) \mid \mathcal{A}$ countable cover of A by cubes with sides parallel to the coordinate axes$\}$, where $\mathrm{vol}(Q)$ denotes the euclidean volume of the cube Q (which is not assumed to be open or closed, i.e. any product of intervals with the same length is a valid cube). Then μ is a measure on $\mathbb{R}^n$, called *Lebesgue measure*. We denote the Lebesgue measure on $\mathbb{R}^n$ by $|\cdot|$ or $\mathcal{L}^n$, and the set $\sigma(\mathcal{L}^n)$ of Lebesgue-measurable sets by $\mathscr{L}_{\mathbb{R}^n}$ (or simply $\mathscr{L}$).
4) Hausdorff measures, to be defined in section 2.

Exercise 1.4.

a) $\mathcal{L}^n$ is invariant by translations, i.e. $\forall x \in \mathbb{R}^n$, $\forall A \subset \mathbb{R}^n$, $\mathcal{L}^n(A + x) = \mathcal{L}^n(A)$.
b) $\mathcal{L}^n$ is homogeneous of degree n with respect to homotheties, i.e. $\forall \lambda > 0$, $\forall A \subset \mathbb{R}^n$, $\mathcal{L}^n(\lambda A) = \lambda^n \mathcal{L}^n(A)$.

Exercise 1.5. Let μ and ν be measures on X and $c > 0$. Then:

a) $\mu + \nu$ is a measure on X and $\sigma(\mu) \cap \sigma(\nu) \subset \sigma(\mu + \nu)$.
b) $c\mu$ is a measure on X and $\sigma(c\mu) = \sigma(\mu)$.

Definition 1.6. Given a set X, $\mathcal{M} \subset 2^X$ is called an *algebra* of subsets of X if it contains the empty set, it is closed under complementation and closed under finite unions. $\mathcal{M}$ is called a *σ-algebra* if it is an algebra closed under countable unions. The sets in $\mathcal{M}$ are called *measurable with respect to* $\mathcal{M}$, or $\mathcal{M}$*-measurable*, or (if clear from the context) simply *measurable*.

Given a σ-algebra $\mathcal{M} \subset 2^X$, we call a set function $\mu : \mathcal{M} \to [0, \infty]$ a *measure on* $\mathcal{M}$ if it satisfies:

M1) $\mu(\emptyset) = 0$,
M2) (*countable additivity*) $\mu(\dot\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu(A_n)$ (we use "$\dot\bigcup$" for disjoint unions).

A measure $\mu : \mathcal{M} \to [0, \infty]$ is called:

- *complete* if $E \in \mathcal{M}$, $\mu(E) = 0$ and $A \subset E$ implies $A \in \mathcal{M}$,
- *finite* if $\mu(X) < \infty$,
- *σ-finite* if there exists a sequence $(E_n)_{n \in \mathbb{N}}$ in $\mathcal{M}$ such that $\cup_{n \in \mathbb{N}} E_n = X$ and $\forall n \in \mathbb{N}, \mu(E_n) < \infty$. More generally, a set $A \subset X$ is said to be σ-finite if it can be covered by countably many measurable sets of finite measure.

Theorem 1.7 (Carathéodory). *If μ is a measure on a set X, then $\sigma(\mu)$ is a σ-algebra and the restriction of μ to $\sigma(\mu)$ is a complete measure on $\sigma(\mu)$.*

Thus, each measure determines a measure on a σ-algebra by restriction to its measurable sets. Conversely, each measure defined on a σ-algebra of subsets of X can be extended to a measure on X:

THEOREM 1.8. *If $\mathcal{M}$ is a σ-algebra of subsets of X and $\mu : \mathcal{M} \to [0, \infty]$ is a measure on $\mathcal{M}$, then the set function:*

$$\begin{array}{rccc} \mu^* : & 2^X & \longrightarrow & [0,\infty] \\ & A & \longmapsto & \inf\{\mu(E) \mid A \subset E \in \mathcal{M}\} \end{array}$$

is a measure which extends μ and such that $\mathcal{M} \subset \sigma(\mu^*)$.

The above theorem is a corollary of *Carathéodory's extension theorem* ([**Fol99**], proposition 1.13). Henceforth, whenever no confusion arises, we shall drop the "$*$" in the notation and denote by the same symbol both the measure μ on $\mathcal{M}$ and its induced measure on X.

DEFINITION 1.9. A measure $\mu : 2^X \to [0, \infty]$ is called:

- *regular*, if $\forall A \subset X$, $\exists E \in \sigma(\mu)$ such that $A \subset E$ and $\mu(A) = \mu(E)$.
- *finite* (respectively, σ-finite) if so is its restriction to $\sigma(\mu)$, cf. definition 1.6. We define similarly sets which are finite or σ-finite with respect to μ.

REMARK 1.10. i) If we depart from a measure $\mu : \mathcal{M} \to [0, \infty]$ defined on a σ-algebra $\mathcal{M} \subset 2^X$ and take its extension $\mu^* : 2^X \to [0, \infty]$ given by theorem 1.8, then the measure $\mu^* : \sigma(\mu^*) \to [0, \infty]$ is an extension of μ. It coincides with the *completion* of μ if μ is σ-finite (hence it is equal to μ if μ is complete and σ-finite). In general, this extension coincides with *saturation* of the completion of μ (see exercise 1.22 in [**Fol99**]).

ii) Similarly, if we depart from a measure $\mu : 2^X \to [0, \infty]$, take the measure on $\sigma(\mu)$ given by the restriction of μ to $\sigma(\mu)$, and then take the extension $\mu^* : 2^X \to [0, \infty]$ of the latter measure given by theorem 1.8, then μ^* is a regular measure which satisfies $\mu \leq \mu^*$. Equality holds iff μ is a regular measure, cf. exercise 1.20 in [**Fol99**].

PROPOSITION 1.11 (continuity properties of measures). *For a measure μ on X, the following properties hold:*

i) (continuity from below) *if $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\sigma(\mu)$, then $\mu(\cup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$,*

ii) (continuity from above) *if $(E_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $\sigma(\mu)$ and $\mu(E_1) < \infty$, then $\mu(\cap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$.*

EXERCISE 1.12. If μ is a regular measure on X, property i) in proposition 1.11 holds for any increasing sequence $(E_n)_{n\in\mathbb{N}}$ of subsets of X (i.e. the sets need not be measurable).

1.1.1. Operations on measures. Three useful ways of obtaining new measures from old are *restrictions, traces and pushforwards*:

DEFINITION 1.13 (Restrictions and traces of measures). Let μ be a measure on a set X and $A \subset X$. We define the:

- *restriction* of μ to A, denoted by $\mu \llcorner A$, as the measure $2^X \to [0,\infty]$ given by $E \mapsto \mu(A \cap E)$.
- *trace* of μ on A, denoted by $\mu|_A$, as the measure $2^A \to [0,\infty]$ given by $E \mapsto \mu(E)$, i.e the restriction to $2^A \subset 2^X$ of the map $\mu : 2^X \to [0,\infty]$.

Note that the restriction $\mu \llcorner A$ is a measure on X, whereas the trace $\mu|_A$ is a measure on A. Moreover, we do not assume A to be μ-measurable.

DEFINITION 1.14 (Pushforward of measures). Let μ be a measure on the set X and $f : X \to Y$ a map into the set Y. We define a measure $2^Y \to [0,\infty]$ on Y by:

$$A \subset Y \mapsto \mu\big(f^{-1}(A)\big),$$

called *pushforward of μ by f* and denoted by $f_\#\mu$.

PROPOSITION 1.15. *Let μ be a measure on the set X, $A \subset X$ and $f : X \to Y$. The following properties hold:*

i) $\sigma(\mu) \subset \sigma(\mu \llcorner A)$.

ii) If $E \in \sigma(\mu)$, then $E \cap A \in \sigma(\mu|_A)$. Besides, if $A \in \sigma(\mu)$, then $\sigma(\mu|_A) = \sigma(\mu) \cap 2^A = \{E \in \sigma(\mu) \mid B \subset A\}$.

iii) For $B \subset Y$, $f^{-1}(B)$ is μ-measurable iff *$\forall A \subset X$, B is $f_\#(\mu \llcorner A)$-measurable.*

PROOF.

i) Let $B \in \sigma(\mu)$. It follows from Carathéodory's condition in 1.2 that, for all $T \subset X$, $\mu \llcorner A(T) = \mu(A \cap T) = \mu(A \cap T \cap B) + \mu\big((A \cap T) \setminus B\big) = \mu \llcorner A(T \cap B) + \mu \llcorner A(T \setminus B)$, hence B is $\mu \llcorner A$-measurable.

ii) Let $B \in \sigma(\mu)$. It follows from Carathéodory's condition that, for all $T \subset X$, $\mu(T) = \mu(T \cap B) + \mu(T \setminus B)$. In particular, for all $T \subset A$, since $T \cap B = T \cap B \cap A$ and $T \setminus B = T \setminus (B \cap A)$: $\mu(T) = \mu(T \cap B) + \mu(T \setminus B) = \mu(T \cap B \cap A) + \mu\big(T \setminus (B \cap A)\big)$, hence $B \cap A$ is $\mu|_A$-measurable.

Besides, if $A \in \sigma(\mu)$ and $B \in \sigma(\mu|_A)$, for all $T \subset X$:

$$\mu(T) \overset{1}{=} \mu(T \cap A) + \mu(T \setminus A) \overset{2}{=}$$
$$= \mu(T \cap A \cap B) + \mu(T \cap A \setminus B) + \mu(T \setminus A) \overset{3}{=} \mu(T \cap B) + \mu(T \setminus B),$$

where we have used the μ-measurability of A in (1), the $\mu|_A$-measurability of B in (2) and again the μ-measurability of A in (3), which allows us to conclude that $\mu(T \cap A \setminus B) + \mu(T \setminus A) = \mu\big((T \setminus B) \cap A\big) + \mu\big((T \setminus B) \setminus A\big) = \mu(T \setminus B)$. Thus $B \in \sigma(\mu)$.

iii) Let $B \subset Y$ such that $f^{-1}(B)$ is μ-measurable. For all $A \subset X$, for all $S \subset Y$:

$$f_\#(\mu \llcorner A)(S) = \mu \llcorner A\big(f^{-1}(S)\big) = \mu\big(A \cap f^{-1}(S)\big) =$$
$$= \mu\big(A \cap f^{-1}(S) \cap f^{-1}(B)\big) + \mu\big(A \cap f^{-1}(S) \setminus f^{-1}(B)\big) =$$
$$= \mu\big(A \cap f^{-1}(S \cap B)\big) + \mu\big(A \cap f^{-1}(S \setminus B)\big) =$$
$$= f_\#(\mu \llcorner A)(S \cap B) + f_\#(\mu \llcorner A)(S \setminus B),$$

hence B is $f_\#(\mu \llcorner A)$-measurable.

Conversely, assume that $\forall A \subset X$, B is $f_\#(\mu \llcorner A)$-measurable. For all $T \subset X$: $\mu\big(T \cap f^{-1}(B)\big) + \mu\big(T \setminus f^{-1}(B)\big) = f_\#(\mu \llcorner T)(B) + f_\#(\mu \llcorner T)(Y \setminus B) = f_\#(\mu \llcorner T)(Y) = \mu \llcorner T(X) = \mu(T)$, hence $f^{-1}(B)$ is μ-measurable.

□

1.1.2. Measures on topological spaces. We now introduce a topology τ on the set X. We shall consider measures on X which interact with the topology, in the sense that they have nice regularity and approximation properties, to be made precise below. This will allow us to obtain theorems which make an interplay between topology and measure theory, one of the key ideas of Geometric Measure Theory.

Recall that, given a subset $S \subset 2^X$, there exists a smallest σ-algebra of subsets of X which contains S, that is, the intersection of the family of σ-algebras that contain S (this family is non-empty, since 2^X is such a σ-algebra). We denote this σ-algebra by $\sigma(S)$, the so-called σ-algebra generated by S.

DEFINITION 1.16. For a topological space (X, τ), we define its *Borel* σ-algebra as the σ-algebra generated by τ, i.e. $\sigma(\tau)$. We denote it by $\mathscr{B}_X$ or $\mathscr{B}(X)$. The elements of $\mathscr{B}_X$ are called *Borel sets*.

We say that a measure μ on X is a *Borel measure* if each Borel set is μ-measurable, i.e. if $\mathscr{B}_X \subset \sigma(\mu)$. A *Borel regular* measure on X is a Borel measure on X which satisfies: $\forall A \subset X$, $\exists E \in \mathscr{B}_X$ such that $A \subset E$ and $\mu(A) = \mu(E)$.

EXERCISE 1.17. Let μ and ν be measures on a topological space X and $c > 0$.

a) If μ and ν are Borel measures on X, so are $\mu + \nu$ and $c\mu$.
b) If μ and ν are Borel regular measures on X, so are $\mu + \nu$ and $c\mu$.

Note that, if $S \subset 2^X$ and $\mathcal{M}$ is a σ-algebra of subsets of X, then $\sigma(S) \subset \mathcal{M}$ iff $S \subset \mathcal{M}$. In particular, if (X, τ) is a topological space and μ is a measure on X, then μ is a Borel measure iff $\tau \subset \sigma(\mu)$, i.e. if each open subset of X is μ-measurable (or, equivalently, if each closed subset of X is μ-measurable). In case the topology be metrizable by a metric d, a simple criterion for a measure μ to be Borelian is given by the theorem below.

THEOREM 1.18 (Carathéodory's criterion). *A measure μ on a metric space (X, d) is Borel* iff

$$\mu(A \cup B) = \mu(A) + \mu(B) \tag{Ca}$$

whenever $A, B \subset X$ satisfy $d(A, B) := \inf\{d(a, b) \mid a \in A, b \in B\} > 0$.

PROOF. If μ is Borel and $d(A, B) > 0$, then $\overline{A} \cap B = \emptyset = A \cap \overline{B}$ and, by the measurability of $\overline{A}$, $\mu(A \cup B) = \mu\big((A \cup B) \cap \overline{A}\big) + \mu\big((A \cup B) \setminus \overline{A}\big) = \mu(A) + \mu(B)$.

Conversely, assume that condition (Ca) holds. In order to prove that μ is Borel, it suffices to prove that every closed set C is measurable. Equivalently, we must show that, for each $T \subset X$ such that $\mu(T) < \infty$, $\mu(T) \geq \mu(T \cap C) + \mu(T \setminus C)$ (what clearly implies Carathéodory's condition in definition 1.2, since the same inequality is trivial if $\mu(T) = \infty$ and the other inequality holds by subadditivity of μ).

For each $i \in \mathbb{N}$, let $C_i := \{x \in X \mid d(x, C) \leq 1/i\}$. Then $d(T \cap C, T \setminus C_i) \geq 1/i > 0$, so that the monotonicity of μ and condition (Ca) imply $\mu(T) \geq \mu\big((T \cap C) \cup (T \setminus C_i)\big) = \mu(T \cap C) + \mu(T \setminus C_i)$. Therefore, it suffices to prove that $\mu(T \setminus C_i) \xrightarrow{i \to \infty} \mu(T \setminus C)$.

For each $j \in \mathbb{N}$, let $T_j := T \cap \{x \in X \mid \frac{1}{j+1} < d(x, C) \leq \frac{1}{j}\}$. Due to the fact that C is closed, $d(x, C) > 0$ iff $x \in X \setminus C$, hence $\forall i \in \mathbb{N}$, $T \setminus C = T \setminus C_i \cup_{j=i}^{\infty} T_j$. Therefore, $\forall i \in \mathbb{N}$, $\mu(T \setminus C) \leq \mu(T \setminus C_i) + \sum_{j=i}^{\infty} \mu(T_j)$. The thesis then follows if we show that $\sum_{j=1}^{\infty} \mu(T_j) < \infty$, since this implies $\sum_{j=i}^{\infty} \mu(T_j) \xrightarrow{i \to \infty} 0$, so that $\mu(T \setminus C) \leq \liminf \mu(T \setminus C_i) \leq \limsup \mu(T \setminus C_i) \leq \mu(T \setminus C)$, where the last inequality holds by monotonicity.

Since $d(T_i, T_j) > 0$ if $i \neq j$ are both odd or both even, it follows from condition (Ca) that, for all $k \in \mathbb{N}$, $\mu(T) \geq \mu(\cup_{j=1}^k T_{2j}) = \sum_{j=1}^k \mu(T_{2j})$ and $\mu(T) \geq \mu(\cup_{j=1}^k T_{2j+1}) = \sum_{j=1}^k \mu(T_{2j+1})$, thus $\sum_{j=1}^\infty \mu(T_{2j}) \leq \mu(T)$ and $\sum_{j=1}^\infty \mu(T_{2j+1}) \leq \mu(T)$, from what we conclude that $\sum_{j=1}^\infty \mu(T_j) \leq 2\mu(T) < \infty$.

$\square$

EXAMPLE 1.19. The Lebesgue measure $\mathcal{L}^n$ is a Borel regular measure on $\mathbb{R}^n$. Indeed:

1) Let d be the euclidean distance in $\mathbb{R}^n$; we show $\mathcal{L}^n$ satisfies the Carathéodory criterion (CA). Given $A, B \subset \mathbb{R}^n$ such that $d(A, B) = \delta > 0$, let $\mathcal{A}$ be a countable cover of $A \cup B$ by cubes with sides parallel to the coordinate axes. Subdividing the sides of each cube in $\mathcal{A}$, if necessary, we may take another countable cover $\mathcal{A}'$ of $A \cup B$ formed by cubes of diameter less than $\delta/2$ and such that $\sum_{Q \in \mathcal{A}'} \operatorname{vol}(Q) = \sum_{Q \in \mathcal{A}} \operatorname{vol}(Q)$. Discarding the cubes of the latter family which do not intersect A or B, we obtain a subfamily $\mathcal{A}''$ which still covers $A \cup B$. In view of the choice of the diameters of the cubes in $\mathcal{A}'$, we can decompose $\mathcal{A}''$ in two disjoint subfamilies $\mathcal{A}'' = \mathcal{A}_1 \dot\cup \mathcal{A}_2$, where the cubes in $\mathcal{A}_1$ cover A and those in $\mathcal{A}_2$ cover B. It then follows that $\mathcal{L}^n(A) + \mathcal{L}^n(B) \leq \sum_{Q \in \mathcal{A}_1} \operatorname{vol}(Q) + \sum_{Q \in \mathcal{A}_2} \operatorname{vol}(Q) = \sum_{Q \in \mathcal{A}''} \operatorname{vol}(Q) \leq \sum_{Q \in \mathcal{A}'} \operatorname{vol}(Q) = \sum_{Q \in \mathcal{A}} \operatorname{vol}(Q)$. By the arbitrariness of the countable cover $\mathcal{A}$ of $A \cup B$ by cubes with sides parallel to the coordinate axes, we conclude that $\mathcal{L}^n(A) + \mathcal{L}^n(B) \leq \mathcal{L}^n(A \cup B)$, and the other inequality holds by finite subadditivity of $\mathcal{L}^n$. Thus, by theorem 1.18, $\mathcal{L}^n$ is a Borel measure.
2) Let $A \subset \mathbb{R}^n$ such that $\mathcal{L}^n(A) < \infty$. We contend that $\exists B \in \mathscr{B}_{\mathbb{R}^n}$ such that $A \subset B$ and $\mathcal{L}^n(A) = \mathcal{L}^n(B)$ (hence $\mathcal{L}^n$ is Borel regular). As a matter of fact, for each $n \in \mathbb{N}$, take a countable cover $\mathcal{A}_n$ of A by cubes with sides parallel to the coordinate axes such that $\mathcal{L}^n(A) + 1/n > \sum_{Q \in \mathcal{A}_n} \operatorname{vol}(Q)$. Take $B := \cap_{n \in \mathbb{N}} \cup_{Q \in \mathcal{A}_n} Q$, so that $A \subset B \in \mathscr{B}_{\mathbb{R}^n}$. Then, for each $n \in \mathbb{N}$, $\mathcal{A}_n$ covers B, so that $\forall n \in \mathbb{N}$, $\mathcal{L}^n(B) \leq \sum_{Q \in \mathcal{A}_n} \operatorname{vol}(Q) < \mathcal{L}^n(A) + 1/n$, whence $\mathcal{L}^n(B) \leq \mathcal{L}^n(A)$, and the other inequality holds by monotonicity of $\mathcal{L}^n$.

REMARK 1.20. Since the Lebesgue measure of each bounded cube Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes is finite (as it is $\leq \operatorname{vol}(Q) < \infty$, by definition; it actually coincides with $\operatorname{vol}(Q)$, but we postpone the proof of this fact to example 1.86, after the introduction of product measures), and since $\mathbb{R}^n$ is a countable union of such cubes, which are Borelian, it follows that the restriction of $\mathcal{L}^n$ to $\mathscr{B}_{\mathbb{R}^n}$ is a σ-finite measure on $\mathscr{B}_{\mathbb{R}^n}$. By the Borel regularity of $\mathcal{L}^n$, cf. example

1.19, the extension given by theorem 1.8 of $\mathcal{L}^n : \mathscr{B}_{\mathbb{R}^n} \to [0,\infty]$ is $\mathcal{L}^n$ itself. Therefore, by remark 1.10.(i), we conclude that the completion of $\mathcal{L}^n : \mathscr{B}_{\mathbb{R}^n} \to [0,\infty]$. is $\mathcal{L}^n : \mathscr{L} = \sigma(\mathcal{L}^n) \to [0,\infty]$.

Remark 1.21. The fact that the inclusions $\mathscr{B}_{\mathbb{R}^n} \subset \mathscr{L} \subset 2^{\mathbb{R}^n}$ are strict can be seen by cardinality arguments. Indeed, card $(\mathscr{B}_{\mathbb{R}^n})$=$\mathfrak{c}$ and card $(\mathscr{L}) = 2^{\mathfrak{c}}$, whence $\mathscr{B}_{\mathbb{R}^n} \subsetneqq \mathscr{L}$. As to the strictness of the other inclusion, it holds a more general result – see theorem 2.2.4 in [**Fed69**].

Definition 1.22. A Borel measure μ on a topological space (X,τ) is called:

- *open σ-finite* if there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of X such that $X = \cup_{n\in\mathbb{N}} U_n$ and $\forall n \in \mathbb{N}$, $\mu(U_n) < \infty$.
- *locally finite* if, for each $x \in X$, there exists an open neighborhood U of x such that $\mu(U) < \infty$.

It is clear that a locally finite Borel measure on a second countable topological space is open σ-finite.

As a rule of thumb, there are two main classes of measures which interact nicely with the topology: 1) locally finite Borel regular measures on separable metric spaces and 2) Radon measures (to be introduced in definition 1.28) on locally compact Hausdorff spaces. For instance, the approximation theorem below holds in the first case (and, by definition 1.28, similar approximation properties also hold for Radon measures). In later developments of the theory we shall be mainly interested in locally compact separable metric spaces (for instance, open subsets of $\mathbb{R}^n$ or, more generally, locally closed subsets of $\mathbb{R}^n$, like embedded submanifolds), for which the aforementioned classes of measures coincide, cf. exercise 1.32.

Theorem 1.23 (approximation by open and closed sets). *Let μ be an open σ-finite Borel regular measure on a topological space (X,τ) for which each closed set is a G_δ (i.e. a countable intersection of open sets). The following approximation properties hold:*

i) (approximation by open sets from the outside) $\forall A \subset X$, $\mu(A) = \inf\{\mu(U) \mid A \subset U \in \tau\}$,

ii) (approximation by closed sets from the inside) $\forall A \in \sigma(\mu)$, $\mu(A) = \sup\{\mu(C) \mid C \subset A, C \text{ closed}\}$.

Remark 1.24. The theorem holds, in particular, for a locally finite Borel regular measure on a separable metric space.

The proof is a consequence of the following lemmas.

Lemma 1.25. *Let X be a set, $S \subset 2^X$ and $\mathcal{F} \subset 2^X$ such that:*

- *$\mathcal{F}$ is closed under countable intersections and countable unions.*
- *If $A \in S$, both A and its complement A^c belong to $\mathcal{F}$.*

Then $\mathcal{F} \supset \sigma(S)$.

Proof. Let $\mathcal{G} := \{A \in \mathcal{F} \mid A^c \in \mathcal{F}\}$. Then:

1) $S \subset \mathcal{G}$.
2) $\mathcal{G}$ is closed under complementation.
3) $\mathcal{G}$ is closed under countable unions. Indeed, if $(A_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{G}$, then $\cup_{n\in\mathbb{N}} A_n \in \mathcal{F}$ and $(\cup_{n\in\mathbb{N}} A_n)^c = \cap_{n\in\mathbb{N}} A_n^c \in \mathcal{F}$.

Therefore, $\mathcal{G}$ is a σ-algebra which contains S, i.e. $\sigma(S) \subset \mathcal{G} \subset \mathcal{F}$. ☐

Corollary 1.26. *If X is a set and $S \subset 2^X$, $\sigma(S)$ is the smallest family of subsets of X closed under countable unions and countable intersections, which contains S and the complements of the elements of S.*

Lemma 1.27. *Let μ be a Borel measure on a topological space (X, τ) for which each closed set is a G_δ. If $B \in \mathscr{B}_X$ and $\mu(B) < \infty$, for all $\epsilon > 0$ there exists a closed set $C \subset B$ such that $\mu(B \setminus C) < \epsilon$.*

Proof. Define $\nu := \mu \llcorner B$. By proposition 1.15, ν is a finite Borel measure.

Let S be the family of all closed subsets of X and $\mathcal{F}$ the family of all ν-measurable sets $A \subset X$ such that $\forall \epsilon > 0$, $\exists C \subset A$ closed with $\nu(A \setminus C) < \epsilon$. We assert that $\mathcal{F} \supset \mathscr{B}_X$; in particular, that implies $B \in \mathcal{F}$, whence the thesis. The assertion follows once we show that $\mathcal{F}$ satisfies the hypotheses of lemma 1.25. Indeed:

- Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ and fix $\epsilon > 0$. For each $n \in \mathbb{N}$, $\exists C_n \subset A_n$ closed such that $\nu(A_n \setminus C_n) < 2^{-n}\epsilon$. Then, since both $\cap_{n\in\mathbb{N}} A_n \setminus \cap_{n\in\mathbb{N}} C_n$ and $\cup_{n\in\mathbb{N}} A_n \setminus \cup_{n\in\mathbb{N}} C_n$ are contained in $\cup_{n\in\mathbb{N}} (A_n \setminus C_n)$, it follows by subadditivity that:
  1) $\nu(\cap_{n\in\mathbb{N}} A_n \setminus \cap_{n\in\mathbb{N}} C_n) < \epsilon$ and $\cap_{n\in\mathbb{N}} C_n$ is closed, thus $\cap_{n\in\mathbb{N}} A_n \in \mathcal{F}$
  2) $\nu(\cup_{n\in\mathbb{N}} A_n \setminus \cup_{n\in\mathbb{N}} C_n) < \epsilon$. Since ν is finite and the sequence of ν-measurable sets $(\cup_{n\in\mathbb{N}} A_n \setminus \cup_{n=1}^k C_n)_{k\in\mathbb{N}}$ decreases to $\cup_{n\in\mathbb{N}} A_n \setminus \cup_{n\in\mathbb{N}} C_n$, by continuity from above 1.11 there exists $k \in \mathbb{N}$ such that $\nu(\cup_{n\in\mathbb{N}} A_n \setminus \cup_{n=1}^k C_n) < \epsilon$. As $\cup_{n=1}^k C_n$ is closed, this shows that $\cup_{n\in\mathbb{N}} A_n \in \mathcal{F}$.
- Since every closed subset of X is a G_δ, taking complements we conclude that every open subset of X is a F_σ, i.e. a countable union of closed sets. Thus, if $C \in S$, then $C \in \mathcal{F}$ and $X \setminus C \in \mathcal{F}$, since $\mathcal{F}$ is closed under countable unions by the previous item.

Hence, by lemma 1.25, $\mathcal{F} \supset \sigma(S) = \mathscr{B}_X$, as asserted. $\square$

PROOF OF THEOREM 1.23. Firstly, we prove part (ii). Let $A \in \sigma(\mu)$. Assume that $\mu(A) < \infty$. Since μ is Borel regular, $\exists B' \in \mathscr{B}_X$ such that $B' \supset A$ and $\mu(B') = \mu(A) < \infty$, hence $\mu(B' \setminus A) = 0$. Use the Borel regularity again to obtain $B'' \in \mathscr{B}_X$ such that $B'' \supset B' \setminus A$ and $\mu(B'') = \mu(B' \setminus A) = 0$. Then $B := B' \setminus B'' \in \mathscr{B}_X$ is such that $B \subset A$ and $\mu(B) = \mu(A)$. Applying lemma 1.27 for a given $\epsilon > 0$, we obtain a closed set $C \subset B$ such that $\mu(B \setminus C) = \mu(A \setminus C) < \epsilon$, which proves part (ii) in case A has finite measure. If $\mu(A) = \infty$, due to the fact that μ is σ-finite, there exists a disjoint sequence $(A_n)_{n \in \mathbb{N}}$ in $\sigma(\mu)$ such that $A = \dot{\cup}_{n \in \mathbb{N}} A_n$ and $\forall n \in \mathbb{N}$, $\mu(A_n) < \infty$. Given $\epsilon > 0$, for each $n \in \mathbb{N}$, apply the case just proved to obtain a closed set $C_n \subset A_n$ such that $\mu(A_n \setminus C_n) = \mu(A_n) - \mu(C_n) < 2^{-n}\epsilon$. Since $\sum_{n=1}^{\infty} \mu(A_n) = \mu(A) = \infty$, it then follows that $\sum_{n=1}^{\infty} \mu(C_n) = \infty$. Thus, for every $M > 0$, there exists $N \in \mathbb{N}$ such that the closed subset $C = \cup_{n=0}^{N} C_n$ of A has measure $\mu(C) = \sum_{n=1}^{N} \mu(C_n) > M$, i.e. $\sup\{\mu(C) \mid C \subset A, C \text{ closed}\} = \infty = \mu(A)$.

In order to prove part (i), we may assume, by Borel regularity, that $A \in \mathscr{B}_X$. Assume that there exists an open set V such that $V \supset A$ and $\mu(V) < \infty$. The thesis in this case follows from part (ii), passing to the complements: for a given $\epsilon > 0$, take a closed set $C \subset V \setminus A$ such that $\mu\big((V \setminus A) \setminus C\big) < \epsilon$. Then $U = V \setminus C$ is an open set which does the job: $U \supset A$ and $\mu(U \setminus A) < \epsilon$, since $U \setminus A = (V \setminus A) \setminus C$.

In the general case, given $A \in \mathscr{B}_X$, we use the hypothesis of μ being open σ-finite to obtain a sequence $(V_n)_{n \in \mathbb{N}}$ of open sets of finite measure such that $A \subset \cup_{n \in \mathbb{N}} V_n$. Fix $\epsilon > 0$. For each $n \in \mathbb{N}$, we may apply the case just proved to the Borel set $A \cap V_n \subset V_n$ to obtain an open set $U_n \supset A \cap V_n$ such that $\mu\big(U_n \setminus (A \cap V_n)\big) < 2^{-n}\epsilon$. Then $U = \cup_{n \in \mathbb{N}} U_n$ is an open set which cointains A and $U \setminus A \subset \cup_{n \in \mathbb{N}} \big(U_n \setminus (A \cap V_n)\big)$, thus $\mu(U \setminus A) < \epsilon$ by countable subadditivity.

$\square$

DEFINITION 1.28. A *Radon measure* on a locally compact Hausdorff topological space (X, τ) is a Borel measure μ on X such that:

R1) (*finiteness on compact sets*) if K is a compact subset of X, then $\mu(K) < \infty$,

R2) (*interior regularity for open sets*) for all $U \subset X$ open, $\mu(U) = \sup\{\mu(K) \mid K \subset U, K \text{ compact}\}$,

R3) (*exterior regularity*) for all $A \subset X$, $\mu(A) = \inf\{\mu(U) \mid A \subset U, U \text{ open}\}$.

REMARK 1.29.

i) Note that, by condition R2 in the definition above, every Radon measure is Borel regular.
ii) A measure $\mu : \mathscr{B}_X \to [0, \infty]$ is called a *Radon measure on* $\mathscr{B}_X$ if its extension $\mu^* : 2^X \to [0, \infty]$ given by theorem 1.8 is a Radon measure as defined above. That is equivalent to saying that μ satisfies R1, R2 and R3 for any Borel set $A \subset X$, what coincides with the usual definition of Radon measures in most Real Analysis textbooks (for instance, in [**Fol99**], section 7.1).

    It is clear that, if we depart from a Radon measure $\mu : 2^X \to [0, \infty]$, its restriction to $\mathscr{B}_X$ is an Radon measure on $\mathscr{B}_X$, whose extension given by theorem 1.8 is the original measure μ, thanks to its Borel regularity. We therefore obtain a bijection between the set of outer Radon measures on X and the set of Radon measures on $\mathscr{B}_X$, which associates each Radon outer measure to its restriction to $\mathscr{B}_X$. By means of this bijection, we may identify Radon outer measures on X and Radon measures on $\mathscr{B}_X$.

Exercise 1.30. If μ and ν are Radon measures on a locally compact Hausdorff space X and $c > 0$, then $\mu + \nu$ and $c\mu$ are Radon measures on X.

Exercise 1.31. If μ is a Radon measure on a locally compact Hausdorff space (X, τ), then μ is inner regular on all σ-finite μ-measurable sets, i.e. property R2 holds for any σ-finite μ-measurable set A in place of U. In particular, if μ is σ-finite, property R2 holds for all μ-measurable sets.

Exercise 1.32. Let X be a locally compact separable metric space. Then μ is a Radon measure on X iff μ is a locally finite Borel regular measure on X. Moreover, if μ is such a measure, then μ is σ-finite, hence it is inner regular on all μ-measurable sets by the previous exercise.

Remark 1.33. It follows from exercise 1.32 that, if X is a locally compact separable metric space and $\mu : \mathscr{B}_X \to [0, \infty]$ is a measure which is finite on compact subsets of X, then the extension of μ to a measure on X given by theorem 1.8 is a Radon measure (since it is a locally finite Borel regular measure on X). In particular, the measure μ on $\mathscr{B}_X$ is Radon, cf. remark 1.29.

Definition 1.34 (Support of a measure on a topological space). Let μ be a measure on a topological space X.

- We say that μ is *concentrated* on a set $A \subset X$ if $\mu(X \setminus A) = 0$.
- The *support* of μ, denoted by spt μ, is the complement of the union of all open sets $V \subset X$ such that $\mu(V) = 0$.

In the situation of the definition above, in general it is not true that μ is concentrated on its support, i.e that $\mu(X \setminus \text{spt } \mu) = 0$. The following proposition gives two sufficient conditions for that property to hold:

PROPOSITION 1.35. *If μ is a measure on a second countable topological space or if μ is a Radon measure on a locally compact Hausdorff topological space, then μ is concentrated on its support. Actually, its support is the smallest closed set on which μ is concentrated.*

PROOF. If μ is a measure on a second countable topological space X, by Lindelöf's theorem we may cover $X \setminus \text{spt } \mu$ by countably many open sets of measure zero, thus $\mu(X \setminus \text{spt } \mu) = 0$. If μ is a Radon measure on a locally compact Hausdorff topological space, for each compact $K \subset X \setminus \text{spt } \mu$, we may cover K with finitely many open sets of measure zero, hence $\mu(K) = 0$. By interior regularity, it follows that $\mu(X \setminus \text{spt } \mu) = \sup\{\mu(K) \mid K \subset X \setminus \text{spt } \mu, K \text{ compact}\} = 0$. In any of the two cases, its clear that spt μ the smallest closed set on which μ is concentrated.

$\square$

In the following propositions, we relate measurability and regularity properties of a measure μ and those of the measures obtained from μ by restriction or pushforward operations.

PROPOSITION 1.36. *Let μ be a measure on the set X and $A \subset X$. The following properties hold:*

i) If X is a metric space, μ is a Borel regular measure on X and either 1) $A \in \mathscr{B}_X$ or 2) $A \in \sigma(\mu)$ and $\mu(A) < \infty$, then $\mu \llcorner A$ is Borel regular.

ii) If X is a locally compact separable metric space, μ a Radon measure on X and either 1) $A \in \mathscr{B}_X$ or 2) $A \in \sigma(\mu)$ and $\mu(A) < \infty$, then $\mu \llcorner A$ is a Radon measure.

PROOF.

i) In both cases $\mu \llcorner A$ is a Borel measure, by proposition 1.15. We must show that it is Borel regular.

   1) Let $A \in \mathscr{B}_X$. Given $T \subset X$, we must show that $\exists B \in \mathscr{B}_X$ such that $B \supset T$ and $\mu \llcorner A(B) = \mu \llcorner A(T)$. Since μ is Borel regular, $\exists B' \in \mathscr{B}_X$ such that $B' \supset A \cap T$ and $\mu(B') = \mu(A \cap T)$. Since $B' \supset A \cap B' \supset A \cap T$, by monotonicity it follows $\mu(A \cap B') = \mu(A \cap T)$. Then, taking $B = B' \cup A^c \in \mathscr{B}_X$, we have $B \supset T$ and $\mu \llcorner A(B) = \mu(A \cap B) = \mu(A \cap B') = \mu(A \cap T) = \mu \llcorner A(T)$.

2) Let $A \in \sigma(\mu)$ with $\mu(A) < \infty$. Since μ is Borel regular, $\exists A' \in \mathscr{B}_X$ such that $A' \supset A$ and $\mu(A') = \mu(A)$. Since $A \in \sigma(\mu)$ has μ-finite measure, by finite additivity it follows that $\mu(A' \setminus A) = 0$, hence $\mu \llcorner A = \mu \llcorner A'$ is Borel regular by the previous item.

ii) By remark 1.29, μ is Borel regular. Hence, by the previous item, $\mu \llcorner A$ is Borel regular. Since μ is locally finite, so is $\mu \llcorner A$. Therefore, from exercise 1.32, we conclude that $\mu \llcorner A$ is Radon.

□

PROPOSITION 1.37. *If both X and Y are separable locally compact metric spaces, f a continuous proper map and μ a Radon measure on X, then $f_{\#}\mu$ is a Radon measure on Y, and* $\operatorname{spt} f_{\#}\mu = f(\operatorname{spt} \mu)$.

PROOF.

1) $f_{\#}\mu$ is a Borel measure. Indeed, if $U \subset Y$ is open, so is $f^{-1}(U)$, since f is continuous. In particular, $f^{-1}(U) \in \mathscr{B}_X \subset \sigma(\mu)$, hence $U \in \sigma(f_{\#}\mu)$ by proposition 1.15.
2) $f_{\#}\mu$ is locally finite. Indeed, if $K \subset Y$ is compact, so is $f^{-1}(K)$, since f is proper. Hence $f_{\#}\mu(K) = \mu\big(f^{-1}(K)\big) < \infty$. As Y is locally compact, the assertion follows.
3) $f_{\#}\mu$ is Borel regular (hence Radon, by the previous items and by exercise 1.32). Indeed, given $T \subset Y$, we apply the exterior regularity of μ on $f^{-1}(T)$ to obtain, for each $n \in \mathbb{N}$, $U_n \subset X$ open such that $U_n \supset f^{-1}(T)$ and $\mu(U_n) \leq \mu\big(f^{-1}(T)\big) + 1/n$. Since f is closed (because it is proper), $V_n = Y \setminus f(X \setminus U_n)$ is open in Y, $T \subset V_n$ and, noting that $f^{-1}\big(Y \setminus f(X \setminus U_n)\big) = X \setminus f^{-1}f(X \setminus U_n) \subset U_n$, $f_{\#}\mu(V_n) \leq \mu(U_n) \leq \mu\big(f^{-1}(T)\big) + 1/n = f_{\#}\mu(T) + 1/n$. Taking $V = \cap_{n\in\mathbb{N}} V_n \in \mathscr{B}_Y$, we then have $T \subset V$ and $f_{\#}\mu(T) = f_{\#}\mu(V)$, what proves the assertion.
4) Finally, we prove that $\operatorname{spt} f_{\#}\mu = f(\operatorname{spt} \mu)$. Firstly, since $0 = f_{\#}\mu(Y \setminus \operatorname{spt} f_{\#}\mu) = \mu\big(X \setminus f^{-1}(\operatorname{spt} f_{\#}\mu)\big)$, and since $X \setminus f^{-1}(\operatorname{spt} f_{\#}\mu)$ is open in X, it follows that $X \setminus f^{-1}(\operatorname{spt} f_{\#}\mu) \subset X \setminus \operatorname{spt} \mu$, hence (taking complements) $\operatorname{spt} \mu \subset f^{-1}(\operatorname{spt} f_{\#}\mu)$, from what we conclude that $f(\operatorname{spt} \mu) \subset \operatorname{spt} f_{\#}\mu$.

   On the other hand, $f_{\#}\mu\big(Y \setminus f(\operatorname{spt} \mu)\big) = \mu\big(X \setminus f^{-1}f(\operatorname{spt} \mu)\big) \leq \mu(X \setminus \operatorname{spt} \mu) = 0$, hence $\operatorname{spt} f_{\#}\mu$ is concentrated on $f(\operatorname{spt} \mu)$. Since f is closed, $f(\operatorname{spt} \mu)$ is a closed subset of Y, thus $\operatorname{spt} f_{\#}\mu \subset f(\operatorname{spt} \mu)$, and we had already proved the other inclusion, whence the thesis.

□

REMARK 1.38. Let X and Y be separable locally compact metric spaces, μ a Radon measure on X and $f : X \to Y$ a Borelian map, i.e.

such that $\forall B \in \mathscr{B}_Y$, $f^{-1}(B) \in \mathscr{B}_X$. If f is not a continuous proper map, $f_{\#}\mu$ may not be a Radon measure on Y (it might not be finite on compact sets, and even if it is, it might not be a Borel regular measure). However, if we add the hypothesis that, for all $K \subset Y$, $\mu\big(f^{-1}(K)\big) < \infty$, we may modify the definition of the pushforward in order to ensure that $f_{\#}\mu$ be a Radon measure on Y. Instead of taking the pushforward by f of the *outer* measure μ (i.e. the pushforward in the sense of definition 1.14), we take the pushforward by f of the measure $\mu : \mathscr{B}_X \to [0, \infty]$, i.e. the measure $f_{\#}\mu$ on $\mathscr{B}_Y$ given by $A \in \mathscr{B}_Y \mapsto \mu\big(f^{-1}(A)\big)$, which is finite on compact sets by the hypothesis assumed on f and μ. Then, by remarks 1.29 and 1.33, $f_{\#}\mu$ is a Radon measure on Y. Both definitions of $f_{\#}\mu$ coincide if f is a proper continuous map.

1.2. Measurable Maps

DEFINITION 1.39 (Measurable spaces and measurable maps). A *measurable space* is a pair $(X, \mathcal{M})$ where X is a set and $\mathcal{M}$ is a σ-algebra of subsets of X. The elements of $\mathcal{M}$ are called *$\mathcal{M}$-measurable* (or simply *measurable*, if $\mathcal{M}$ is clear from the context) subsets of X.

Given measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, a map $f : X \to Y$ is called *measurable with respect to $\mathcal{M}$ and $\mathcal{N}$* (or simply *measurable*, if $\mathcal{M}$ and $\mathcal{N}$ are clear from the context) if, $\forall A \in \mathcal{N}$, $f^{-1}(A) \in \mathcal{M}$.

If X (or Y) is a topological space, we shall tacitly assume that the σ-algebra $\mathcal{M}$ is the Borel σ-algebra $\mathscr{B}_X$, unless another σ-algebra is explicitly specified. Thus, for instance:

- For X and Y topological spaces, a map $f : X \to Y$ is called *Borelian* or *Borel measurable* if it is measurable with respect to $\mathscr{B}_X$ and $\mathscr{B}_Y$.
- For $X = \mathbb{R}$ and Y a topological space (in particular, for $Y = \mathbb{R}$ or $\mathbb{C}$), a map $f : X \to Y$ is called *Lebesgue measurable* if it is measurable with respect to $\mathscr{L}$ and $\mathscr{B}_Y$, where $\mathscr{L}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$.

DEFINITION 1.40 (μ-measurable maps). Let μ be a measure on the set X and Y a topological space. A function $f : \operatorname{dom} f \subset X \to Y$ is called *measurable with respect to μ* if the following conditions hold:

i) its domain covers almost all of X, i.e. $\mu(X \setminus \operatorname{dom} f) = 0$,
ii) for all $B \in \mathscr{B}_Y$, $f^{-1}(B)$ is μ-measurable.

Due to the fact that every subset of null measure of X is μ-measurable, a map $f : \operatorname{dom} f \subset X \to Y$ is measurable with respect to μ in the sense of the definition above iff any extension of f to a a map $X \to Y$ is measurable with respect to $\sigma(\mu)$ and $\mathscr{B}_Y$ in the sense of definition 1.39.

Moreover, if f is μ-measurable, any other function which coincides with f except for a set of null measure is also μ-measurable.

We list below some of the main properties of measurable maps.

THEOREM 1.41 (Properties of measurable maps). *Let $(X, \mathcal{M})$, $(Y, \mathcal{N})$, $(Z, \mathcal{O})$ be measurable spaces. The following properties hold:*

i) *$f : X \to Y$ is measurable* iff *given $S \subset 2^Y$ such that $\sigma(S) = \mathcal{N}$, for all $B \in S$, $f^{-1}(B) \in \mathcal{M}$.*
ii) *If $f : X \to Y$ and $g : Y \to Z$ are both measurable maps, so is $g \circ f$.*
iii) *If X and Y are topological spaces and $f : X \to Y$ is continuous, then it is Borelian.*
iv) *For $Y = \overline{\mathbb{R}}$, if $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable maps $X \to \overline{\mathbb{R}}$, the following maps $X \to \overline{\mathbb{R}}$ are measurable: $\inf_{n\in\mathbb{N}} f_n$, $\sup_{n\in\mathbb{N}} f_n$, $\liminf f_n$, $\limsup f_n$. In particular, if $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent, the limit function is measurable. More generally, if Y is a metric space and $(f_n)_{n\in\mathbb{N}}$ is a pointwise convergent sequence of measurable maps $X \to Y$, the limit function is measurable.*

PROOF.

i) The implication ($\Rightarrow$) is clear. On the other hand, $\mathcal{N}' := \{T \subset Y \mid f^{-1}(T) \in \mathcal{M}\}$ is clearly a σ-algebra of subsets of Y. Thus, if $S \subset \mathcal{N}'$, $\mathcal{N} = \sigma(S) \subset \mathcal{N}'$, i.e. f is measurable.
ii) $\forall A \in \mathcal{O}$, $(g \circ f)^{-1}(A) = f^{-1}\big(g^{-1}(A)\big) \in \mathcal{M}$.
iii) Since f is continuous, $\forall U \in \tau_Y$, $f^{-1}(U) \in \tau_X \subset \sigma(\tau_X) = \mathscr{B}_X$. As $\sigma(\tau_Y) = \mathscr{B}_Y$, it suffices to apply part i to $S = \tau_Y$.
iv) Let $g := \inf_{n\in\mathbb{N}} f_n$. For all $\alpha \in \mathbb{R}$, $g^{-1}([\alpha, \infty]) = \cap_{n\in\mathbb{N}} f_n^{-1}([\alpha, \infty]) \in \mathcal{M}$. Since $S := \{[\alpha, \infty] \mid \alpha \in \mathbb{R}\}$ generates $\mathscr{B}_{\overline{\mathbb{R}}}$, it follows from part i that g is measurable. Similarly, $\sup_{n\in\mathbb{N}} f_n$ is measurable, and so are $\liminf f_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} f_n$ and $\limsup f_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} f_n$.

   Finally, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable maps $X \to Y$ pointwise convergent to f. Then, for each open set $U \subset Y$, $f^{-1}(U) = \cup_{i\in\mathbb{N}} \cup_{j\in\mathbb{N}} \cap_{n\geq j} f_n^{-1}(\{x \in U \mid d(x, Y \setminus U) \geq 1/i\}) \in \mathcal{M}$, hence f is measurable.

□

COROLLARY 1.42. *If $f, g : X \to \overline{\mathbb{R}}$ are both measurable, so are $\max\{f, g\}$ and $\min\{f, g\}$. In particular, both $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$ are measurable.*

DEFINITION 1.43 (σ-algebra induced by a family of maps). Let X be a set, $(Y_\alpha, \mathcal{N}_\alpha)_{\alpha\in A}$ a family of measurable spaces and $(X \xrightarrow{f_\alpha} Y_\alpha)_{\alpha\in A}$ a family of maps defined on X. The smallest σ-algebra on X for which $\forall \alpha \in A$, f_α is measurable (i.e. the intersection of the family of σ-algebras which make all f_α's measurable maps) is called *σ-algebra induced by* $(f_\alpha)_{\alpha\in A}$, denoted by $\sigma\big((f_\alpha)_{\alpha\in A}\big)$.

PROPOSITION 1.44. *With the notation from definition 1.43, let* $\mathcal{M} = \sigma\big((f_\alpha)_{\alpha\in A}\big)$.

i) If $\forall\alpha \in A$, $\mathcal{N}_\alpha = \sigma(S_\alpha)$, *then* $\mathcal{M} = \sigma\big(\{V \subset X \mid \exists\alpha \in A, \exists D \in S_\alpha, V = f_\alpha^{-1}(D)\}\big)$.

ii) If $(Z, \mathcal{O})$ *is a measurable space, then a map* $g : Z \to X$ *is measurable with respect to* $\mathcal{O}$ *and* $\mathcal{M}$ iff $\forall\alpha \in A$, $f_\alpha \circ g$ *is measurable.*

PROOF.

i) Let $\mathcal{N} := \sigma\big(\{V \subset X \mid \exists\alpha \in A, \exists D \in S_\alpha, V = f_\alpha^{-1}(D)\}\big)$. It follows from theorem 1.41.i that $\forall\alpha \in A$, f_α is measurable with respect to $\mathcal{N}$ and $\mathcal{N}_\alpha$. Hence, $\mathcal{M} \subset \mathcal{N}$, and the other inclusion follows from the fact that $\{V \subset X \mid \exists\alpha \in A, \exists D \in S_\alpha, V = f_\alpha^{-1}(D)\} \subset \mathcal{M}$.

ii) $(\Rightarrow)$ follows from theorem 1.41 part ii. Conversely, if $\forall\alpha \in A$, $f_\alpha \circ g$ is measurable, then $\forall\alpha \in A$, $\forall D \in \mathcal{N}_\alpha$, $g^{-1}f_\alpha^{-1}(D) = (f_\alpha \circ g)^{-1}(D) \in \mathcal{O}$, thus g is measurable by the previous item and by theorem 1.41 part i.

$\square$

Particular cases of the above construction are:

Product σ-algebra: Let $(X_\alpha, \mathcal{M}_\alpha)_{\alpha\in A}$ be a family of measurable spaces. On the product $X = \prod_{\alpha\in A} X_\alpha$, the σ-algebra induced by the family of projections $(\mathsf{pr}_\alpha : X \to X_\alpha)_{\alpha\in A}$ is called *product σ-algebra*, denoted by $\otimes_{\alpha\in A}\mathcal{M}_\alpha$.

Pullback: Let $(Y, \mathcal{N})$ be a measurable space and $f : X \to Y$. The σ-agebra on X induced by $\{f\}$ is called *pullback* of $\mathcal{N}$, denoted by $f^*\mathcal{N}$.

Note that $f^*\mathcal{N} = \{f^{-1}(V) : V \in \mathcal{N}\}$. In particular, if $X \subset Y$ e $f = i$ is the inclusion $X \to Y$, the *pullback* $i^*\mathcal{N}$ coincides with $\{B \cap X : B \in \mathcal{N}\}$, called *restriction* or *trace* of $\mathcal{N}$ on X, usually denoted by $\mathcal{N}|_X$. In this situation, if $X \in \mathcal{N}$, then $\mathcal{N}|_X = \{B \in \mathcal{N} \mid B \subset X\}$.

REMARK 1.45. As an application of proposition 1.44, note that:

- a map taking values on a product of measurable spaces endowed with the product σ-algebra is measurable iff each of its components is measurable.
- if a map f takes values on a measurable space $(Y, \mathcal{N})$ and has its image contained in a subset X, than f is measurable iff it is measurable as a map taking values on X endowed with the trace σ-algebra.

- If X is a topological space and $A \subset X$, $\mathscr{B}_X|_A = \mathscr{B}_A$, i.e. the trace σ-algebra of $\mathscr{B}_X$ on A coincides with the Borel σ-algebra of A endowed with the relative topology.

EXERCISE 1.46. If X is a locally compact separable metric space, μ a Radon measure on X and $A \subset X$ is locally compact in the relative topology (i.e. A is a locally closed subspace), then $\mu|_A$ is a Radon measure on A.

PROPOSITION 1.47 (Product of Borel σ-algebras). *Let $(X_\alpha, \tau_\alpha)_{\alpha\in A}$ be a family of topological spaces and $X = \prod_{\alpha\in A} X_\alpha$ endowed with the product topology. Then:*

i) $\otimes_{\alpha\in A} \mathscr{B}_{X_\alpha} \subset \mathscr{B}_X$.
ii) Equality holds in the previous item if A is countable and $\forall\alpha \in A, \tau_\alpha$ is second countable.

PROOF. For each $\alpha \in A$, the projection $\prod_{\alpha\in A} X_\alpha \to X_\alpha$ is continuous. Hence, by theorem 1.41.iii, it is measurable with respect to $\mathscr{B}_X$ and $\mathscr{B}_{X_\alpha}$. It then follows from proposition 1.44.ii that the identity $X \to X$ is measurable with respect to $\mathscr{B}_X$ and $\otimes_{\alpha\in A} \mathscr{B}_{X_\alpha}$, i.e. $\otimes_{\alpha\in A} \mathscr{B}_{X_\alpha} \subset \mathscr{B}_X$.

On the other hand, assume that A is countable and $\forall\alpha \in A, \tau_\alpha$ is second countable, so that the product topology on X is second countable. We may take a countable base for this topology formed by rectangles $\prod_{\alpha\in A} U_\alpha$ where each U_α is open in X_α and, except for finetely many α's, $U_\alpha = X_\alpha$. Since each such rectangle is measurable with respect to $\otimes_{\alpha\in A} \mathscr{B}_{X_\alpha}$, it follows that every open set in the product topology, being a countable union of such rectangles, is measurable with respect to $\otimes_{\alpha\in A} \mathscr{B}_{X_\alpha}$, thus $\mathscr{B}_X \subset \otimes_{\alpha\in A} \mathscr{B}_{X_\alpha}$. ☐

COROLLARY 1.48. *For any $n \in \mathbb{N}$, $\mathscr{B}_{\mathbb{R}^n} = \otimes_1^n \mathscr{B}_{\mathbb{R}}$. In particular, if $(X, \mathcal{M})$ is a measure space, a map $f = (f_1, \dots, f_n) : X \to \mathbb{R}^n$ is measurable* iff *each component f_i is measurable, $1 \leq i \leq n$.*

It follows from the corollary above, identifying $\mathbb{C} \equiv \mathbb{R}^2$ as metric spaces, that a function $f : X \to \mathbb{C}$ is measurable iff both $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable.

COROLLARY 1.49. *Let $(X, \mathcal{M})$ be a measurable space, Y a topological space, $f_1, \dots, f_n : X \to \mathbb{R}$ measurable maps and $\Phi : \mathbb{R}^n \to Y$ Borelian. Then $\Phi(f_1, \dots, f_n) : X \to Y$ is measurable. In particular, sums, products and differences of measurable maps $X \to \mathbb{R}$ are measurable.*

So is the quotient of measurable maps, as long as the denominator is never zero, but see example 1.51, below.

The following is a useful criterion for testing measurability in terms of countable measurable covers.

PROPOSITION 1.50. *Let $(X, \mathcal{M})$, $(Y, \mathcal{N})$ be measurable spaces, and $(A_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{M}$ such that $\cup_{n\in\mathbb{N}} A_n = X$. Then a map $f : X \to Y$ is measurable* iff *$\forall n \in \mathbb{N}$, $f|_{A_n} : A_n \to Y$ is measurable, where each A_n is endowed with the trace σ-algebra.*

PROOF. For each $B \in \mathcal{N}$, $f^{-1}(B) = \cup_{n\in\mathbb{N}} f|_{A_n}^{-1}(B) \in \mathcal{M}$, since $\forall n \in \mathbb{N}$, $\mathcal{M}|_{A_n} \subset \mathcal{M}$, due to the fact that $\forall n \in \mathbb{N}$, $A_n \in \mathcal{M}$. $\square$

The following example shows how the proposition above may be applied:

EXAMPLE 1.51.

1) Let $\operatorname{sgn} : \mathbb{C} \to \mathbb{C}$ be defined by $\operatorname{sgn} z := z/|z|$ if $z \neq 0$ and $\operatorname{sgn} 0 := 0$. Taking $X = \mathbb{C}$, $A_1 = \{0\}$ and $A_2 = \mathbb{C} \setminus \{0\}$ in proposition 1.50, it is clear that sgn is measurable (note that the trace σ-algebra on A_2 coincides with its Borel σ-algebra as a metric subspace of $\mathbb{C}$, and that $\operatorname{sgn}|_{A_2}$ is continuous). Thus, if $(X, \mathcal{M})$ is a measurable space, each measurable function $f : X \to \mathbb{C}$ admits a polar decomposition $f = \operatorname{sgn} f \cdot |f|$, where each factor is measurable.
2) Let $+ : \overline{\mathbb{R}}\times\overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be arbitrarily defined on $\{(+\infty, -\infty), (-\infty, +\infty)\}$ and in the usual way on the complement of this set. Taking $A_1 = \{(+\infty, -\infty), (-\infty, +\infty)\}$, $A_2 = \mathbb{R} \times \{+\infty\} \cup \{+\infty\} \times \mathbb{R}$, $A_3 = \mathbb{R} \times \{-\infty\} \cup \{-\infty\} \times \mathbb{R}$ and $A_4 = \mathbb{R} \times \mathbb{R}$ in preposition 1.50, is clear that $+$ is Borelian. Thus, if $(X, \mathcal{M})$ is a measurable space and $f, g : X \to \overline{\mathbb{R}}$ are measurable maps, so is $f + g$. We can treat similarly the difference, product and quotient of extended real valued measurable maps.

We now focus our attention in a class of measurable maps which will be used to develop the integration theory of a measure μ on a set X.

DEFINITION 1.52 (Simple functions). Let X and Y be measurable spaces. A function $\varphi : X \to Y$ is called *simple* if it is measurable and has finite image.

In the next section we shall be concerned with simple functions taking values in $\mathbb{R}$ or $\mathbb{C}$.

PROPOSITION 1.53 (properties of simple functions). *Let $(X, \mathcal{M})$ be a measure space.*

i) The set of all $\mathbb{C}$-valued simple functions on X is a subalgebra of $\mathbb{C}^X$.

ii) If $f : X \to [0,\infty]$ *is measurable, there exists an increasing sequence* $(\varphi_n)_{n\in\mathbb{N}}$ *of simple functions* $X \to [0,\infty)$ *which converges pointwise to* f *and such that the convergence is uniform on each part where* f *is bounded.*

iii) If $f : X \to \mathbb{C}$ *is measurable, there exists a sequence* $(\varphi_n)_{n\in\mathbb{N}}$ *of simple functions* $X \to \mathbb{C}$ *which converges pointwise to* f*, and such that* $\forall n \in \mathbb{N}$*,* $|\varphi_n| \leq |\varphi_{n+1}| \leq |f|$ *and the convergence is uniform on each part where* f *is bounded.*

Exercise 1.54. Let $(X, \mathcal{M})$ be a measurable space, $f : X \to [0,\infty]$ measurable, $(r_n)_{n\in\mathbb{N}}$ a sequence in $(0,\infty)$ such that $r_n \to 0$ e $\sum_{i=1}^{\infty} r_n = \infty$. Then there exists a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ such that $\sum_{k=1}^{n} r_k\chi_{A_k}$ increases pointwise to f.

Hint. Define $(A_k)_{k\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$ inductively by: 1) $A_1 := \{x \in X \mid r_1 \leq f(x)\}$ and $g_1 := r_1\chi_{A_1}$; 2) $A_k := \{x \in X \mid g_{k-1}(x)+r_k \leq f(x)\}$ and $g_k := g_{k-1} + r_k\chi_{A_k}$.

Exercise 1.55. Let $(X, \mathcal{M})$ be a measurable space, Y a separable metric space and $f : X \to Y$ a measurable map. Then there exists a sequence of simple functions $X \to Y$ which converges pointwise to f.

We end this section with a definition of support for measurable functions on a topological space endowed with a Borel measure which is often more natural from a measure-theoretic point of view than the usual definition of support.

Definition 1.56 (Support and essential support). Let X be a topological space endowed with a Borel measure μ and f a measurable function on X.

i) The *support of* f, denoted by $\operatorname{spt} f$, is the complement in X of the union of all open sets on which f is null[1].

ii) The *essential support of* f, denoted by $\operatorname{ess\,spt} f$, is the complement in X of the union of all open sets on which f is μ-a.e. null.

Remark 1.57 (Support and essential support). With the notation from the previous definition:

1) It is clear that $\operatorname{ess\,spt} f \subset \operatorname{spt} f$.
2) If f is continuous on X and $\operatorname{spt} \mu = X$, then $\operatorname{ess\,spt} f = \operatorname{spt} f$.
3) If X is second countable, or if X is locally compact Hausdorff and μ is Radon, then $\operatorname{ess\,spt} f$ is the complement of the biggest open set on which f is null μ-a.e.

[1] Actually this definition makes sense for arbitrary functions on topological spaces, not necessarily endowed with measures.

4) We adopt the convention that, henceforth, "support of f" means "essential support of f", which will be denoted accordingly by "spt f".

1.3. Integration Theory

Up to the end of this section, we fix a measure μ on the set X. The restriction of μ to $\sigma(\mu)$ yields a classical *measure space* $(X, \sigma(\mu), \mu)$, for which an integration theory is developed in standard *Real Analysis* textbooks. For the sake of completeness, we list some definitions and theorems below and refer the reader elsewhere for more details.

In the theory of integration described below we consider measurable functions on X taking values in $\overline{\mathbb{R}}$ or $\mathbb{C}$. We denote by $\mathsf{L}^+(\mu)$ the set of μ-measurable functions on X taking values in $[0, \infty]$.

Definition 1.58. For a simple function $\varphi \in \mathsf{L}^+(\mu)$, i.e. for φ simple and taking values in $[0, \infty)$, let $\operatorname{Im} \varphi = \{a_1, \dots, a_n\}$, with $a_i \neq a_j$ if $i \neq j$, so that $\varphi = \sum_{i=1}^n a_i \chi_{\varphi^{-1}(a_i)}$ (which is the so-called *standard form* or *standard representation* of the simple function φ). We define the *integral of* φ with respect to μ by:

$$\int \varphi \, \mathrm{d}\mu := \sum_{i=1}^n a_i \mu\big(\varphi^{-1}(a_i)\big) \in [0, \infty],$$

where we use the convention $0 \cdot \infty := 0$.

For an arbitrary $f \in \mathsf{L}^+(\mu)$, we now define:

$$\int f \, \mathrm{d}\mu := \sup\{\int \varphi \, \mathrm{d}\mu \mid \varphi \in \mathsf{L}^+ \text{ simple }, \varphi \leq f\} \in [0, \infty].$$

One can check that, whenever $f, g \in \mathsf{L}^+(\mu)$ and $c \in [0, \infty)$, $\int (f + g) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$ and $\int cf \, \mathrm{d}\mu = c \int f \, \mathrm{d}\mu$.

For a μ-measurable function f taking values in $\overline{\mathbb{R}}$, we consider the positive and negative parts of f, i.e. $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$; according to corollary 1.42, they are both measurable and satisfy $f = f^+ - f^-$, $|f| = f^+ + f^-$. We say that f is *integrable* if $\int f^+ \, \mathrm{d}\mu < \infty$ or $\int f^- \, \mathrm{d}\mu < \infty$; if so, we define $\int f \, \mathrm{d}\mu := \int f^+ \, \mathrm{d}\mu - \int f^+ \, \mathrm{d}\mu \in \overline{\mathbb{R}}$. We say that f is *summable* if both $\int f^+ \, \mathrm{d}\mu < \infty$ and $\int f^- \, \mathrm{d}\mu < \infty$ (or, equivalently, if $\int |f| \, \mathrm{d}\mu < \infty$), i.e. if f is integrable and $\int f \, \mathrm{d}\mu \in \mathbb{R}$.

As it is usual, henceforth we omit the "μ" in the notation whenever the measure is clear from the context. For a measurable set $E \subset X$, we define $\int_E f := \int \chi_E f$.

Finally, a μ-measurable $\mathbb{C}$-valued function f is called *summable* if $\int |f| < \infty$ (or, equivalently, if both real and imaginary parts of f are summable). For such a function, we define $\int f := \int \operatorname{Re} f + i \int \operatorname{Im} f \in \mathbb{C}$. We denote by $\mathsf{L}^1(\mu)$ the set of summable functions $f : X \to \mathbb{C}$.

If μ is the counting measure on a set X and f is an integrable function on X, we use the notation $\sum_{x\in X} f(x)$ for $\int f \, \mathrm{d}\mu$, called *unordered sum* of f.

Exercise 1.59. Let If μ be the counting measure on a set X and $f : X \to [0, \infty]$. Then

$$\sum_{x\in X} f(x) = \sup\Big\{\sum_{x\in F} f(x) \mid F \subset X \text{ finite}\Big\}.$$

Moreover, if $\sum_{x\in X} f(x) < \infty$, then $\{x \in X \mid f(x) > 0\}$ is countable.

Warning. Some authors use the nomenclature "almost integrable" for what we have called "integrable" and "integrable" for what we have called "summable".

We summarize the main properties of the integral defined above in the theorems that follow. As it is usual, we say that a property P which refers to points of X holds μ-*almost everywhere* (or simply *almost everywhere* if the measure is clear from the context), with notation "P μ-a.e." or "P a.e. $[\mu]$", if the set of the points at which P does not hold has measure zero.

Theorem 1.60 (properties of the integral). *The following properties for the integral defined in 1.58 hold:*

i) $\mathsf{L}^1(\mu)$ *is a complex vector space and the integral is a linear functional on it.*
ii) *If* $f \in \mathsf{L}^+$, *then* $\int f = 0$ iff $f = 0$ μ*-a.e.*
iii) *If* f *and* g *are integrable and* $f = g$ *a.e., then* $\int f = \int g$. *If* $f \le g$ *a.e., then* $\int f \le \int g$.
iv) (integral triangle inequality) *If* $f \in \mathsf{L}^1$, *then* $|\int f| \le \int |f|$.
v) $\forall f \in \mathsf{L}^1(\mu)$, $\|f\|_1 := \int |f| \, \mathrm{d}\mu$ *defines a seminorm on* $\mathsf{L}^1(\mu)$.

It follows from ii, above, that the linear subspace $N := \{f \in \mathsf{L}^1(\mu) \mid \|f\|_1 = 0\}$ of $\mathsf{L}^1(\mu)$ consists of the measurable functions on X which are null almost everywhere. The elements of the quotient $\mathsf{L}^1(\mu)/N$ are, therefore, classes of equivalence of summable functions which coincide almost everywhere, and $\|\cdot\|_1$ is a norm on this quotient. The fact that this norm is complete (so that $\mathsf{L}^1(\mu)/N$ is a Banach space) is a consequence of the convergence theorems 1.62 and 1.64 stated below.

Remark 1.61. As it is usual, we shall, henceforth, overload the notation "$\mathsf{L}^1(\mu)$", which will be used both with its original meaning and also to denote the aforementioned quotient space. That is, whenever we write "$f \in \mathsf{L}^1(\mu)$", it may signify, depending on the context, that f is a summable function or that f is a class of equivalence of summable functions which coincide almost everywhere. A similar remark applies to the L^p spaces, to be introduced in subsection 1.3.1, below.

THEOREM 1.62 (monotone convergence theorem). *Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathsf{L}^+(\mu)$, which converges μ-a.e. to $f \in \mathsf{L}^+(\mu)$. Then $\int f_n \to \int f$.*

THEOREM 1.63 (Fatou's lemma). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathsf{L}^+(\mu)$. Then $\int \liminf f_n \leq \liminf \int f_n$.*

THEOREM 1.64 (dominated convergence theorem). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathsf{L}^1(\mu)$ dominated by a summable function g, i.e such that $\forall n \in \mathbb{N}$, $|f_n| \leq g$. If $(f_n)_{n\in\mathbb{N}}$ converges pointwise almost everywhere to f, then $f \in \mathsf{L}^1(\mu)$ and $\int f_n \to \int f$.*

COROLLARY 1.65. *With the same hypothesis, $f_n \to f$ in $\mathsf{L}^1(\mu)$.*

PROOF. $|f_n - f|$ converges pointwise almost everywhere to zero and the convergence is dominated by $2g$, hence $\|f_n - f\|_1 = \int |f_n - f| \to 0$. □

Thus, dominated pointwise almost everywhere convergence implies convergence in L^1. On the other hand, without additional hypotheses we cannot recover pointwise almost everywhere convergence from L^1 convergence, but we can ensure the pointwise almost everywhere convergence of a subsequence, i.e. if $(f_n)_n$ converges to f in $\mathsf{L}^1(\mu)$, there exists a subsequence of (f_n) which converges pointwise almost everywhere to f.

The following improved version of the dominated convergence theorem often comes in handy:

THEOREM 1.66 (generalized dominated convergence theorem). *Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be sequences in $\mathsf{L}^1(\mu)$ such that:*

i) $\forall n \in \mathbb{N}$, $|f_n| \leq g_n$ μ-a.e.
ii) $f_n \to f$ pointwise a.e. and $g_n \to g$ pointwise a.e.
iii) $\int g_n \to \int g < \infty$.

Then $f \in \mathsf{L}^1(\mu)$ and $\int f_n \to \int f$.

An important application of the dominated convergence theorem is related to the study of continuity and differentiability of functions defined by integrals. For instance, the following proposition is a direct consequence of the dominated convergence theorem:

PROPOSITION 1.67 (differentiation under the integral sign using the dominated convergence theorem). *Let $I \subset \mathbb{R}$ be a nondegenerate interval and $f : X \times I \to \mathbb{R}$ such that $\forall t \in I$, $f(\cdot, t) \in \mathsf{L}^1(\mu)$. Let $F : I \to \mathbb{R}$ defined by $F(t) := \int f(x,t)\,\mathrm{d}\mu(x)$.*

i) Let $t_0 \in I$ and suppose that $\forall x \in X$, $\exists \lim_{t\to t_0} f(x,t)$ and there exists $g \in \mathsf{L}^1(\mu)$ such that $|f(x,t)| \leq g(x)$ for all (x,t). Then

$$\lim_{t\to t_0} \int f(x,t)\,\mathrm{d}\mu(x) = \int \lim_{t\to t_0} f(t,x)\,\mathrm{d}\mu(x).$$

In particular, F is continuous at t_0 if $\forall x \in X$, $f(x,\cdot)$ is continuous in t_0.

ii) Suppose that exists $\frac{\partial f}{\partial t}$ and there exists $g \in \mathsf{L}^1(\mu)$ such that $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq g(x)$ for all (x,t). Then F is differentiable and

$$F'(t) = \int \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}\mu(x).$$

Similar statements hold if we replace the parameter interval I by an open subset of $\mathbb{R}^k$.

EXERCISE 1.68 (upper and lower integrals). We call $\varphi : X \to \mathbb{C}$ an *extended simple function* if it is μ-measurable and its image is countable. For a function $f : X \to [0,\infty]$, not necessarily measurable, we define:

- (*upper integral*) $\int^* f\,\mathrm{d}\mu := \inf\{\int \varphi\,\mathrm{d}\mu \mid \varphi \in \mathsf{L}^+ \text{ extended simple}, \varphi \geq f \text{ a.e.}\} \in [0,\infty]$,
- (*lower integral*) $\int_* f\,\mathrm{d}\mu := \sup\{\int \varphi\,\mathrm{d}\mu \mid \varphi \in \mathsf{L}^+ \text{ extended simple}, \varphi \leq f \text{ a.e.}\} \in [0,\infty]$.

Prove that:

a) If $\int_* f = \int^* f < \infty$, then f is μ-measurable and $\int f$ coincides with both upper and lower integrals (hence $f \in \mathsf{L}^1$).

b) If f is μ-measurable, then $\int_* f = \int^* f = \int f$.

c) The monotone convergence theorem holds for the upper integral, i.e. if $(f_n)_{n\in\mathbb{N}}$ is a sequence of positive functions (not necessarily measurable) which increases μ-a.e. to a function f (not necessarily measurable), then $\int^* f_n \to \int^* f$. Similarly, Fatou's lemma also holds for the upper integral.

d) If $(f_n)_{n\in\mathbb{N}}$ is a sequence of positive functions (not necessarily measurable) such that $\int^* f_n \to 0$, there exists a subsequence of $(f_n)_{n\in\mathbb{N}}$ which converges pointwise almost everywhere to zero.

EXERCISE 1.69. Let μ be a measure on a set X and $A \subset X$.

a) If $f \in \mathsf{L}^+(\mu)$, then $f \in \mathsf{L}^+(\mu \llcorner A)$, $f|_A \in \mathsf{L}^+(\mu|_A)$ and $\int f\,\mathrm{d}(\mu \llcorner A) = \int f|_A\,\mathrm{d}(\mu|_A)$.

b) If $A \in \sigma(\mu)$, both integrals in the previous item coincide with $\int_A f \, \mathrm{d}\mu$.

Exercise 1.70. Let μ be a measure on the set X and $f : X \to Y$ a map into the set Y. If $g \in \mathsf{L}^+(f_\# \mu)$, then $g \circ f \in \mathsf{L}^+(\mu)$ and $\int g \, \mathrm{d}(f_\# \mu) = \int g \circ f \, \mathrm{d}\mu$.

Remark 1.71. In the previous exercise, if X and Y are topological spaces and $f : X \to Y$ is a Borelian map, the same same statement holds for a Borelian function $g \geq 0$ on Y if we take the alternative definition of the pushforward from remark 1.38, i.e. we take the push-forward by f of the measure μ on $\mathscr{B}_X$, which is a measure $f_\# \mu$ on $\mathscr{B}_Y$, and then we take the extension of this measure given by theorem 1.7 (the alternative definition may be more convenient in this situation because it yields a Borel regular measure). In fact, both definitions of $f_\# \mu$ coincide on $\mathscr{B}_Y$, and the integrals depend only on the measures on the Borel sets, i.e. they depend only on $\mu : \mathscr{B}_X \to [0, \infty]$ and $f_\# \mu : \mathscr{B}_Y \to [0, \infty]$.

1.3.1. L^p spaces.

Definition 1.72. Let f be a $\mathbb{C}$-valued measurable function on X. We define:

- For real $0 < p < \infty$, $\|f\|_p := (\int |f|^p \, \mathrm{d}\mu)^{1/p} \in [0, \infty]$.
- For $p = \infty$, $\|f\|_p := \inf\{C \in \mathbb{R} \mid |f| \leq C \ \mu - a.e. \text{ on } X\} \in [0, \infty]$ (note that $\inf \emptyset = +\infty$).

For $0 < p \leq \infty$, we define $\mathsf{L}^\mathsf{p}(\mu) := \{f : X \to \mathbb{C} \ \mu - \text{measurable} \mid \|f\|_p < \infty\}$.

For $p \in [1, \infty]$, we define its *conjugate exponent* $p' \in [1, \infty]$ by $\frac{1}{p} + \frac{1}{p'} = 1$ (thus $p' = \infty$ for $p = 0$ and $p' = 1$ for $p = \infty$).

For each real $0 < p < \infty$ or $p = \infty$, one can readily check that $\mathsf{L}^\mathsf{p}(\mu)$ is a vector space over $\mathbb{C}$. For $1 \leq p \leq \infty$, it follows from theorem 1.75, stated below, that $\|\cdot\|_p$ is a seminorm on $\mathsf{L}^\mathsf{p}(\mu)$.

Theorem 1.73 (Hölder's inequality). *For any $p \in [1, \infty]$, f, g $\mathbb{C}$-valued measurable functions on X, the following inequality holds:*

$$\|fg\|_1 \leq \|f\|_p \|g\|_{p'}.$$

In particular, $fg \in \mathsf{L}^1(\mu)$ if $f \in \mathsf{L}^\mathsf{p}(\mu)$ and $g \in \mathsf{L}^{\mathsf{p}'}(\mu)$.

Theorem 1.74 (Generalized Hölder's inequality). *Let $p_1, \ldots, p_k \in [1, \infty]$ such that $\sum_{i=1}^k \frac{1}{p_i} = \frac{1}{r} \leq 1$ and $f_1, \ldots, f_k$ $\mathbb{C}$-valued measurable* *functions on X. Then*

$$\|\prod_{i=1}^{k} f_i\|_r \leq \prod_{i=1}^{k} \|f\|_{p_i}.$$

In particular, $\prod_{i=1}^{k} f_i \in \mathsf{L}^{\mathsf{r}}(\mu)$ *if* $f_i \in \mathsf{L}^{\mathsf{p}_i}(\mu)$ *for* $1 \leq i \leq k$.

THEOREM 1.75 (Minkowski's inequality). *For any* $p \in [1, \infty]$, f, g $\mathbb{C}$*-valued measurable functions on* X*, the following inequality holds:*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

For $1 \leq p \leq \infty$, the linear subspace $N := \{f \in \mathsf{L}^{\mathsf{p}}(\mu) \mid \|f\|_p = 0\}$ of $\mathsf{L}^{\mathsf{p}}(\mu)$ consists of the measurable functions on X which are null almost everywhere. Therefore, the quotient $\mathsf{L}^{\mathsf{p}}(\mu)/N$ consists of classes of equivalence of functions in $\mathsf{L}^{\mathsf{p}}(\mu)$ which coincide almost everywhere, and $\|\cdot\|_p$ is a norm on this quotient, which is complete by the following theorem. As in remark 1.61, we shall henceforth overload the notation "$\mathsf{L}^{\mathsf{p}}(\mu)$", which will be used both with its original meaning and also to denote the aforementioned quotient space.

THEOREM 1.76. *For* $1 \leq p \leq \infty$, $\mathsf{L}^{\mathsf{p}}(\mu)$ *is a Banach space. For* $p = 2$*, it is a Hilbert space, since* $\|\cdot\|_2$ *is induced by the Hermitian inner product* $\langle f, g \rangle := \int f \bar{g} \, \mathrm{d}\mu$ *(where* $\bar{\cdot}$ *denotes complex conjugation), whenever* $f, g \in \mathsf{L}^2(\mu)$.

For $1 \leq p < \infty$, the theorem above is a consequence of the convergence theorems for the integral 1.62, 1.63, 1.64.

We now state a basic interpolation theorem which may be derived by a convenient application of Hölder's inequality.

THEOREM 1.77 (Basic interpolation for L^{p} spaces). *If* $0 < p < q < r \leq \infty$*, then* $\mathsf{L}^{\mathsf{p}}(\mu) \cap \mathsf{L}^{\mathsf{r}}(\mu) \subset \mathsf{L}^{\mathsf{q}}(\mu)$ *and, for all measurable* f *on* X, $\|f\|_q \leq \|f\|_p^{\lambda} \|f\|_r^{1-\lambda}$*, where* $\lambda \in (0, 1)$ *is defined by*

$$\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}.$$

The following density theorem is a consequence of the regularity properties of Radon measures (1.28, 1.31). For a locally compact Hausdorff space X, we denote by $\mathsf{C}_{\mathsf{c}}(X)$ the space of continuous functions on X with compact support.

PROPOSITION 1.78. *If* μ *is a Radon measure on a locally compact Hausdorff space* X *and* $1 \leq p < \infty$*, then* $\mathsf{C}_{\mathsf{c}}(X)$ *is dense in* $\mathsf{L}^{\mathsf{p}}(\mu)$.

Proof. Since $\mathsf{L^p}$ simple functions are dense in $\mathsf{L^p}$ (as it can be readily checked by means of proposition 1.53 and theorem 1.64), it suffices to prove that such functions may be arbitrarily approximated in the $\mathsf{L^p}$ norm by continuous functions with compact support. Besides, since any $\mathsf{L^p}$ simple function is a finite linear combination of characteristic functions of measurable sets of finite measure, it suffices to show that, given $E \in \sigma(\mu)$ with $\mu(E) < \infty$ and $\epsilon > 0$, there exists $\phi \in \mathsf{C_c}(X)$ such that $\|\phi - \chi_E\|_p < \epsilon$. Indeed, take a compact set $K \subset E$ and an open set $U \supset E$ such that $\mu(U \setminus K) < \delta$, with $\delta > 0$ to be chosen later. Applying Urysohn's lemma, choose $\phi \in \mathsf{C_c}(X)$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on a neighborhood of K and spt $\phi \subset U$. Therefore, $\chi_K \leq \phi \leq \chi_U$ and $\chi_K \leq \chi_E \leq \chi_U$, so that $|\phi - \chi_E| \leq \chi_U - \chi_K$, what implies $\|\phi - \chi_E\|_p \leq \|\chi_U - \chi_K\|_p = \delta^{1/p}$. Taking $\delta^{1/p} < \epsilon$, the thesis is achieved. □

We now identify the dual space of the Banach space $\mathsf{L^p}(\mu)$. For fixed $p \in [1, \infty]$, $q = p'$ the conjugate exponent of p and $f \in \mathsf{L^p}(\mu)$, let $\Phi_p(f) : \mathsf{L^q}(\mu) \to \mathbb{C}$ be defined by $g \mapsto \int fg \, \mathrm{d}\mu$. It follows from Hölder's inequality 1.73 that $\Phi_p(f)$ is well defined, $\Phi_p(f) \in \mathsf{L^q}(\mu)'$ and $\|\Phi_p(f)\|_q \leq \|f\|_p$. Actually, in "almost all" situations the last inequality is an equality, and Φ_p is an isometry of $\mathsf{L^p}(\mu)$ onto $\mathsf{L^q}(\mu)$:

Theorem 1.79 (Riesz representation theorem). *With the notation above, if $1 < p < \infty$, Φ_p is an isometry of $\mathsf{L^p}(\mu)$ onto $\mathsf{L^q}(\mu)'$, so that we may identify by means of this isometry $\mathsf{L^q}(\mu)' \equiv \mathsf{L^p}(\mu)$.*

- *For $p = \infty$, if μ is σ-finite, Φ_∞ is an isometry of $\mathsf{L^\infty}(\mu)$ onto $\mathsf{L^1}(\mu)'$, so that $\mathsf{L^1}(\mu)' \equiv \mathsf{L^\infty}(\mu)$.*
- *For $p = 1$, Φ_1 is an isometry of $\mathsf{L^1}(\mu)$ into $\mathsf{L^\infty}(\mu)'$, but in general it is not onto, i.e. in general the dual of $\mathsf{L^\infty}(\mu)$ is bigger than $\mathsf{L^1}(\mu)$.*

We end this subsection with a criterion for compacity in $\mathsf{L^p}(\mathcal{L}^n)$.

For a function f defined on $\mathbb{R}^n$ and $x, y \in \mathbb{R}^n$, we adopt the usual notation for translations:

$$\tau_y f(x) := f(x - y)$$

Theorem 1.80 (Kolmogorov-Riesz-Fréchet). *Let $1 \leq p < \infty$ and $\mathcal{F}$ be a bounded subset of $\mathsf{L^p}(\mathcal{L}^n)$ such that*

$$\lim_{h \to 0} \|\tau_h f - f\|_p = 0$$

uniformly in $f \in \mathcal{F}$. Then, for each $\Omega \in \sigma(\mathcal{L}^n)$ with finite measure, the closure of $\mathcal{F}|_\Omega$ in $\mathsf{L^p}(\mathcal{L}^n|_\Omega)$ is compact.

Here, $\mathcal{F}|_\Omega := \{f|_\Omega \mid f \in \mathcal{F}\}$.

1.3.2. Change of variables formula. We state in the next two theorems the version of the change of variables formula for the Lebesgue integral which is usually presented in Real Analysis textbooks. That formula will be generalized in chapter 5 by the area and coarea formulas.

THEOREM 1.81 (linear change of variables for the Lebesgue integral). *Let $T \in \mathrm{GL}(n, \mathbb{R})$.*

i) If $A \in \mathscr{L}_{\mathbb{R}^n}$, then $T \cdot A \in \mathscr{L}_{\mathbb{R}^n}$ and $\mathcal{L}^n(T \cdot A) = |\det T| \mathcal{L}^n(A)$;

ii) If $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue-measurable, so is $f \circ T$, and, if $f \geq 0$ or $f \in \mathsf{L}^1$,

$$\int f \,\mathrm{d}\mathcal{L}^n = \int f \circ T |\det T| \,\mathrm{d}\mathcal{L}^n.$$

THEOREM 1.82 (C^1-change of variables formula for the Lebesgue integral). *Let $U \subset \mathbb{R}^n$ open and $\phi : U \to \mathbb{R}^n$ be a C^1 diffeomorphism onto its image $V := \phi(U)$ (which is an open subset of $\mathbb{R}^n$). If f is a Lebesgue-measurable function on V, $f \circ \phi$ is a Lebesgue-measurable function on U; besides, if $f \geq 0$ or $f \in \mathsf{L}^1$, then*

$$\int_V f \,\mathrm{d}\mathcal{L}^n = \int_U f \circ \phi(x) |\det \mathsf{D}\phi(x)| \,\mathrm{d}\mathcal{L}^n(x).$$

1.4. Product measures and Fubini-Tonelli's theorem

If $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces, there exists a standard construction, based on Carathéodory's extension theorem, which yields a measure $\mu \times \nu$ on the product σ-algebra $\mathcal{M} \otimes \mathcal{N} \subset 2^{X \times Y}$, called *product measure* of μ and ν. If both μ and ν are σ-finite, $\mu \times \nu$ is characterized by the property of being the unique measure on $\mathcal{M} \times \mathcal{N}$ such that , for all *measurable rectangles* $A \times B \in \mathcal{M} \times \mathcal{N}$, $\mu \times \nu(A \times B) = \mu(A)\nu(B)$. The main tool used in the study and computation of integrals with respect to $\mu \times \nu$ is the classical Fubini-Tonelli's theorem, which relates integrals with respect to $\mu \times \nu$ to iterated integrals with respect to μ and ν.

We now describe how to make an analogous construction for the product of outer measures μ on a set X and ν on a set Y. We may define the product $\mu \times \nu$ as the extension given by theorem 1.8 of the product (in the sense of the previous paragraph) $\mu|_{\sigma(\mu)} \otimes \nu|_{\sigma(\nu)}$. That is equivalent to the definition below.

DEFINITION 1.83 (product measure). We define, for all $E \subset X \times Y$, $\mu \times \nu(E) := \inf\{\sum_{n \in \mathbb{N}} \mu(A_n)\nu(B_n) \mid \forall n \in \mathbb{N},\, A_n \in \sigma(\mu), B_n \in \sigma(\nu), E \subset \cup_{n \in \mathbb{N}} A_n \times B_n\} \in [0, \infty]$. Recall that we use the convention $0 \cdot \infty = 0$. We call $\mu \times \nu$ the *product measure* of μ and ν.

We make a similar definition for any finite number of measures.

The theorem below, which may be obtained as a direct consequence of classical Fubini-Tonelli's theorem for products of measures on σ-algebras, ensures that $\mu \times \nu$ it is indeed a measure. Note that $\mu \times \nu$ is a regular measure, but we do not assume the regularity of μ or ν. We use the following:

NOTATION.

- For $E \subset X \times Y$ and $(x_0, y_0) \in X \times Y$, $E_{x_0} := \{y \in Y \mid (x_0, y) \in E\}$ (the x_0-*section* of E) and $E_{y_0} := \{x \in X \mid (x, y_0) \in E\}$ (the y_0-*section* of E).
- For a function f defined on $\operatorname{dom} f \subset X \times Y$ and $(x_0, y_0) \in X \times Y$, f_{x_0} (the x_0-*section* of f) and f_{y_0} (the y_0-*section* of f) are the functions defined, respectively, on $(\operatorname{dom} f)_{x_0}$ and $(\operatorname{dom} f)_{y_0}$ by $y \mapsto f(x_0, y)$ and $x \mapsto f(x, y_0)$.

THEOREM 1.84 (Fubini-Tonelli's for outer measures, [**Fed69**], [**EG91**]). *With the notation from the previous definition, $\mu \times \nu : 2^{X \times Y} \to [0, \infty]$ is a regular measure. Moreover:*

i) If $A \in \sigma(\mu)$ and $B \in \sigma(\nu)$, then $A \times B \in \sigma(\mu \times \nu)$ and $\mu \times \nu(A \times B) = \mu(A)\nu(B)$.

ii) If $E \in \sigma(\mu \times \nu)$ is σ-finite with respect to $\mu \times \nu$, then, for μ-almost every $x \in X$, $E_x \in \sigma(\nu)$, and for ν-almost every $y \in Y$, $E_y \in \sigma(\mu)$. The functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E_y)$ are measurable, and the measure of E may be computed by:

$$\mu \times \nu(E) = \int \nu(E_x) \,\mathrm{d}\mu(x) = \int \mu(E_y) \,\mathrm{d}\nu(y).$$

iii) If f is an integrable function defined on $\operatorname{dom} f \subset X \times Y$ *such that $\{f \neq 0\}$ is σ-finite with respect to $\mu \times \nu$ (what holds, in particular, if f is summable), then, for μ-almost every $x \in X$, f_x is ν-integrable, and for ν-almost every $y \in Y$, f_y is μ-integrable. The almost everywhere defined functions $x \mapsto \int f_x \,\mathrm{d}\nu$ and $y \mapsto \int f_y \,\mathrm{d}\mu$ are integrable, and $\int f \,\mathrm{d}(\mu \times \nu)$ may be computed by iterated integrals:*

$$\int f \,\mathrm{d}(\mu \times \nu) = \int \Big(\int f_x \,\mathrm{d}\nu \Big) \,\mathrm{d}\mu(x) = \int \Big(\int f_y \,\mathrm{d}\mu \Big) \,\mathrm{d}\nu(y).$$

REMARK 1.85. If μ and ν are both σ-finite, then so is $\mu \times \nu$, so that the σ-finiteness hypotheses in parts ii and iii above are automatically fulfilled. Moreover, every positive measurable function is integrable, so that part iii holds for such functions (what corresponds to classical Tonelli's theorem).

EXAMPLE 1.86. We show that the Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$ coincides with the product measure $(\mathcal{L}^1)^n = \mathcal{L}^1 \times \cdots \times \mathcal{L}^1$. Fix $A \subset \mathbb{R}^n$.

1) As defined in example 1.3, $\mathcal{L}^n(A) = \inf\{\sum_{Q\in\mathcal{A}} \mathrm{vol}(Q) \mid \mathcal{A}$ countable cover of A by cubes with sides parallel to the coordinate axes$\}$. Since any cube $Q \in \mathbb{R}^n$ is a product of intervals (hence a product of $\mathcal{L}^1$-measurable sets) and since, for each such cube, the euclidean volume $\mathrm{vol}(Q)$ coincides with $(\mathcal{L}^1)^n(Q)$ (by the fact that the length of an interval coincides with its Lebesgue measure), we immediately conclude from definition 1.83 that $(\mathcal{L}^1)^n(A) \leq \mathcal{L}^n(A)$.
2) In the definition of $\mathcal{L}^n(A)$, we may use rectangles (i.e. products of arbitrary intervals) instead of cubes (products of intervals with the same side length), without modifying $\mathcal{L}^n(A)$. Indeed, it suffices to show that, for any such rectangle $R = \prod_{j=1}^n I_j$ and $\epsilon > 0$, R may be covered by a countable family $\mathcal{A}$ of cubes such that $\sum_{Q\in\mathcal{A}} \mathrm{vol}(Q) \leq \mathrm{vol}(R) + \epsilon$. In order to accomplish that, assume $\mathrm{vol}(R) < \infty$ (otherwise we are done) and, for $m \in \mathbb{N}$ (to be chosen later), cover each interval I_j by countably many disjoint intervals $(I_{j,m}^k)_{k\in\mathbb{N}}$ with side lengths equal to $1/m$ so that $\sum_{k\in\mathbb{N}} \mathcal{L}^1(I_{j,m}^k) - \mathcal{L}^1(I_j) < 1/m$. Then $\mathcal{A}_m := \left(\prod I_{1,m}^{k_1} \times \cdots \times I_{n,m}^{k_n}\right)_{k_1,\ldots,k_n\in\mathbb{N}}$ is a countable cover of R by cubes and $\sum_{Q\in\mathcal{A}_m} \mathrm{vol}(Q) = \prod_{j=1}^n \left(\sum_{k\in\mathbb{N}} \mathcal{L}^1(I_{j,m}^k)\right) \overset{m\to\infty}{\longrightarrow} \mathrm{vol}(R)$; thus, for m sufficiently large, $\mathcal{A} = \mathcal{A}_m$ does the job.
3) In view of the previous item, to prove the remaining inequality $(\mathcal{L}^1)^n(A) \geq \mathcal{L}^n(A)$, it suffices to show that, given $B = \prod_{j=1}^n B_j$ with $(\forall 1 \leq j \leq n)\, B_j \in \mathscr{L}_{\mathbb{R}}$, for all $\epsilon > 0$, there exists a countable family $\mathcal{A}$ of rectangles which covers B and such that $\sum_{Q\in\mathcal{A}} \mathrm{vol}(Q) \leq \prod_{1\leq j\leq n} \mathcal{L}^1(B_j) + \epsilon$. We assume that $\prod_{1\leq j\leq n} \mathcal{L}^1(B_j) < \infty$, otherwise the inequality is trivial. Recall that $\mathcal{L}^1$ is a Borel regular measure, as we have seen in example 1.19; actually, it is a Radon measure, by exercise 1.32. It then follows from theorem 1.23 that, for any $m \in \mathbb{N}$ and for $1 \leq j \leq n$, there exist open sets $U_{j,m} \subset \mathbb{R}$ such that $B_j \subset U_{j,m}$ and $\mathcal{L}^1\left(U_{j,m} \setminus B_j\right) < 1/m$. Since each open set in $\mathbb{R}$ is a countable disjoint union of open intervals, there exists a countable family $(I_{j,m}^k)_{k\in\mathbb{N}}$ of disjoint open intervals such that $U_{j,m} = \dot\bigcup_{k\in\mathbb{N}} I_{j,m}^k$; take $\mathcal{A}_m := \left(\prod I_{1,m}^{k_1} \times \cdots \times I_{n,m}^{k_n}\right)_{k_1,\ldots,k_n\in\mathbb{N}}$. Then $\mathcal{A}_m$ is a countable family of rectangles which covers B and $\sum_{R\in\mathcal{A}_m} \mathrm{vol}(R) = \prod_{j=1}^n \mathcal{L}^1(U_{j,m}) \overset{m\to\infty}{\longrightarrow} \prod_{1\leq j\leq n} \mathcal{L}^1(B_j)$; thus, for m sufficiently large, $\mathcal{A} = \mathcal{A}_m$ does the job.

EXERCISE 1.87 (Layer-cake formula, [**LL01**]). Let μ be a σ-finite measure on X and ν a Radon measure on $[0,\infty)$. Define $\phi : [0,\infty) \to \mathbb{R}$

by $\phi(t) = \nu\big([0,t)\big)$. Then, for every $f \in \mathsf{L}^+(\mu)$:

$$\int \phi \circ f \,\mathrm{d}\mu = \int_{[0,\infty)} \mu(\{f > t\}) \,\mathrm{d}\nu(t).$$

In particular, if $p > 0$ and $\nu = pt^{p-1}\,\mathrm{d}t$, it follows:

$$\int f(x)^p \,\mathrm{d}\mu(x) = p \int_0^\infty \mu(\{f > t\}) t^{p-1} \,\mathrm{d}t.$$

Hint. Compute the integral on the first member by means of Fubini-Tonelli's theorem.

We close this section with a useful generalization of Minkowski's inequality 1.75 which may be obtained as a corollary of Fubini-Tonelli's theorem:

Theorem 1.88 (Minkowski's inequality for integrals). *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and f a $\mathcal{M} \otimes \mathcal{N}$-measurable function on $X \times Y$.*

i) If $f \geq 0$ and $1 \leq p < \infty$, then

$$\Big[\int \Big(\int f(x,y)\,\mathrm{d}\nu(y)\Big)^p \mathrm{d}\mu(x)\Big]^{1/p} \leq \int \Big[\int f(x,y)^p \,\mathrm{d}\mu(x)\Big]^{1/p} \mathrm{d}\nu(y).$$

ii) If $p \in [1, \infty]$, the inequality below holds if the second member makes sense and is finite. That is, if for ν-a.e. $y \in Y$ $f(\cdot, y) \in \mathsf{L}^\mathsf{p}(\mu)$ and the a.e. defined ν-measurable function $y \mapsto \|f(\cdot,y)\|_p$ is ν-summable, then for μ-a.e. $x \in X$, the function $f(x,\cdot)$ is ν-summable, the a.e. defined μ-measurable function $x \mapsto \int f(x,y)\,\mathrm{d}\nu(y)$ is in $\mathsf{L}^\mathsf{p}(\mu)$ and

$$\|\int f(\cdot,y)\,\mathrm{d}\nu(y)\|_p \leq \int \|f(\cdot,y)\|_p \,\mathrm{d}\nu(y).$$

1.5. Signed measures and Lebesgue-Radon-Nikodym theorems

In this subsection we are concerned with measures on σ-algebras (i.e. we don't consider outer measures). We recall the notion of "signed measure" and some important decomposition theorems which may be used to relate the properties and the integration theory of one measure on a σ-algebra to the corresponding properties of another measure on the same σ-algebra.

Definition 1.89 (signed measures). A *charge* or *signed measure* on a measurable space $(X, \mathcal{M})$ is a set function $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ such that

SM1) $\nu(\emptyset) = 0$;

SM2) $\operatorname{Im}\nu \subset [-\infty, \infty)$ or $\operatorname{Im}\nu \subset (-\infty, \infty]$ (i.e. ν omits $-\infty$ or $+\infty$);

SM3) ν is σ-additive, i.e. for all countable disjoint family $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$,
$$\nu(\cup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \nu(A_n),$$
with the meaning that $n \mapsto \mu(A_n)$ is summable with respect to the counting measure on $\mathbb{N}$ and the sum is $\mu(\cup_{n\in\mathbb{N}} A_n)$.

We say that a signed measure ν is *finite* if $\operatorname{Im}\nu \subset \mathbb{R}$ (i.e. if ν omits both $-\infty$ and $+\infty$). We say that ν is *σ-finite* if there exists a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ such that $\cup_{n\in\mathbb{N}} A_n = X$ and $\forall n \in \mathbb{N}$, $\nu(A_n) \in \mathbb{R}$.

EXAMPLE 1.90.

1) Let μ_1 and μ_2 be measures on $(X, \mathcal{M})$ such that μ_1 or μ_2 is finite. Then $\nu = \mu_1 - \mu_2$ is a signed measure on $(X, \mathcal{M})$.
2) Let μ be a measure on $(X, \mathcal{M})$ and $f : X \to \overline{\mathbb{R}}$ an integrable function (in the sense of definition 1.58). Then $f\mu : \mathcal{M} \to \overline{\mathbb{R}}$ given by $A \mapsto \int_A f \,\mathrm{d}\mu$ is a signed measure.

REMARK 1.91.

1) The second example is a particular case of the first, since $f\mu = f^+\mu - f^-\mu$. We will see in theorem 1.94 and in exercise 1.96 that every signed measure on $(X, \mathcal{M})$ may be written in both forms 1) and 2).
2) Note that every measure on $(X, \mathcal{M})$ is a signed measure. As it is usual, for clarity reasons, sometimes we call a measure on $(X, \mathcal{M})$ a *positive measure*, to contrast with "signed measure".

DEFINITION 1.92 (absolute continuity and mutual singularity). Let μ and ν be positive measures on a measurable space $(X, \mathcal{M})$. We say that:

1) μ is *absolutely continuous* with respect to ν (notation: $\mu \ll \nu$) if $\forall A \in \mathcal{M}$, $\nu(A) = 0$ implies $\mu(A) = 0$.
2) μ and ν are *mutually singular* (notation: $\mu \perp \nu$) if there exists $A \in \mathcal{M}$ such that μ is concentrated on A and ν is concentrated on $X \setminus A$.

EXERCISE 1.93. Let μ be a finite positive measure and ν a positive measure on a measurable space $(X, \mathcal{M})$. Then $\mu \ll \nu$ iff $\forall \epsilon > 0$, $\exists \delta > 0$, $\forall A \in \mathcal{M}$, $\nu(A) < \delta$ implies $\mu(A) < \epsilon$.

THEOREM 1.94 (Jordan decomposition theorem). *Let ν be a signed measure on a measure space $(X, \mathcal{M})$. Then there are unique positive measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

DEFINITION 1.95 (positive part, negative part and total variation of a signed measure). With the notation from theorem 1.94, we call ν^+ the *positive part* of ν and ν^- the *negative part* of ν.

The positive measure $|\nu| := \nu^+ + \nu^-$ is called the *total variation* of ν.

EXERCISE 1.96. If ν is a signed measure on a measure space $(X, \mathcal{M})$, there exists a Borelian $|\nu|$-integrable function $f : X \to \mathbb{R}$ such that $|f| \equiv 1$ and $\nu = f|\nu|$.

DEFINITION 1.97 (integration with respect to a signed measure). Let ν be a signed measure on a measure space $(X, \mathcal{M})$ and $f : X \to \overline{\mathbb{R}}$ a measurable function. We say that f is *summable with respect to* ν if it is summable with respect to $|\nu|$ and we use the notation $\mathsf{L}^1(\nu) := \mathsf{L}^1(|\nu|)$. For such f, we define

$$\int f \, \mathrm{d}\nu := \int f \, \mathrm{d}\nu^+ - \int f \, \mathrm{d}\nu^-.$$

Note that the integral defined above satisfies the usual linearity and convergence properties, which are inherited from the corresponding properties of the integrals with respect to ν^+ and ν^-.

DEFINITION 1.98 (absolute continuity and mutual singularity, bis). Let ν be a signed measure and μ a positive measure on a measurable space $(X, \mathcal{M})$. We say that:

1) $\nu \ll \mu$ if $|\nu| \ll \mu$.
2) $\nu \perp \mu$ if $|\nu| \perp \mu$.

EXERCISE 1.99. Let ν be a signed measure on a measurable space $(X, \mathcal{M})$. The following properties hold:

a) For all $A \in \mathcal{M}$, $|\nu(A)| \leq |\nu|(A)$.
b) ν is finite (respectively, σ-finite) iff ν^+ and ν^- are finite (respectively, σ-finite) iff $|\nu|$ is finite (respectively, σ-finite). If ν is finite, $\operatorname{Im}\nu$ is a bounded subset of $\mathbb{R}$.
c) For all $A \in \mathcal{M}$,
$$|\nu|(A) = \sup\{\sum_{n\in\mathbb{N}} |\nu(A_n)| \mid \forall n \in \mathbb{N}, A_n \in \mathcal{M} \text{ and } \dot{\bigcup_{n\in\mathbb{N}}} A_n = A\}.$$
d) $\mathsf{L}^1(|\nu|) = \mathsf{L}^1(\nu^+) \cap \mathsf{L}^1(\nu^-)$.
e) For all $f \in \mathsf{L}^1(\nu)$, $|\int f \, \mathrm{d}\nu| \leq \int |f| \, \mathrm{d}|\nu|$.

EXERCISE 1.100. Let ν_1 and ν_2 be signed measures on a measurable space $(X, \mathcal{M})$ such that both omit $-\infty$ or both omit $+\infty$, μ a positive measure $(X, \mathcal{M})$ and $c \in \mathbb{R}$. Then:

a) $c\nu_1$ and $\nu_1 + \nu_2$ are signed measures.
b) $|c\nu_1| = |c||\nu_1|$ and $|\nu_1 + \nu_2| \le |\nu_1| + |\nu_2|$, with equality if $|\nu_1| \perp |\nu_2|$.
c) If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$, then $\nu_1 + \nu_2 \perp \mu$. If both ν_1 and ν_2 are positive measures, then $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$ iff $\nu_1 + \nu_2 \perp \mu$.
d) If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then $\nu_1 + \nu_2 \ll \mu$. If both ν_1 and ν_2 are positive measures, then $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$ iff $\nu_1 + \nu_2 \ll \mu$.
e) $\mathsf{L}^1(\nu_1) \cap \mathsf{L}^1(\nu_2) \subset \mathsf{L}^1(\nu_1 + \nu_2)$ and, $\forall f \in \mathsf{L}^1(\nu_1) \cap \mathsf{L}^1(\nu_2)$, $\int f \,\mathrm{d}(\nu_1 + \nu_2) = \int f \,\mathrm{d}\nu_1 + \int f \,\mathrm{d}\nu_2$.
f) If $c \neq 0$, $\mathsf{L}^1(c\nu_1) = \mathsf{L}^1(\nu_1)$ and, $\forall f \in \mathsf{L}^1(\nu_1)$, $\int f \,\mathrm{d}(c\nu_1) = c \int f \,\mathrm{d}\nu_1$.

THEOREM 1.101 (Lebesgue decomposition theorem). *Let ν be a signed measure and μ a positive measure on a measurable space $(X, \mathcal{M})$, both σ-finite. Then there exist unique signed measures ν_s and ν_a on $(X, \mathcal{M})$ such that $\nu_s \perp \mu$, $\nu_a \ll \mu$ and $\nu = \nu_s + \nu_a$.*

DEFINITION 1.102. With the notation from the previous theorem, we call ν_s the *singular part of ν*, ν_a the *absolutely continuous part of ν* and $\nu = \nu_s + \nu_a$ the *Lebesgue decomposition of ν* with respect to μ.

THEOREM 1.103 (Radon-Nikodym theorem). *Let μ be a positive measure and ν a σ-finite signed measure on a measurable space $(X, \mathcal{M})$, such that $\nu \ll \mu$. Then there exists a μ-integrable function $f : X \to \overline{\mathbb{R}}$, unique up to μ-null sets in $\mathcal{M}$, such that $\nu = f\mu$, i.e. for all $A \in \mathcal{M}$,*

$$\nu(A) = \int_A f \,\mathrm{d}\mu.$$

DEFINITION 1.104 (Radon-Nikodym derivative). With the notation from the previous theorem, we cal f (or any measurable function which coincides μ-a..e with f) the *Radon-Nikodym derivative* of ν with respect to μ and denote it by $\frac{d\nu}{d\mu}$.

Note that, in the situation of example 1.90.2), i.e. if $\nu = f\mu$ where μ is a positive measure on $(X, \mathcal{M})$ and $f : X \to \overline{\mathbb{R}}$ is μ-integrable, then $\nu \ll \mu$. Hence, if ν is σ-finite, it follows from the uniqueness of the Radon-Nikodym derivative stated in theorem 1.103 that $f = \frac{d\nu}{d\mu}$ (equality here and in similar statements below means that f is in the equivalence class of $\frac{d\nu}{d\mu}$ modulo μ-a.e. null functions).

The Radon-Nikodym derivative has the properties suggested by the notation $\frac{d\nu}{d\mu}$. We list those properties in exercise 1.105 and proposition 1.107 below.

EXERCISE 1.105. Let $(X, \mathcal{M})$ be a measurable space and μ a positive measure on $(X, \mathcal{M})$.

a) If ν is a σ-finite signed measure on $(X, \mathcal{M})$ and $\nu \ll \mu$, then

$$\frac{d|\nu|}{d\mu} = \left|\frac{d\nu}{d\mu}\right|.$$

In other words, if $f : X \to \overline{\mathbb{R}}$ is μ-integrable and $\nu = f\mu$, then $|\nu| = |f|\mu$. Moreover, ν is finite iff $\frac{d\nu}{d\mu} \in \mathsf{L}^1(\mu)$.

b) Let ν_1 and ν_2 be σ-finite signed measures on $(X, \mathcal{M})$ such that both omit $-\infty$ or both omit $+\infty$ and $c \in \mathbb{R}$. Suppose that $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$. Then

$$\frac{d(c\nu_1)}{d\mu} = c\frac{d\nu_1}{d\mu} \quad \text{and} \quad \frac{d(\nu_1 + \nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}.$$

c) If ν is a σ-finite signed measure on $(X, \mathcal{M})$ and $\nu \ll \mu$, then $\frac{d\nu}{d\mu} = \frac{d(\nu^+)}{d\mu} - \frac{d(\nu^-)}{d\mu}$, $\frac{d(\nu^+)}{d\mu} = \left(\frac{d\nu}{d\mu}\right)^+$ and $\frac{d(\nu^-)}{d\mu} = \left(\frac{d\nu}{d\mu}\right)^-$.

Remark 1.106. Let ν be a signed measure on $(X, \mathcal{M})$. It is immediate from defintion 1.98 that $\nu \ll |\nu|$. If ν is σ-finite, it then follows from exercise 1.105.a) with $|\nu|$ in place of μ that $\frac{d\nu}{d|\nu|} = \pm 1$ $|\nu|$-a.e. on X.

Proposition 1.107 (chain rule for the Radon-Nikodym derivative). *Let λ, ν and μ be positive measures on a measurable space $(X, \mathcal{M})$ with λ and ν σ-finite and such that $\lambda \ll \nu \ll \mu$. Then:*

i) For every $f : X \to [0, \infty]$ measurable,

$$\int f \,\mathrm{d}\nu = \int f \frac{d\nu}{d\mu} \,\mathrm{d}\mu.$$

ii) $\lambda \ll \mu$ and

$$\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \cdot \frac{d\nu}{d\mu}.$$

1.6. Convolutions

In this section we consider integrals with respect to the Lebesgue measure in $\mathbb{R}^n$, which we often denote by $\mathrm{d}x$, $\mathrm{d}y$, etc. We recall the basic properties of convolutions and mollifiers, which will be extensively used in subsequent chapters.

Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be $\mathcal{L}^n$-measurable functions. We define the convolution $f * g$ by:

$$f * g(x) := \int f(x - y) g(y) \,\mathrm{d}y,$$

whenever the integral makes sense at least for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$. That occurs mainly in two cases:

- one of the functions is essentially bounded, the other belongs to $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n)$ and one of them has compact support;
- one of the functions belong to $\mathsf{L}^\mathsf{p}(\mathcal{L}^n)$ and the other belongs to $\mathsf{L}^\mathsf{q}(\mathcal{L}^n)$, with $\frac{1}{p}+\frac{1}{q}\geq 1$, cf. proposition 1.108.g) below.

Broadly speaking, whenever defined, the convolution product is commutative and associative, and inherits the regularity properties from both factors. The latter property is widely explored in techniques which involve approximation of functions by means of mollifiers.

We summarize the main properties of the convolution product in the propositions below. For a function f defined on $\mathbb{R}^n$ and $x, y\in\mathbb{R}^n$, we use the notation:

$$\tau_y f(x) := f(x-y) \quad \text{and} \quad \check{f}(x) := f(-x)$$

as well as the standard multi-index notation for partial derivatives, i.e. given $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n_+$,

$$\partial^\alpha f(x) := \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\cdots\Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n} f(x),$$
$$|\alpha| := \alpha_1+\cdots+\alpha_n.$$

Proposition 1.108 (properties of the convolution product). *For $f, g, h:\mathbb{R}^n\to\mathbb{C}$ such that the convolution products are defined:*

a) $f*g$ *is* $\mathcal{L}^n$*-measurable.*
b) $f*g=g*f$.
c) $(f*g)*h=f*(g*h)$.
d) $\operatorname{spt} f*g\subset\overline{\operatorname{spt} f+\operatorname{spt} g}$.
e) $(f\check{*}g)=\check{f}*\check{g}$.
f) $\forall y\in\mathbb{R}^n$, $\tau_y(f*g)=(\tau_y f)*g=f*(\tau_y g)$.
g) (Young's inequality) If $p,q,r\in[1,\infty]$ *with* $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$*, then*

$$\|f*g\|_r\leq\|f\|_p\|g\|_q.$$

Thus, $f*g\in\mathsf{L}^\mathsf{r}(\mathcal{L}^n)$ *if* $f\in\mathsf{L}^\mathsf{p}(\mathcal{L}^n)$ *and* $g\in\mathsf{L}^\mathsf{q}(\mathcal{L}^n)$*. In particular, for* $p=1$ *and* $q=r\in[1,\infty]$, $\|f*g\|_q\leq\|f\|_1\|g\|_q$.

If p *and* q *are conjugate exponents, i.e. if we take* p,q,r *above with* $r=\infty$*, then* $f*g(x)$ *exists for every* $x\in\mathbb{R}^n$ *and* $f*g$ *is bounded and uniformly continuous; besides, if both* p *and* q *are finite, then* $f*g\in\mathsf{C}_0(\mathbb{R}^n,\mathbb{C})$.

h) If $f\in\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n)$ *and* $g\in\mathsf{L}^\infty(\mathcal{L}^n)$ *with* $\operatorname{spt} g$ *compact, then* $f*g\in\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n)$.

i) *If f and g satisfy the hypothesis of one of the two previous items and $\varphi \in \mathsf{C}_\mathsf{c}(\mathbb{R}^n, \mathbb{C})$, then*

$$\int f * g(x)\varphi(x)\,\mathrm{d}x = \iint f(x)g(y)\varphi(x+y)\,\mathrm{d}x\,\mathrm{d}y = \int f(x)\check{g} * \varphi(x)\,\mathrm{d}x.$$

j) *Let $0 \le k \le \infty$ and $\alpha \in \mathbb{Z}^n$ a multi-index with $|\alpha| \le k$. If $f \in \mathsf{C}^\mathsf{k}_\mathsf{b}(\mathbb{R}^n)$ and $g \in \mathsf{L}^1(\mathcal{L}^n)$, or $f \in \mathsf{C}^\mathsf{k}(\mathbb{R}^n)$, $g \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n)$ and one of them has compact support, then $f * g \in \mathsf{C}^\mathsf{k}(\mathbb{R}^n)$ and $\partial^\alpha(f * g) = (\partial^\alpha f) * g$.*

Recall our convention adopted in remark 1.57, i.e. for measurable functions "support" means "essential support".

The proof of the proposition above can be found in standard real analysis textbooks, but we offer a proof of part j) as an application of the dominated convergence theorem.

Proof of part j). We prove the assertion for $k = 1$ and $\alpha = e_j$, $1 \le j \le n$. The general case follows by induction using the same argument.

1) Suppose $f \in \mathsf{C}^1_\mathsf{b}$ and $g \in \mathsf{L}^1$. We have

$$\frac{\partial}{\partial x_j}\left[f(x-y)g(y)\right] = \partial_{x_j} f(x-y)g(y)$$

hence $\forall x, y \in \mathbb{R}^n$, $\left|\frac{\partial}{\partial x_j}\left[f(x-y)g(y)\right]\right| \le \|\partial_{x_j} f\|_u |g(x)|$. Since $g \in \mathsf{L}^1$, we may differentiate under the integral sign using the dominated convergence theorem 1.67.ii:

$$\partial_{x_j}(f * g)(x) = \int \partial_{x_j} f(x-y)g(y)\,\mathrm{d}y = (\partial_{x_j} f) * g(x).$$

Moreover, since $\partial_{x_j} f \in \mathsf{L}^\infty$ and $g \in \mathsf{L}^1$, it follows from the last statement in part g) that $(\partial_{x_j} f) * g$ is continuous (actually it is uniformly continuous). Since that holds for all $1 \le j \le n$, we conclude that $f * g \in \mathsf{C}^1$, as asserted.

2) Suppose that $f \in \mathsf{C}^1(\mathbb{R}^n)$, $g \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n)$ and one of them, say f, has compact support (the case $\operatorname{spt} g \Subset \mathbb{R}^n$ is similar).

Fix $x_0 \in \mathbb{R}^n$ and $r > 0$. Let K be the compact set $\mathbb{B}(x_0, r) - \operatorname{spt} f$ (so that, for $x \in \mathbb{B}(x_0, r)$, $x - y \in \operatorname{spt} f$ implies $y \in K$). We have, for all $(x, y) \in \mathbb{U}(x_0, r) \times \mathbb{R}^n$, $\left|\frac{\partial}{\partial x_j}\left[f(x-y)g(y)\right]\right| = |\partial_{x_j} f(x-y)||g(y)| \le \|\partial_{x_j} f\|_u\, \chi_K |g|$. Since $\chi_K |g| \in \mathsf{L}^1(\mathcal{L}^n)$, we may apply proposition 1.67 with $\mathbb{U}(x_0, r)$ in place of I and $(\mathbb{R}^n, \mathcal{L}^n)$ in place of (X, μ), yielding, for all $x \in \mathbb{U}(x_0, r)$,

$$\partial_{x_j}(f * g)(x) = \int \partial_{x_j} f(x-y)g(y)\,\mathrm{d}y = (\partial_{x_j} f) * g(x).$$

Since $(\partial_{x_j} f) * g$ coincides on $\mathbb{U}(x_0, r)$ with $(\partial_{x_j} f) *(\chi_K g)$ (where $K = \mathbb{B}(x_0, r) - \operatorname{spt} f$, as above), $\partial_{x_j} f \in \mathsf{L}^\infty$ and $\chi_K g \in \mathsf{L}^1$, we conclude that $(\partial_{x_j} f) * g$ is continuous on $\mathbb{U}(x_0, r)$. As that holds for $1 \leq j \leq n$, it follows that $f * g$ is continuously differentiable on $\mathbb{U}(x_0, r)$; since $x_0 \in \mathbb{R}^n$ and $r > 0$ were arbitrarily taken, we are done.

□

LEMMA 1.109. *If $f \in \mathsf{C}_0(\mathbb{R}^n)$, then f is uniformly continuous.*

PROOF. Fix $\epsilon > 0$ and let $K \subset \mathbb{R}^n$ compact such that $|f| \leq \epsilon$ on K^c. Since $f|_K$ is uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| \leq \epsilon$ whenever $x, y \in K$ with $\|x - y\| < \delta$. If $x, y \in \mathbb{R}^n$ and $\|x - y\| < \delta$, we have:

1) if $x, y \in K$, $|f(x) - f(y)| \leq \epsilon$;
2) if $x, y \in K^c$, $|f(x) - f(y)| \leq 2\epsilon$;
3) if $x \in K$ and $y \in K^c$, the closed segment $[x, y]$ intersects ∂K, hence there exists $z \in \partial K \subset K$ such that $\|z-x\| < \delta$. Since, by continuity, $|f| \leq \epsilon$ in ∂K, we have $|f(x)-f(y)| \leq |f(x)-f(z)|+|f(z)-f(y)| \leq |f(x) - f(z)| + |f(z)| + |f(y)| \leq 3\epsilon$.

Hence, for all $x, y \in \mathbb{R}^n$ with $\|x - y\| < \delta$, $|f(x) - f(y)| \leq 3\epsilon$. □

LEMMA 1.110. *If $1 \leq p < \infty$, translation is continuous in the L^p norm, i.e. for fixed $f \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n)$, the map $\mathbb{R}^n \to \mathsf{L}^\mathsf{p}(\mathcal{L}^n)$ given by $y \mapsto \tau_y f$ is continuous.*

PROOF. Fix $z \in \mathbb{R}^n$. We must prove that $\lim_{y \to z} \|\tau_y f - \tau_z f\|_p = 0$. Fix $\epsilon > 0$. Since $\mathsf{C}_\mathsf{c}(\mathbb{R}^n)$ is dense in $\mathsf{L}^\mathsf{p}(\mathcal{L}^n)$ (by proposition 1.78), there exists $g \in \mathsf{C}_\mathsf{c}(\mathbb{R}^n)$ such that $\|f - g\|_p \leq \epsilon$. Then $\|\tau_y f - \tau_z f\|_p \leq \|\tau_y(f - g)\|_p + \|\tau_y g - \tau_z g\|_p + \|\tau_z(g - f)\|_p \leq 2\epsilon + \|\tau_y g - \tau_z g\|_p$. Since $\mathsf{C}_\mathsf{c}(\mathbb{R}^n) \subset \mathsf{C}_0(\mathbb{R}^n)$, it follows from lemma 1.109 that g is uniformly continuous; hence

$$\|\tau_y g - \tau_z g\|_p \leq \|\tau_y g - \tau_z g\|_u \, \mathcal{L}^n(\operatorname{spt} g) \overset{y \to z}{\to} 0.$$

We therefore conclude that $\limsup_{y \to z} \|\tau_y f - \tau_z f\|_p \leq 2\epsilon$, whence the thesis, since $\epsilon > 0$ was arbitrarily taken. □

In the next theorem we use the following notation: for $\phi : \mathbb{R}^n \to \mathbb{C}$ and $t > 0$, we define $\phi_t : \mathbb{R}^n \to \mathbb{C}$ by

$$\phi_t(x) := t^{-n} \phi(t^{-1} x). \tag{1.1}$$

Note that, if $\phi \in \mathsf{L}^1(\mathcal{L}^n)$, it follows from theorem 1.81 that $\phi_t \in \mathsf{L}^1(\mathcal{L}^n)$ and $\int \phi \, \mathrm{d}\mathcal{L}^n = \int \phi_t \, \mathrm{d}\mathcal{L}^n$.

THEOREM 1.111 (mollifiers, part I). *Let* $\phi \in \mathsf{L}^1(\mathcal{L}^n)$ *with* $\int \phi \, \mathrm{d}\mathcal{L}^n = a$ *and* $f : \mathbb{R}^n \to \mathbb{C}$*.*

i) If $1 \le p < \infty$ *and* $f \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n)$*, then* $\phi_t * f \overset{t \to 0}{\to} af$ *in* $\mathsf{L}^\mathsf{p}(\mathcal{L}^n)$*.*
ii) If f *is uniformly continuous and either* (1) f *is bounded or* (2) $\operatorname{spt} \phi$ *is compact, then* $\phi_t * f \overset{t \to 0}{\to} af$ *uniformly in* $\mathbb{R}^n$*.*
iii) If f *is continuous on an open set* $U \subset \mathbb{R}^n$ *and either* (1) $f \in \mathsf{L}^\infty(\mathcal{L}^n)$ *or* (2) $f \in \mathsf{L}^\infty_{\mathsf{loc}}(\mathcal{L}^n)$ *and* $\operatorname{spt} \phi$ *is compact, then* $\phi_t * f \overset{t \to 0}{\to} af$ *uniformly on compact subsets of* U*.*

PROOF.

i) $\forall t > 0$, $\forall x \in \mathbb{R}^n$,

$$\begin{aligned} f * \phi_t(x) - af(x) &= \int [f(x-y) - f(x)]\phi_t(y) \, \mathrm{d}y \overset{z = t^{-1}y}{=} \\ &= \int [f(x - tz) - f(x)]\phi(z) \, \mathrm{d}z = \\ &= \int [\tau_{tz} f(x) - f(x)]\phi(z) \, \mathrm{d}z. \end{aligned}$$

Thus, by Minkowski's inequality for integrals 1.88,

$$\|f * \phi_t - af\|_p \le \int \|\tau_{tz} f - f\|_p |\phi(z)| \, \mathrm{d}z.$$

For any sequence $t_n \to 0$, $\|\tau_{t_n z} f - f\|_p |\phi(z)|$ converges pointwise to 0, by force of lemma 1.110, and $\|\tau_{t_n z} f - f\|_p |\phi(z)| \le 2\|f\|_p |\phi(z)|$. Since $\phi \in \mathsf{L}^1(\mathcal{L}^n)$, we may therefore apply the dominated convergence theorem 1.64 to conclude that $\int \|\tau_{t_n z} f - f\|_p |\phi(z)| \, \mathrm{d}z \overset{n \to \infty}{\to} 0$, hence $\|f * \phi_{t_n} - af\|_p \to 0$. Since the sequence $t_n \to 0$ was arbitrarily taken, it follows that $\|f * \phi_t - af\|_p \to 0$ as $t \to 0$, as asserted.

ii) By the same computation from the previous item, $\forall t > 0$, $\forall x \in \mathbb{R}^n$,

$$f * \phi_t(x) - af(x) = \int [\tau_{tz} f(x) - f(x)]\phi(z) \, \mathrm{d}z.$$

Thus $\|f * \phi_t - af\|_u \le \int \|\tau_{tz} f - f\|_u |\phi(z)| \, \mathrm{d}z$. Since f is uniformly continuous, we have $\|\tau_{tz} f - f\|_u \overset{t \to 0}{\to} 0$ for all $z \in \mathbb{R}^n$. If (1) holds, then $\|\tau_{tz} f - f\|_u |\phi(z)| \le 2\|f\|_u |\phi(z)|$ for all $z \in \mathbb{R}^n$; if (2) holds, by the compacity of $\operatorname{spt} \phi$ and by the uniform continuity of f we may take $\delta > 0$ such that for all $0 < t < \delta$ and all $z \in \operatorname{spt} \phi$, $\|\tau_{tz} f - f\|_u \le 1$, whence $\|\tau_{tz} f - f\|_u |\phi(z)| \le |\phi(z)|$ for all $0 < t < \delta$ and all $z \in \mathbb{R}^n$. In either case, the dominated convergence theorem ensures that $\|f * \phi_t - af\|_u \to 0$ as $t \to 0$, as asserted.

iii) Fix $\epsilon > 0$. Let $K \subset U$ compact and $C \subset \mathbb{R}^n$ compact such that $\int_{\mathbb{R}^n \setminus C} |\phi| \, \mathrm{d}\mathcal{L}^n \leq \epsilon$ (which exists, since $\phi \in \mathsf{L}^1$). Take $\eta > 0$ such that $\eta \cdot \sup\{|x| \mid x \in C\} < d(K, U^c)$; then $K' := \{x - ty \mid x \in K, y \in C, |t| \leq \eta\}$ is a compact subset of U which contains K. Since $f|_U$ is continuous, by compacity it follows that $f|_{K'}$ is uniformly continuous; therefore, there exists $0 < \delta < \min\{\eta, 1\}$ such that $\sup\{|f(x - ty) - f(x)| \mid x \in K, y \in C, |t| < \delta\} < \epsilon$. Hence, if $|t| < \delta$, $\forall x \in K$:

$$\begin{aligned}
|f * \phi_t(x) - af(x)| &= \Big| \int [\tau_{ty} f(x) - f(x)] \phi(y) \, \mathrm{d}y \Big| \leq \\
&\leq \int |\tau_{ty} f(x) - f(x)| |\phi(y)| \, \mathrm{d}y = \\
&= \int_{\mathbb{R}^n \setminus C} |\tau_{ty} f(x) - f(x)| |\phi(y)| \, \mathrm{d}y + \int_C |\tau_{ty} f(x) - f(x)| |\phi(y)| \, \mathrm{d}y \leq \\
&\leq 2\epsilon \|f\|_{\mathsf{L}^\infty(K'')} + \epsilon \|\phi\|_1.
\end{aligned}$$

where $K'' = \mathbb{R}^n$ in case (1) or $K'' = K + \mathbb{B}(0, \sup\{|y| \mid y \in \operatorname{spt} \phi\})$ in case (2) (recall that $\delta < 1$).

Since $\epsilon > 0$ was arbitrarily taken, we therefore conclude that $\|(f * \phi_t - af)|_K\|_u = \sup\{|f * \phi_t(x) - af(x)| \mid x \in K\} \to 0$ as $t \to 0$.

□

In the applications of the previous theorem, the most important cases are $a = 1$ and $a = 0$. For $a = 1$, we call $(\phi_t)_{>0}$ an *approximate identity* or *mollifier*, since it can be used to approximate a function f on $\mathbb{R}^n$ by means of the convolutions $\phi_t * f$, which have the same class of regularity as ϕ, in the appropriate topology of the function space f belongs to.

For instance, we may take $(\phi_t)_{>0}$ given by the following definition:

Definition 1.112 (standard mollifier in $\mathbb{R}^n$). Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be the smooth function given by

$$\phi(x) := \begin{cases} c \exp\left(\frac{1}{\|x\|^2 - 1}\right) & \text{if } \|x\| < 1 \\ 0 & \text{if } \|x\| \geq 1, \end{cases}$$

where c is chosen so that $\int_{\mathbb{R}^n} \phi(x) \, \mathrm{d}x = 1$. The family $(\phi_t)_{>0}$ induced by ϕ by means of (1.1) is called *standard mollifier* in $\mathbb{R}^n$.

Note that $\operatorname{spt} \phi = \mathbb{B}(0, 1)$, so that $\forall t > 0$, $\operatorname{spt} \phi_t = \mathbb{B}(0, t)$.

Exercise 1.113. Show that ϕ in the definition above is a C^∞ function on $\mathbb{R}^n$.

Exercise 1.114 (differentiable Urysohn's lemma). If $K \subset U \subset \mathbb{R}^n$ with K compact and U open, there exists $f \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ such that $0 \leq f \leq 1$, $f \equiv 1$ on K and $\operatorname{spt} f \subset U$.

Hint. Let $K' := K + \mathbb{B}(0, \frac{1}{2}d(K, U^c))$ and $(\phi_t)_{>0}$ the standard mollifier in $\mathbb{R}^n$. Take $f = \phi_t * \chi_{K'}$ for a convenient choice of t.

Exercise 1.115 (approximation in $\mathsf{L}^1_{\mathsf{loc}}$). Let $1 \leq p < \infty$, $f \in \mathsf{L}^\mathsf{p}_{\mathsf{loc}}(\mathcal{L}^n)$ and $(\phi_t)_{>0}$ the standard mollifier in $\mathbb{R}^n$. Then $\phi_t * f \to f$ in $\mathsf{L}^\mathsf{p}_{\mathsf{loc}}(\mathcal{L}^n)$, i.e. for all $K \subset \mathbb{R}^n$ compact, $\|\phi_t * f - f\|_{\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_K)} \to 0$.

Hint. For each $K \subset \mathbb{R}^n$ compact, let $K' := K + \mathbb{B}_1$ and $\widetilde{f} := \chi_{K'} \cdot f$. Then $\widetilde{f} \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n)$ and, by theorem 1.111, $\phi_t * \widetilde{f} \to \widetilde{f}$ in $\mathsf{L}^\mathsf{p}(\mathcal{L}^n)$.

Remark 1.116. The previous exercise means that $\phi_t * f$ converges to f in the Fréchet topology of $\mathsf{L}^\mathsf{p}_{\mathsf{loc}}(\mathcal{L}^n)$.

We will resume this discussion on mollifiers and approximations in chapter 6.

1.7. Lusin's and Egorov's Theorems

Theorem 1.117 (Lusin, [**Fed69**]). *Let μ be a Borel regular measure on a metric space X (respectively, a Radon measure on a locally compact Hausdorff space X), Y a separable metric space, $f : \operatorname{dom} f \subset X \to Y$ a μ-measurable map. Then, for each $A \in \sigma(\mu)$ with $\mu(A) < \infty$ and for each $\epsilon > 0$, there exists a closed (respectively, compact) set $C \subset A$ such that $\mu(A \setminus C) < \epsilon$ and $f|_C$ is continuous.*

Proof. We may assume $\operatorname{dom} f = X$ (otherwise, replace A by $A \cap \operatorname{dom} f$ and extend f arbitrarily to X). For each $i \in \mathbb{N}$, there exists a countable disjoint sequence $(Y_{i,j})_{j\in\mathbb{N}}$ in $\mathscr{B}_Y$ such that $\forall j \in \mathbb{N}$, $\operatorname{diam} Y_{i,j} < 1/i$ and $\dot\cup_{j\in\mathbb{N}} Y_{i,j} = Y$. To obtain such a sequence, cover Y with countably many balls $(B_{i,j})$ with diameter less than $1/i$ (what is possible, since Y is separable) and then take $\forall j \in \mathbb{N}$, $Y_{i,j} := B_{i,j} \setminus \cup_{n=1}^{j-1} B_{i,n}$. Let $\forall j \in \mathbb{N}$, $A_{i,j} := A \cap f^{-1}(Y_{i,j}) \in \sigma(\mu)$, so that $A = \dot\cup_{j\in\mathbb{N}} A_{i,j}$. By theorem 1.23 (respectively, exercise 1.31), there exists a closed (respectively, compact) set $C_{i,j} \subset A_{i,j}$ such that $\forall j \in \mathbb{N}, \mu(A_{i,j} \setminus C_{i,j}) < 2^{-i-j}\epsilon$. Then $\mu(A \setminus \cup_{j\in\mathbb{N}} C_{i,j}) < 2^{-i}\epsilon$ and, by proposition 1.11.ii (continuity from above for μ), there exists $J(i) \in \mathbb{N}$ such that $\mu(A \setminus \cup_{j=1}^{J(i)} C_{i,j}) < 2^{-i}\epsilon$. Now, for each $1 \leq j \leq J(i)$, choose $y_{i,j} \in Y_{i,j}$ and define the function g_i on the closed (respectively, compact) set $C_i := \cup_{j=1}^{J(i)} C_{i,j}$ by $g_i|_{C_{i,j}} \equiv y_{i,j}$. It is then clear that $g_i : C_i \to Y$ is continuous and, due to the fact that $\operatorname{diam} Y_{i,j} < 1/i$, we have (denoting by d the metric on Y) $\sup\{d\big(f(x), g_i(x)\big) \mid x \in C_i\} \leq 1/i$. Take

$C := \cap_{i \in \mathbb{N}} C_i$, so that C is closed (respectively, compact) and $\mu(A \setminus C) \le \sum_{i \in \mathbb{N}} \mu(A \setminus C_i) \le \epsilon$. As $\forall i \in \mathbb{N}$, $C \subset C_i$, we have $\forall i \in \mathbb{N}$, $\sup\{d\big(f(x), g_i(x)\big) \mid x \in C\} \le 1/i$, hence $(g_i|_C)_{i \in \mathbb{N}}$ converges uniformly to $f|_C$, so $f|_C$ is continuous. $\square$

COROLLARY 1.118. *With the same hypotheses, if μ is σ-finite, f coincides μ-a.e. with a Borelian map $X \to Y$.*

PROOF. Let $(A_i)_{i \in \mathbb{N}}$ be a sequence in $\sigma(\mu)$ of disjoint sets such that $\forall i \in \mathbb{N}$, $\mu(A_i) < \infty$ and $X = \dot{\cup}_{i \in \mathbb{N}} A_i$. For each $i, j \in \mathbb{N}$, we may apply theorem 1.117 to obtain a closed set $C_{i,j} \subset A_i$ such that $\mu(A_i \setminus C_{i,j}) < 1/j$ and $f|_{C_{i,j}}$ is continuous. Then $B := \cup_{i,j \in \mathbb{N}} C_{i,j}$ is a Borel set such that $\mu(X \setminus C) = 0$ and, for each $E \in \mathscr{B}_Y$, $(f|_B)^{-1}(E) = \cup_{i,j \in \mathbb{N}} (f|_{C_{i,j}})^{-1}(E) \in \mathscr{B}_X$. Choose $y_0 \in Y$ and define $F : X \to Y$ by $F \equiv y_0$ on $X \setminus B$ and $F = f$ on B. Then F is Borelian and $F = f$ μ-a.e. $\square$

COROLLARY 1.119. *Let μ be a σ-finite Borel regular measure on a metric space X (respectively, a σ-finite Radon measure on a locally compact Hausdorff space X), $Y = \mathbb{R}$ or $\mathbb{C}$, $f : \operatorname{dom} f \subset X \to Y$ a μ-measurable function. Then there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathsf{C}(X)$ (respectively, in $\mathsf{C_c}(X)$) which converges pointwise μ-a.e. to f. Moreover, if $\|f\|_\infty < \infty$, we can take this sequence $(f_n)_{n \in \mathbb{N}}$ so that $\forall n \in \mathbb{N}$, $\|f_n\|_\infty \le \|f\|_\infty$.*

PROOF. We may assume, modifying f on a set of measure zero if necessary, that $\operatorname{dom} f = X$ and $\forall x \in X, |f(x)| \le \|f\|_\infty$. Let $(A_i)_{i \in \mathbb{N}}$ be a sequence in $\sigma(\mu)$ of disjoint sets such that $\forall i \in \mathbb{N}$, $\mu(A_i) < \infty$ and $X = \dot{\cup}_{i \in \mathbb{N}} A_i$. For each $i, j \in \mathbb{N}$, we may apply theorem 1.117 to obtain a closed (respectively, compact) set $C_{i,j} \subset A_i$ such that $\mu(A_i \setminus C_{i,j}) < 1/j$ and $f|_{C_{i,j}}$ is continuous. For each $n \in \mathbb{N}$, let C_n be the closed (respectively, compact) set $\cup_{i,j=1}^n C_{i,j}$. Since the last union is finite, $f|_{C_n}$ is continuous; use Tietze's extension theorem to extend $f|_{C_n}$ to a function $f_n \in \mathsf{C}(X)$ (respectively, $f_n \in \mathsf{C_c}(X)$). Note that, if $\|f\|_\infty < \infty$, we may and do take the extension f_n so that $\|f_n\|_\infty \le \|f\|_\infty$. The sequence $(f_n)_{n \in \mathbb{N}}$ thus defined converges pointwise to f on $\cup_{n \in \mathbb{N}} C_n = \cup_{i,j \in \mathbb{N}} C_{i,j}$. Since $\mu(X \setminus \cup_{i,j \in \mathbb{N}} C_{i,j}) = 0$, we are done. $\square$

DEFINITION 1.120. Let μ be a measure on the set X, Y a metric space and $A \subset X$. We say that a sequence $(f_n)_{n \in \mathbb{N}}$ of μ-measurable Y-valued functions on X converges *almost uniformly* on A to a μ-measurable function $f : \operatorname{dom} f \subset X \to Y$ if, for all $\epsilon > 0$, there exists $B \in \sigma(\mu)$ such that $\mu(A \setminus B) < \epsilon$ and $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f on B.

Note that, both in the definition above and in the theorem below, we do not assume A to be μ-measurable.

THEOREM 1.121 (Egorov). *Let μ be a measure on the set X, Y a separable metric space, $A \subset X$ with $\mu(A) < \infty$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of Y-valued measurable functions on X which converges pointwise μ-a.e. on A to a μ-measurable function $f : \operatorname{dom} f \subset X \to Y$. Then $(f_n)_{n\in\mathbb{N}}$ converges almost uniformly to f on A.*

PROOF. Fix $\epsilon > 0$ and denote by d the metric on Y. For $i, j \in \mathbb{N}$, define $C_{i,j} := \cup_{n\geq j}\{x \in X \mid d\big(f_n(x), f(x)\big) \geq 1/i\}$. Note that each $C_{i,j}$ is μ-measurable, since $d : Y \times Y \to \mathbb{R}$ is continuous (hence Borelian) and, for fixed $n \geq j$, $x \in \operatorname{dom} f_n \cap \operatorname{dom} f \mapsto \big(f_n(x), f(x)\big) \in Y \times Y$ is μ-measurable (it is in this point that we use the separability of Y, what ensures $\mathscr{B}_{Y\times Y} = \mathscr{B}_Y \otimes \mathscr{B}_Y$), so that $x \mapsto d\big(f_n(x), f(x)\big)$ is μ-measurable and $C_{i,j}$ is a countable union of μ-measurable sets. Hence, for fixed $i \in \mathbb{N}$, it follows from proposition 1.15.i that $(C_{i,j})_{j\in\mathbb{N}}$ is a decreasing sequence of $\mu \llcorner A$-measurable sets. Since $\mu(A) < \infty$, and since $\cap_{j\in\mathbb{N}} C_{i,j}$ has $\mu \llcorner A$-measure zero due to the fact that $(f_n)_{n\in\mathbb{N}}$ converges μ-a.e. on A to f, we may apply the continuity from above 1.11 for the measure $\mu \llcorner A$ to conclude that $\mu \llcorner A(C_{i,j}) \xrightarrow{j\to\infty} 0$. Therefore, for fixed $i \in \mathbb{N}$, there exists $J(i) \in \mathbb{N}$ such that $\mu \llcorner A(C_{i,J(i)}) < 2^{-i}\epsilon$. Let $B := X \setminus \cup_{i\in\mathbb{N}} C_{i,J(i)}$. Then $B \in \sigma(\mu)$, $\mu(A \setminus B) = \mu \llcorner (\cup_{i\in\mathbb{N}} C_{i,J(i)}) \leq \sum_{i\in\mathbb{N}} \mu \llcorner (C_{i,J(i)}) < \epsilon$ and $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on B. $\square$

We close this section with an application of Egorov's theorem. Recall that, for a Banach space X, the *weak topology* of X is the topology induced by its dual X', i.e. the weakest topological vector space topology on X which makes all elements of X' continuous.

THEOREM 1.122 (theorem 1.35 in [**AFP00**]). *Let μ be a measure on the set X and $1 < p < \infty$. If $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathsf{L}^\mathsf{p}(\mu)$ which converges pointwise almost everywhere to a function f, then $f \in \mathsf{L}^\mathsf{p}(\mu)$ and $(f_n)_{n\in\mathbb{N}}$ converges weakly to f.*

CHAPTER 2

Hausdorff Measures

2.1. Carathéodory's construction

Lebesgue measure on $\mathbb{R}^n$ is not adequate to study "lower dimensional" objects, such as embedded k-dimensional manifolds or, more generally, their measure-theoretic cousins, the k-rectifiable sets. For that purpose, we shall introduce *Hausdorff measures* and *dimension*, a class of Borel regular measures whose origins may be traced back to [**Hau18**] and [**Car14**].

In order to construct Hausdorff measures, we depart from an abstract construction on a metric space X which may be used to generate a plethora of Borel measures on X with nice geometric flavor. We shall be concerned only with Hausdorff measures here, but the interested reader may consult, for instance, [**Fed69**], [**Mat95**] or [**KP08**] for other such measures as well.

Let X be a metric space, $\mathcal{F} \subset 2^X$ and $\zeta : \mathcal{F} \to [0, \infty]$. Roughly speaking, the idea is to "measure" the elements of $\mathcal{F}$ by means of the *method* or *gauge* ζ and use that to define a Borel measure on X, abstracting the geometric idea underlying the construction of the Lebesgue measure. We define such measure in two steps:

1) For $0 < \delta \leq \infty$ we define $\forall A \subset X$,

$$\psi_\delta(A) := \inf\{\sum_{S \in \mathcal{G}} \zeta(S) \mid \mathcal{G} \subset \mathcal{F} \cap \{S \mid \operatorname{diam} S \leq \delta\}, \mathcal{G} \text{ countable cover of } A\}.$$

   Note that $\inf \emptyset = \infty$, so that $\psi_\delta(A) = \infty$ if there is no countable cover of A by elements of $\mathcal{F}$ with diameter $\leq \delta$. Moreover, if $A = \emptyset$, $\mathcal{G} = \emptyset \subset \mathcal{F}$ is such a cover and, since the sum over the empty family is zero (as we defined the sum by means of the integral with respect to counting measure at the end of definition 1.58), we conclude that $\psi_\delta(\emptyset) = 0$. Besides, it is straightforward to check that ψ_δ is monotone and countably subbaditive, i.e. it is a measure according to definition 1.1. In general, the measures $(\psi_\delta)_\delta$ thus defined are not Borel, but we can fabricate a Borel measure out of them, as we do in the second step of the construction:
2) Define, for each $A \subset X$, $\psi(A) := \sup\{\psi_\delta(A) \mid 0 < \delta \leq \infty\} \in [0, \infty]$.

Note that, for fixed $A \subset X$, $\{\psi_\delta(A)\}_\delta$ is decreasing in δ, so that the sup in the definition above coincides with $\lim_{\delta\to 0}\psi_\delta(A)$.

Definition 2.1. With the notation above, we call ψ the *result of Carathéodory's construction from the gauge ζ on $\mathcal{F}$*, and we call ψ_δ the *size δ approximating measure.*

We prove in the next proposition that ψ is actually a Borel measure.

Proposition 2.2. *Let X be a metric space and ψ be the result of Carathéodory's construction from the gauge ζ on $\mathcal{F} \subset 2^X$. Then ψ is a Borel measure. Besides, if $\mathcal{F} \subset \mathscr{B}_X$, ψ is Borel regular.*

Proof. We denote by d the metric on X. That ψ is a measure follows directly from the fact that, for each $\delta \in (0,\infty]$, ψ_δ is a measure. In order to prove that ψ is Borel, we verify the Carathéodory's criterion 1.18. Let $A, B \subset X$ such that $d(A,B) = \delta > 0$ and $\psi(A\cup B) < \infty$; we must show that $\psi(A\cup B) \geq \psi(A)+\psi(B)$. Indeed, for any $\eta < \delta/2$, cover $A \cup B$ by a countable family $\mathcal{G} \subset \mathcal{F}$ whose elements have diameters $\leq \eta$. Note that no element of $\mathcal{G}$ intersects both A an B, thanks to the triangle inequality. Thus, discarding the elements of $\mathcal{G}$ which do not meet A or B, we obtain a subcover $\mathcal{G}' \subset \mathcal{G}$ of $A\cup B$ that may be written as a disjoint union $\mathcal{G}' = \mathcal{G}_1 \dot\cup \mathcal{G}_2$, where $\mathcal{G}_1$ covers A and $\mathcal{G}_2$ covers B. Then $\sum_{S\in\mathcal{G}}\zeta(S) \geq \sum_{S\in\mathcal{G}'}\zeta(S) = \sum_{S\in\mathcal{G}_1}\zeta(S) + \sum_{S\in\mathcal{G}_2}\zeta(S) \geq \psi_\eta(A) + \psi_\eta(B)$. By the arbitrariness of $\mathcal{G} \subset \mathcal{F}$ whose elements have diameters $\leq \eta$, we conclude that $\psi_\eta(A\cup B) \geq \psi_\eta(A) + \psi_\eta(B)$, for all $\eta < \delta/2$. Hence, taking $\eta \to 0$, it follows that $\psi(A\cup B) \geq \psi(A)+\psi(B)$, as asserted.

Assume now that $\mathcal{F} \subset \mathscr{B}_X$. We contend that ψ is Borel regular. Indeed, let $A \subset X$ such that $\psi(A) < \infty$; we must prove the existence of $B \in \mathscr{B}_X$ such that $B \supset A$ and $\psi(B) = \psi(A)$. For each $\delta > 0$, we can take $B_\delta \in \mathscr{B}_X$ such that $B_\delta \supset A$ and $\psi_\delta(B_\delta) = \psi_\delta(A)$: choose, for each $n \in \mathbb{N}$, a countable cover $\mathcal{G}_n \subset \mathcal{F}$ of A whose elements have diameters $\leq \delta$, such that $\sum_{S\in\mathcal{G}_n}\zeta(S) < \psi_\delta(A) + 1/n$, and then put $B_\delta := \cap_{n\in\mathbb{N}} \cup_{S\in\mathcal{G}_n} S$. Define $B := \cap_{n\in\mathbb{N}} B_{1/n} \in \mathscr{B}_X$. Then $B \supset A$ and, for each $n \in \mathbb{N}$, $\psi_{1/n}(A) \leq \psi_{1/n}(B) \leq \psi_{1/n}(B_{1/n}) = \psi_{1/n}(A)$, so that $\psi_{1/n}(A) = \psi_{1/n}(B)$, and taking $n \to \infty$ yields $\psi(A) = \psi(B)$. $\square$

Definition 2.3. Let X be a metric space and m a nonnegative real number. Take $\mathcal{F} = 2^X$ and $\zeta : 2^X \to [0,\infty]$ given by

$$\zeta(S) := \alpha(m)\frac{(\operatorname{diam} S)^m}{2^m},$$

where $\alpha(m) = \frac{\pi^{m/2}}{\Gamma(m/2+1)}$ (i.e. the euclidean volume of $\mathbb{B}^m$ if m integer). The result of Carathéodory's construction from the gauge ζ on 2^X is called *Hausdorff m-dimensional measure* on X, denoted by $\mathcal{H}^m$. We use the notation $\mathcal{H}^m_\delta$ for the size δ approximation of $\mathcal{H}^m$.

PROPOSITION 2.4 (immediate properties of Hausdorff measure). *Let X be a metric space and m a nonnegative real number. The following properties hold for $\mathcal{H}^m$:*

1) *The Hausdorff measure is compatible with the operation of taking traces. That is, if X is a metric space and $A \subset X$, the trace of $\mathcal{H}^m$ on A coincides with the m-dimensional Hausdorff measure on A (as a metric subspace of X).*
2) *The Hausdorff measure is invariant by isometries. That is, if Y is another metric space and $f : X \to Y$ is an isometry onto Y, then the pushforward $f_\# \mathcal{H}^m$ coincides with the Hausdorff m-dimensional measure on Y.*
3) *If Y is another metric space and $f : X \to Y$ has Lipschitz constant $\operatorname{Lip} f < \infty$, then $\forall A \subset X$, $\mathcal{H}^m\big(f(A)\big) \le (\operatorname{Lip} f)^m \mathcal{H}^m(A)$.*
4) *$\mathcal{H}^m$ also coincides with the result of Carathéodory's construction from ζ (same gauge as in definition 2.3) on $\mathcal{F}' = \{$closed subsets of $X\}$ or $\mathcal{F}'' = \{$open subsets of $X\}$. If X is a normed vector space, we may also take $\mathcal{F}''' = \{$closed convex subsets of $X\}$.*
5) *$\mathcal{H}^m$ is a Borel regular measure on X.*
6) *$\mathcal{H}^0$ coincides with the counting measure on X.*

PROOF. Properties 1) and 2) are immediate; property 2) is also a direct consequence of 3) since, if f is an isometry onto Y, then $\operatorname{Lip} f = \operatorname{Lip} f^{-1} = 1$.

3) Let $A \subset X$, $\delta \in (0, \infty]$ and $\mathcal{G}$ a countable cover of A by subsets of diameter $\le \delta$. Then $f(\mathcal{G}) := \{f(S) \mid S \in \mathcal{G}\}$ is a countable cover of $f(A)$ by subsets of diameter $\le \delta \cdot \operatorname{Lip} f$ since, for each $S \subset X$, $\operatorname{diam} f(S) \le \operatorname{diam} S \cdot \operatorname{Lip} f$. Hence,
$$\begin{aligned}\mathcal{H}^m_{\delta \operatorname{Lip} f}\big(f(A)\big) &\le \sum_{S\in\mathcal{G}} \alpha(m) 2^{-m} \big(\operatorname{diam}\, f(S)\big)^m \le \\ &\le (\operatorname{Lip} f)^m \sum_{S\in\mathcal{G}} \alpha(m) 2^{-m} (\operatorname{diam}\, S)^m.\end{aligned}$$
By the arbitrariness of $\mathcal{G}$, taking the infimum we conclude that $\mathcal{H}^m_{\delta \operatorname{Lip} f}\big(f(A)\big) \le (\operatorname{Lip} f)^m \mathcal{H}^m_\delta(A)$; thus, taking $\delta \to 0$, the thesis follows.
4) It is clear that we may use $\mathcal{F}' = \{$closed subsets of $X\}$ instead of $\mathcal{F} = 2^X$, since the diameters are not affected by taking closures. If X is a normed vector space, the same argument works for $\mathcal{F}''' = \{$closed

convex subsets of $X\}$, since diameters are not affected by taking closed convex hulls either.

As to the remaining case, let ψ be the result of Carathéodory's construction from ζ on $\mathcal{F}'' = \{\text{open subsets of } X\}$. Since $\mathcal{F}'' \subset \mathcal{F} = 2^X$, it is clear that $\mathcal{H}^m \leq \psi$. To prove the reverse inequality, define, for each $S \subset X$ and $\delta > 0$, $S_\delta := \cup_{x \in S} \mathbb{U}(x, \delta/2)$; note that S_δ is open (for it is a union of open balls) and $\operatorname{diam} S_\delta \leq \operatorname{diam} S + \delta$. Given $\delta > 0$ and $A \subset X$ such that $\mathcal{H}^m(A) < \infty$, for any countable cover $\mathcal{G} = \{S_n \mid n \in \mathbb{N}\}$ of A by subsets of diameter $\leq \delta$ such that $\sum_{n\in\mathbb{N}} \alpha(m) 2^{-m} (\operatorname{diam} S_n)^m < \infty$, and for any $0 < \epsilon \leq \delta$, $\mathcal{G}_\epsilon := \{(S_n)_{2^{-n}\epsilon} \mid n \in \mathbb{N}\} \subset \mathcal{F}'''$ is a countable open cover of A by subsets of diameter $\leq 2\delta$. Therefore, $\psi_{2\delta}(A) \leq \sum_{n\in\mathbb{N}} \alpha(m) 2^{-m} \big(\operatorname{diam} (S_n)_{2^{-n}\epsilon}\big)^m \leq \sum_{n\in\mathbb{N}} \alpha(m) 2^{-m} (\operatorname{diam} S_n + 2^{-n}\epsilon)^m$. Taking $\epsilon \to 0$ along any decreasing sequence, we may apply theorem 1.64 to conclude that the last sum converges to $\sum_{n\in\mathbb{N}} \alpha(m) 2^{-m} (\operatorname{diam} S_n)^m$, so that $\psi_{2\delta}(A) \leq \sum_{n\in\mathbb{N}} \alpha(m) 2^{-m} (\operatorname{diam} S_n)^m$. By the arbitrariness of the cover $\mathcal{G}$, taking the infimum we conclude that $\psi_{2\delta}(A) \leq \mathcal{H}^m_\delta(A)$; thus, taking $\delta \to 0$, it follows that $\psi(A) \leq \mathcal{H}^m(A)$, as asserted.

5) Since $\mathcal{F}' \subset \mathscr{B}_X$, it follows from the previous item and from proposition 2.2 that $\mathcal{H}^m$ is a Borel regular measure.
6) It is clear that, $\forall x \in X, \forall \delta \in (0, \infty]$, $\mathcal{H}^0_\delta(\{x\}) = 1$. Thus, $\mathcal{H}^0(\{x\}) = 1$. Since $\mathcal{H}^0$ is a Borel measure, we conclude that the measure of each finite set coincides with its cardinality and the measure of each infinity set is ∞, i.e. $\mathcal{H}^0$ is the counting measure on X.

$\square$

Corollary 2.5. *If X, Y are metric spaces, m a nonnegative real number and $f : X \to Y$ is an isometry into Y, then $\forall A \subset X$, $\mathcal{H}^m\big(f(A)\big) = \mathcal{H}^m(A)$.*

Proof. By proposition 2.4.(1), we may substitute $Y' := \operatorname{Im} f$ for Y without modifying $\mathcal{H}^m\big(f(A)\big)$. Since $f : X \to Y'$ is an isometry onto Y', the thesis follows from proposition 2.4.(2). $\square$

Exercise 2.6 ($\mathcal{H}^m$–null sets). Let X be a metric space, $A \subset X$ and $0 < m < \infty$. The following statements are equivalent:

1) $\mathcal{H}^m(A) = 0$.
2) $\exists \delta \in (0, \infty]$ such that $\mathcal{H}^m_\delta(A) = 0$.
3) $\forall \epsilon > 0$, $\exists (E_n)_{n\in\mathbb{N}}$ cover of A such that $\sum_{n\in\mathbb{N}} (\operatorname{diam} E_n)^m < \epsilon$.

The next proposition is a preparation for the introduction of the notion of *Hausdorff dimension*.

PROPOSITION 2.7. *Let X be a metric space, $A \subset X$ and $0 \leq s < t < \infty$. If $\mathcal{H}^s(A) < \infty$ then $\mathcal{H}^t(A) = 0$.*

PROOF. For each $\delta > 0$, since $\mathcal{H}^s_\delta(A) \leq \mathcal{H}^s(A) < \infty$, there exists a countable cover $\mathcal{G}$ of A by subsets of diameter $\leq \delta$ such that $\sum_{S\in\mathcal{G}} \alpha(s)2^{-s}(\mathrm{diam}\ S)^s < \mathcal{H}^s(A) + 1$. Then

$$\mathcal{H}^t_\delta(A) \leq \sum_{S\in\mathcal{G}} \alpha(t)2^{-t}(\mathrm{diam}\ S)^t = \frac{\alpha(t)2^{-t}}{\alpha(s)2^{-s}} \sum_{S\in\mathcal{G}} \alpha(s)2^{-s}(\mathrm{diam}\ S)^{t-s}(\mathrm{diam}\ S)^s \leq$$

$$\leq 2^{s-t}\frac{\alpha(t)}{\alpha(s)}\delta^{t-s} \sum_{S\in\mathcal{G}} \alpha(s)2^{-s}(\mathrm{diam}\ S)^s \leq 2^{s-t}\frac{\alpha(t)}{\alpha(s)}\delta^{t-s}\big(\mathcal{H}^s(A)+1\big),$$

and taking $\delta \to 0$ we conclude that $\mathcal{H}^t(A) = 0$. □

As a corollary, if $0 \leq s < t < \infty$ and $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = \infty$. It then follows that $\inf\{m \in [0,\infty) \mid \mathcal{H}^m(A) = 0\} = \sup\{m \in [0,\infty) \mid \mathcal{H}^m(A) = \infty\} \in [0,\infty]$.

DEFINITION 2.8. Let X be a metric space and $A \subset X$. The extended real number $\inf\{m \in [0,\infty) \mid \mathcal{H}^m(A) = 0\} = \sup\{m \in [0,\infty) \mid \mathcal{H}^m(A) = \infty\} \in [0,\infty]$ is called *Hausdorff dimension* of A, denoted by $\mathcal{H}$-dim A.

With the notation above, note that $\forall m > \mathcal{H}\text{-dim}\ A$, $\mathcal{H}^m(A) = 0$, and $\forall m < \mathcal{H}\text{-dim}\ A$, $\mathcal{H}^m(A) = \infty$. For $m = \mathcal{H}\text{-dim}\ A$, nothing can be said about $\mathcal{H}^m(A)$, i.e. it can be zero, strictly positive or ∞. On the other hand, if $\exists m \in [0,\infty)$ such that $0 < \mathcal{H}^m(A) < \infty$, then $\mathcal{H}\text{-dim}\ A = m$.

EXERCISE 2.9 (properties of Hausdorff dimension). Let X be a metric space.

a) If $Y \subset X$ is a metric subspace of X and $A \subset Y$, the Hausdorff dimension of A as a subset of the metric space Y is the same for A as a subset of the metric space X.
b) The Hausdorff dimension is invariant by isometries, i.e. if Y is a metric space, $f : X \to Y$ an isometry into Y and $A \subset X$, then $\mathcal{H}\text{-dim}\ A = \mathcal{H}\text{-dim}\ f(A)$.
c) Let X, Y be metric spaces and $f : X \to Y$ be a Lipschitz map. For all $A \subset X$, $\mathcal{H}\text{-dim}\ f(A) \leq \mathcal{H}\text{-dim}\ A$. In particular, if f is bi-Lipschitz onto its image (i.e. f is Lipschitz and has a Lipschitz inverse $f^{-1} : \mathrm{Im}\, f \to X$), then $\forall A \subset X$, $\mathcal{H}\text{-dim}\ f(A) = \mathcal{H}\text{-dim}\ A$.
d) (monotonicity) If $A \subset B \subset X$, $\mathcal{H}\text{-dim}\ A \leq \mathcal{H}\text{-dim}\ B$.
e) (stability with respect to countable unions) If $A = \cup_{n\in\mathbb{N}} A_n \subset X$, then $\mathcal{H}\text{-dim}\ A = \sup\{\mathcal{H}\text{-dim}\ A_n \mid n \in \mathbb{N}\}$.

We will show next that, for $X = \mathbb{R}^n$, the Hausdorff n-dimensional measure $\mathcal{H}^n$ coincides with the Lebesgue n-dimensional measure $\mathcal{L}^n$. In particular, that implies $\mathcal{H}$-dim $\mathbb{R}^n = n$ (use the stability with respect to countable unions of the Hausdorff dimension, cf. exercise 2.9, to $\mathbb{R}^n = \cup_{n\in\mathbb{N}} C_n$, where each C_n is a cube with finite Lebesgue measure). More generally, we will show in exercise 5.42 with the help of the *area formula* that, for any smooth embedded k-submanifold $\mathsf{M} \subset \mathbb{R}^n$, the measure induced by the Riemannian metric on M coincides with the trace $\mathcal{H}^k|_{\mathsf{M}}$, which implies $\mathcal{H}$-dim $M = k$.

In order to prove that $\mathcal{H}^n = \mathcal{L}^n$ in $\mathbb{R}^n$, we need to establish some preliminaries which are of interest on their own right.

2.2. Vitali's Covering Theorem

NOTATION. For a closed ball $B = \mathbb{B}(x, r)$ in $\mathbb{R}^n$ and $0 < t < \infty$, we define

$$tB := \mathbb{B}(x, tr).$$

In a general metric space, however, the center and radius of a ball are not uniquely determined (take, for instance, $[0, \infty)$ as a metric subspace of $\mathbb{R}$ and look at the balls centered at 0). In this case, for a closed ball $B \subset X$, in order to define tB, we could choose once and for all a center x and a radius r and proceed like above, i.e. we might consider x and r as part of the given data when we speak of a ball; instead, we prefer to proceed like in [**Mat95**] and define, let us say, for $t = 5$:

(2.1)
$$5B := \cup\{B' \subset X \text{ closed ball} \mid B' \cap B \neq \emptyset, \text{diam } B' \leq 2 \,\text{diam } B\},$$

which clearly coincides with the previous definition in case $X = \mathbb{R}^n$.

THEOREM 2.10 (5–times covering lemma). *Let X be a metric space and $\mathcal{F} \subset 2^X$ a family of nondegenerate closed balls in X such that $\sup\{\text{diam } B \mid B \in \mathcal{F}\} < \infty$. Then there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\cup_{B\in\mathcal{F}} B \subset \cup_{B\in\mathcal{G}} 5B$.*

PROOF. Let $R := \sup\{\text{diam } B \mid B \in \mathcal{F}\} < \infty$. Since the balls in $\mathcal{F}$ are nondegenerate, i.e. have strictly positive diameter, for any $B \in \mathcal{F}$, diam $B \in (0, R] = \dot\cup_{j\in\mathbb{N}}(\frac{R}{2^j}, \frac{R}{2^{j-1}}]$. Thus, putting $\forall j \in \mathbb{N}, \mathcal{F}_j := \{B \in \mathcal{F} \mid \text{diam } B \in (\frac{R}{2^j}, \frac{R}{2^{j-1}}]\}$, we have $\dot\cup_{j\in\mathbb{N}} \mathcal{F}_j = \mathcal{F}$.

We now define inductively $(\mathcal{G}_j)_{j\in\mathbb{N}}$ by: 1) $\mathcal{G}_1$ is a maximal disjoint subfamily of $\mathcal{F}_1$, obtained by an application of Zorn's lemma to the set of all disjoint subfamilies of $\mathcal{F}_1$ partially ordered by inclusion; 2) Once defined $\mathcal{G}_1 \subset \mathcal{F}_1, \ldots, \mathcal{G}_{j-1} \subset \mathcal{F}_{j-1}$, we take a maximal disjoint subfamily $\mathcal{G}_j$ of $\mathcal{F}'_j := \{B \in \mathcal{F}_j \mid \forall B' \in \cup_{i=1}^{j-1}\mathcal{G}_i, B \cap B' = \emptyset\}$, obtained by an application of Zorn's lemma to the set of all disjoint subfamilies of $\mathcal{F}'_j \subset \mathcal{F}_j$ partially ordered by inclusion.

We contend that $\mathcal{G} := \cup_{j\in\mathbb{N}}\mathcal{G}_j \subset \mathcal{F}$ satisfies the thesis of the theorem. Indeed, it is clear, by construction, that $\mathcal{G}$ is a disjoint subfamily of $\mathcal{F}$. On the other hand, for any $B \in \mathcal{F}_j$, there exists $B' \in \cup_{i=1}^{j}\mathcal{G}_i$ such that $B \cap B' \neq \emptyset$, otherwise $\mathcal{G}_j \,\dot\cup \{B\} \supsetneqq \mathcal{G}_j$ would be a disjoint subfamily of $\mathcal{F}'_j$, violating the maximality of $\mathcal{G}_j$. Since $\operatorname{diam} B \leq \frac{R}{2^{j-1}} = 2\frac{R}{2^j}$ and $\frac{R}{2^j} < \operatorname{diam} B'$, it follows that $\operatorname{diam} B < 2 \operatorname{diam} B'$, so that $B \subset 5B'$. $\square$

REMARK 2.11. With the notation from theorem 2.10:

1) Note that, if X is separable, then $\mathcal{G}$ is countable (since any disjoint family of sets with nonempty interiors in X is countable).
2) We have actually proved a stronger statement than the thesis: there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that, for any $B \in \mathcal{F}$, $\exists B' \in \mathcal{G}$ with $B \cap B' \neq \emptyset$ and $\operatorname{diam} B < 2 \operatorname{diam} B'$ (thus $B \subset 5B'$).

DEFINITION 2.12. Let X be a metric space, $\mathcal{F}$ a collection of balls in X and $A \subset X$. We say that $\mathcal{F}$ is a *fine cover* A, or that $\mathcal{F}$ *covers A finely*, if $\mathcal{F}$ is a cover of A such that, $\forall x \in A$, $\inf\{\operatorname{diam} B \mid x \in B \in \mathcal{F}\} = 0$.

COROLLARY 2.13. *Let X be a metric space, $A \subset X$, $\mathcal{F} \subset 2^X$ a family of nondegenerate closed balls of X which covers A finely. Then there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that, for all $F \subset \mathcal{F}$ finite, $A \setminus \cup_{B\in F} B \subset \cup_{B\in\mathcal{G}\setminus F} 5B$.*

PROOF. Since the cover $\mathcal{F}$ is fine, we may assume that $\sup\{\operatorname{diam} B \mid B \in \mathcal{F}\} \leq 1$; otherwise, discard the balls in $\mathcal{F}$ with diameter > 1, so that the remaining balls still cover A finely. Take $\mathcal{G} \subset \mathcal{F}$ as in remark 2.11.2. Let $x \in A \setminus \cup_{B\in F} B$. Since F is finite, $\cup_{B\in F} B$ is closed, hence there exists $r > 0$ such that $\mathbb{U}(x, r) \cap \cup_{B\in F} B = \emptyset$. Since $\mathcal{F}$ covers A finely, there exists $B \in \mathcal{F}$ such that $x \in B$ and $\operatorname{diam} B < r$, so that $B \subset \mathbb{U}(x, r)$, thus $B \cap \cup_{B\in F} B = \emptyset$. By remark 2.11.2, there exists $B' \in \mathcal{G}$ such that $B' \cap B \neq \emptyset$ (hence $B' \notin F$) and $\operatorname{diam} B < 2 \operatorname{diam} B'$, so that $x \in B \subset 5B'$. Therefore, $A \setminus \cup_{B\in F} B \subset \cup_{B\in\mathcal{G}\setminus F} 5B$, as asserted. $\square$

COROLLARY 2.14 (Vitali's covering theorem for the Lebesgue measure). *Let $A \subset \mathbb{R}^n$ and $\mathcal{F}$ a collection of nondegenerate closed balls in $\mathbb{R}^n$ which covers A finely. Then, for every $\epsilon > 0$, there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{L}^n(\cup\mathcal{G}) \leq \mathcal{L}^n(A) + \epsilon$ and $\mathcal{L}^n(A \setminus \cup\mathcal{G}) = 0$.*

Note that we do not assume A to be $\mathcal{L}^n$-measurable.

PROOF. Assume first that $\mathcal{L}^n(A) < \infty$. We may assume that $\mathcal{L}^n(A) > 0$, otherwise the thesis is trivial. Fix $0 < \delta < 5^{-n}$ (so that $1-5^{-n}+\delta < 1$) with $\delta\mathcal{L}^n(A) < \epsilon$. Since $\mathcal{L}^n$ is Borel regular, by theorem 1.23 there exists an open set $U \supset A$ such that $\mathcal{L}^n(U) < (1+\delta)\mathcal{L}^n(A)$. We will show that there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ whose balls are contained in U and $\mathcal{L}^n(A \setminus \cup\mathcal{G}) = 0$, whence the thesis (since $\mathcal{L}^n(\cup\mathcal{G}) \leq \mathcal{L}^n(U) < (1+\delta)\mathcal{L}^n(A) < \mathcal{L}^n(A) + \epsilon$).

Fix $\theta \in (1 - 5^{-n} + \delta, 1)$.

1) Put $\mathcal{F}_U := \{B \in \mathcal{F} \mid B \subset U, \text{diam } B \leq 1\}$. Since $\mathcal{F}$ covers A finely, it is clear that $\mathcal{F}_U$ is still a fine cover of A. Applying theorem 2.10 to $\mathcal{F}_U$, there exists a disjoint subfamily $\mathcal{G}_U \subset \mathcal{F}_U$ such that $A \subset \cup\mathcal{F}_U \subset \cup_{B\in\mathcal{G}_U} 5B$. Since $\mathcal{G}_U$ is disjoint (hence countable, by remark 2.11), it follows that $\mathcal{L}^n(A) \leq \mathcal{L}^n(\cup_{B\in\mathcal{G}_U} 5B) \leq \sum_{B\in\mathcal{G}_U} \mathcal{L}^n(5B) = 5^n \sum_{B\in\mathcal{G}_U} \mathcal{L}^n(B) = 5^n\mathcal{L}^n(\cup\mathcal{G}_U)$. Thus $\mathcal{L}^n(A\setminus\cup\mathcal{G}_U) \leq \mathcal{L}^n(U\setminus\cup\mathcal{G}_U) = \mathcal{L}^n(U) - \mathcal{L}^n(\cup\mathcal{G}_U) \leq (1+\delta-5^{-n})\mathcal{L}^n(A) < \theta\mathcal{L}^n(A)$; moreover, since $\mathcal{L}^n(A) < \infty$, we may apply the continuity from above 1.11 for the Borel measure $\mathcal{L}^n \llcorner A$ to obtain a finite subfamily $\mathcal{G}_1 \subset \mathcal{G}_U$ such that $\mathcal{L}^n(A \setminus \cup\mathcal{G}_1) < \theta\mathcal{L}^n(A)$.
2) Given $2 \leq j \in \mathbb{N}$, assume we have defined finite disjoint subfamilies $\mathcal{G}_1 \subset \cdots \subset \mathcal{G}_{j-1} \subset \mathcal{F}$ such that, for $1 \leq i \leq j-1$, the balls of $\mathcal{G}_i$ are contained in U and $\mathcal{L}^n(A \setminus \cup\mathcal{G}_i) < \theta^i\mathcal{L}^n(A)$. If $\mathcal{L}^n(A \setminus \cup\mathcal{G}_{j-1}) = 0$, we stop and the thesis follows with $\mathcal{G} := \mathcal{G}_{j-1}$; otherwise, we reapply the argument of the previous item to the open set $U' := U \setminus \cup\mathcal{G}_{j-1}$ in place of U and to $A' := A \setminus \cup\mathcal{G}_{j-1} \subset U'$ in place of A (reducing the open set U', if necessary, we may assume that $\mathcal{L}^n(U') < (1+\delta)\mathcal{L}^n(A')$): take $\mathcal{F}_{U'} := \{B \in \mathcal{F} \mid B \subset U', \text{diam } B \leq 1\}$, which is a fine cover of A', and use theorem 2.10 to extract a disjoint subfamily $\mathcal{G}_{U'} \subset \mathcal{F}_{U'}$ such that $A' \subset \cup\mathcal{F}_{U'} \subset \cup_{B\in\mathcal{G}_{U'}} 5B$, so that $\mathcal{L}^n(A' \setminus \cup\mathcal{G}_{U'}) \leq (1+\delta-5^{-n})\mathcal{L}^n(A')$. Then, applying once more the continuity from above for the Borel measure $\mathcal{L}^n \llcorner A'$, there exists a finite set $\mathcal{G}'_j \subset \mathcal{G}_{U'}$ such that $\mathcal{L}^n(A' \setminus \cup\mathcal{G}'_j) < \theta\mathcal{L}^n(A') < \theta^j\mathcal{L}^n(A)$. Put $\mathcal{G}_j := \mathcal{G}_{j-1} \cup \mathcal{G}'_j$. Then $\mathcal{G}_j \subset \mathcal{F}$ is a finite disjoint family whose balls are contained in U, and $A \setminus \cup\mathcal{G}_j = A' \setminus \cup\mathcal{G}'_j$ has Lebesgue measure $< \theta^j\mathcal{L}^n(A)$.
3) We have thus inductively defined an increasing sequence $(\mathcal{G}_j)_{j\in\mathbb{N}}$ such that, for each $j \in \mathbb{N}$, $\mathcal{G}_j$ is a finite disjoint subfamily of $\mathcal{F}$ whose balls are contained in U, with $\mathcal{L}^n(A\setminus\cup\mathcal{G}_j) < \theta^j\mathcal{L}^n(A)$. Define $\mathcal{G} := \cup_{i\in\mathbb{N}}\mathcal{G}_i$; then $\mathcal{G}$ is a disjoint subfamily of $\mathcal{F}$ whose balls are contained in U, with $\forall j \in \mathbb{N}$, $\mathcal{L}^n(A \setminus \cup\mathcal{G}) \leq \theta^j\mathcal{L}^n(A) \xrightarrow{j\to\infty} 0$. Hence $\mathcal{L}^n(A \setminus \cup\mathcal{G}) = 0$, which concludes the proof in the case $\mathcal{L}^n(A) < \infty$.

If $\mathcal{L}^n(A) = \infty$, we take $\forall k \in \mathbb{N}$, $V_k := \{x \in \mathbb{R}^n \mid k-1 < \|x\| < k\}$. Given $\epsilon > 0$, for each $k \in \mathbb{N}$ we take an open set $U_k \supset A \cap V_k$ such that $\mathcal{L}^n\big(U_k \setminus (A \cap V_k)\big) < 2^{-k}\epsilon$; substituting $U_k \cap V_k$ for U_k, we may assume $U_k \subset V_k$. We now apply the first part of the proof to find, for each $k \in \mathbb{N}$, a disjoint subfamily $\mathcal{G}_k \subset \mathcal{F}$ whose balls are contained in $U_k \subset V_k$ and $\mathcal{L}^n(A \cap V_k \setminus \cup\mathcal{G}_k) = 0$. Then, since the V_k's are pairwise disjoint, $\mathcal{G} := \cup_{k\in\mathbb{N}}\mathcal{G}_k$ is a disjoint subfamily of $\mathcal{F}$ such that $\mathcal{L}^n(A \setminus \cup\mathcal{G}) \leq \mathcal{L}^n\big(\cup_{k\in\mathbb{N}}(A \cap V_k \setminus \cup\mathcal{G}_k)\big) + \mathcal{L}^n(\cup_{k\in\mathbb{N}}\{x \in \mathbb{R}^n \mid \|x\| = k-1\}) = 0$. Besides, $\mathcal{L}^n(\cup\mathcal{G}) = \sum_{k\in\mathbb{N}} \mathcal{L}^n(\cup\mathcal{G}_k) < \sum_{k\in\mathbb{N}} \mathcal{L}^n(A \cap V_k) + 2^{-k}\epsilon = \mathcal{L}^n(A) + \epsilon$. $\square$

COROLLARY 2.15 (filling open sets with balls with respect to Lebesgue measure). *Let $U \subset \mathbb{R}^n$ be an open set and $\mathcal{F}$ a family of nondegenerate closed balls contained in U which covers U finely (for instance, if $\mathcal{F}$ is the family of all nondegenerate closed balls contained in U, or the family of all such balls with diameters bounded by a fixed $\delta > 0$). Then there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{L}^n(U \setminus \cup\mathcal{G}) = 0$.*

PROOF. Apply the previous corollary with $A = U$. $\square$

2.3. Steiner Symmetrization

We briefly study in this subsection some properties of the operation called *Steiner symmetrization*, introduced by Jakob Steiner in 1836 ([Ste38]). This operation will be used to prove the isodiametric inequality in 2.19, our key ingredient to show that $\mathcal{L}^n = \mathcal{H}^n$ in $\mathbb{R}^n$.

DEFINITION 2.16. Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$ and identify $\mathbb{R}^{n-1} \equiv \langle e_1, \dots, e_{n-1}\rangle$, $\mathbb{R} \equiv \langle e_n\rangle$, so that $\mathbb{R}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R}$. We define the *Steiner symmetrization* with respect to $\mathbb{R}^{n-1}$ to be the map $\mathsf{S}_{\mathsf{e_n}} : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$ defined by (see figure 1):

$$\mathsf{S}_{\mathsf{e_n}}(A) := \bigcup_{\{x' \in \mathbb{R}^{n-1} \mid A_{x'} \neq \emptyset\}} \{(x', x_n) \mid |x_n| \leq \frac{1}{2}\mathcal{L}^1(A_{x'})\},$$

where we have used the notation for sections established in 1.4.

Given $a \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$, we define similarly the Steiner symmetrization S_a with respect to the $(n-1)$-dimensional subspace $\langle a\rangle^\perp$: take any orthogonal map $\phi \in \mathrm{O}(n)$ such that $\phi(a) = e_n$ (hence $\phi(\langle a\rangle^\perp) = \mathbb{R}^{n-1}$) and put $\mathsf{S}_\mathsf{a} := \phi^{-1} \circ \mathsf{S}_{\mathsf{e_n}} \circ \phi$.

PROPOSITION 2.17 (properties of Steiner symmetrization). *Let $a \in \mathbb{S}^{n-1}$.*

i) $\forall A \subset \mathbb{R}^n$, diam $\mathsf{S}_\mathsf{a}(A) \leq$ diam A.
ii) If $A \subset \mathbb{R}^n$ is $\mathcal{L}^n$-measurable, then so is $\mathsf{S}_\mathsf{a}(A)$ and $\mathcal{L}^n(A) = \mathcal{L}^n\big(\mathsf{S}_\mathsf{a}(A)\big)$.

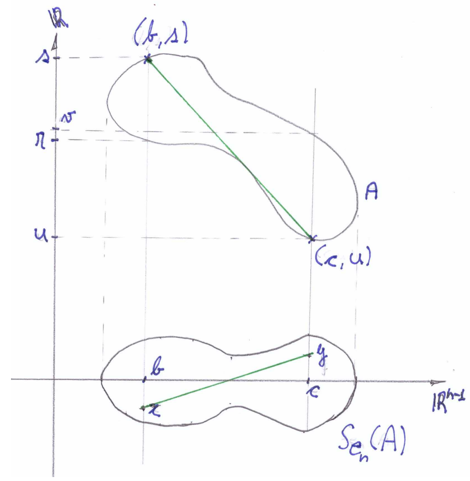

FIGURE 1. Steiner Symmetrization

PROOF. Since diameters and Lebesgue measure are preserved by isometries, it suffices to prove the proposition for $a = e_n$.

i) First we make a reduction: it suffices to prove the thesis for closed sets. Indeed, assuming the thesis for closed sets, since the Steiner symmetrization is clearly monotone with respect to set inclusion, it follows for arbitrary $A \subset \mathbb{R}^n$ that diam $\mathsf{S}_{\mathsf{e_n}}(A) \leq$ diam $\mathsf{S}_{\mathsf{e_n}}(\overline{A}) \leq$ diam $\overline{A} =$ diam A.

   So, assume that A is closed. Given $x, y \in \mathsf{S}_{\mathsf{e_n}}(A)$, we will exhibit $x', y' \in A$ such that $\|x - y\| \leq \|x' - y'\|$, what clearly implies diam $\mathsf{S}_{\mathsf{e_n}}(A) \leq$ diam (A). Indeed, let (see figure 1 for the notation) $x = (b, x_n)$, $y = (c, y_n)$, $r := \inf\{t \mid (b,t) \in A\}$, $s := \sup\{t \mid (b,t) \in A\}$, $u := \inf\{t \mid (c,t) \in A\}$ and $v := \sup\{t \mid (c,t) \in A\}$. Note that, since A is closed, $(b,r), (b,s), (c,u), (c,v) \in A$. Note also that, since $A_b \subset [r,s]$ and $A_c \subset [u,v]$, we have $s - r \geq \mathcal{L}^1(A_b)$ and $v - u \geq \mathcal{L}^1(A_c)$.

   Up to relabeling the points, we may assume that $s - u \geq v - r$ (like it is the case in the figure). It then follows that:

$$
\begin{aligned}
s - u &\geq \frac{1}{2}(s-u) + \frac{1}{2}(v-r) = \\
&= \frac{1}{2}(s-r) + \frac{1}{2}(v-u) \geq \\
&\geq \frac{1}{2}\mathcal{L}^1(A_b) + \frac{1}{2}\mathcal{L}^1(A_c) \geq \\
&\geq |x_n| + |y_n| \geq |x_n - y_n|.
\end{aligned}
$$

Thus, $\|x-y\|^2 = |x_n - y_n|^2 + \|b-c\|^2 \le (s-u)^2 + \|b-c\|^2 = \|(b,s)-(c,u)\|^2$, thus the assertion is proved with $x' = (b,s), y' = (c,u) \in A$.

ii) If $n = 1$, $\mathsf{S}_{\mathsf{e_n}}(A)$ is a closed set and it is clear that $\mathcal{L}^1(A) = \mathcal{L}^1\big(\mathsf{S}_{\mathsf{e_n}}(A)\big)$. If $n \ge 2$, let $f : \mathbb{R}^{n-1} \to [0,\infty]$ be given by $f(x) = \frac{1}{2}\mathcal{L}^1(A_x)$. It follows from Fubini-Tonelli's theorem 1.84 that f is $\mathcal{L}^{n-1}$-measurable; hence, by lemma 2.18, $S := \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid -f(x) \le t \le f(x)\}$ is $\mathcal{L}^n$-measurable. Then $\mathsf{S}_{\mathsf{e_n}}(A) = S \setminus \{(x,0) \mid A_x = \emptyset\}$ is $\mathcal{L}^n$-measurable and the fact that $\mathcal{L}^n(A) = \mathcal{L}^n\big(\mathsf{S}_{\mathsf{e_n}}(A)\big)$ is a consequence of Fubini-Tonelli's theorem.

$\square$

LEMMA 2.18. *Let $f : \mathbb{R}^n \to [0,\infty]$ be $\mathcal{L}^n$-measurable. Then* $\operatorname{hyp} f := \{(x,t) \in \mathbb{R}^n \times [0,\infty) \mid t \le f(x)\} \subset \mathbb{R}^{n+1}$ *is $\mathcal{L}^{n+1}$-measurable.*

PROOF. Let $\theta : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be defined by $\theta(x,y) = x - y$, with $\infty - \infty := 0$, $-\infty - (-\infty) := 0$. Then θ is measurable with respect to $\mathscr{B}_{\overline{\mathbb{R}}\times\overline{\mathbb{R}}}$ and $\mathscr{B}_{\overline{\mathbb{R}}}$ (see proposition 1.50 and example 1.51). On the other hand, $(f,\iota) : \mathbb{R}^n \times \mathbb{R} \to \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ given by $(x,y) \mapsto \big(f(x), y\big)$ is measurable with respect to $\mathscr{L}_{\mathbb{R}^{n+1}}$ and $\mathscr{B}_{\overline{\mathbb{R}}} \otimes \mathscr{B}_{\overline{\mathbb{R}}}$, since each component is measurable (see remark 1.45). Since $\mathscr{B}_{\overline{\mathbb{R}}} \otimes \mathscr{B}_{\overline{\mathbb{R}}} = \mathscr{B}_{\overline{\mathbb{R}}\times\overline{\mathbb{R}}}$ (by proposition 1.47), it follows that the composite $f - \iota$ is measurable with respect to $\mathscr{L}_{\mathbb{R}^{n+1}}$ and $\mathscr{B}_{\overline{\mathbb{R}}}$, whence $\operatorname{hyp} f = (f-\iota)^{-1}([0,\infty]) \cap \mathbb{R}^n \times [0,\infty)$ is $\mathcal{L}^{n+1}$-measurable. $\square$

2.4. The isodiametric inequality; $\mathcal{L}^n = \mathcal{H}^n$

THEOREM 2.19 (isodiametric inequality). *The Lebesgue measure of any subset of $\mathbb{R}^n$ is at most the measure of an euclidean ball with the same diameter. That is, for all $A \subset \mathbb{R}^n$,*

$$\mathcal{L}^n(A) \le \alpha(n)\Big(\frac{\operatorname{diam} A}{2}\Big)^n.$$

PROOF. We assume that $\operatorname{diam} A < \infty$, otherwise the thesis is trivial.

1) Let $(e_1,\dots,e_n)$ be the standard basis of $\mathbb{R}^n$. Define $\mathsf{S}_0 := \mathsf{S}_{\mathsf{e_n}} \circ \mathsf{S}_{\mathsf{e_{n-1}}} \circ \cdots \circ \mathsf{S}_{\mathsf{e_1}}$. Note that the same properties stated in proposition 2.17 for the Steiner symmetrization also hold for S_0 (just apply that proposition n times in a row).
2) We contend that, for all $B \subset \mathbb{R}^n$, for $1 \le j \le n$, $\mathsf{S}_0(B)$ is symmetric with respect to the hyperplane $\langle e_j\rangle^\perp$; that is, denoting by $R_j : \mathbb{R}^n \to \mathbb{R}^n$ the reflection with respect to $\langle e_j\rangle^\perp$, $R_j\big(\mathsf{S}_0(B)\big) = \mathsf{S}_0(B)$. Indeed, for $1 \le j \le n$, let $B_j := \mathsf{S}_{\mathsf{e_j}} \circ \cdots \circ \mathsf{S}_{\mathsf{e_1}}(B)$.

i) By definition 2.16, it is clear that $B_1 = \mathsf{S}_{\mathsf{e}_1}(B)$ is invariant by R_1.

ii) Assume that, given $2 \leq j \leq n$, B_{j-1} is invariant by R_i for $1 \leq i \leq j-1$. We will show that $B_j = \mathsf{S}_{\mathsf{e}_j}(B_{j-1})$ is invariant by R_i for $1 \leq i \leq j$. That is clear for $i = j$, by definition 2.16. For $i < j$, since B_{j-1} is invariant by R_i, we have, denoting by P_j the orthogonal projection on $\mathbb{R}^{n-1} \equiv \langle e_j \rangle^{\perp}$ and by $P_j^{\perp}$ the orthogonal projection on $\mathbb{R} \equiv \langle e_j \rangle$, $\forall x \in \mathbb{R}^{n-1}$:

$$\begin{aligned} B_{j-1} \cap P_j^{-1}(x) = R_i(B_{j-1}) \cap P_j^{-1}(x) &\overset{R_i \circ P_j = P_j \circ R_i}{=} \\ &= R_i\big(B_{j-1} \cap P_j^{-1}(R_i^{-1} \cdot x)\big) \end{aligned}$$

As $R_i^{-1} = R_i$ and $P_j^{\perp} \circ R_i = P_j^{\perp}$, it then follows that, $\forall x \in \mathbb{R}^{n-1}$:

$$\begin{aligned} (B_{j-1})_x &= P_j^{\perp}\big(B_{j-1} \cap P_j^{-1}(x)\big) = \\ &= P_j^{\perp} \circ R_i\big(B_{j-1} \cap P_j^{-1}(R_i \cdot x)\big) = \\ &= P_j^{\perp}\big(B_{j-1} \cap P_j^{-1}(R_i \cdot x)\big) = \\ &= (B_{j-1})_{R_i \cdot x}. \end{aligned}$$

By the arbitrariness of $x \in \mathbb{R}^{n-1}$, the equality above implies, in view of definition 2.16, that $R_i\big(\mathsf{S}_{\mathsf{e}_j}(B_{j-1})\big) = \mathsf{S}_{\mathsf{e}_j}(B_{j-1})$, i.e. $B_j = \mathsf{S}_{\mathsf{e}_j}(B_{j-1})$ is invariant by R_i, as asserted.

Our contention is therefore proved.

3) From the contention in the previous item, it follows that, given $B \subset \mathbb{R}^n$, $\mathsf{S}_0(B)$ is invariant by $R_n \circ R_{n-1} \circ \cdots \circ R_1$, i.e. $\mathsf{S}_0(B)$ is symmetric with respect to the origin. Thus, $\forall x \in \mathsf{S}_0(B)$, $-x \in \mathsf{S}_0(B)$, so that $2\|x\| \leq \text{diam } \mathsf{S}_0(B) \leq \text{diam } B$, i.e. $\mathsf{S}_0(B) \subset \mathbb{B}(0, \frac{\text{diam } B}{2})$.

4) It follows from the previous item applied to $B = \overline{A}$ that:

$$\begin{aligned} \mathcal{L}^n(A) \leq \mathcal{L}^n(\overline{A}) = \mathcal{L}^n\big(\mathsf{S}_0(\overline{A})\big) \leq \mathcal{L}^n\Big(\mathbb{B}(0, \frac{\text{diam } \overline{A}}{2})\Big) = \\ = \alpha(n)\Big(\frac{\text{diam } \overline{A}}{2}\Big)^n = \alpha(n)\Big(\frac{\text{diam } A}{2}\Big)^n. \end{aligned}$$

$\square$

Exercise 2.20. Show an example of a set $A \subset \mathbb{R}^n$ which is not contained in any ball with diameter diam A.

Theorem 2.21. *For all $\delta \in (0, \infty]$ and $n \in \mathbb{N}$, $\mathcal{H}^n = \mathcal{H}^n_\delta = \mathcal{L}^n$ in $\mathbb{R}^n$.*

Proof. Fix $\delta \in (0, \infty]$ and $A \subset \mathbb{R}^n$.

1) Claim 1: $\mathcal{H}^n_\delta(A) \geq \mathcal{L}^n(A)$. Indeed, let $\mathcal{F}$ be a countable cover of A by subsets of $\mathbb{R}^n$ with diameters $\leq \delta$. For each $S \in \mathcal{F}$, it follows from the isodiametric inequality 2.19 that $\mathcal{L}^n(S) \leq \alpha(n)2^{-n}(\text{diam } S)^n$. Hence, $\sum_{S\in\mathcal{F}} \alpha(n)2^{-n}(\text{diam } S)^n \geq \sum_{S\in\mathcal{F}} \mathcal{L}^n(S) \geq \mathcal{L}^n(A)$, where the last inequality is due to the countable subadditivity of $\mathcal{L}^n$. Taking the infimum of all such covers $\mathcal{F}$ yields the claim.
2) Claim 2: for all $B \subset \mathbb{R}^n$, $\mathcal{L}^n(B) = 0$ implies $\mathcal{H}^n(B) = 0$. To prove the claim, fix $\epsilon > 0$ and take $\mathcal{F}$ a countable cover of B by cubes of sides parallel to the coordinate axes such that $\sum_{Q\in\mathcal{F}} \text{vol}(Q) < \epsilon$; such a cover exists, in view of the definition of the Lebesgue measure in example 1.3. Since, for each cube $Q \in \mathcal{F}$, $\text{vol}(Q) = \left(\frac{\text{diam } Q}{\sqrt{n}}\right)^n$, we conclude that $\sum_{Q\in\mathcal{F}}(\text{diam } Q)^n < n^{n/2}\epsilon$. Hence, by the arbitrariness of the $\epsilon > 0$ fixed, the claim follows from exercise 2.6.
3) Claim 3: $\mathcal{H}^n_\delta(A) \leq \mathcal{L}^n(A)$. Assume that $\mathcal{L}^n(A) < \infty$ (otherwise the claim is trivial) and take a countable cover $\mathcal{F}$ of A by cubes of sides parallel to the coordinate axes such that $\sum_{Q\in\mathcal{F}} \text{vol}(Q) < \mathcal{L}^n(A) + \epsilon$. For each $Q \in \mathcal{F}$, we may apply corollary 2.15 to Q^{o} to obtain a countable disjoint family $(B^k_Q)_{k\in\mathbb{N}}$ such that each B^k_Q is a nondegenerate closed ball with diameter $\leq \delta$ contained in Q^{o} and $\mathcal{L}^n(Q^{\text{o}} \setminus \cup_{k\in\mathbb{N}} B^k_Q) = 0$. Since $\mathcal{L}^n(\partial Q) = 0$, it then follows that $\mathcal{L}^n(Q \setminus \cup_{k\in\mathbb{N}} B^k_Q) = 0$; hence, by claim 2, $\mathcal{H}^n(Q \setminus \cup_{k\in\mathbb{N}} B^k_Q) = 0$, so $\mathcal{H}^n_\delta(Q \setminus \cup_{k\in\mathbb{N}} B^k_Q) = 0$. Since, by finite subadditivity, $\mathcal{H}^n_\delta(Q) \leq \mathcal{H}^n_\delta(Q \setminus \cup_{k\in\mathbb{N}} B^k_Q) + \mathcal{H}^n_\delta(\cup_{k\in\mathbb{N}} B^k_Q) = \mathcal{H}^n_\delta(\cup_{k\in\mathbb{N}} B^k_Q)$, it follows that $\mathcal{H}^n_\delta(Q) = \mathcal{H}^n_\delta(\cup_{k\in\mathbb{N}} B^k_Q)$. Therefore,

$$\begin{aligned}
\mathcal{H}^n_\delta(A) &\leq \sum_{Q\in\mathcal{F}} \mathcal{H}^n_\delta(Q) = \sum_{Q\in\mathcal{F}} \mathcal{H}^n_\delta(\cup_{k\in\mathbb{N}} B^k_Q) \overset{\text{countable subadditivity}}{\leq} \\
&\leq \sum_{Q\in\mathcal{F}} \sum_{k\in\mathbb{N}} \mathcal{H}^n_\delta(B^k_Q) \overset{\text{diam } B^k_Q \leq \delta}{\leq} \\
&\leq \sum_{Q\in\mathcal{F}} \sum_{k\in\mathbb{N}} \alpha(n)2^{-n}(\text{diam } B^k_Q)^n = \sum_{Q\in\mathcal{F}} \sum_{k\in\mathbb{N}} \mathcal{L}^n(B^k_Q) = \\
&= \sum_{Q\in\mathcal{F}} \mathcal{L}^n(\cup_{k\in\mathbb{N}} B^k_Q) = \sum_{Q\in\mathcal{F}} \mathcal{L}^n(Q) = \sum_{Q\in\mathcal{F}} \text{vol}(Q) < \mathcal{L}^n(A) + \epsilon.
\end{aligned}$$

Thus, by the arbitrariness of ϵ, claim 3 is proved.
4) By claims 1 and 3, $\mathcal{H}^n_\delta(A) = \mathcal{L}^n(A)$. Since that holds for all $\delta > 0$, it follows that $\mathcal{L}^n(A) = \mathcal{H}^n(A)$, hence the thesis follows.

$\square$

Corollary 2.22. *$\mathcal{H}$-dim $\mathbb{R}^n = n$.*

Proof. Apply the stability with respect to countable unions of the Hausdorff dimension, cf. exercise 2.9.e), to $\mathbb{R}^n = \cup_{k\in\mathbb{N}} C_k$, where each C_k is a nondegenerate cube with finite Lebesgue measure, i.e. $0 < \mathcal{H}^n(C_k) < \infty$, so that $\forall k \in \mathbb{N}$, $\mathcal{H}$-dim $C_k = n$. □

Exercise 2.23. If E is a k-dimensional subspace of a normed space X, then $\mathcal{H}$-dim $E = k$.

CHAPTER 3

Differentiation of Measures

The main reference for this chapter is [Sim83].

3.1. Densities

Up to the end of this section we fix a metric space (X, d).

Definition 3.1 (upper and lower n-dimensional densities). Let $A \subset X$, $x \in X$, $n > 0$ real and μ a measure on X. We define:

1) the *n-dimensional upper density of A at x with respect to μ*:

$$\Theta^{*n}(\mu, A, x) := \limsup_{r \to 0} \frac{\mu\big(A \cap \mathbb{B}(x,r)\big)}{\alpha(n) r^n} \in [0, \infty].$$

2) the *n-dimensional lower density of A at x with respect to μ*:

$$\Theta_*^{n}(\mu, A, x) := \liminf_{r \to 0} \frac{\mu\big(A \cap \mathbb{B}(x,r)\big)}{\alpha(n) r^n} \in [0, \infty].$$

If $\Theta^{*n}(\mu, A, x) = \Theta_*^n(\mu, A, x)$, we denote their common value by $\Theta^n(\mu, A, x)$ and call it *density of A at x with respect to μ.*

For $A = X$, we use the notations $\Theta^{*n}(\mu, x)$, $\Theta_*^n(\mu, x)$ and $\Theta^n(\mu, x)$ for $\Theta^{*n}(\mu, X, x)$, $\Theta_*^n(\mu, X, x)$ and $\Theta^n(\mu, X, x)$, respectively.

Note that we don't assume A to be measurable.

Remark 3.2. With the notation above:

1) Note that $\Theta^{*n}(\mu, A, x) = \Theta^{*n}(\mu \llcorner A, x)$ and $\Theta_*^n(\mu, A, x) = \Theta_*^n(\mu \llcorner A, x)$.
2) If $U \subset X$ is an open set and $x \in U$, $\Theta^{*n}(\mu, A, x) = \Theta^{*n}(\mu \llcorner U, A, x)$ and $\Theta_*^n(\mu, A, x) = \Theta_*^n(\mu \llcorner U, A, x)$.

Lemma 3.3. *If μ is a locally finite Borel measure on X, $A \subset X$, $x \in X$ and $n > 0$ real, then, the definitions of $\Theta^{*n}(\mu, A, x)$ or $\Theta_*^n(\mu, A, x)$ do not change if we use open balls instead of closed balls.*

Proof. Recall that, for $f : (0, \infty) \to \overline{\mathbb{R}}$,

$$\limsup_{r \to 0} f(r) := \inf_{r>0} \sup_{0<\rho<r} f(\rho),$$

$$\liminf_{r \to 0} f(r) := \sup_{r>0} \inf_{0<\rho<r} f(\rho).$$

Put, for $r > 0$,

$$f(r) := \frac{\mu\big(A \cap \mathbb{U}(x,r)\big)}{\alpha(n)r^n} \text{ and } g(r) := \frac{\mu\big(A \cap \mathbb{B}(x,r)\big)}{\alpha(n)r^n}.$$

Since μ is locally finite, there exists $r_0 > 0$ such that $\mu\big(\mathbb{U}(x,r_0)\big) < \infty$. In order to prove the lemma, it suffices to show that, $\forall 0 < r < r_0$,

$$\sup_{0<\rho<r} f(\rho) = \sup_{0<\rho<r} g(\rho) \text{ and } \inf_{0<\rho<r} f(\rho) = \inf_{0<\rho<r} g(\rho)$$

That is a consequence of the following claims:

1) Claim 1: $\forall \rho \in (0,r)$, $g(\rho)$ may be arbitrarily approximated by elements of $\{f(\rho) \mid 0 < \rho < r\}$. Indeed, for a given $\rho \in (0,r)$, $\mathbb{B}(x,\rho) = \cap_{k\in\mathbb{N}}\mathbb{U}(x,\rho+1/k)$; for sufficiently large k, $\rho + 1/k < r$, hence $\mu\big(\mathbb{U}(x,\rho+1/k)\big) < \infty$. That allows us to apply the continuity from above 1.11 to the Borel measure $\mu \llcorner A$, which ensures $\mu\big(A \cap \mathbb{U}(x,\rho+1/k)\big) \to \mu\big(A \cap \mathbb{B}(x,\rho)\big)$ as $k \to \infty$. Thus, $f(\rho+1/k) \to g(\rho)$, as asserted.
2) Claim 2: $\forall \rho \in (0,r)$, $f(\rho)$ may be arbitrarily approximated by elements of $\{g(\rho) \mid 0 < \rho < r\}$. Indeed, for a given $\rho \in (0,r)$, $\mathbb{U}(x,\rho) = \cup\{\mathbb{B}(x,\rho-1/k) \mid k \in \mathbb{N}, 1/k < \rho\}$. Applying the continuity from below 1.11 to the Borel measure $\mu \llcorner A$, it follows that $\mu\big(A \cap \mathbb{B}(x,\rho-1/k)\big) \to \mu\big(A \cap \mathbb{U}(x,\rho)\big)$ as $k \to \infty$. Thus, $g(\rho-1/k) \to f(\rho)$, as asserted.

□

PROPOSITION 3.4. *If μ is a locally finite Borel measure on X, $A \subset X$ and $n > 0$ real, then the functions $X \to [0,\infty]$ given by $x \in X \mapsto \Theta^{*n}(\mu,A,x)$ and $x \in X \mapsto \Theta^n_*(\mu,A,x)$ are Borelian.*

PROOF.

1) Firstly, note that, for fixed $r > 0$, the function $X \to [0,\infty]$ given by $x \mapsto \mu\big(A \cap \mathbb{U}(x,r)\big)$ is lower semicontinuous (hence Borelian). Indeed, let $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ a sequence in X convergent to x. For all $k \in \mathbb{N}$ such that $r - 1/k > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$, $\mathbb{U}(x_n,r) \supset \mathbb{B}(x,r-1/k)$. Hence $\forall n \geq n_0$, $\mu\big(A\cap\mathbb{U}(x_n,r)\big) \geq \mu\big(A\cap\mathbb{B}(x,r-1/k)\big)$, whence $\liminf_{n\to\infty} \mu\big(A\cap\mathbb{U}(x_n,r)\big) \geq \mu\big(A\cap\mathbb{B}(x,r-1/k)\big)$. Applying the continuity from below 1.11 for the Borel measure $\mu \llcorner A$, we conclude that $\mu\big(A\cap\mathbb{B}(x,r-1/k)\big) \to \mu\big(A\cap\mathbb{U}(x,r)\big) \leq \liminf_{n\to\infty} \mu\big(A\cap \mathbb{U}(x_n,r)\big)$, which shows the asserted lower semicontinuity at x.
2) Claim: Given $r > 0$, the functions $\psi_r, \psi^r : X \to [0,\infty]$ given by:

$$x \mapsto \inf_{0<\rho<r} \frac{\mu\big(A \cap \mathbb{U}(x,\rho)\big)}{\alpha(n)\rho^n} \text{ and } x \mapsto \sup_{0<\rho<r} \frac{\mu\big(A \cap \mathbb{U}(x,\rho)\big)}{\alpha(n)\rho^n},$$

respectively, are Borelian. Indeed, it suffices to show that, $\forall x \in X$,

$$\inf_{0<\rho<r} \frac{\mu\big(A \cap \mathbb{U}(x,\rho)\big)}{\alpha(n)\rho^n} = \inf\{\frac{\mu\big(A \cap \mathbb{U}(x,\rho)\big)}{\alpha(n)\rho^n} \mid 0<\rho<r, \rho \in \mathbb{Q}\} \text{ and}$$

$$\sup_{0<\rho<r} \frac{\mu\big(A \cap \mathbb{U}(x,\rho)\big)}{\alpha(n)\rho^n} = \sup\{\frac{\mu\big(A \cap \mathbb{U}(x,\rho)\big)}{\alpha(n)\rho^n} \mid 0<\rho<r, \rho \in \mathbb{Q}\},$$

in which case the asserted measurability follows from the previous item and from theorem 1.41.(iv). In order to show the equalities above, it is enough to prove that, for all $0 < \rho < r$, $\frac{\mu\left(A\cap\mathbb{U}(x,\rho)\right)}{\alpha(n)\rho^n}$ may be arbitrarily approximated by elements of the set $\{\frac{\mu\left(A\cap\mathbb{U}(x,\rho)\right)}{\alpha(n)\rho^n} \mid 0 < \rho < r, \rho \in \mathbb{Q}\}$. For that purpose, take a sequence of rationals $(\rho_k)_{k\in\mathbb{N}}$ in $(0,\rho)$ such that $\rho_k \uparrow \rho$; then $\alpha(n)\rho_k^n \to \alpha(n)\rho^n$ and, applying the continuity from below to the Borel measure $\mu \llcorner A$, $\mu\big(A \cap \mathbb{U}(x,\rho_k)\big) \uparrow \mu\big(A \cap \mathbb{U}(x,\rho)\big)$, hence $\frac{\mu\left(A\cap\mathbb{U}(x,\rho_k)\right)}{\alpha(n)\rho_k^n} \to \frac{\mu\left(A\cap\mathbb{U}(x,\rho)\right)}{\alpha(n)\rho^n}$, as asserted.

3) Due to the fact that μ is a locally finite Borel measure, it follows from lemma 3.3 that $\Theta^{*n}(\mu, A, \cdot) = \inf_{r\in\mathbb{Q}_+^*} \psi^r$ and $\Theta_*^n(\mu, A, \cdot) = \sup_{r\in\mathbb{Q}_+^*} \psi_r$. Thus, from the claim in the previous item and from theorem 1.41.(iv), we conclude that both $\Theta^{*n}(\mu, A, \cdot)$ and $\Theta_*^n(\mu, A, \cdot)$ are Borelian.

$\square$

Corollary 3.5. *If μ is a locally finite Borel measure on X, $A \subset X$ and $n > 0$ real, then the set $Y := \{x \in X \mid \Theta^{*n}(\mu, A, x) = \Theta_*^n(\mu, A, x)\}$ is Borel measurable and $\Theta^n(\mu, A, \cdot) : Y \to [0, \infty]$ is Borelian.*

Theorem 3.6 (comparison density theorem). *Let μ be a Borel measure on a metric space X, $n > 0$ real, $t \geq 0$ and $A \subset A_1 \subset X$. If $\forall x \in A$, $\Theta^{*n}(\mu, A_1, x) \geq t$ then $t\mathcal{H}^n(A) \leq \mu(A_1)$.*

Proof. We assume that $t > 0$ and $\mu(A_1) < \infty$, otherwise the thesis is trivial.

Fix $0 < \tau < t$ and $\delta > 0$. Since, $\forall x \in A$, $\Theta^{*n}(\mu, A_1, x) = \limsup_{r\to 0} \frac{\mu\left(A_1\cap\mathbb{B}(x,r)\right)}{\alpha(n)r^n} \geq t > \tau$, it follows that $\forall x \in A$, $\forall r > 0$, $\exists 0 < \rho < r$ such that $\frac{\mu\left(A_1\cap\mathbb{B}(x,\rho)\right)}{\alpha(n)\rho^n} > \tau$. It then follows that $\mathcal{F} := \{B \mid \exists x \in A, \exists r > 0, B = \mathbb{B}(x,r), \frac{\mu\left(A_1\cap\mathbb{B}(x,r)\right)}{\alpha(n)r^n} > \tau, 2r \leq \delta\}$ is a fine cover of A. Take a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$, given by corollary 2.13, such that, for all $F \subset \mathcal{F}$ finite,

$$A \setminus \cup_{B\in F} B \subset \cup_{B\in\mathcal{G}\setminus F} 5B. \tag{3.1}$$

Due to the fact that $\mu \llcorner A_1$ is a Borel measure, for all $F \subset \mathcal{G}$ finite, $\sum_{B\in F} \mu(A_1 \cap B) = \mu\big(A_1 \cap (\cup_{B\in F} B)\big) \le \mu(A_1) < \infty$. Therefore, by exercise 1.59, $\sum_{B\in\mathcal{G}} \mu(A_1 \cap B) = \sup\{\sum_{B\in F} \mu(A_1 \cap B) \mid F \subset \mathcal{G}, F \text{ finite}\} \le \mu(A_1) < \infty$; since, for all $B \in \mathcal{G} \subset \mathcal{F}$, $\mu(A_1 \cap B) > 0$, it follows that $\mathcal{G}$ is countable. Let $(B_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathcal{G}$. For each $k \in \mathbb{N}$, $B_k \in \mathcal{F}$, hence there exists $x_k \in A$ and $r_k > 0$ such that $B_k = \mathbb{B}(x_k, r_k)$ and $\tau\alpha(n)r_k^n < \mu(A_1 \cap B_k)$, so that

$$\tau \sum_{k=1}^{\infty} \alpha(n) r_k^n \le \sum_{k=1}^{\infty} \mu(A_1 \cap B_k) = \\ = \mu\big(A_1 \cap (\cup_{k\in\mathbb{N}} B_k)\big) \le \mu(A_1) < \infty.$$

On the other hand, for all $N \in \mathbb{N}$, it follows from (3.1) that $A \subset (\cup_{k=1}^N B_k) \cup (\cup_{k\ge N+1} 5B_k)$. Since, for each $k \in \mathbb{N}$, diam $B_k \le 2r_k \le \delta$ and diam $5B_k \le 5\,\text{diam } B_k \le 10r_k \le 5\delta$, we then conclude that

$$\mathcal{H}^n_{5\delta}(A) \le \sum_{k=1}^{N} \alpha(n)2^{-n}(\text{diam } B_k)^n + \sum_{k=N+1}^{\infty} \alpha(n)2^{-n}(\text{diam } 5B_k)^n \le \\ \le \sum_{k=1}^{N} \alpha(n) r_k^n + 5^n \sum_{k=N+1}^{\infty} \alpha(n) r_k^n.$$

Thus, taking $N \to \infty$, it follows $\mathcal{H}^n_{5\delta}(A) \le \sum_{k=1}^{\infty} \alpha(n) r_k^n$, hence $\tau\mathcal{H}^n_{5\delta}(A) \le \tau \sum_{k=1}^{\infty} \alpha(n) r_k^n \le \mu(A_1)$. Taking $\delta \to 0$, we obtain $\tau\mathcal{H}^n(A) \le \mu(A_1)$. Finally, since $\tau \in (0,t)$ was arbitrarily taken, making $\tau \to t$ in the last inequality yields the thesis. □

THEOREM 3.7 (upper density theorem). *Let μ be a Borel regular measure on a metric space X, $n > 0$ real and $B \in \sigma(\mu)$ with $\mu(B) < \infty$. Then $\Theta^{*n}(\mu, B, x) = 0$ for $\mathcal{H}^n$-a.e. $x \in X \setminus B$.*

PROOF. Let $C \subset B$ be a closed set and $t > 0$. Define $A^t := \{x \in X \setminus B \mid \Theta^{*n}(\mu, B, x) \ge t\}$ and $A_1^t := X \setminus C \supset A^t$. Since $A_1^t = X \setminus C$ is an open set, it follows from remark 3.2 that, for all $x \in A^t \subset A_1^t$, $\Theta^{*n}(\mu \llcorner B, A_1^t, x) = \Theta^{*n}(\mu \llcorner A_1^t, B, x) = \Theta^{*n}(\mu, B, x) \ge t$. Thus, we may apply theorem 3.6 with the Borel measure $\mu \llcorner B$ in place of μ, A^t in place of A and A_1^t in place of A_1, yielding $t\mathcal{H}^n(A^t) \le \mu \llcorner B(A_1^t) = \mu(B \setminus C)$. By the arbitrariness of C, it follows that $t\mathcal{H}^n(A^t) \le \inf\{\mu(B \setminus C) \mid C \subset B, C \text{ closed}\}$. On the other hand, it follows from proposition 1.36 that $\mu \llcorner B$ is a finite Borel regular measure, to which theorem 1.23 may be applied to approximate B by closed sets contained in B, which yields $\inf\{\mu(B \setminus C) \mid C \subset B, C \text{ closed}\} = 0$. As $t > 0$ was arbitrarily taken, it follows that $\forall t > 0$, $\mathcal{H}^n(A^t) = 0$. Since $\{x \in X \setminus B \mid \Theta^{*n}(\mu, B, x) > 0\} = \cup_{k\in\mathbb{N}} A^{1/k}$, we conclude that $\mathcal{H}^n(\{x \in X \setminus B \mid \Theta^{*n}(\mu, B, x) > 0\}) = 0$, whence the thesis. $\square$

Exercise 3.8. If μ is an open σ-finite Borel regular measure on a metric space X, the thesis in theorem 3.7 holds for all $B \in \sigma(\mu)$, i.e. the hypothesis of $\mu(B)$ being finite may be dropped.

Corollary 3.9 (density theorem for the Lebesgue measure). *If $B \subset \mathbb{R}^n$ is $\mathcal{L}^n$-measurable, then $\Theta^n(\mathcal{L}^n, B, x)$ exists for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$, $\Theta^n(\mathcal{L}^n, B, x) = 1$ for $\mathcal{L}^n$-a.e. $x \in B$ and $\Theta^n(\mathcal{L}^n, B, x) = 0$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n \setminus B$.*

Proof. Note that, if $f, g : (0,\infty) \to [0,\infty]$, then

$$\begin{aligned}\liminf_{r\to 0} f(r) + \liminf_{r\to 0} g(r) \le \liminf_{r\to 0}(f+g)(r) \le \\ \le \liminf_{r\to 0} f(r) + \limsup_{r\to 0} g(r) \le \\ \le \limsup_{r\to 0}(f+g)(r) \le \limsup_{r\to 0} f(r) + \limsup_{r\to 0} g(r).\end{aligned}$$

Fix $x \in \mathbb{R}^n$. Applying the inequalities above to $f(r) = \frac{\mathcal{L}^n\left(B\cap\mathbb{B}(x,r)\right)}{\alpha(n)r^n}$ and $g(r) = \frac{\mathcal{L}^n\left((\mathbb{R}^n\setminus B)\cap\mathbb{B}(x,r)\right)}{\alpha(n)r^n}$, and taking into consideration that $f(r) + g(r) \equiv 1$, it follows that, for all $x \in \mathbb{R}^n$:

$$\Theta^{*n}(\mathcal{L}^n, B, x) + \Theta^n_*(\mathcal{L}^n, \mathbb{R}^n \setminus B, x) = 1, \tag{3.2}$$

and the same holds with $\mathbb{R}^n \setminus B$ in place of B.

On the other hand, theorems 2.21, 3.7, exercise 3.8 and the fact that $0 \le \Theta^n_*(\mathcal{L}^n, B, \cdot) \le \Theta^{*n}(\mathcal{L}^n, B, \cdot) \le 1$ imply that $\Theta^n(\mathcal{L}^n, B, x) = 0$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n \setminus B$. The same holds for $\mathbb{R}^n \setminus B$ in place of B, i.e. $\Theta^n(\mathcal{L}^n, \mathbb{R}^n \setminus B, x) = 0$ for $\mathcal{L}^n$-a.e. $x \in B$. The last equality implies, in view of (3.2), that $\Theta^n(\mathcal{L}^n, B, x) = 1$ for $\mathcal{L}^n$-a.e. $x \in B$. $\square$

Exercise 3.10. Any convex subset X of $\mathbb{R}^n$ is $\mathcal{L}^n$-measurable.

Hint. Use corollary 3.9 to prove that ∂X has null Lebesgue measure.

Exercise 3.11. Let X be a metric space, $n > 0$ real and $A \subset X$ be $\mathcal{H}^n$-measurable with $\mathcal{H}^n(A) < \infty$. Then $\Theta^{*n}(\mathcal{H}^n, A, x) \le 1$ for $\mathcal{H}^n$-a.e. $x \in A$.

Hint. For each $t > 1$, put $A_t := \{x \in A \mid \Theta^{*n}(\mathcal{H}^n, A, x) \ge t\}$. Given $\epsilon > 0$, take an open set $U \supset A_t$ such that $\mathcal{H}^n(U \cap A) < \mathcal{H}^n(A_t) + \epsilon$

(why such an open set exists?) and apply theorem 3.6 with $\mathcal{H}^n \llcorner A$ in place of μ, A_t in place of A and $A \cap U$ in place of A_1.

3.2. Differentiation Theorems

In the first part of this section we extend theorems 3.6 and 3.7 to the situation in which we define upper and lower densities of a Borel measure μ on a metric space X with respect to another Borel measure ν on X, with convenient regularity and finiteness assumptions.

Definition 3.12 (upper and lower densities of a measure relative another). Let X be a metric space, μ and ν measures on X, and $x \in X$. We define the *upper* and *lower density of μ relative to ν at x* by, respectively:

$$\Theta^{*\nu}(\mu, x) := \limsup_{r \to 0} \frac{\mu\big(\mathbb{B}(x,r)\big)}{\nu\big(\mathbb{B}(x,r)\big)} \in [0, \infty],$$

$$\Theta_*^{\nu}(\mu, x) := \liminf_{r \to 0} \frac{\mu\big(\mathbb{B}(x,r)\big)}{\nu\big(\mathbb{B}(x,r)\big)} \in [0, \infty],$$

where we adopt the extended arithmetic rules $\frac{0}{0} := 0$, $\frac{\infty}{\infty} := 0$. If $\Theta^{*\nu}(\mu, x) = \Theta_*^{\nu}(\mu, x)$, we say that the *density of μ relative to ν at x* exists and denote it by $\Theta^{\nu}(\mu, x) := \Theta^{*\nu}(\mu, x) = \Theta_*^{\nu}(\mu, x)$.

Note that:

- if $\exists r > 0$, $\mu\big(\mathbb{B}(x,r)\big) = 0$, then $\Theta^{*\nu}(\mu, x) = \Theta_*^{\nu}(\mu, x) = 0$.
- if $\nexists r > 0$, $\mu\big(\mathbb{B}(x,r)\big) = 0$ and $\exists r > 0$, $\nu\big(\mathbb{B}(x,r)\big) = 0$, then $\Theta^{*\nu}(\mu, x) = \Theta_*^{\nu}(\mu, x) = \infty$.

In particular, if $x \notin \operatorname{spt} \mu \cap \sup \nu$, the upper and lower densities at x assume value 0 or ∞.

Remark 3.13. If $X = \mathbb{R}^n$, $A \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$ and μ a measure on $\mathbb{R}^n$, the n-dimensional upper and lower densities of A at x with respect to μ, defined in 3.1, are special cases of the above definition: $\Theta^{*n}(\mu, A, x) = \Theta^{*\mathcal{L}^n}(\mu \llcorner A, x)$ and $\Theta_*^n(\mu, A, x) = \Theta_*^{\mathcal{L}^n}(\mu \llcorner A, x)$.

Lemma 3.14. *If μ and ν are locally finite Borel measures on a metric space X, and $x \in X$, then the definitions of $\Theta^{*\nu}(\mu, x)$ or $\Theta_*^{\nu}(\mu, x)$ do not change if we use open balls instead of closed balls.*

Proof. It is an adaptation of the proof of lemma 3.3, analyzing separately the case in which $x \neq \operatorname{spt} \mu \cap \operatorname{spt} \nu$.

Define, for $r > 0$,

$$f(r) := \frac{\mu\big(\mathbb{U}(x,r)\big)}{\nu\big(\mathbb{U}(x,r)\big)} \text{ and } g(r) := \frac{\mu\big(\mathbb{B}(x,r)\big)}{\nu\big(\mathbb{B}(x,r)\big)}.$$

1) If $\exists r_0 > 0$, $\mu\big(\mathbb{B}(x,r_0)\big) = 0$, then $\forall 0 < r < r_0$, $f(r) = g(r) = 0$ and the thesis is trivial in this case.
2) If $\nexists r > 0$, $\mu\big(\mathbb{B}(x,r)\big) = 0$ and $\exists r_0 > 0$, $\nu\big(\mathbb{B}(x,r_0)\big) = 0$, then $\forall 0 < r < r_0$, $f(r) = g(r) = \infty$ and the thesis is also trivial in this case.
3) If neither of the previous cases holds, then $x \in \operatorname{spt}\mu \cap \operatorname{spt}\nu$. Since both μ and ν are locally finite, there exists $r_0 > 0$ such that $\mu\big(\mathbb{B}(x,r_0)\big) < \infty$ and $\nu\big(\mathbb{B}(x,r_0)\big) < \infty$. That is, for all $0 < r < r_0$, $0 < \mu\big(\mathbb{B}(x,r)\big) < \infty$ and $0 < \nu\big(\mathbb{B}(x,r)\big) < \infty$. In order conclude the proof of the lemma, it suffices to show that, $\forall 0 < r < r_0$,
$$\sup_{0<\rho<r} f(\rho) = \sup_{0<\rho<r} g(\rho) \text{ and } \inf_{0<\rho<r} f(\rho) = \inf_{0<\rho<r} g(\rho)$$
That is a consequence of the following claims:
   i) Claim 1: $\forall \rho \in (0,r)$, $g(\rho)$ may be arbitrarily approximated by elements of $\{f(\rho) \mid 0 < \rho < r\}$. Indeed, for a given $\rho \in (0,r)$, $\mathbb{B}(x,\rho) = \cap_{k\in\mathbb{N}}\mathbb{U}(x,\rho+1/k)$; for sufficiently large k, $\rho+1/k < r$, hence $\mu\big(\mathbb{U}(x,\rho+1/k)\big) < \infty$ and $\nu\big(\mathbb{U}(x,\rho+1/k)\big) < \infty$. That allows us to apply the continuity from above 1.11 to the Borel measures μ and ν, which ensures $\mu\big(\mathbb{U}(x,\rho+1/k)\big) \to \mu\big(\mathbb{B}(x,\rho)\big)$ and $\nu\big(\mathbb{U}(x,\rho+1/k)\big) \to \nu\big(\mathbb{B}(x,\rho)\big) > 0$ as $k \to \infty$. Thus, $f(\rho+1/k) \to g(\rho)$, as asserted.
   ii) Claim 2: $\forall \rho \in (0,r)$, $f(\rho)$ may be arbitrarily approximated by elements of $\{g(\rho) \mid 0 < \rho < r\}$. Indeed, for a given $\rho \in (0,r)$, $\mathbb{U}(x,\rho) = \cup\{\mathbb{B}(x,\rho-1/k) \mid k \in \mathbb{N}, 1/k < \rho\}$. Applying the continuity from below 1.11 to the Borel measures μ and ν, it follows that $\mu\big(\mathbb{B}(x,\rho-1/k)\big) \to \mu\big(\mathbb{U}(x,\rho)\big)$ and $\nu\big(\mathbb{B}(x,\rho-1/k)\big) \to \nu\big(\mathbb{U}(x,\rho)\big) > 0$ as $k \to \infty$. Thus, $g(\rho-1/k) \to f(\rho)$, as asserted.

□

PROPOSITION 3.15. *Let μ and ν be locally finite Borel measures on a metric space X, with ν finite on all closed balls of X. Then the functions $X \to [0,\infty]$ given by $x \in X \mapsto \Theta^{*\nu}(\mu,x)$ and $x \in X \mapsto \Theta_*^{\nu}(\mu,x)$ are Borelian.*

PROOF. We adapt the proof or proposition 3.4.

1) Let $U_0 := \{x \in X \mid \exists r > 0, \nu\big(\mathbb{B}(x,r)\big) = 0\} = X \setminus \operatorname{spt}\nu$ and $V_0 := \{x \in X \mid \exists r > 0, \mu\big(\mathbb{B}(x,r)\big) = 0\} = X \setminus \operatorname{spt}\mu$. We will apply proposition 1.50 to $A_1 = V_0$, $A_2 = U_0 \setminus V_0$ and $A_3 = X \setminus (U_0 \cup V_0)$. Note that, since U_0 and V_0 are open sets, $(A_i)_{1\le i\le 3}$ is a Borel partition of X. As $\Theta^{*\nu}(\mu,\cdot)$ and $\Theta_*^{\nu}(\mu,\cdot)$ are constant on A_1 (equal to 0) and A_2 (equal to ∞), their restrictions to A_1 and

A_2, endowed with the respective trace σ-algebras, are measurable. It remains to show that the restrictions of $\Theta^{*\nu}(\mu,\cdot)$ and $\Theta_*^{\nu}(\mu,\cdot)$ to $A_3 = \operatorname{spt}\mu \cap \operatorname{spt}\nu$ are measurable, endowing A_3 with the trace σ-algebra $\mathscr{B}_X|_{A_3}$.

2) Note that, for fixed $r>0$, the functions $X \to [0,\infty]$ given by $x \mapsto \mu\big(\mathbb{U}(x,r)\big)$ and $x \mapsto \nu\big(\mathbb{U}(x,r)\big)$ are lower semicontinuous (hence Borelian). Indeed, let $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ a sequence in X convergent to x. For all $k \in \mathbb{N}$ such that $r - 1/k > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$, $\mathbb{U}(x_n, r) \supset \mathbb{B}(x, r-1/k)$. Hence $\forall n \geq n_0$, $\mu\big(\mathbb{U}(x_n,r)\big) \geq \mu\big(\mathbb{B}(x,r-1/k)\big)$, whence $\liminf_{n\to\infty}\mu\big(\mathbb{U}(x_n,r)\big) \geq \mu\big(\mathbb{B}(x,r-1/k)\big)$. Applying the continuity from below 1.11 to the Borel measure μ, we conclude that $\mu\big(\mathbb{B}(x,r-1/k)\big) \to \mu\big(\mathbb{U}(x,r)\big) \leq \liminf_{n\to\infty}\mu\big(\mathbb{U}(x_n,r)\big)$, what shows the asserted lower semicontinuity at x for μ, and the same argument holds for ν.

   It then follows that the quotient $\frac{\mu\left(\mathbb{U}(\cdot,r)\right)}{\nu\left(\mathbb{U}(\cdot,r)\right)} : X \to [0,\infty]$ is Borelian (see example 1.51.2), so its restriction to A_3 is measurable with respect to the trace σ-algebra.

3) Claim: Given $r>0$, the functions $\psi_r, \psi^r : X \to [0,\infty]$ given by:

$$x \mapsto \inf_{0<\rho<r} \frac{\mu\big(\mathbb{U}(x,\rho)\big)}{\nu\big(\mathbb{U}(x,\rho)\big)} \text{ and } x \mapsto \sup_{0<\rho<r} \frac{\mu\big(\mathbb{U}(x,\rho)\big)}{\nu\big(\mathbb{U}(x,\rho)\big)},$$

respectively, have measurable restrictions to A_3. Indeed, it suffices to show that, $\forall x \in A_3$,

$$\inf_{0<\rho<r} \frac{\mu\big(\mathbb{U}(x,\rho)\big)}{\nu\big(\mathbb{U}(x,\rho)\big)} = \inf\{\frac{\mu\big(\mathbb{U}(x,\rho)\big)}{\nu\big(\mathbb{U}(x,\rho)\big)} \mid 0<\rho<r, \rho\in\mathbb{Q}\} \text{ and}$$
$$\sup_{0<\rho<r} \frac{\mu\big(\mathbb{U}(x,\rho)\big)}{\nu\big(\mathbb{U}(x,\rho)\big)} = \sup\{\frac{\mu\big(\mathbb{U}(x,\rho)\big)}{\nu\big(\mathbb{U}(x,\rho)\big)} \mid 0<\rho<r, \rho\in\mathbb{Q}\},$$

in which case the asserted measurability follows from the previous item and from theorem 1.41.(iv). In order to show the equalities above, it is enough to prove that, for all $x \in A_3$ and $0<\rho<r$, $\frac{\mu\left(\mathbb{U}(x,\rho)\right)}{\nu\left(\mathbb{U}(x,\rho)\right)}$ may be arbitrarily approximated by elements of the set $\{\frac{\mu\left(\mathbb{U}(x,\rho)\right)}{\nu\left(\mathbb{U}(x,\rho)\right)} \mid 0<\rho<r, \rho\in\mathbb{Q}\}$. For that purpose, take a sequence of rationals $(\rho_k)_{k\in\mathbb{N}}$ in $(0,\rho)$ such that $\rho_k \uparrow \rho$; then, applying the continuity from below to the Borel measures μ and ν, $\mu\big(\mathbb{U}(x,\rho_k)\big) \uparrow \mu\big(\mathbb{U}(x,\rho)\big)$ and $\nu\big(\mathbb{U}(x,\rho_k)\big) \uparrow \nu\big(\mathbb{U}(x,\rho)\big)$. Since $x \in A_3 \subset \operatorname{spt}\nu$ and

ν is finite on balls, we have $0 < \nu\big(\mathbb{U}(x,\rho)\big) < \infty$. Hence $\frac{\mu\left(\mathbb{U}(x,\rho_k)\right)}{\nu\left(\mathbb{U}(x,\rho_k)\right)} \to \frac{\mu\left(\mathbb{U}(x,\rho)\right)}{\nu\left(\mathbb{U}(x,\rho)\right)}$, as asserted.

4) Due to the fact that μ and ν are locally finite Borel measures, it follows from lemma 3.14 that $\Theta^{*\nu}(\mu,\cdot)|_{A_3} = \inf_{r\in\mathbb{Q}_+^*} \psi^r|_{A_3}$ and $\Theta_*^{\nu}(\mu,\cdot)|_{A_3} = \sup_{r\in\mathbb{Q}_+^*} \psi_r|_{A_3}$. Thus, from the claim in the previous item and from theorem 1.41.(iv), we conclude that both $\Theta^{*\nu}(\mu,\cdot)|_{A_3}$ and $\Theta_*^{\nu}(\mu,\cdot)|_{A_3}$ are measurable.

□

Exercise 3.16. Show that, in proposition 3.15, the hypothesis of ν being finite on all closed balls of X may be replaced by the hypothesis of X being separable.

Hint. Adapt the argument above. Prove that, for each $x \in A_3$, there exists an open neighborhood $x \in U \subset X$ and $r_0 > 0$ such that, for all $0 < r < r_0$ the restrictions of ψ_r and ψ^r to $U \cap A_3$ are measurable.

Corollary 3.17. *Let μ and ν be locally finite Borel measures on a metric space X, with ν finite on all closed balls of X. Then the set $Y := \{x \in X \mid \Theta^{*\nu}(\mu,x) = \Theta_*^{\nu}(\mu,x)\}$ is Borel measurable and $\Theta^{\nu}(\mu,\cdot) : Y \to [0,\infty]$ is Borelian.*

In order to obtain similar versions of the comparison 3.6 and upper density 3.7 theorems to the situation in which the densities of a Borel measure μ are taken with respect to another Borel measure ν, we need ν to satisfy the "symmetric Vitali property" introduced below. The idea is to abstract the Vitali property of the Lebesgue measure stated in corollary 2.14.

Definition 3.18. Let X be a metric space, $\mathcal{F}$ a collection of balls in X and $A \subset X$. We say that $\mathcal{F}$ is a *strongly fine cover* A, or that *$\mathcal{F}$ covers A finely in the strong sense*, if $\mathcal{F}$ is a cover of A such that, $\forall x \in A$, $\inf\{r > 0 \mid \mathbb{B}(x,r) \in \mathcal{F}\} = 0$.

It is clear that every strongly fine cover of A is a fine cover of A in the sense of definition 2.12, but the converse does not hold.

Definition 3.19 (symmetric Vitali property (SVP)). We say that a measure μ on a metric space X satisfies the *symmetric Vitali property* if, for all $A \subset X$ with $\mu(A) < \infty$ and for all $\mathcal{F}$ strongly fine cover of A by nondegenerate closed balls, there exists a countable disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\mu(A \setminus \cup\mathcal{G}) = 0$.

Note that A is not assumed to be μ-measurable.

REMARK 3.20.

1) It is clear that, if a measure μ on a metric space X has SVP, so does any restriction of μ, i.e. $\forall Y \subset X$, $\mu \llcorner Y$ has SVP.
2) If a measure μ on a metric space X is σ-finite and has SVP, then μ is concentrated on its support, i.e. $\mu(X \setminus \mathrm{spt}\ \mu) = 0$. Indeed, let $X = \cup_{k\in\mathbb{N}} A_k$, with $\forall k \in \mathbb{N}$, $A_k \in \sigma(\mu)$ and $\mu(A_k) < \infty$. For each $k \in \mathbb{N}$, the family of nondegenerate closed balls $\mathcal{F} = \{\mathbb{B}(x,r) \mid x \in X \setminus \mathrm{spt}\ \mu, r > 0, \mu\big(\mathbb{B}(x,r)\big) = 0\}$ covers $A_k \setminus \mathrm{spt}\ \mu$ finely in the strong sense. Hence, there exists a countable disjoint subfamily $\mathcal{G}_k \subset \mathcal{F}$ such that $\mu\big((A_k \setminus \mathrm{spt}\ \mu) \setminus \cup\mathcal{G}_k\big) = 0$; since $\mu(\cup\mathcal{G}_k) = 0$ (because $\mathcal{G}_k$ is countable and each $B \in \mathcal{G}_k$ has null measure), we conclude that $\mu(A_k \setminus \mathrm{spt}\ \mu) = 0$. Therefore $X \setminus \mathrm{spt}\ \mu = \cup_{k\in\mathbb{N}}(A_k \setminus \mathrm{spt}\ \mu)$ has μ-measure zero.

We list in the propositions below some sufficient conditions in order for a measure to satisfy the symmetric Vitali property.

PROPOSITION 3.21 (doubling property implies SVP). *Let X be a separable metric space and μ a finite Borel regular measure on X. Assume that μ satisfies the* doubling property*:*

$$\exists C > 0,\ \forall B \subset X \textit{ nondegenerate closed ball},\ \mu(5B) \le C\mu(B),$$

where $5B$ is given by (2.1). Then μ has the symmetric Vitali property.

PROOF. Let $A \subset X$ and $\mathcal{F}$ a fine cover of A by nondegenerate closed balls. By corollary 2.13, there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that, for all $F \subset \mathcal{F}$ finite, $A \setminus \cup_{B\in F} B \subset \cup_{B\in\mathcal{G}\setminus F} 5B$. Since X is separable, $\mathcal{G}$ is countable; let $(B_n)_{n\in\mathbb{N}}$ be an enumeration of $\mathcal{G}$. Then, for all $N \in \mathbb{N}$,

$$A \setminus \cup_{n=1}^N B_n \subset \cup_{n\ge N+1} 5B_n.$$

Hence, $\mu(A \setminus \cup_{n=1}^N B_n) \le C \sum_{n=N+1}^\infty \mu(B_n) \overset{N\to\infty}{\longrightarrow} 0$, since $\sum_{n=1}^\infty \mu(B_n) = \mu(\cup\mathcal{G}) \le \mu(X) < \infty$. Thus, applying the continuity from above 1.11 to the finite measure μ, it follows that $\mu(A \setminus \cup\mathcal{G}) = 0$. $\square$

REMARK 3.22. We have actually proved that, if μ is a finite Borel regular measure on X with the doubling property, then the symmetric Vitali property holds in a stronger sense, i.e. given $A \subset X$ with $\mu(A) < \infty$, the symmetric Vitali property holds for arbitrary fine covers of A, not necessarily in the strong sense.

PROPOSITION 3.23 (Borel measures on subsets of $\mathbb{R}^n$ satisfy SVP). *Let X be a metric subspace of $\mathbb{R}^n$ and μ a Borel measure on X. Then μ satisfies the symmetric Vitali property.*

In order to prove this proposition, we will need the following covering theorem:

THEOREM 3.24 (Besicovitch covering theorem). *For each $n \in \mathbb{N}$, there exists a natural constant $N = N(n)$, depending only on n, which satisfies the following property: if $\mathcal{F}$ is any family of nondegenerate closed balls in $\mathbb{R}^n$ with $\sup\{\text{diam } B \mid B \in \mathcal{F}\} < \infty$ and A is the set of centers of the balls in $\mathcal{F}$, then exist $\mathcal{G}_1, \dots, \mathcal{G}_N$ such that, for $1 \leq i \leq N$, $\mathcal{G}_i$ is a disjoint subfamily of $\mathcal{F}$ and $\cup_{i=1}^N \mathcal{G}_i$ covers A.*

For the proof of this theorem, we refer, for instance, to [**EG91**], [**KP08**], [**Mat95**] or [**Fed69**].

COROLLARY 3.25. *Let μ be a Borel measure in $\mathbb{R}^n$, $A \subset \mathbb{R}^n$ with $\mu(A) < \infty$ and $\mathcal{F}$ a family of nondegenerate closed balls which covers A finely in the strong sense. Then, for any open set $U \supset A$, there exists a countable disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\cup\mathcal{G} \subset U$ and $\mu(A \setminus \cup\mathcal{G}) = 0$.*

PROOF. It is an adaptation of the argument used to prove corollary 2.14, using Besicovitch covering theorem instead of the 5-times covering lemma 2.10.

Let $N = N(n)$ be the constant given by theorem 3.24 and fix $\theta \in (1 - \frac{1}{N}, 1)$. We may assume that $\mu(A) > 0$, otherwise the thesis is trivial. Let $U \supset A$ be an open set.

1) Put $\mathcal{F}_U := \{B \in \mathcal{F} \mid B \subset U, \text{diam } B \leq 1\}$. Since $\mathcal{F}$ covers A finely in the strong sense, it is clear that $\mathcal{F}_U$ is still a strongly fine cover of A; in particular, A is contained in the set of centers of the balls in $\mathcal{F}$. Applying theorem 3.24 to $\mathcal{F}_U$, we may take disjoint subfamilies $\mathcal{G}_U^1, \dots \mathcal{G}_U^N \subset \mathcal{F}_U$ such that $A \subset \cup_{i=1}^N(\cup\mathcal{G}_U^i)$. Hence, by subadditivity, $\mu(A) \leq \sum_{i=1}^N \mu\big(A \cap (\cup\mathcal{G}_U^i)\big)$. We therefore conclude that there exists $1 \leq i \leq N$ such that $\mu\big(A \cap (\cup\mathcal{G}_U^i)\big) \geq \frac{1}{N}\mu(A) > (1-\theta)\mu(A)$. Since $\mathcal{G}_U^i$ is a countable family (by remark 2.11) of closed balls, we may apply the continuity from below 1.11 to the Borel measure $\mu \llcorner A$ to obtain a finite subfamily $\mathcal{G}_1 \subset \mathcal{G}_U^i$ such that $\mu\big(A \cap (\cup\mathcal{G}_1)\big) > (1-\theta)\mu(A)$. But, since $\cup\mathcal{G}_1$ is Borelian (hence μ-measurable), we have
$$\mu(A) = \mu\big(A \cap (\cup\mathcal{G}_1)\big) + \mu(A \setminus \cup\mathcal{G}_1),$$
and the fact that $\mu(A) < \infty$ allows us to conclude that $\mu(A \setminus \cup\mathcal{G}_1) < \theta\mu(A)$.
2) Given $2 \leq j \in \mathbb{N}$, assume we have defined finite disjoint subfamilies $\mathcal{G}_1 \subset \cdots \subset \mathcal{G}_{j-1} \subset \mathcal{F}$ such that, for $1 \leq i \leq j-1$, the balls of $\mathcal{G}_i$ are contained in U and $\mu(A \setminus \cup\mathcal{G}_i) < \theta^i\mu(A)$. We reapply the argument of the previous item to the open set $U' := U \setminus \cup\mathcal{G}_{j-1}$

in place of U and to $A' := A \setminus \cup\mathcal{G}_{j-1} \subset U'$ in place of A: take $\mathcal{F}_{U'} := \{B \in \mathcal{F} \mid B \subset U', \operatorname{diam} B \leq 1\}$, which is a strongly fine cover of A', and use theorem 3.24 as in the previous item to find a disjoint subfamily $\mathcal{G}_{U'} \subset \mathcal{F}_{U'}$ such that $\mu\big(A' \cap (\cup\mathcal{G}_{U'})\big) \geq \frac{1}{N}\mu(A') > (1-\theta)\mu(A')$. Then, applying the continuity from below to the Borel measure $\mu \,\llcorner A'$, there exists a finite set $\mathcal{G}'_j \subset \mathcal{G}_{U'}$ such that $\mu\big(A' \cap (\cup\mathcal{G}'_j)\big) > (1-\theta)\mu(A')$. As in the previous item, the μ-measurability of $\mathcal{G}'_j$ and the fact that $\mu(A') < \infty$ imply that $\mu(A' \setminus \cup\mathcal{G}'_j) < \theta\mu(A') < \theta^j\mu(A)$. Put $\mathcal{G}_j := \mathcal{G}_{j-1} \cup \mathcal{G}'_j$. Then $\mathcal{G}_j \subset \mathcal{F}$ is a finite disjoint family whose balls are contained in U, and $A \setminus \cup\mathcal{G}_j = A' \setminus \cup\mathcal{G}'_j$ satisfies $\mu(A \setminus \cup\mathcal{G}_j) < \theta^j\mu(A)$.

3) We have thus inductively defined an increasing sequence $(\mathcal{G}_j)_{j\in\mathbb{N}}$ such that, for each $j \in \mathbb{N}$, $\mathcal{G}_j$ is a finite disjoint subfamily of $\mathcal{F}$ whose balls are contained in U, with $\mu(A \setminus \cup\mathcal{G}_j) < \theta^j\mu(A)$. Define $\mathcal{G} := \cup_{i\in\mathbb{N}}\mathcal{G}_i$; then $\mathcal{G}$ is a disjoint subfamily of $\mathcal{F}$ whose balls are contained in U, with $\forall j \in \mathbb{N}$, $\mu(A \setminus \cup\mathcal{G}) \leq \theta^j\mu(A) \overset{j\to\infty}{\longrightarrow} 0$. Hence $\mu(A \setminus \cup\mathcal{G}) = 0$, which concludes the proof.

□

Proof of proposition 3.23. If $X = \mathbb{R}^n$, the thesis follows directly from corollary 3.25.

In the general case, let ρ denote the metric on X induced by the euclidean metric d on $\mathbb{R}^n$ and $\iota : X \to \mathbb{R}^n$ the inclusion. We use superscripts ρ and d for balls in X and $\mathbb{R}^n$, respectively, so that $\forall x \in X$, $\mathbb{B}(x,r)^\rho = \mathbb{B}(x,r)^d \cap X$. Given $A \subset X$ with $\mu(A) < \infty$ and $\mathcal{F} \subset 2^X$ a strongly fine cover of A by nondegenerate closed balls, let $\mathcal{F}' := \{B \text{ nondegenerate closed ball in } \mathbb{R}^n \mid B \cap X \in \mathcal{F}\}$. Then $\mathcal{F}' \subset 2^{\mathbb{R}^n}$ is a strongly fine cover of A in $\mathbb{R}^n$; indeed, $\forall x \in A$, $\forall \delta > 0$, the fact that $\mathcal{F}$ is a strongly fine cover for A in X ensures the existence of $0 < r < \delta$ such that $\mathbb{B}(x,r)^d \cap X = \mathbb{B}(x,r)^\rho \in \mathcal{F}$, hence $\mathbb{B}(x,r)^d \in \mathcal{F}'$ by definition. Since, by proposition 1.15.(iii), the pushforward measure $\iota_\#\mu$ is a Borel measure on $\mathbb{R}^n$, it follows from the case already proved that there exists a countable disjoint subfamily $\mathcal{G}' \subset \mathcal{F}'$ such that $\iota_\#\mu(A \setminus \cup\mathcal{G}') = 0$. Define $\mathcal{G} := \{B \cap X \mid B \in \mathcal{G}'\}$; then $\mathcal{G}$ is a countable disjoint subfamily of $\mathcal{F}$ and, since $\forall B \in \mathcal{G}'$, $A \setminus B = A \setminus (B \cap X)$, it follows that $A \setminus \cup\mathcal{G} = A \setminus \cup\mathcal{G}'$, thus $\mu(A \setminus \cup\mathcal{G}) = \iota_\#\mu(A \setminus \cup\mathcal{G}') = 0$, which concludes the proof. □

Theorem 3.26 (general comparison density theorem). *Let μ and ν be open σ-finite Borel regular measures on a metric space X such that ν has the symmetric Vitali property, $t \geq 0$ and $A \subset X$. If $\forall x \in A$, $\Theta^{*\nu}(\mu, x) \geq t$ then $t\nu(A) \leq \mu(A)$.*

Proof. We assume that $t > 0$, otherwise the thesis is trivial.

Firstly, assume that $\nu(A) < \infty$.

Fix $0 < \tau < t$ and an open set $U \supset A$. Since, $\forall x \in A$, $\Theta^{*\nu}(\mu, x) = \limsup_{r\to 0} \frac{\mu\left(\mathbb{B}(x,r)\right)}{\nu\left(\mathbb{B}(x,r)\right)} \geq t > \tau$, it follows that $\forall x \in A$, $\forall r > 0$, $\exists 0 < \rho < r$ such that $\frac{\mu\left(\mathbb{B}(x,\rho)\right)}{\nu\left(\mathbb{B}(x,\rho)\right)} > \tau$, so that $\mu\big(\mathbb{B}(x, \rho)\big) > \tau\nu\big(\mathbb{B}(x, \rho)\big)$ (note that, in order for the quotient to be $> \tau$, according to our extended arithmetic convention in 3.12, the numerator cannot be 0 and the denominator cannot be ∞). It then follows that $\mathcal{F} := \{B \mid \exists x \in A, \exists r > 0, B = \mathbb{B}(x, r), \mu\big(\mathbb{B}(x, \rho)\big) > \tau\nu\big(\mathbb{B}(x, \rho)\big), B \subset U\}$ is a strongly fine cover of A. Since $\nu(A) < \infty$ and ν has the symmetric Vitali property, we may take a countable disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\nu(A \setminus \cup\mathcal{G}) = 0$. Therefore, by countable subadditivity,

$$\begin{aligned}\tau\nu(A) &\leq \tau\Big[\sum_{B\in\mathcal{G}} \nu(B) + \nu(A \setminus \cup\mathcal{G})\Big] \leq \\ &\leq \sum_{B\in\mathcal{G}} \mu(B) = \mu(\cup\mathcal{G}) \leq \mu(U).\end{aligned}$$

Since μ is open σ-finite Borel regular, theorem 1.23 may be applied and yields $\mu(A) = \inf\{\mu(U) \mid U \supset A \text{ open}\} \geq \tau\nu(A)$ and, taking $\tau \to t$, the thesis follows in case $\nu(A) < \infty$.

If $\nu(A) = \infty$, the fact that ν is open σ-finite allows us to take a countable disjoint family $(B_k)_{k\in\mathbb{N}}$ of Borel sets in X such that $\dot\bigcup_{k\in\mathbb{N}} B_k = X$ and $\forall k \in \mathbb{N}$, $\nu(B_k) < \infty$. Thus, for all $k \in \mathbb{N}$, the case already proved applies to $A \cap B_k$, which yields $\mu(A \cap B_k) \geq t\nu(A \cap B_k)$. By the fact that both $\mu \llcorner A$ and $\nu \llcorner A$ are Borel measures, it then follows that $\mu(A) = \sum_{k\in\mathbb{N}} \mu(A \cap B_k) \geq t \sum_{k\in\mathbb{N}} \nu(A \cap B_k) = t\nu(A)$. $\square$

Corollary 3.27. *Let μ and ν be open σ-finite Borel regular measures on a metric space X such that ν has the symmetric Vitali property. Then $\Theta^{*\nu}(\mu, x) < \infty$ for ν-a.e. $x \in X$.*

Proof. Let $I := \{x \in X \mid \Theta^{*\nu}(\mu, x) = \infty\}$. We must show that $\nu(I) = 0$. Since μ is open σ-finite, we may take a sequence of open sets $(U_k)_{k\in\mathbb{N}}$ such that $\cup_{k\in\mathbb{N}} U_k = X$ and $\forall k \in \mathbb{N}$, $\mu(U_k) < \infty$.

Fix $k \in \mathbb{N}$ and $t > 0$, and let $A^k_t := \{x \in U_k \mid \Theta^{*\nu}(\mu, x) \geq t\}$. Applying theorem 3.26 with A^k_t in place of A, it follows that $t\nu(A^k_t) \leq \mu(A^k_t) \leq \mu(U_k) < \infty$. Since $I \cap U_k = \cap_{t>0} A^k_t$, we then conclude that $\forall t > 0$, $\nu(I \cap U_k) \leq \nu(A^k_t) \leq t^{-1}\mu(U_k) \xrightarrow{t\to\infty} 0$, hence $\nu(I \cap U_k) = 0$. As $I = \cup_{k\in\mathbb{N}}(I \cap U_k)$, the thesis follows from the countable subadditivity of ν. $\square$

THEOREM 3.28 (general upper density theorem). *Let μ be a Borel regular measure on a metric space X, ν an open σ-finite Borel regular measure on X with the symmetric Vitali property, and $A \in \sigma(\mu)$ with $\mu(A) < \infty$. Then $\Theta^{*\nu}(\mu \llcorner A, x) = 0$ for ν-a.e. $x \in X \setminus A$.*

PROOF. For each $t > 0$, let $S_t := \{x \in X \setminus A \mid \Theta^{*\nu}(\mu \llcorner A, x) \geq t\}$. By proposition 1.36.(i), $\mu \llcorner A$ is a finite Borel regular measure on X; hence, we may apply theorem 3.26 with $\mu \llcorner A$ in place of μ, ν and S_t in place of A, which yields $t\nu(S_t) \leq \mu \llcorner A(S_t) = \mu(A \cap S_t) = 0$, since $A \cap S_t = \emptyset$. Thus, $\nu(S_t) = 0$, whence $\nu(\{x \in X \setminus A \mid \Theta^{*\nu}(\mu \llcorner A, x) > 0\}) = \nu(\cup_{n\in\mathbb{N}} S_{1/n}) = 0$. □

THEOREM 3.29 (general density theorem). *Let μ be an open σ-finite Borel regular measure on a metric space X with symmetric Vitali property and $A \in \sigma(\mu)$. Then the density $\Theta^{\mu}(\mu \llcorner A, \cdot)$ coincides μ-a.e. on X with χ_A, i.e.*

$$\Theta^{\mu}(\mu \llcorner A, x) = \lim_{r\to 0} \frac{\mu\big(A \cap \mathbb{B}(x,r)\big)}{\mu\big(\mathbb{B}(x,r)\big)} = \begin{cases} 1 & \text{for } \mu\text{-a.e. } x \in A, \\ 0 & \text{for } \mu\text{-a.e. } x \in X \setminus A. \end{cases}$$

PROOF. It is an adaptation of the proof of corollary 3.9, using theorem 3.28 instead of theorem 3.7.

Firstly, we prove the case in which $\mu(X) < \infty$. Since μ is concentrated on its support, by remark 3.20.2), it suffices to show that $\Theta^{\mu}(\mu \llcorner A, \cdot)$ coincides μ-a.e. on $\operatorname{spt}\mu$ with χ_A. Fix $x \in \operatorname{spt}\mu$ and define $f, g : (0,\infty) \to [0,\infty]$ by $f(r) = \frac{\mu\left(A\cap\mathbb{B}(x,r)\right)}{\mu\left(\mathbb{B}(x,r)\right)}$ and $g(r) = \frac{\mu\left((X\setminus A)\cap\mathbb{B}(x,r)\right)}{\mu\left(\mathbb{B}(x,r)\right)}$. Due to the fact that $f(r) + g(r) \equiv 1$ and that $\liminf_{r\to 0}(f+g)(r) \leq \liminf_{r\to 0} f(r) + \limsup_{r\to 0} g(r) \leq \limsup_{r\to 0}(f+g)(r)$, it follows that

$$\Theta^{*\mu}(\mu \llcorner A, x) + \Theta^{\mu}_{*}(\mu \llcorner (X \setminus A), x) = 1, \tag{3.3}$$

and the same holds with $X \setminus A$ in place of A.

On the other hand, since μ is a finite Borel regular measure and $A \in \sigma(\mu)$, we may apply theorem 3.28 with $\nu = \mu$, which yields $\Theta^{*\mu}(\mu \llcorner A, x) = 0$ for μ-a.e. $x \in X \setminus A$. Then the fact that $0 \leq \Theta^{\mu}_{*}(\mu \llcorner A, \cdot) \leq \Theta^{*\mu}(\mu \llcorner A, \cdot) \leq 1$ implies that $\Theta^{\mu}(\mu \llcorner A, x) = 0$ for μ-a.e. $x \in X \setminus A$. The same holds for $X \setminus A$ in place of A, i.e. $\Theta^{\mu}(\mu \llcorner (X \setminus A), x) = 0$ for μ-a.e. $x \in A$. The last equality implies, in view of (3.3), that $\Theta^{\mu}(\mu \llcorner A, x) = 1$ for μ-a.e. $x \in A \cap \operatorname{spt}\mu$, which concludes the proof in case $\mu(X) < \infty$.

In the general case, since μ is open σ-finite, we may cover X with countably many open sets $(U_k)_{k\in\mathbb{N}}$ such that $\forall k \in \mathbb{N}$, $\mu(U_k) < \infty$. For fixed $k \in \mathbb{N}$, it follows from proposition 1.36.(i) and from remark 3.20.(1) that $\mu \llcorner U_k$ is a finite Borel regular measure with SVP, to which the case already proved yields $\Theta^{\mu \llcorner U_k}(\mu \llcorner (U_k \cap A), \cdot) = \chi_A$ $(\mu \llcorner U_k)$-a.e. on X. Since U_k is open, the functions $\Theta^{\mu \llcorner U_k}(\mu \llcorner (U_k \cap A), \cdot)$ and $\Theta^\mu(\mu \llcorner A, \cdot)$ coincide on U_k; hence, $\Theta^\mu(\mu \llcorner A, \cdot)$ coincides with χ_A μ-a.e. on U_k. As $\cup_{k\in\mathbb{N}} U_k = X$, we conclude that $\Theta^\mu(\mu \llcorner A, \cdot)$ coincides with χ_A μ-a.e. on X, as asserted. $\square$

Corollary 3.30 (general Lebesgue differentiation theorem). *Let μ be an open σ-finite Borel regular measure on a metric space X with symmetric Vitali property and $f : X \to \mathbb{C}$ a μ-measurable function satisfying one of the following conditions:*

i) $f \in \mathsf{L}^1(\mu)$ or
ii) X is separable and $f \in \mathsf{L}^1_{\mathsf{loc}}(\mu)$, i.e. $\forall x \in X$, $\exists r > 0$, $\int_{\mathbb{B}(x,r)} |f| \, \mathrm{d}\mu < \infty$.

Then, for μ-a.e. $x \in X$:

$$\lim_{r\to 0} \frac{1}{\mu\big(\mathbb{B}(x,r)\big)} \int_{\mathbb{B}(x,r)} f \, \mathrm{d}\mu = f(x). \tag{3.4}$$

Proof. Note that, since $f = [(\operatorname{Re} f)^+ - (\operatorname{Re} f)^-] + i[(\operatorname{Im} f)^+ - (\operatorname{Im} f)^-]$, it suffices to prove the thesis for positive functions, i.e. we may assume $f : X \to [0, \infty)$. Moreover, the fact that μ is open σ-finite ensures the existence of a sequence $(U_k)_{k\in\mathbb{N}}$ of open sets such that $\cup_{k\in\mathbb{N}} U_k = X$ and $\forall k \in \mathbb{N}$, $\mu(U_k) < \infty$; we may also assume that $\int f \, \mathrm{d}(\mu \llcorner U_k) = \int_{U_k} f \, \mathrm{d}\mu < \infty$ in case hypothesis (ii) holds. Therefore, if we prove the thesis for finite Borel regular measures with SVP and hypothesis (i), it will follow that, for each $k \in \mathbb{N}$, (3.4) holds with $\mu \llcorner U_k$ in place of μ. In particular, for all $k \in \mathbb{N}$, (3.4) holds (with μ) for μ-a.e. $x \in U_k$, hence it holds for μ-a.e. $x \in X = \cup_{k\in\mathbb{N}} U_k$. We may then assume that $\mu(X) < \infty$ and that hypothesis (i) holds, i.e. $f \in \mathsf{L}^1(\mu)$.

Fix $k \in \mathbb{N}$. The fact that μ is a finite Borel regular measure allows us to apply Lusin's theorem 1.117, which yields a closed subset $F_k \subset X$ such that $\mu(X \setminus F_k) < 1/k$ and $f|_{F_k}$ is continuous. Since μ has the symmetric Vitali property, so does $\mu \llcorner F_k$; hence, by remark 3.20, $\mu \llcorner F_k$ is concentrated on its support. In particular, if $N_k := F_k \cap (X \setminus \operatorname{spt} \mu \llcorner F_k) = \{x \in F_k \mid \exists r > 0, \mu\big(F_k \cap \mathbb{B}(x,r)\big) = 0\}$, then $\mu(N_k) = 0$. For each $x \in F_k \setminus N_k$, for each $r > 0$, we have $0 < \mu\big(F_k \cap \mathbb{B}(x,r)\big) \leq$

$\mu\big(\mathbb{B}(x,r)\big)$, so that

$$\begin{aligned}\frac{1}{\mu\big(\mathbb{B}(x,r)\big)}\int_{F_k\cap\mathbb{B}(x,r)} f\,\mathrm{d}\mu &= \\ &= \frac{\mu\big(F_k\cap\mathbb{B}(x,r)\big)}{\mu\big(\mathbb{B}(x,r)\big)}\cdot\frac{1}{\mu\big(F_k\cap\mathbb{B}(x,r)\big)}\int_{F_k\cap\mathbb{B}(x,r)} f\,\mathrm{d}\mu.\end{aligned}\tag{3.5}$$

From the general density theorem 3.29, for μ-a.e. $x\in F_k$, $\frac{\mu\left(F_k\cap\mathbb{B}(x,r)\right)}{\mu\left(\mathbb{B}(x,r)\right)}\xrightarrow{r\to 0} 1$, and the continuity of $f|_{F_k}$ ensures $\forall x\in F_k$, $\frac{1}{\mu\left(F_k\cap\mathbb{B}(x,r)\right)}\int_{F_k\cap\mathbb{B}(x,r)} f\,\mathrm{d}\mu\xrightarrow{r\to 0} f(x)$. It then follows from (3.5) that, adjoining a μ-null set to N_k if necessary, $\forall x\in F_k\setminus N_k$,

$$\frac{1}{\mu\big(\mathbb{B}(x,r)\big)}\int_{F_k\cap\mathbb{B}(x,r)} f\,\mathrm{d}\mu\xrightarrow{r\to 0} f(x).\tag{3.6}$$

We contend that, for μ-a.e. $x\in F_k$,

$$\frac{1}{\mu\big(\mathbb{B}(x,r)\big)}\int_{\mathbb{B}(x,r)\setminus F_k} f\,\mathrm{d}\mu\xrightarrow{r\to 0} 0.\tag{3.7}$$

Indeed, let ν be the Borel regular measure on X given by $f\,\mathrm{d}\mu$, i.e. the extension of the measure $A\in\mathscr{B}_X\mapsto\int_A f\,\mathrm{d}\mu$ given by theorem 1.8. Since $F_k^c=X\setminus F_k$ has finite ν-measure (because $f\in\mathsf{L}^1(\mu)$), we can apply the general upper density theorem 3.28 with ν in place of μ, μ in place of ν and F_k^c in place of A, thus proving our contention.

It then follows from (3.6) and (3.7) that, adjoining another μ-null set to N_k if necessary, $\forall x\in F_k\setminus N_k$, (3.4) holds. Therefore, as $k\in\mathbb{N}$ was arbitrarily taken and $X\setminus\cup_{k\in\mathbb{N}}F_k$ is μ-null, (3.4) holds for x in the complement of the μ-null set $(\cup_{k\in\mathbb{N}}N_k)\cup(X\setminus\cup_{k\in\mathbb{N}}F_k)$ and we are done. □

COROLLARY 3.31 (Lebesgue Points). *Let X be a separable metric space, μ an open σ-finite Borel regular measure on X with symmetric Vitali property, $1\le p<\infty$ and $f\in\mathsf{L}^p_{\mathsf{loc}}(\mu)$, i.e. $\forall x\in X,\exists r>0,\int_{\mathbb{B}(x,r)}|f|^p\,\mathrm{d}\mu<\infty$. Then, for μ-a.e. $x\in X$,*

$$\lim_{r\to 0}\frac{1}{\mu\big(\mathbb{B}(x,r)\big)}\int_{\mathbb{B}(x,r)}|f(y)-f(x)|^p\,\mathrm{d}\mu(y)=0.\tag{3.8}$$

PROOF. Let $\{r_i\mid i\in\mathbb{N}\}$ be a countable dense subset of $\mathbb{C}$. It follows from corollary 3.30 that, for every $i\in\mathbb{N}$, there exists a μ-null set A_i such that, for all $x\in A_i^c$,

$$\lim_{r\to 0}\frac{1}{\mu\big(\mathbb{B}(x,r)\big)}\int_{\mathbb{B}(x,r)}|f-r_i|^p\,\mathrm{d}\mu=|f(x)-r_i|^p.$$

Then the above equality holds for all $i \in \mathbb{N}$ and for all x in the complement of the μ-null set $A := \cup_{i\in\mathbb{N}} A_i$.

Fix $x \in A^c$ and $\epsilon > 0$. There exists $i \in \mathbb{N}$ such that $|f(x) - r_i| < \epsilon$. Then

$$\begin{aligned}
&\frac{1}{\mu\big(\mathbb{B}(x,r)\big)} \int_{\mathbb{B}(x,r)} |f(y) - f(x)|^p \,\mathrm{d}\mu(y) \le \\
&\quad\le \frac{2^{p-1}}{\mu\big(\mathbb{B}(x,r)\big)} \int_{\mathbb{B}(x,r)} \big(|f(y) - r_i|^p + |r_i - f(x)|^p\big) \,\mathrm{d}\mu(y) \le \\
&\quad\le \frac{2^{p-1}}{\mu\big(\mathbb{B}(x,r)\big)} \int_{\mathbb{B}(x,r)} |f(y) - r_i|^p \,\mathrm{d}\mu(y) + 2^{p-1}|f(x) - r_i|^p,
\end{aligned}$$

so that

$$\limsup_{r\to 0} \frac{1}{\mu\big(\mathbb{B}(x,r)\big)} \int_{\mathbb{B}(x,r)} |f(y) - f(x)|^p \,\mathrm{d}\mu(y) \le 2\cdot 2^{p-1}|f(x) - r_i|^p < 2^p \epsilon^p.$$

Since $\epsilon > 0$ was arbitrarily taken, the thesis follows. ☐

DEFINITION 3.32 (Lebesgue Points). With the same notation from the previous corollary, a point $x \in X$ for which (3.8) holds is called *Lebesgue point of f with respect to μ.*

It is clear that every point of continuity of f is a Lebesgue point of f.

If $X = \mathbb{R}^n$ and $\mu = \mathcal{L}^n$, the limit in (3.8) can be taken along all closed balls B containing x (not necessarily centered at x) with diam $B \to 0$:

COROLLARY 3.33 (Lebesgue points with noncentered balls). *Let $1 \le p < \infty$ and $f \in \mathsf{L}^{\mathsf{p}}_{\mathsf{loc}}(\mathcal{L}^n)$. Then, for each Lebesgue point x of f with respect to $\mathcal{L}^n$ (in particular, for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$),*

$$\lim_{B\downarrow\{x\}} \frac{1}{\mathcal{L}^n(B)} \int_B |f(y) - f(x)|^p \,\mathrm{d}\mathcal{L}^n(y) = 0,$$

where the limit is taken over all closed balls B containing x with diam $B \to 0$.

PROOF. For each closed ball B containing x, we have:

$$\begin{aligned}
&\frac{1}{\mathcal{L}^n(B)} \int_B |f(y) - f(x)|^p \,\mathrm{d}\mathcal{L}^n(y) \le \\
&\quad\le \frac{1}{\mathcal{L}^n(B)} \int_{\mathbb{B}(x,\mathrm{diam}\ B)} |f(y) - f(x)|^p \,\mathrm{d}\mathcal{L}^n(y) = \\
&\quad= 2^n \frac{1}{\mathcal{L}^n\big(\mathbb{B}(x,\mathrm{diam}\ B)\big)} \int_{\mathbb{B}(x,\mathrm{diam}\ B)} |f(y) - f(x)|^p \,\mathrm{d}\mathcal{L}^n(y)
\end{aligned}$$

and then the thesis follows from corollary 3.31. □

Our next step is to prove a version of the general comparison density theorem 3.26 for lower densities.

Firstly we introduce for Borel outer measures the notions of absolute continuity and mutual singularity which were introduced in 1.92 for measures on a σ-algebra $\mathcal{M}$.

DEFINITION 3.34 (absolute continuity and mutual singularity). Let μ and ν be Borel measures on a topological space X. We say that:

1) μ is *absolutely continuous* with respect to ν (notation: $\mu \ll \nu$) if $\forall A \subset X$, $\nu(A) = 0$ implies $\mu(A) = 0$.
2) μ and ν are *mutually singular* (notation: $\mu \perp \nu$) if there exists $A \in \mathscr{B}_X$ such that μ is concentrated on A and ν is concentrated on $X \setminus A$.

REMARK 3.35. Note that $\mu \perp \nu$ iff $\mu|_{\mathscr{B}_X} \perp \nu|_{\mathscr{B}_X}$ in the sense of definition 1.92. Besides, it is clear that, if μ is a Borel measure and ν is a Borel regular measure on a topological space X, then $\mu \ll \nu$ iff $\forall A \in \mathscr{B}_X$, $\nu(A) = 0$ implies $\mu(A) = 0$. Thus, if ν is Borel regular, then $\mu \ll \nu$ iff $\mu|_{\mathscr{B}_X} \ll \nu|_{\mathscr{B}_X}$ in the sense of definition 1.92.

We now prove a version of the Lebesgue decomposition theorem 1.101 for outer measures. The lemma below may be obtained as a direct consequence of the previous remark and theorem 1.101, but we give a direct proof.

LEMMA 3.36 (Lebesgue decomposition theorem). *Let μ be a σ-finite Borel measure and ν a Borel regular measure on a metric space X. Then there exists $B \in \mathscr{B}_X$ such that ν is concentrated on B^c and $\mu \llcorner B^c \ll \nu$, so that*

$$\mu = \mu \llcorner B + \mu \llcorner B^c, \quad \mu \llcorner B \perp \nu,\ \mu \llcorner B^c \ll \nu. \tag{LD}$$

Moreover:

1) $B \in \mathscr{B}_X$ satisfying (LD) is unique up to μ-null sets, i.e. if $B' \in \mathscr{B}_X$ also satisfies (LD), then $B \,\Delta\, B'$ is μ-null.
2) the decomposition (LD) is unique in the sense that, if $\mu = \mu_s + \mu_a$ with $\mu_s \perp \nu$ and $\mu_a \ll \nu$, then $\mu_s = \mu \llcorner B$ and $\mu_a = \mu \llcorner B^c$.

DEFINITION 3.37. With the notation above, we call $\mu \llcorner B$ the *singular part* and $\mu \llcorner B^c$ the *absolutely continuous part* of μ with respect to ν.

PROOF. 1) Assume μ finite. Let $\mathcal{F} := \{F \in \mathscr{B}_X \mid \nu(F) = 0\}$ and $\alpha := \sup\{\mu(F) \mid F \in \mathcal{F}\}$, so that $0 \leq \alpha \leq \mu(X) < \infty$. We contend that this sup is attained. Indeed, take a sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $\mu(F_n) \to \alpha$ and define $B := \cup_{n\in\mathbb{N}} F_n$. Then $B \in \mathscr{B}_X$ and $\nu(B) = 0$ (since each F_n is ν-null), so that $B \in \mathcal{F}$. Since $\forall n \in \mathbb{N}$, $\mu(F_n) \le \mu(B)$ and $\mu(F_n) \to \alpha$, it follows $\alpha \le \mu(B)$, hence $\alpha = \mu(B)$.

We contend that $\mu \llcorner B^c \ll \nu$. Indeed, if that is not the case, the Borel regularity of ν and remark 3.35 imply the existence of $A \in \mathscr{B}_X$ such that $\nu(A) = 0$ and $\mu \llcorner B^c(A) > 0$; hence $B \cup A \in \mathcal{F}$ and $\mu(B \cup A) = \mu(B) + \mu(B^c \cap A) = \alpha + \mu \llcorner B^c(A) > \alpha$, which contradicts the definition of α. The contention is then proved, so that ν is concentrated on B^c and $\mu \llcorner B^c \ll \nu$. Besides, since $\mu \llcorner B$ is trivially concentrated on B, we have $\mu \llcorner B \perp \nu$ and the decomposition (LD) holds.

2) In the general case, since μ is σ-finite, we can take an increasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\sigma(\mu)$ such that $\cup_{n\in\mathbb{N}} A_n = X$ and $\forall n \in \mathbb{N}$, $\mu(A_n) < \infty$. For each $n \in \mathbb{N}$, $\mu \llcorner A_n$ is a finite Borel measure, to which the previous item can be applied, yielding $B_n \in \mathscr{B}_X$ such that ν is concentrated on B_n^c and $\mu \llcorner A_n \cap B_n^c \ll \nu$. Let $B := \cup_{n\in\mathbb{N}} B_n \in \mathscr{B}_X$. Then $\nu(B) = 0$, i.e. ν is concentrated on B^c; we contend that $\mu \llcorner B^c \ll \nu$, which yields the validity of decomposition (LD). Indeed, if $F \in \mathscr{B}_X$ is ν-null, $\forall n \in \mathbb{N}$, $\mu \llcorner A_n \cap B_n^c(F) = \mu(A_n \cap B_n^c \cap F) = 0$, hence $\mu\big(A_n \cap (\cap_{n\in\mathbb{N}} B_n^c) \cap F\big) = 0$. Since $\big(A_n \cap (\cap_{n\in\mathbb{N}} B_n^c) \cap F\big)_{n\in\mathbb{N}}$ is a sequence in $\sigma(\mu)$ which increases to $\big((\cap_{n\in\mathbb{N}} B_n^c) \cap F\big) = B^c \cap F$, the continuity from below 1.11 applied to μ yields $\mu \llcorner B^c(F) = \mu(B^c \cap F) = 0$, which proves our contention.
3) Let $B' \in \mathscr{B}_X$ such that (LD) also holds with B' in place of B. Then $\mu \llcorner B \setminus B'$ is concentrated on B and absolutely continuous with respect to ν (which is null on B), hence $\mu \llcorner B \setminus B'$ is the null measure. Similarly, $\mu \llcorner B' \setminus B$ is null, and so is $\mu \llcorner B \,\Delta\, B' = \mu \llcorner B \setminus B' + \mu \llcorner B' \setminus B$, i.e. $\mu(B \,\Delta\, B') = 0$, as asserted.
4) Let $\nu = \mu_s + \mu_a$ with $\mu_s \perp \nu$ and $\mu_a \ll \nu$. Let $B' \in \mathscr{B}_X$ such that μ_s is concentrated on B' and ν is concentrated on $(B')^c$. Then $\mu_a \ll \nu$ is also concentrated on $(B')^c$; thus, for all $A \subset X$, $\mu_s(A) = \mu_s(A \cap B') = \mu(A \cap B') = \mu \llcorner B'(A)$ and $\mu_a(A) = \mu_a\big(A \cap (B')^c\big) = \mu\big(A \cap (B')^c\big) = \mu \llcorner (B')^c(A)$. That is, $\mu_a = \mu \llcorner B'$ and $\mu_s = \mu \llcorner (B')^c$. In particular, (LD) holds with B' in place of B; by the previous item, it follows that $\mu(B \,\Delta\, B') = 0$, hence $\mu_s = \mu \llcorner B' = \mu \llcorner B$ and $\mu_a = \mu \llcorner (B')^c = \mu \llcorner B^c$, as asserted.

$\square$

THEOREM 3.38 (comparison theorem for lower densities). *Let μ and ν be open σ-finite Borel regular measures on a metric space X, $t \geq 0$ and $A \subset X$ with $\forall x \in A$, $\Theta_*^\nu(\mu, x) \leq t$.*

i) If μ has SVP, then $\mu(A) \leq t\,\nu(A)$.
ii) If ν has SVP and B is given by lemma 3.36, so that (LD) holds, then $\mu(A \setminus B) \leq t\,\nu(A)$.

PROOF. It is similar to the proof of theorem 3.26.

(1) Firstly, we make a reduction: it is enough to prove the case in which both $\mu(A)$ and $\nu(A)$ are finite. Indeed, suppose that the thesis holds in that case. In the general case, since both μ and ν are σ-finite, there exists a disjoint sequence $(B_k)_{k\in\mathbb{N}}$ of Borel sets in X such that $\mu(B_k) < \infty$, $\nu(B_k) < \infty$ and $X = \dot\bigcup_{k\in\mathbb{N}} B_k$. Thus, for each $k \in \mathbb{N}$, the thesis holds for $B_k \cap A$ in place of A, so that, for all $k \in \mathbb{N}$, $\mu(B_k \cap A) \leq t\,\nu(B_k \cap A)$ in case i) and $\mu(B_k \cap A \setminus B) \leq t\,\nu(B_k \cap A)$ in case ii). By the fact that both $\mu \llcorner A$ and $\nu \llcorner A$ are Borel measures, it then follows that $\mu(A) = \sum_{k\in\mathbb{N}} \mu(B_k \cap A) \leq t \sum_{k\in\mathbb{N}} \nu(B_k \cap A) = t\,\nu(A)$ in case i), and $\mu(A \setminus B) = \sum_{k\in\mathbb{N}} \mu(B_k \cap A \setminus B) \leq t \sum_{k\in\mathbb{N}} \nu(B_k \cap A) = t\,\nu(A)$ in case ii).
   Assume, therefore, $\mu(A) < \infty$ and $\nu(A) < \infty$.
(2) It is enough to prove part i) for $A \subset \operatorname{spt} \mu$. Indeed, suppose that the thesis holds in that case. Since μ is σ-finite and has SVP, it follows from remark 3.20 that μ is concentrated on its support; thus, for arbitrary A it follows that $\mu(A) = \mu(A \cap \operatorname{spt} \mu) \leq t\,\nu(A \cap \operatorname{spt} \mu) \leq t\,\nu(A)$ and we are done.
   Assume, therefore, $A \subset \operatorname{spt} \mu$. Fix $\tau > t$ and an open set $U \supset A$. Since, $\forall x \in A$, $\Theta_*^\nu(\mu, x) = \liminf_{r\to 0} \frac{\mu\left(\mathbb{B}(x,r)\right)}{\nu\left(\mathbb{B}(x,r)\right)}$ $\leq t < \tau$, it follows that $\forall x \in A$, $\forall r > 0$, $\exists 0 < \rho < r$ such that $\frac{\mu\left(\mathbb{B}(x,\rho)\right)}{\nu\left(\mathbb{B}(x,\rho)\right)} < \tau$, so that $\mu\big(\mathbb{B}(x,\rho)\big) < \tau\nu\big(\mathbb{B}(x,\rho)\big)$ (note that $\mu\big(\mathbb{B}(x,\rho)\big) > 0$, since $x \in \operatorname{spt} \mu$; hence, in order for the quotient to be $< \tau$, according to our extended arithmetic convention in 3.12, the denominator cannot be 0). It then follows that $\mathcal{F} := \{B \mid \exists x \in A, \exists r > 0, B = \mathbb{B}(x,r), \mu\big(\mathbb{B}(x,r)\big) < \tau\nu\big(\mathbb{B}(x,r)\big), B \subset U\}$ is a strongly fine cover of A. Since $\mu(A) < \infty$ and μ has the symmetric Vitali property, we may take a countable disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\mu(A \setminus$

$\cup\mathcal{G}) = 0$. Therefore, by countable subadditivity,

$$\mu(A) \leq \mu(A \setminus \cup\mathcal{G}) + \sum_{B\in\mathcal{G}} \mu(B) < \tau \sum_{B\in\mathcal{G}} \nu(B) =$$
$$= \tau\nu(\cup\mathcal{G}) \leq \tau\nu(U)$$

Since ν is open σ-finite Borel regular, theorem 1.23 may be applied and yields $\tau\nu(A) = \inf\{\tau\nu(U) \mid U \supset A \text{ open}\} \geq \mu(A)$ and, taking $\tau \to t$, part i) is proved.

(3) We now prove ii). Take $B \in \mathscr{B}_X$ given by lemma 3.36, so that (LD) holds. Note that, since the measure $\mu \llcorner B^c$ is absolutely continuous with respect to ν, it clearly has SVP; besides, it is trivially open σ-finite, it is Borel regular by proposition 1.36.(i), and $\forall x \in A$:

$$\Theta^\nu_*(\mu \llcorner B^c, x) = \liminf_{r\to 0} \frac{\mu \llcorner B^c\big(\mathbb{B}(x,r)\big)}{\nu\big(\mathbb{B}(x,r)\big)} \leq$$
$$\leq \liminf_{r\to 0} \frac{\mu\big(\mathbb{B}(x,r)\big)}{\nu\big(\mathbb{B}(x,r)\big)} = \Theta^\nu_*(\mu, x) \leq t.$$

We may therefore apply part i) with $\mu \llcorner B^c$ in place of μ, yielding $\mu(A \setminus B) = \mu \llcorner B^c(A) \leq t\,\nu(A)$, as asserted.

□

THEOREM 3.39 (differentiation theorem for Borel measures on metric spaces). *Let μ and ν be open σ-finite Borel regular measures on a metric space X. Suppose that X is separable or that ν is finite on closed balls of X.*

i) The set $Y := \{x \in X \mid \Theta^{*\nu}(\mu, x) = \Theta^\nu_*(\mu, x)\}$ is Borel measurable and $\Theta^\nu(\mu, \cdot) : Y \to [0, \infty]$ is Borelian.

ii) If ν has SVP, $Y_f := \{x \in Y \mid \Theta^\nu(\mu, x) < \infty\}$ is a Borel measurable subset of X whose complement is ν-null.

iii) If both μ and ν have SVP, $\mu(Y^c) = \nu(Y^c) = 0$.

PROOF.

1) If ν is finite on closed balls, part (i) follows from corollary 3.17; if X is separable, part (i) is a direct consequence of exercise 3.16.

2) We now prove part (iii). It suffices to prove the case in which $\mu(X) < \infty$ and $\nu(X) < \infty$. Indeed, assuming that the thesis holds in this case, in the general case we can take a sequence $(U_k)_{k\in\mathbb{N}}$ of open sets such that $X = \cup_{k\in\mathbb{N}} U_k$ and $\forall k \in \mathbb{N}$, $\mu(U_k) < \infty$ and $\nu(U_k) < \infty$. The thesis then holds, for each $k \in \mathbb{N}$, for the finite Borel regular measures with SVP $\mu \llcorner U_k$ and $\nu \llcorner U_k$

in place of μ and ν, respectively. Thus, for each $k \in \mathbb{N}$, since $\Theta^{*\nu}(\mu, \cdot) = \Theta^{*\nu \llcorner U_k}(\mu \llcorner U_k, \cdot)$ and $\Theta^{\nu}_{*}(\mu, \cdot) = \Theta^{\nu \llcorner U_k}_{*}(\mu \llcorner U_k, \cdot)$ on the open set U_k, it follows that $\mu(Y^c \cap U_k) = 0 = \nu(Y^c \cap U_k)$, whence $\mu(Y^c) = 0 = \nu(Y^c)$, as asserted.

Assume, therefore, that both μ and ν are finite. Let $0 < a < b < \infty$ and $Y_{a,b} := \{x \in X \mid \Theta^{\nu}_{*}(\mu, x) \leq a \text{ and } \Theta^{*\nu}(\mu, x) \geq b\}$. Since both μ and ν are open σ-finite Borel regular measures with SVP, we may apply theorems 3.26 and 3.38 to conclude that $b\nu(Y_{a,b}) \leq \mu(Y_{a,b})$ and $\mu(Y_{a,b}) \leq a\nu(Y_{a,b})$, so that $\mu(Y_{a,b}) \leq a\nu(Y_{a,b}) \leq \frac{a}{b}\mu(Y_{a,b})$. Since $\frac{a}{b} < 1$ and $\mu(Y_{a,b}) < \infty$, it follows that $\mu(Y_{a,b}) = 0$, hence $\nu(Y_{a,b}) = 0$. As $Y^c = \cup\{Y_{a,b} \mid 0 < a < b < \infty, a \in \mathbb{Q}, b \in \mathbb{Q}\}$, it follows that $\mu(Y^c) = \nu(Y^c) = 0$, as asserted.

3) We prove part (ii). It is clear from part (i) that $Y_f = \{x \in Y \mid \Theta^{\nu}(\mu, x) < \infty\}$ is Borel measurable. By the same reduction made in the previous item, we may assume that both μ and ν are finite. Take $B \in \mathscr{B}_X$ given by lemma 3.36, so that (LD) holds. Note that $\mu \llcorner B^c$ is a finite Borel regular measure on X with SVP. Applying part (iii) with $\mu \llcorner B^c$ in place of μ, we conclude that $\{x \in X \mid \Theta^{*\nu}(\mu \llcorner B^c, x) \neq \Theta^{\nu}_{*}(\mu \llcorner B^c, x)\}$ is ν-null. On the other hand, as $\mu(B) < \infty$, we may apply theorem 3.28 with B in place of A, yielding $\Theta^{*\nu}(\mu \llcorner B, x) = 0$ for ν-a.e. $x \in B^c$. That implies $\Theta^{*\nu}(\mu \llcorner B^c, x) = \Theta^{*\nu}(\mu, x)$ and $\Theta^{\nu}_{*}(\mu \llcorner B^c, x) = \Theta^{\nu}_{*}(\mu, x)$ for ν-a.e. $x \in B^c$. It then follows that $Y^c \cap B^c = \{x \in B^c \mid \Theta^{*\nu}(\mu, x) \neq \Theta^{\nu}_{*}(\mu, x)\}$ differs from $\{x \in B^c \mid \Theta^{*\nu}(\mu \llcorner B^c, x) \neq \Theta^{\nu}_{*}(\mu \llcorner B^c, x)\}$ by a ν-null set; thus, since the latter set is ν-null, so is the former. Since ν is concentrated on B^c, we then conclude that $\nu(Y^c) = \nu(Y^c \cap B^c) = 0$. Finally, by corollary 3.27, $F := \{x \in X \mid \Theta^{*\nu}(\mu, x) < \infty\}$ has ν-null complement; as $Y_f = Y \cap F$, we conclude that $Y_f^c = Y^c \cup F^c$ is ν-null, as asserted.

□

THEOREM 3.40 (Lebesgue-Besicovitch-Radon-Nikodym differentiation theorem). *Let μ and ν be open σ-finite Borel regular measures on a metric space X. Suppose that X is separable or that ν is finite on closed balls of X, and that ν has SVP.*

i) Let $\mu = \mu_s + \mu_a$ be the Lebesgue decomposition of μ with respect to ν, i.e. $\mu_s = \mu \llcorner B$ and $\mu_a = \mu \llcorner B^c$, where $B \in \mathscr{B}_X$ is given by lemma 3.36. Then, for all $A \in \mathscr{B}_X$,

$$\mu_a(A) = \int_A \Theta^{\nu}(\mu, x) \, \mathrm{d}\nu(x), \tag{3.9}$$

so that, for all $A \in \mathscr{B}_X$, $\mu(A) = \int_A \Theta^{\nu}(\mu, x) \, \mathrm{d}\nu(x) + \mu_s(A)$.

ii) If μ also has SVP, in lemma 3.36 we can take $B' = \{x \in X \mid \Theta^\nu(\mu, x) = \infty\}$ in place of B.

Proof. i) Note that the integral in (3.9) makes sense, since, by theorem 3.39.(ii), $\Theta^\nu(\mu, \cdot)$ is a positive Borel measurable function defined on the complement of a ν-null set.

Let $\lambda = \Theta^\nu(\mu, \cdot)\,\mathrm{d}\nu$ be the Borel regular measure on X defined by the second member in (3.9), i.e. the extension of the measure $A \in \mathscr{B}_X \mapsto \int_A \Theta^\nu(\mu, x)\,\mathrm{d}\nu(x)$ given by theorem 1.8. We must show that $\mu_a = \lambda$. Since both measures are Borel regular, it suffices to show that they coincide on Borel sets.

Let $A \in \mathscr{B}_X$, fix $t > 1$ and take Y_f given by 3.39.(ii), so that $\nu(B \cup Y_f^c) = 0$. We contend that both λ and μ_a are concentrated on $S := \{x \in B^c \cap Y_f \mid \Theta^\nu(\mu, x) > 0\} \in \mathscr{B}_X$. Indeed, since $S^c = B \cup Y_f^c \cup \{x \in B^c \cap Y_f \mid \Theta^\nu(\mu, x) = 0\}$, it is clear that $\lambda(S^c) = 0$. On the other hand applying theorem 3.38.(ii) with $\{x \in B^c \cap Y_f \mid \Theta^\nu(\mu, x) = 0\}$ in place of A and $t = 0$, it follows that $\mu_a\big(\{x \in B^c \cap Y_f \mid \Theta^\nu(\mu, x) = 0\}\big) = \mu\big(\{x \in B^c \cap Y_f \mid \Theta^\nu(\mu, x) = 0\}\big) = 0$ and, as $\mu_a(B \cup Y_f^c) = 0$ (because $\mu_a \ll \nu$), we conclude that $\mu_a(S^c) = 0$.

Define, $\forall k \in \mathbb{Z}$, $A_k := \{x \in A \cap S \mid t^k \le \Theta^\nu(\mu, x) < t^{k+1}\} \in \mathscr{B}_X$. Since $A \cap S = \dot\bigcup_{k\in\mathbb{Z}} A_k$, we have:

$$\begin{aligned} \mu_a(A) &= \mu_a(A \cap S) = \mu(A \cap S) = \sum_{k\in\mathbb{Z}} \mu(A_k), \\ \lambda(A) &= \lambda(A \cap S) = \sum_{k\in\mathbb{Z}} \lambda(A_k). \end{aligned} \tag{3.10}$$

On the other hand, for all $k \in \mathbb{Z}$:

$$\begin{aligned} & t^k \nu(A_k) \overset{3.26}{\le} \mu(A_k) \overset{3.38.(ii)}{\le} t^{k+1}\nu(A_k), \\ & t^k \nu(A_k) \le \lambda(A_k) \le t^{k+1}\nu(A_k). \end{aligned} \tag{3.11}$$

From (3.10) and (3.11) we then conclude that:

$$\mu_a(A) = \sum_{k\in\mathbb{Z}} \mu(A_k) \le t \sum_{k\in\mathbb{Z}} t^k \nu(A_k) \le t\lambda(A)$$
$$\lambda(A) = \sum_{k\in\mathbb{Z}} \lambda(A_k) \le t \sum_{k\in\mathbb{Z}} t^k \nu(A_k) \le t\mu_a(A).$$

Since $t > 1$ was arbitrarily taken, we can make $t \downarrow 1$ to conclude $\mu_a(A) \le \lambda(A)$ and $\lambda(A) \le \mu_a(A)$, hence $\mu_a(A) = \lambda(A)$, as asserted.

ii) Since $B' \subset Y_f^c$, it follows from theorem 3.39.(ii) that $\nu(B') = 0$, i.e. ν is concentrated on $(B')^c$. Therefore, it is enough to prove that $\mu \llcorner (B')^c \ll \nu$. Indeed, let $A \subset X$ such that $\nu(A) = 0$. We must show that $\mu \llcorner (B')^c(A) = \mu\big((B')^c \cap A\big) = 0$. Since $(B')^c = \{x \in X \mid \Theta^\nu_*(\mu, x) < \infty\} = \cup_{n\in\mathbb{N}}\{x \in X \mid \Theta^\nu_*(\mu, x) \leq n\}$, it suffices to show that $\forall n \in \mathbb{N}$, $\mu\big(A \cap \{x \in X \mid \Theta^\nu_*(\mu, x) \leq n\}\big) = 0$. But, as μ has SVP, we may apply theorem 3.38.(i), which yields $\mu\big(A \cap \{x \in X \mid \Theta^\nu_*(\mu, x) \leq n\}\big) \leq n\nu\big(A \cap \{x \in X \mid \Theta^\nu_*(\mu, x) \leq n\}\big) \leq n\nu(A) = 0$, whence the thesis.

□

Corollary 3.41. *With the same hypothesis from theorem 3.40, $\Theta^\nu(\mu, \cdot)$ coincides ν-a.e. with the Radon-Nikodym derivative* $\frac{d(\mu_a|_{\mathscr{B}_X})}{d(\nu|_{\mathscr{B}_X})}$.

Proof. It is a direct consequence of (3.9) and the uniqueness of the Radon-Nikodym derivative stated in theorem 1.103. □

CHAPTER 4

$\mathbb{R}^n$-valued Radon Measures

4.1. Linear functionals on spaces of continuous functions

In this section we fix a locally compact Hausdorff space X, which will be assumed σ-compact, unless otherwise specified. We aim to study the representation of continuous linear functionals on certain spaces of continuous functions on X by means of vector valued Radon measures.

NOTATION. We denote by

- $\mathsf{C_c}(X, \mathbb{R}^n)$ the space of continuous functions $f : X \to \mathbb{R}^n$ with $\operatorname{spt} f$ compact;
- $\mathsf{C_0}(X, \mathbb{R}^n)$ the space of continuous functions $f : X \to \mathbb{R}^n$ which vanish at infinity, i.e. such that $\forall \epsilon > 0$, $\exists K \subset X$ compact such that $\|f\| < \epsilon$ on $X \setminus K$.
- $\mathsf{C_b}(X, \mathbb{R}^n)$ the space of bounded continuous functions $f : X \to \mathbb{R}^n$.

Endowed with the norm of uniform convergence, i.e. $\|f\|_u := \sup\{\|f(x)\| \mid x \in X\}$, $\mathsf{C_b}(X, \mathbb{R}^n)$ is a Banach space. As it can be readily verified by means of Urysohn's lemma for locally compact Hausdorff spaces (see lemma 4.5, below), $\mathsf{C_0}(X, \mathbb{R}^n)$ is the closure of $\mathsf{C_c}(X, \mathbb{R}^n)$ in $\mathsf{C_b}(X, \mathbb{R}^n)$; in particular, $\mathsf{C_0}(X, \mathbb{R}^n)$ is itself a Banach space with the norm of uniform convergence.

DEFINITION 4.1 ($\mathbb{R}^n$-valued Radon measures). We say that a linear functional $\mu : \mathsf{C_c}(X, \mathbb{R}^n) \to \mathbb{R}$ is an *$\mathbb{R}^n$-valued Radon measure* on X if, for each compact $K \subset X$, the restriction of μ to $\mathsf{C_c^K}(X, \mathbb{R}^n) := \{f \in \mathsf{C_c}(X, \mathbb{R}^n) \mid \operatorname{spt} f \subset K\}$, endowed with $\|\cdot\|_u$, is linear continuous; that is, if $\exists C_K \geq 0$ such that

$$\text{(LF cont)} \qquad \sup\{\mu \cdot f \mid f \in \mathsf{C_c^K}(X, \mathbb{R}^n), \|f\|_u \leq 1\} \leq C_K.$$

If the condition above holds with a constant $C \geq 0$ which does not depend on K, i.e. if μ is linear continuous on $\mathsf{C_c}(X, \mathbb{R}^n)$ endowed with $\|\cdot\|_u$, we call μ a *finite $\mathbb{R}^n$-valued Radon measure* on X.

REMARK 4.2.

1) We will identify $\mathbb{R}^n$-valued Radon measures on X with set functions on X, as the name "measure" indicates, after we prove Riesz representation theorem for Radon measures 4.9.
2) The definition adopted for an $\mathbb{R}^n$-valued Radon measure on X is equivalent to saying that $\mu : \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n) \to \mathbb{R}$ is linear continuous with respect to the natural topological vector space topology on $\mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$, which is an inductive limit of Fréchet spaces (an *LF space* for short). It is actually the *countable strict inductive limit* (thanks to the σ-compactness of X) of the Banach spaces $\left\{\left(\mathsf{C}_\mathsf{c}^\mathsf{K}(X, \mathbb{R}^n), \|\cdot\|_u\right) \mid K \subset X \text{ compact}\right\}$; its topology is the strongest locally convex topology on $\mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$ which makes all inclusions $\mathsf{C}_\mathsf{c}^\mathsf{K}(X, \mathbb{R}^n) \to \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$ continuous, for $K \subset X$ compact. With such a topology, given a locally convex space Y, a linear map $\mathsf{C}_\mathsf{c}(X, \mathbb{R}^n) \to Y$ is continuous iff $\forall K \subset X$ compact, its restriction to $\mathsf{C}_\mathsf{c}^\mathsf{K}(X, \mathbb{R}^n)$ is continuous (as we defined in the case $Y = \mathbb{R}$). We don't suppose the reader to have any prior knowledge on locally convex spaces, but if he or she wants to delve into some of the details which may be left behind the scenes, we suggest: [**Con90**], chapter IV, for a brief overview of locally convex spaces; [**Osb14**], for a gentle introduction to locally convex spaces; [**K̈69**], [**SW99**], [**Tre06**] or [**Bou87**] for the heavy stuff.
3) For those fluent in locally convex spaces: the LF topology of $\mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$ introduced in the previous item coincides with the product topology of the LF spaces $\mathsf{C}_\mathsf{c}(X, \mathbb{R})$, i.e. we may identify $\mathsf{C}_\mathsf{c}(X, \mathbb{R}^n) \equiv \mathsf{C}_\mathsf{c}(X, \mathbb{R})^n$ as topological vector spaces. Indeed, the continuity of $\mathsf{C}_\mathsf{c}(X, \mathbb{R}^n) \to \mathsf{C}_\mathsf{c}(X, \mathbb{R})^n$, $f \mapsto (f_1, \ldots, f_n)$, is clear; the continuity of its inverse can be verified using the facts that it maps bounded sets to bounded sets, $\mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$ is locally convex and $\mathsf{C}_\mathsf{c}(X, \mathbb{R})^n$ is bornological.

   If X is an open set in some Euclidean space, $\mathsf{C}_\mathsf{c}^\infty(X, \mathbb{R})^n$ with its LF topology (i.e. the topology induced by the family of Fréchet spaces $\{f \in \mathsf{C}_\mathsf{c}^\infty(X, \mathbb{R})^n \mid \operatorname{spt} f \subset K\}$, for each $K \subset X$ compact) has a continuous dense inclusion in $\mathsf{C}_\mathsf{c}(X, \mathbb{R}^n) \equiv \mathsf{C}_\mathsf{c}(X, \mathbb{R})^n$. That means that the dual of $\mathsf{C}_\mathsf{c}(X, \mathbb{R})^n$ may be identified with a linear subspace of the dual of $\mathsf{C}_\mathsf{c}^\infty(X, \mathbb{R})^n$, i.e. every $\mathbb{R}^n$-valued Radon measure on X is an $\mathbb{R}^n$-valued Schwartz distribution on X.

Exercise 4.3 ($\mathbb{R}^n$-valued Radon measures on open sets of Euclidean spaces).

a) Let X be a locally compact separable metric space and $(U_k)_{k \in \mathbb{N}}$ be an increasing sequence of relatively compact open subsets of X such that $\cup_{k \in \mathbb{N}} U_k = X$. Then a linear map $\mu : \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n) \to \mathbb{R}$ is continuous, i.e. it is an $\mathbb{R}^n$-valued Radon measure on X, iff $\forall k \in \mathbb{N}$, $\mu|_{\left(\mathsf{C_c}(U_k,\mathbb{R}^n),\|\cdot\|_u\right)}$ is continuous (we identify $\mathsf{C_c}(U_k,\mathbb{R}^n)$ with the linear subspace of $\mathsf{C_c}(X,\mathbb{R}^n)$ formed by the functions with support in U_k).

b) Let U be an open subset of $\mathbb{R}^m$. Then $\mathsf{C_c^\infty}(U,\mathbb{R}^n)$ is sequentially dense in $\left(\mathsf{C_c}(U,\mathbb{R}^n),\|\cdot\|_u\right)$.

Hint. Use the standard mollifier 1.112, proposition 1.108 and theorem 1.111.

c) Let X be an open subset of $\mathbb{R}^m$ and $(U_k)_{k\in\mathbb{N}}$ be an increasing sequence of relatively compact open subsets of X such that $\cup_{k\in\mathbb{N}}U_k = X$. Let $\mu : \mathsf{C_c^\infty}(X,\mathbb{R}^n) \to \mathbb{R}$ be a linear map such that $\forall k \in \mathbb{N}$, $\mu|_{\left(\mathsf{C_c^\infty}(U_k,\mathbb{R}^n),\|\cdot\|_u\right)}$ is continuous. Then μ may be uniquely extended to a continuous linear map $\mathsf{C_c}(X,\mathbb{R}^n) \to \mathbb{R}$.

Hint. Use the two previous items.

Remark 4.4 ($\mathbb{R}^n$-valued Radon measures on open sets of Euclidean spaces). In view of part c) of the previous exercise, we may identify $\mathbb{R}^n$-valued Radon measures on open subsets X of Euclidean spaces with linear functionals $\mu : \mathsf{C_c^\infty}(X,\mathbb{R}^n) \to \mathbb{R}$ such that, for each compact subset $K \subset X$, the restriction of μ to $\{f \in \mathsf{C_c^\infty}(X,\mathbb{R})^n \mid \operatorname{spt} f \subset K\}$ is continuous with respect to the topology of uniform convergence (i.e. given by the norm $\|\cdot\|_u$).

We recall more preliminaries from *Real Analysis* in order to prove the version of Riesz representation theorem for $\mathbb{R}^n$-valued Radon measures 4.9 below.

Notation. Let X be a locally compact Hausdorff space, $U \subset X$ open and f a function on X. The notation $f \prec U$ means that $0 \leq f \leq 1$, $f \in \mathsf{C_c}(X,\mathbb{R})$ and $\operatorname{spt} f \subset U$.

Lemma 4.5 (Urysohn's lemma for LCH). *If X is a locally compact Hausdorff space, $U \subset X$ open and $K \subset U$ compact, then there exists $f \in \mathsf{C_c}(X,\mathbb{R})$ such that $\chi_K \leq f \prec U$.*

Theorem 4.6 (Tietze's extension theorem for LCH). *If X is a locally compact space, $K \subset X$ compact and $f : K \to \mathbb{R}$ continuous, then f admits a continuous extension $\widetilde{f} : X \to \mathbb{R}$. Moreover, we may take $\widetilde{f}$ with compact support and, if f is bounded, we may also take $\widetilde{f}$ such that $\|\widetilde{f}\|_u = \|f\|_u$.*

Theorem 4.7 (Riesz representation theorem for positive linear functionals). *Let X be a locally compact Hausdorff space and $L : \mathsf{C_c}(X,\mathbb{R}) \to$*

$\mathbb{R}$ *a positive linear functional, i.e.* L *is linear and* $L \cdot f \geq 0$ *whenever* $f \geq 0$. *Then there exists a unique Radon measure* η *on* X *which represents* L, *i.e.* $\forall f \in \mathsf{C_c}(X, \mathbb{R})$, $L \cdot f = \int f \, \mathrm{d}\eta$. *Moreover, on open sets* η *is given by*

$$\eta(U) = \sup\{L \cdot f \mid f \prec U\}.$$

For the proof of 4.5, 4.6 (which are direct consequences of the corresponding versions of those theorems for normal spaces) and 4.7 we refer the reader, for instance, to [**Fol99**] or [**Rud87**].

REMARK 4.8. Every positive linear functional on $\mathsf{C_c}(X, \mathbb{R})$ is an $\mathbb{R}$-valued Radon measure on X, i.e. positivity implies continuity on $\mathsf{C_c}(X, \mathbb{R})$. Indeed, given $K \subset X$ compact, take $\Phi \in \mathsf{C_c}(X, \mathbb{R})$ given by lemma 4.5 such that $\chi_K \leq \Phi \prec X$. For all $f \in \mathsf{C_c^K}(X, \mathbb{R})$ with $f \neq 0$, we have $\frac{|f|}{\|f\|_u} \leq \Phi$, so that $\Phi \pm \frac{f}{\|f\|_u} \geq 0$ and $\Phi \pm \frac{f}{\|f\|_u} \in \mathsf{C_c}(X, \mathbb{R})$. Hence $0 \leq L\big(\Phi \pm \frac{f}{\|f\|_u}\big) = L(\Phi) \pm \frac{L(f)}{\|f\|_u}$, which implies $|L(f)| \leq L(\Phi)\|f\|_u$. The continuity condition (LF cont) is then satisfied with $C_K := L(\Phi)$.

THEOREM 4.9 (Riesz representation theorem for Radon measures). *Let* X *be a* σ*-compact locally compact Hausdorff space and* $\mu : \mathsf{C_c}(X, \mathbb{R}^n) \to \mathbb{R}$ *an* $\mathbb{R}^n$*-valued Radon measure on* X. *Then there exists a unique Radon measure* λ *on* X *and a Borel measurable map* $\nu : X \to \mathbb{R}^n$ *unique up to* λ*-null sets such that* $\|\nu\| = 1$ λ*-a.e. on* X *and* $\forall f \in \mathsf{C_c}(X, \mathbb{R}^n)$,

$$\mu \cdot f = \int \langle f, \nu \rangle \, \mathrm{d}\lambda, \tag{4.1}$$

where $\langle \cdot, \cdot \rangle$ *denotes the Euclidean inner product in* $\mathbb{R}^n$. *Moreover,*

i) $\forall U \subset X$ *open,*

$$\lambda(U) = \sup\{\mu \cdot f \mid f \in \mathsf{C_c}(X, \mathbb{R}^n), \|f\| \prec U\}. \tag{4.2}$$

ii) μ *is a finite* $\mathbb{R}^n$*-valued Radon measure* iff λ *is a finite Radon measure; if that is the case,* $\|\mu\|_{\mathsf{C_0}(X,\mathbb{R}^n)^*} = \lambda(X)$.

REMARK 4.10. Note that, in (4.2), $\sup\{\mu \cdot f \mid f \in \mathsf{C_c}(X, \mathbb{R}^n), \|f\| \prec U\} = \sup\{|\mu \cdot f| \mid f \in \mathsf{C_c}(X, \mathbb{R}^n), \|f\| \prec U\}$. Indeed, if $f \in \mathsf{C_c}(X, \mathbb{R}^n)$ and $\|f\| \prec U$, so does $-f$, and $\mu \cdot (-f) = -\mu \cdot f$, hence either $\mu \cdot f$ or $\mu \cdot (-f)$ coincides with $|\mu \cdot f|$.

PROOF.

1) (Existence) Let $\mathsf{C_c^+}(X) := \{f \in \mathsf{C_c}(X, \mathbb{R}) \mid f \geq 0\}$. Define $L : \mathsf{C_c^+}(X) \to [0, \infty)$ by $f \mapsto \sup\{\mu \cdot \phi \mid \phi \in \mathsf{C_c}(X, \mathbb{R}^n), \|\phi\| \leq f\}$. Note that L is well-defined, i.e. the sup is indeed ≥ 0 (since $\mu \cdot 0 = 0$) and finite, due to the continuity condition (LF cont): if $f \neq 0$, $\forall \phi \in \mathsf{C_c}(X, \mathbb{R}^n)$ with $\|\phi\| \leq f$, we have $\psi := \frac{\phi}{\|f\|_u} \in \mathsf{C_c^{spt\, f}}(X, \mathbb{R}^n)$

and $\|\psi\|_u = \frac{\|\phi\|_u}{\|f\|_u} \le 1$, hence $\mu \cdot \psi \le C_{\mathrm{spt}\, f}$, i.e. $\mu \cdot \phi \le C_{\mathrm{spt}\, f} \|f\|_u$, showing that $L(f) \le C_{\mathrm{spt}\, f} \|f\|_u$.

We contend that L is additive and 1-homogeneous, i.e. $\forall f, g \in \mathsf{C}^+_{\mathsf{c}}(X)$, $\forall c \ge 0$, $L(f+g) = L(f) + L(g)$ and $L(cf) = cL(f)$. The 1-homogeneity is clear, since, for $c = 0$ the equality is trivial and for $c > 0$ and $f \in \mathsf{C}^+_{\mathsf{c}}(X)$, we have $\|\phi\| \le cf$ iff $\|c^{-1}\phi\| \le f$, so that $\{\mu \cdot \phi \mid \phi \in \mathsf{C}_{\mathsf{c}}(X, \mathbb{R}^n) \mid \|\phi\| \le cf\} = c\{\mu \cdot \phi \mid \phi \in \mathsf{C}_{\mathsf{c}}(X, \mathbb{R}^n) \mid \|\phi\| \le f\}$. To prove the additivity, let $f, g \in \mathsf{C}^+_{\mathsf{c}}(X)$. If $\phi, \psi \in \mathsf{C}_{\mathsf{c}}(X, \mathbb{R}^n)$ satisfy $\|\phi\| \le f$ and $\|\psi\| \le g$, then $\phi + \psi \in \mathsf{C}_{\mathsf{c}}(X, \mathbb{R}^n)$ and $\|\phi + \psi\| \le f + g$, so that $\mu \cdot \phi + \mu \cdot \psi = \mu \cdot (\phi + \psi) \le L(f+g)$, and taking the sup over all such ϕ and ψ we conclude that $L(f) + L(g) \le L(f+g)$. It remains to prove the reverse inequality. Given $\phi \in \mathsf{C}_{\mathsf{c}}(X, \mathbb{R}^n)$ such that $\|\phi\| \le f + g$, define:

$$\phi_1 := \begin{cases} \frac{f}{f+g}\phi & \text{if} f+g > 0, \\ 0 & \text{if} f+g = 0, \end{cases} \quad \text{and } \phi_2 := \begin{cases} \frac{g}{f+g}\phi & \text{if} f+g > 0, \\ 0 & \text{if} f+g = 0. \end{cases}$$

Note that, for $i = 1, 2$, ϕ_i is clearly continuous on the open set $\{f + g > 0\}$; besides if $x_0 \in X$ is such that $(f+g)(x_0) = 0$ and $\epsilon > 0$ is given, there exists an open neighborhood V of x_0 on which $f + g < \epsilon$, hence $\|\phi\| < \epsilon$ on V, whence $\|\phi_i\| < \epsilon$ on V, thus proving the continuity of ϕ_i at x_0. Hence ϕ_i is continuous and $\{\|\phi_i\| > 0\} \subset \{f + g > 0\}$; taking closures we conclude that $\mathrm{spt}\, \phi_i \subset \mathrm{spt}\,(f+g) \Subset X$. Then $\phi_1, \phi_2 \in \mathsf{C}_{\mathsf{c}}(X, \mathbb{R}^n)$, $\|\phi_1\| \le f$, $\|\phi_2\| \le g$ and $\phi_1 + \phi_2 = \phi$, so that $\mu \cdot \phi = \mu \cdot \phi_1 + \mu \cdot \phi_2 \le L(f) + L(g)$. Taking the sup over all such ϕ, we conclude that $L(f+g) \le L(f) + L(g)$ and our contention is proved.

We now extend L to a positive linear functional on $\mathsf{C}_{\mathsf{c}}(X, \mathbb{R})$. For $f \in \mathsf{C}_{\mathsf{c}}(X, \mathbb{R})$, we may write $f = f^+ - f^-$, where $f^+ = \max\{f, 0\} \in \mathsf{C}^+(X)$ and $f^- = \max\{-f, 0\} \in \mathsf{C}^+(X)$; define $L \cdot f := L(f^+) - L(f^-) \in \mathbb{R}$ (which coincides with the original definition if $f = f^+ \in \mathsf{C}^+_{\mathsf{c}}(X)$). If $c \in \mathbb{R}$ and $f \in \mathsf{C}_{\mathsf{c}}(X, \mathbb{R})$, the fact that $L \cdot (cf) = cL \cdot f$ follows from the definition of L and from the equalities $(cf)^+ = cf^+$ and $(cf)^- = cf^-$ if $c \ge 0$; $(cf)^+ = -cf^-$ and $(cf)^- = -cf^+$ if $c < 0$. On the other hand, if $f, g \in \mathsf{C}_{\mathsf{c}}(X, \mathbb{R})$ and $h = f + g$, then $h^+ + f^- + g^- = h^- + f^+ + g^+$; applying L to both members and using the additivity of L on $\mathsf{C}^+_{\mathsf{c}}(X)$, it follows that $L \cdot h = L \cdot f + L \cdot g$, thus proving the linearity of L. If $f \in \mathsf{C}_{\mathsf{c}}(X, \mathbb{R})$ and $f \ge 0$, then $f = f^+ \in \mathsf{C}^+_{\mathsf{c}}(X)$, so that $L \cdot f = L(f^+) \ge 0$; therefore L is a positive linear functional on $\mathsf{C}_{\mathsf{c}}(X, \mathbb{R})$.

We may then apply theorem 4.7 to L, which ensures the existence of a unique Radon measure η on X which represents L. Thus, for every $f \in \mathsf{C}_\mathsf{c}^+(X)$, we have

$$L \cdot f = \sup\{\mu \cdot \phi \mid \phi \in \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n), \|\phi\| \leq f\} = \int f \, \mathrm{d}\eta. \tag{4.3}$$

For $1 \leq i \leq n$, define $\mu_i : \mathsf{C}_\mathsf{c}(X, \mathbb{R}) \to \mathbb{R}$ by $\mu_i \cdot f := \mu \cdot (f e_i)$. Since $\|\pm f e_i\| = |f| \in \mathsf{C}_\mathsf{c}^+(X)$, it follows from (4.3) that, for all $f \in \mathsf{C}_\mathsf{c}(X, \mathbb{R})$, $\mu_i \cdot (\pm f) \leq \sup\{\mu \cdot \phi \mid \phi \in \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n), \|\phi\| \leq |f|\} = \int |f| \, \mathrm{d}\eta$, so that $|\mu_i \cdot f| \leq \int |f| \, \mathrm{d}\eta$. Thus, since $\mathsf{C}_\mathsf{c}(X, \mathbb{R})$ is dense on $\mathsf{L}^1(\eta)$ (by proposition 1.78; we consider L^p spaces of real valued functions), μ_i extends to a bounded linear function on $\mathsf{L}^1(\eta)$, still denoted by μ_i. As η is σ-finite (because η is a Radon measure on X and X is σ-compact), we may apply Riesz representation theorem 1.79 for the dual of L^1 to conclude that there exists $g_i \in \mathsf{L}^\infty(\eta)$ which represents μ_i, i.e. such that $\forall f \in \mathsf{L}^1(\eta)$ (in particular, $\forall f \in \mathsf{C}_\mathsf{c}(X, \mathbb{R})$), $\mu_i \cdot f = \int f g_i \, \mathrm{d}\eta$. It follows from corollary 1.118 that g_i coincides η-a.e. with a Borelian function; since this Borelian function is essentially bounded, it may be modified in η-null Borel set, yielding a bounded Borelian function in the same L^∞ equivalence class. We may therefore assume that g_i is a bounded Borelian function.

For all $f = \sum_{i=1}^n f_i e_i \in \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$,

$$\begin{aligned} \mu \cdot f &= \sum_{i=1}^n \mu \cdot (f_i e_i) = \sum_{i=1}^n \mu_i \cdot f_i = \\ &= \sum_{i=1}^n \int f_i g_i \, \mathrm{d}\eta = \int \langle f, g \rangle \, \mathrm{d}\eta, \end{aligned}$$

where $g = (g_1, \ldots, g_n) : X \to \mathbb{R}^n$ is a bounded Borelian map. To complete the proof of the existence part of the theorem, we now take $\nu := \frac{g}{\|g\|}$ in the Borel set where $g \neq 0$ and 0 on its complement, and $\lambda := \|g\|\eta$, i.e. the extension given by theorem 1.8 of the measure on $\mathscr{B}_X$ defined by $A \in \mathscr{B}_X \mapsto \int_A \|g\| \, \mathrm{d}\eta$. Then $\nu : X \to \mathbb{R}^n$ is Borelian, by proposition 1.50 with $A_1 := \{g \neq 0\}$ and $A_2 := A_1^c$, and $\|\nu\| = 1$ λ-a.e., since $\lambda(A_2) = 0$. The fact that the measure $A \in \mathscr{B}_X \mapsto \int_A \|g\| \, \mathrm{d}\eta$ is a Radon measure on $\mathscr{B}_X$ is a consequence of lemma 4.11, below, with $\|g\|$ in place of f and η in place of μ. It then follows from remark 1.29.(ii) that λ is a Radon measure on X. Since, for all $f \in \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$, $\mu \cdot f = \int \langle f, g \rangle \, \mathrm{d}\eta = \int \langle f, \nu \rangle \, \mathrm{d}\lambda$, i.e. (4.1) holds, so the existence part is proved.

2) (Uniqueness and proof of (4.2)) Suppose that (4.1) holds with a Radon measure λ and a Borel measurable map $\nu : X \to \mathbb{R}^n$ with $\|\nu\| = 1$ λ-almost everywhere. Modifying ν on a λ-null Borel set, if necessary, we may assume that equality holds everywhere, i.e.

$\forall x \in X$, $\|\nu(x)\| = 1$. Given $U \subset X$ open, denote by $|\mu|(U)$ the second member of (4.2), i.e.

$$|\mu|(U) := \sup\{\mu \cdot f \mid f \in \mathsf{C_c}(X, \mathbb{R}^n), \|f\| \prec U\}.$$

For all $f \in \mathsf{C_c}(X, \mathbb{R}^n)$ such that $\|f\| \prec U$, we have $\mu \cdot f = \int \langle f, \nu \rangle \, \mathrm{d}\lambda \overset{\operatorname{spt} f \subset U}{=} \int_U \langle f, \nu \rangle \, \mathrm{d}\lambda \leq \int_U \|f\| \, \mathrm{d}\lambda \leq \lambda(U)$; hence, taking the sup in the first member, we conclude that

$$|\mu|(U) \leq \lambda(U). \tag{4.4}$$

We now prove the reverse inequality (hence the equality) in (4.4), which then implies (4.2). Firstly, assume that $\lambda(U) < \infty$. Fix $\epsilon > 0$. We may apply Lusin's theorem 1.117 to obtain a compact set $K \subset U$ such that $\lambda(U \setminus K) < \epsilon$ and $\nu|_K$ continuous. Then we may apply Tietze's extension theorem 4.6 to each component of $\nu|_K$, yielding $f : X \to \mathbb{R}^n$ continuous such that $f|_K = \nu|_K$. Multiplying f by a convenient cut function, we may assume $\operatorname{spt} f \subset U$ and $\|f\|_u \leq 1 + \epsilon$. Indeed, since $\|\nu\| \equiv 1$, by continuity of f we may take an open neighborhood $V \subset U$ of K such that $\|f|_V\|_u \leq 1 + \epsilon$, and then we take $\phi \in \mathsf{C_c}(X, \mathbb{R})$ given by Urysohn's lemma 4.5 such that $\chi_K \leq \phi \prec V$, so that $\phi f \in \mathsf{C_c}(X, \mathbb{R}^n)$, $\operatorname{spt} \phi f \subset V \subset U$ and $\|\phi f\|_u \leq 1 + \epsilon$; we then substitute $f\phi$ for f. It therefore follows that:

i) $\int \langle f, \nu \rangle \, \mathrm{d}\lambda = \int_U \langle f, \nu \rangle \, \mathrm{d}\lambda = \int_{U \setminus K} \langle f, \nu \rangle \, \mathrm{d}\lambda + \int_K \langle f, \nu \rangle \, \mathrm{d}\lambda$. Since $\int_K \langle f, \nu \rangle \, \mathrm{d}\lambda = \int_K \langle \nu, \nu \rangle \, \mathrm{d}\lambda = \lambda(K) > \lambda(U) - \epsilon$ and $|\int_{U \setminus K} \langle f, \nu \rangle \, \mathrm{d}\lambda| \leq \int_{U \setminus K} \|f\| \, \mathrm{d}\lambda \leq \|f\|_u \, \lambda(U \setminus K) \leq (1+\epsilon)\epsilon$, it follows that $\int \langle f, \nu \rangle \, \mathrm{d}\lambda \geq -(1 + \epsilon)\epsilon + \lambda(U) - \epsilon = \lambda(U) - \epsilon(2 + \epsilon)$.

ii) On the other hand, $\int \langle f, \nu \rangle \, \mathrm{d}\lambda = \mu \cdot f = \|f\|_u \, \mu \cdot \frac{f}{\|f\|_u} \leq (1 + \epsilon)|\mu|(U)$, since $\|f\|_u \leq 1 + \epsilon$ and $\frac{f}{\|f\|_u}$ is one of the competitors in the definition of $|\mu|(U)$, i.e. $\frac{f}{\|f\|_u} \in \mathsf{C_c}(X, \mathbb{R}^n)$ and $\frac{\|f\|}{\|f\|_u} \prec U$.

From (i) and (ii) above we conclude that $\lambda(U) \leq \epsilon(2 + \epsilon) + (1 + \epsilon)|\mu|(U)$. Since $\epsilon > 0$ was arbitrarily taken, we may send $\epsilon \to 0$ to conclude that $\lambda(U) \leq |\mu|(U)$, thus proving the reverse inequality (hence the equality) in (4.4) if $\lambda(U) < \infty$.

In the general case, for an arbitrary open set $U \subset X$, due to σ-compactness of X, we may take an increasing sequence $(U_n)_{n \in \mathbb{N}}$ of open subsets of X such that $\forall n \in \mathbb{N}$, $U_n \Subset U$ and $\cup_{n \in \mathbb{N}} U_n = U$ (to obtain such a sequence, take a sequence $(K_n)_{n \in \mathbb{N}}$ of compact sets which increases to U, then for each $n \in \mathbb{N}$ take an open set V_n such that $K_n \subset V_n \Subset U$ and define $U_n := \cup_{i=1}^n V_i$). Since, for each $n \in \mathbb{N}$, $\lambda(U_n) < \infty$ (because $\overline{U_n}$ is compact and λ is Radon, hence finite on compact sets), we may apply the case already proved to conclude that $\lambda(U_n) = |\mu|(U_n)$. Applying the continuity from below 1.11 to λ, it then follows that $\lambda(U) = \sup_{n\in\mathbb{N}}|\mu|(U_n) = \sup_{n\in\mathbb{N}}\sup\{\mu \cdot f \mid f \in \mathsf{C_c}(X, \mathbb{R}^n), \|f\| \prec U_n\}$. We contend that the second member in the latter equality is $|\mu|(U) = \sup\{\mu \cdot f \mid f \in \mathsf{C_c}(X, \mathbb{R}^n), \|f\| \prec U\}$, which yields the asserted equality in (4.4). Indeed, for each $n \in \mathbb{N}$, $|\mu|(U_n) \leq |\mu|(U)$, thus $\sup_{n\in\mathbb{N}}|\mu|(U_n) \leq |\mu|(U)$. On the other hand, let f be one of the competitors in the definition of $|\mu|(U)$, i.e. $f \in \mathsf{C_c}(X, \mathbb{R}^n)$ and $\|f\| \prec U$. Since spt $f \subset U$ is compact, it can be covered by finitely many of the U_n's; thus, since $(U_n)_{n\in\mathbb{N}}$ is increasing, there exists $n \in \mathbb{N}$ such that spt $f \subset U_n$. It therefore follows that $\|f\| \prec U_n$, i.e. f is one of the competitors in the definition of $|\mu|(U_n)$, hence $\mu \cdot f \leq |\mu|(U_n) \leq \sup_{n\in\mathbb{N}}|\mu|(U_n)$. Taking the sup of $\mu \cdot f$ over all such f, we conclude that $|\mu|(U) \leq \sup_{n\in\mathbb{N}}|\mu|(U_n)$, hence the equality holds, thus proving our contention.

We have thus proved that (4.2) holds, so that λ is uniquely determined on open sets by $|\mu|$. Since Radon measures are uniquely determined by their values on open sets, we have proved the uniqueness of λ.

We now prove the uniqueness of ν. Suppose that $\nu' : X \to \mathbb{R}^n$ is another Borelian map such that $\|\nu'\| = 1$ λ-a.e. and (4.1) holds with ν' in place of ν. Modifying both ν and ν' on a λ-null Borel set, we may assume that $\|\nu\| \equiv 1$ and $\|\nu'\| \equiv 1$. Then, for all $f \in \mathsf{C_c}(X, \mathbb{R}^n)$, $\int\langle f, \nu-\nu'\rangle \,\mathrm{d}\lambda = 0$. Let $U \subset X$ open with $\lambda(U) < \infty$ and fix $\epsilon > 0$. Once again we apply Lusin's theorem 1.117 to obtain a compact set $K \subset U$ such that $\lambda(U \setminus K) < \epsilon$ and $(\nu - \nu')|_K$ continuous. Then we may apply Tietze's extension theorem 4.6 to each component of $(\nu - \nu')|_K$, yielding $f : X \to \mathbb{R}^n$ continuous such that $f|_K = (\nu - \nu')|_K$. Since $\|\nu - \nu'\|_u \leq 2$, as before, multiplying f by a convenient cut function if necessary, we may assume spt $f \subset U$ and $\|f\|_u \leq 2 + \epsilon$. Thus

$$
\begin{aligned}
0 = \int \langle f, \nu - \nu'\rangle \,\mathrm{d}\lambda &= \int_U \langle f, \nu - \nu'\rangle \,\mathrm{d}\lambda = \\
&= \int_{U\setminus K} \langle f, \nu - \nu'\rangle \,\mathrm{d}\lambda + \int_K \|\nu - \nu'\|^2 \,\mathrm{d}\lambda.
\end{aligned}
$$

Since $|\int_{U\setminus K}\langle f, \nu - \nu'\rangle \,\mathrm{d}\lambda| \leq \|f\|_u\|\nu - \nu'\|_u\lambda(U \setminus K) \leq (2 + \epsilon) \cdot 2\epsilon$ and $\int_K\|\nu - \nu'\|^2 \,\mathrm{d}\lambda \geq \int_U\|\nu - \nu'\|^2 \,\mathrm{d}\lambda - 4\epsilon$, it then follows that $\int_U\|\nu - \nu'\|^2 \,\mathrm{d}\lambda \leq 4\epsilon + (2 + \epsilon) \cdot 2\epsilon$. Hence, sending $\epsilon \to 0$ we conclude that $\int_U\|\nu - \nu'\|^2 \,\mathrm{d}\lambda = 0$, which implies $\nu = \nu'$ λ-a.e. on U. As X is σ-compact, we may cover X with countably many relatively compact open sets $(U_n)_{n\in\mathbb{N}}$, which are λ-finite, since λ is Radon (that is, X is open σ-finite). Therefore, as $\forall n \in \mathbb{N}$, $\nu = \nu'$ λ-a.e. on U_n, it follows that $\nu = \nu'$ λ-a.e. on $\cup_{n\in\mathbb{N}} U_n = X$, thus proving that ν is unique up to λ-null sets, as asserted.

It remains to prove assertion ii). Indeed, by (4.2), the norm of μ as a linear function on the normed space $\big(\mathsf{C_c}(X, \mathbb{R}^n), \|\cdot\|_u\big)$ is given by $\|\mu\| = \sup\{\mu \cdot f \mid f \in \mathsf{C_c}(X, \mathbb{R}^n), \|f\|_u \leq 1\} = \lambda(X)$, so that μ is a bounded linear functional iff $\lambda(X) < \infty$. If that is the case, as $\mathsf{C_c}(X, \mathbb{R}^n)$ is dense on $\big(\mathsf{C_0}(X, \mathbb{R}^n), \|\cdot\|_u\big)$, μ extends to a unique bounded linear functional on $\mathsf{C_0}(X, \mathbb{R}^n)$ with $\|\mu\|_{\mathsf{C_0}(X,\mathbb{R}^n)^*} = \lambda(X)$, thus proving ii).

$\square$

Lemma 4.11. *Let X be a locally compact Hausdorff space, $f : X \to [0, \infty)$ bounded Borelian and μ a σ-finite Radon measure on $\mathscr{B}_X$ (in the sense of remark 1.29.ii). Then $\lambda := f\mu : \mathscr{B}_X \to [0, \infty]$ given by $A \mapsto \int_A f \, \mathrm{d}\mu$ is a Radon measure on $\mathscr{B}_X$.*

Proof. It is clear that $\lambda = f\mu$ is a measure on $\mathscr{B}_X$ which is finite on compact subsets of X, since f is bounded and μ is Radon (hence finite on such subsets). We must show that λ is outer regular on Borel sets and inner regular on open subsets of X (actually it is inner regular on all Borel sets). That is a consequence of the σ-compactness of μ and of the fact that $\lambda \ll \mu$:

1) Let $B \in \mathscr{B}_X$. Note that, if μ is finite on B, so is $\lambda = f\mu$, since f is bounded.

   Assume that μ finite on B. For each $n \in \mathbb{N}$, since $\mu(B) = \inf\{\mu(U) \mid U \supset B \text{ open}\}$ and $\mu(B) < \infty$, we may take an open set $U_n \supset B$ such that $\mu(U_n \setminus B) < \frac{1}{n}$. Substituting U_n with $\cap_{i=1}^n U_i$, we may assume that the sequence $(U_n)_{n\in\mathbb{N}}$ thus defined is decreasing. Then $\cap_{n\in\mathbb{N}} U_n \supset B$ and $\mu\big((\cap_{n\in\mathbb{N}} U_n) \setminus B\big) = 0$; since $\lambda \ll \mu$, it then follows that λ is null on $(\cap_{n\in\mathbb{N}} U_n) \setminus B$, so that $\lambda(B) = \lambda(\cap_{n\in\mathbb{N}} U_n)$. On the other hand, since $\mu(U_1) < \infty$ (because μ is finite both on B and $U_1 \setminus B$), as noted above we also have $\lambda(U_1) < \infty$; applying the continuity from above 1.11 to λ, we then conclude that $\inf_{n\in\mathbb{N}} \lambda(U_n) = \lim \lambda(U_n) = \lambda(\cap_{n\in\mathbb{N}} U_n) = \lambda(B)$, thus proving the outer regularity of λ on B if $\mu(B) < \infty$.

   In the general case, using the fact that μ is σ-finite, we may write $B = \cup_{n\in\mathbb{N}} B_n$ as a countable union of Borel sets with finite μ-measure (hence with finite λ-measure). Given $\epsilon > 0$, for each $n \in \mathbb{N}$, we may choose, in view of the fact that $\lambda(B_n) < \infty$ and that λ is outer regular on B_n by the case proved above, an open set

$U_n \supset B_n$ such that $\lambda(U_n \setminus B_n) < 2^{-n}\epsilon$. Put $U := \cup_{n\in\mathbb{N}} U_n$, so that $U \supset B$ open. As $U \setminus B \subset \cup_{n\in\mathbb{N}}(U_n \setminus B_n)$, it follows by countable subadditivity that $\lambda(U \setminus B) < \epsilon$, thus proving the exterior regularity of λ on B.

2) Let $B \in \mathscr{B}_X$. We will show that λ is inner regular on B.

Assume that $\mu(B) < \infty$. Since μ is Radon, it follows from exercise 1.31 that μ is inner regular on B; as $\mu(B) < \infty$, for each $n \in \mathbb{N}$, there exists $K_n \subset B$ compact such that $\mu(B \setminus K_n) < \frac{1}{n}$. Substituting K_n with $\cup_{i=1}^n K_i$, we may assume that $(K_n)_{n\in\mathbb{N}}$ thus defined is increasing. Then $\cup_{n\in\mathbb{N}} K_n \subset B$ and $\mu(B \setminus \cup_{n\in\mathbb{N}} K_n) = 0$; as $\lambda \ll \mu$, it then follows $\lambda(B \setminus \cup_{n\in\mathbb{N}} K_n) = 0$. Thus, applying the continuity from below 1.11 to λ, we conclude that $\lambda(B) = \lambda(\cup_{n\in\mathbb{N}} K_n) = \lim \lambda(K_n) = \sup_{n\in\mathbb{N}} \lambda(K_n)$, which proves the interior regularity of λ on B if $\mu(B) < \infty$.

In the general case, by the fact that μ is σ-finite, we may take an increasing sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathscr{B}_X$ such that $\cup_{n\in\mathbb{N}} B_n = B$ and $\forall n \in \mathbb{N}$, $\mu(B_n) < \infty$ (thus $\lambda(B_n) < \infty$). By the case proved above, for each $n \in \mathbb{N}$, λ is inner regular on B_n; hence, since $\lambda(B_n) < \infty$, we may take a compact $K_n \subset B_n$ such that $\lambda(B_n \setminus K_n) < \frac{1}{n}$. Therefore, $\lim \lambda(K_n) = \lim \lambda(B_n) = \lambda(B)$, where in the last equality we have applied the continuity from below to λ, showing that λ is inner regular on B, as asserted.

□

In theorem 4.9, we may drop the σ-compactness hypothesis on X if μ is a finite $\mathbb{R}^n$-valued Radon measure. That is, we obtain the following version of the theorem:

THEOREM 4.12 (Riesz representation theorem for finite Radon measures). *Let X be a locally compact Hausdorff space and $\mu : \mathsf{C_c}(X, \mathbb{R}^n) \to \mathbb{R}$ a finite $\mathbb{R}^n$-valued Radon measure on X. Then there exists a unique finite Radon measure λ on X and a Borel measurable map $\nu : X \to \mathbb{R}^n$ unique up to λ-null sets such that $\|\nu\| = 1$ λ-a.e. on X and $\forall f \in \mathsf{C_c}(X, \mathbb{R}^n)$,*

$$\mu \cdot f = \int \langle f, \nu \rangle \, \mathrm{d}\lambda,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$. Moreover,

i) $\forall U \subset X$ *open,*

$$\lambda(U) = \sup\{\mu \cdot f \mid f \in \mathsf{C_c}(X, \mathbb{R}^n), \|f\| \prec U\}.$$

ii) $\|\mu\|_{\mathsf{C_0}(X,\mathbb{R}^n)^*} = \lambda(X)$.

The proof is the same, as:

i) In the existence part, we only used the σ-compactness condition to ensure that the restriction of λ to $\mathscr{B}_X$ is σ-finite (in order to be able to apply Riesz representation theorem 1.79 for the dual of $\mathsf{L}^1(\lambda)$ and lemma 4.11), but λ is finite in case of μ finite (since, if μ is finite, the positive linear functional L defined in the beginning of the proof is bounded, hence $\lambda(X) = \|L\| < \infty$ by the formula to compute the measure which represents the linear functional on open sets given in theorem 4.7).
ii) In the uniqueness part and in the proof of (4.2), we used the σ-compactness condition only in case $\lambda(U) = \infty$; but, as pointed in the previous item, λ is finite in case of μ finite, so that we don't need the σ-compactness either.

Definition 4.13 (total variation and polar decomposition). Let μ be an $\mathbb{R}^n$-valued Radon measure on a σ-compact locally compact Hausdorff space X. With the same notation of theorem 4.9, λ is called the *total variation of* μ, and the pair (ν, λ) is called the *polar decomposition of* μ. Henceforth, we will use the notation $|\mu| := \lambda$ to denote the total variation of μ, and

$$\mu = \nu|\mu|$$

with the meaning that $(\nu, |\mu|)$ is the polar decomposition of μ.

Example 4.14. Let X be a σ-compact locally compact Hausdorff space.

1) Let μ be a locally finite Borel measure on X. Then μ induces a positive linear functional $\hat{\mu}$ on $\mathsf{C}_\mathsf{c}(X, \mathbb{R})$ (which is necessarily continuous, by remark 4.8), given by $\hat{\mu} \cdot f := \int f \, \mathrm{d}\mu$. If μ is a Radon measure, then $\hat{\mu} = 1 \cdot \mu$ is the polar decomposition of $\hat{\mu}$ (by the uniqueness of the polar decomposition); in particular, μ is the total variation of $\hat{\mu}$.
2) Similarly, let ν be a signed measure on $\mathscr{B}_X$ whose total variation $|\nu|$ is locally finite. Then ν induces a continuous linear functional $\hat{\nu}$ on $\mathsf{C}_\mathsf{c}(X, \mathbb{R})$ given by $\hat{\nu} \cdot f := \int f \, \mathrm{d}\nu$. Indeed, it is clear that $\hat{\nu}$ is a well-defined linear functional on $\mathsf{C}_\mathsf{c}(X, \mathbb{R})$, and the continuity follows from the triangle inequality 1.99.e): $\forall K \subset X$ compact and $\forall f \in \mathsf{C}^\mathsf{K}_\mathsf{c}(X)$, $|\hat{\nu} \cdot f| \leq \int |f| \, \mathrm{d}|\nu| = \int_K |f| \, \mathrm{d}|\nu| \leq |\nu|(K)\|f\|_u$, hence $\hat{\nu}$ is bounded on $\mathsf{C}^\mathsf{K}_\mathsf{c}(X)$.

   We may take Borelian function $h : X \to \mathbb{R}$ such that $|h| \equiv 1$ and $\nu = h|\nu|$. Indeed, since $\nu^+ \perp \nu^-$, we may take disjoint Borel sets P and N such that $X = P \dot\cup N$, ν^+ is concentrated on P and ν^- is concentrated on N. Thus, $\nu^+ = \chi_P|\nu|$ and $\nu^- = \chi_N|\nu|$, so that $\nu = \nu^+ - \nu^- = (\chi_P - \chi_N)|\nu|$ and we take $h := \chi_P - \chi_N$. If $|\nu|$ is Radon, it follows from the uniqueness of the polar decomposition of $\hat{\nu}$ that its polar decomposition is $\hat{\nu} = h|\nu|$ (we identify $|\nu|$ with a Radon outer measure on X, cf. remark 1.29). In particular, $|\hat{\nu}| = |\nu|$. Besides, it follows from the uniqueness of the Jordan decomposition 1.94 that, as measures on $\mathscr{B}_X$, $\nu^+ = h^+|\nu|$ and $\nu^- = h^-|\nu|$. Since either ν^+ or ν^- if finite, we conclude that either $h^+ \in \mathsf{L}^1(|\nu|)$ or $h^- \in \mathsf{L}^1(|\nu|)$, i.e. h is $|\nu|$-integrable. Thus, in order for a continuous linear functional μ on $\mathsf{C_c}(X, \mathbb{R})$ to be induced by a signed measure on $\mathscr{B}_X$ whose total variation is Radon, it is necessary that μ have polar decomposition $\mu = h|\mu|$ with h $|\mu|$-integrable (which means that not every continuous linear functional on $\mathsf{C_c}(X, \mathbb{R})$ is obtained in this way if X is not compact).

3) Let $X = \mathbb{R}$ and I be the positive linear functional defined on $\mathsf{C_c}(X, \mathbb{R})$ by the Riemann integral, i.e. $I \cdot f := \int_a^b f(x)\, \mathrm{d}x$ for $a < b$ such that $\operatorname{spt} f \subset [a, b]$. The polar decomposition of I is $I = 1 \cdot \mathcal{L}^n$. In particular, that could have been taken as the definition of the Lebesgue measure, i.e. it is the total variation of the positive linear functional induced by the Riemann integral.

PROPOSITION 4.15 (properties of the total variation, part I). *Let μ and ν be $\mathbb{R}^n$-valued Radon measures on a σ-compact locally compact Hausdorff space X and $c \in \mathbb{R}$. Then:*

i) $|\mu + \nu| \le |\mu| + |\nu|$, *with equality if* $|\mu| \perp |\nu|$.
ii) $|c\mu| = |c||\mu|$.

That is, the total variation of $\mathbb{R}^n$-valued Radon measures has the same properties stated in 1.100.b) for the total variation of signed measures on a σ-algebra of subsets of X.

PROOF. Let $U \subset X$ open. It follows from (4.2) and remark 4.10 that:

1) $|\mu+\nu|(U) = \sup\{\mu \cdot f + \nu \cdot f \mid f \in \mathsf{C_c}(X, \mathbb{R}^n), \|f\| \le 1\} \le \sup\{\mu \cdot f \mid f \in \mathsf{C_c}(X, \mathbb{R}^n), \|f\| \le 1\} + \sup\{\nu \cdot f \mid f \in \mathsf{C_c}(X, \mathbb{R}^n), \|f\| \le 1\} = |\mu|(U) + |\nu|(U)$.
2) $|c\mu|(U) = \sup\{|c||\mu \cdot f| \mid f \in \mathsf{C_c}(X, \mathbb{R}^n), \|f\| \le 1\} = |c| \sup\{|\mu \cdot f| \mid f \in \mathsf{C_c}(X, \mathbb{R}^n), \|f\| \le 1\} = |c||\mu|(U)$.

For an arbitrary set $A \subset X$, we now use the outer regularity on A of the Radon measures $|\mu + \nu|$, $|\mu|$ and $|\nu|$. Note that, for arbitrary open sets U, V containing A, the open set $U \cap V$ contains A and $|\mu|(U) +$

$|\nu|(V) \geq |\mu|(U \cap V) + |\nu|(U \cap V)$, which justifies the equality $(*)$ below:

$$\begin{aligned}
|\mu+\nu|(A) &= \inf\{|\mu+\nu|(U) \mid U \supset A \text{ open}\} \overset{\text{by 1)}}{\leq} \\
&\leq \inf\{|\mu|(U) + |\nu|(U) \mid U \supset A \text{ open}\} \overset{*}{=} \\
&= \inf\{|\mu|(U) + |\nu|(V) \mid U, V \supset A \text{ open}\} = \\
&= \inf\{|\mu|(U) \mid U \supset A \text{ open}\} + \inf\{|\nu|(V) \mid V \supset A \text{ open}\} = \\
&= |\mu|(A) + |\nu|(A),
\end{aligned}$$

which proves the inequality in part i).

Similarly,

$$\begin{aligned}
|c\mu|(A) &= \inf\{|c\mu|(U) \mid U \supset A \text{ open}\} \overset{\text{by 2)}}{=} \\
&= |c| \inf\{|\mu|(U) \mid U \supset A \text{ open}\} = |c||\mu|(A),
\end{aligned}$$

thus proving part ii).

It remains to prove the equality in part i) if $|\mu| \perp |\nu|$. Indeed, in that case, there exist disjoint Borel sets $A, B \subset X$ such that $X = A \dot{\cup} B$, $|\mu|$ concentrated on A and $|\nu|$ concentrated on B. Let $(n_\mu, |\mu|)$ and $(n_\nu, |\nu|)$ be the polar decompositions of μ and ν, respectively. We have, for all $f \in \mathsf{C_c}(X, \mathbb{R}^n)$:

$$\begin{aligned}
(\mu+\nu) \cdot f = \mu \cdot f + \nu \cdot f &= \int \langle f, n_\mu \rangle \,\mathrm{d}|\mu| + \int \langle f, n_\nu \rangle \,\mathrm{d}|\nu| = \\
&= \int \langle f, \chi_A n_\mu + \chi_B n_\nu \rangle \,\mathrm{d}(|\mu| + |\nu|).
\end{aligned}$$

Since $\|\chi_A n_\mu + \chi_B n_\nu\| = 1$ $(|\mu| + |\nu|)$-a.e., it follows that the polar decomposition of $\mu+\nu$ is $(\chi_A n_\mu + \chi_B n_\nu, |\mu|+|\nu|)$; in particular, $|\mu+\nu| = |\mu| + |\nu|$, as asserted. □

Definition 4.16 (integration with respect to $\mathbb{R}^n$-valued Radon measures). Let μ be an $\mathbb{R}^n$-valued Radon measure on a σ-compact locally compact Hausdorff space X, with polar decomposition $\mu = \nu|\mu|$.

i) A vector Borelian map $f : X \to \mathbb{R}^n$ is called *summable with respect to* μ if it is summable with respect to $|\mu|$, i.e. if $f \in \mathsf{L}^1(|\mu|, \mathbb{R}^n) \equiv \mathsf{L}^1(|\mu|)^n$. For such f, we define

$$\int f \cdot \mathrm{d}\mu := \int \langle f, \nu \rangle \,\mathrm{d}|\mu| \in \mathbb{R}.$$

ii) An scalar Borelian map $f : X \to \mathbb{R}$ is called *summable with respect to* μ if it is summable with respect to $|\mu|$, i.e. if $f \in \mathsf{L}^1(|\mu|)$. For such f, we define

$$\int f \,\mathrm{d}\mu := \int f\nu \,\mathrm{d}|\mu| = \Big(\int f\nu_1 \,\mathrm{d}|\mu|, \dots, \int f\nu_n \,\mathrm{d}|\mu|\Big) \in \mathbb{R}^n.$$

REMARK 4.17.

1) Note that both integrals in the definition above make sense, since $|\langle f, \nu\rangle| \le \|f\| \in \mathsf{L}^1(|\mu|)$ if f vector-valued and $\|f\nu\| = |f| \in \mathsf{L}^1(|\mu|)$ if f scalar-valued.
2) Since $|\mu|$ is a Radon measure, we have $\mathsf{C}_\mathsf{c}(X, \mathbb{R}^n) \subset \mathsf{L}^1(|\mu|, \mathbb{R}^n)$; the inclusion is actually dense, in view of proposition 1.78. It is clear that the integral defined in i) extends $\mu : \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n) \to \mathbb{R}$, i.e. $\forall f \in \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$,
$$\int f \cdot \,\mathrm{d}\mu = \mu \cdot f.$$
3) The integrals defined above satisfy the usual linearity and convergence properties, which are inherited from the corresponding properties for the integral with respect to $|\mu|$. So are following versions of the triangle inequality:
$$|\int f \cdot \,\mathrm{d}\mu| \le \int \|f\| \,\mathrm{d}|\mu| \quad \text{and} \quad \|\int f \,\mathrm{d}\mu\| \le \int |f| \,\mathrm{d}|\mu|,$$
for $f \in \mathsf{L}^1(|\mu|, \mathbb{R}^n)$ or $f \in \mathsf{L}^1(|\mu|)$, respectively.

We now aim to identify $\mathbb{R}^n$-valued Radon measures with set functions. Firstly we introduce the notion of $\mathbb{R}^n$-valued measures as set functions.

DEFINITION 4.18 ($\mathbb{R}^n$-valued measure on a σ-algebra). Let X be a set and $\mathcal{M}$ a σ-algebra of subsets of X. We say that a map $\mu : \mathcal{M} \to \mathbb{R}^n$ is an $\mathbb{R}^n$-valued measure on $\mathcal{M}$ if

VM1) $\mu(\emptyset) = 0$;

VM2) μ is σ-additive, i.e. for all countable disjoint family $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$,
$$\mu(\cup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n),$$
with the meaning that the series is absolutely convergent (or, equivalently, that each component of $n \mapsto \mu(A_n)$ is summable with respect to the counting measure on $\mathbb{N}$) and the sum is $\mu(\cup_{n\in\mathbb{N}} A_n)$.

DEFINITION 4.19 ($\mathbb{R}^n$-valued Radon measures as set functions). Let X be a σ-compact locally compact Hausdorff space. We denote by $\mathscr{B}_X^c$ the set of Borel subsets of X which are relatively compact. We define:

i) a *finite $\mathbb{R}^n$-valued Radon measure set function* on X is an $\mathbb{R}^n$-valued measure on $\mathscr{B}_X$ in the sense of definition 4.18.

ii) an *$\mathbb{R}^n$-valued Radon measure set function* on X is a set function $\mu : \mathscr{B}^c_X \to \mathbb{R}^n$ such that, for all $K \subset X$ compact, its restriction to $\mathscr{B}_K \subset \mathscr{B}^c_X$ is a finite $\mathbb{R}^n$-valued Radon measure set function on K, i.e. $\mu|_{\mathscr{B}_K} : \mathscr{B}_K \to \mathbb{R}^n$ is an $\mathbb{R}^n$-valued measure on $\mathscr{B}_K$ in the sense of definition 4.18.

We denote by $\mathcal{M}(X)^n$ or $\mathcal{M}(X, \mathbb{R}^n)$ the set of finite $\mathbb{R}^n$-valued Radon measure set functions on X and by $\mathcal{M}_{\mathrm{loc}}(X)^n$ or $\mathcal{M}_{\mathrm{loc}}(X, \mathbb{R}^n)$ the set of $\mathbb{R}^n$-valued Radon measure set functions on X. It is clear that those are real linear spaces, i.e. $\mathcal{M}(X, \mathbb{R}^n)$ is a linear subspace of $(\mathbb{R}^n)^{\mathscr{B}_X}$ and $\mathcal{M}_{\mathrm{loc}}(X, \mathbb{R}^n)$ is a linear subspace of $(\mathbb{R}^n)^{\mathscr{B}^c_X}$.

Remark 4.20.

1) The nomenclature established in the previous definition is provisional. That is, for a moment we want to use different names for $\mathbb{R}^n$-valued Radon measures as linear functionals on spaces of continuous functions and for $\mathbb{R}^n$-valued Radon measures as set functions. However, we will see shortly that, if X is second countable, i.e. if X is a locally compact separable metrizable space (which is the case of interest in subsequent developments), a (finite) $\mathbb{R}^n$-valued Radon measure set function on X may be canonically identified with a (finite) $\mathbb{R}^n$-valued Radon measure on X (the latter in the sense of definition 4.1), and conversely. Making these identifications, we will treat those objects as one and the same thing, so that we may abandon this provisional nomenclature.
2) Each $\mu \in \mathcal{M}(X, \mathbb{R}^n)$ determines an element of $\mathcal{M}_{\mathrm{loc}}(X, \mathbb{R}^n)$ by restriction of $\mu : \mathscr{B}_X \to \mathbb{R}^n$ to $\mathscr{B}^c_X$. The fact that X is σ-compact allows to decompose each $B \in \mathscr{B}_X$ as a countable disjoint union of elements of $\mathscr{B}^c_X$; thus, by σ-additivity, μ is uniquely determined by its restriction to $\mathscr{B}^c_X$, i.e. the association $\mu \in \mathcal{M}(X, \mathbb{R}^n) \mapsto \mu|_{\mathscr{B}^c_X} \in \mathcal{M}_{\mathrm{loc}}(X, \mathbb{R}^n)$ is linear 1-1 and allows us to identify $\mathcal{M}(X, \mathbb{R}^n)$ with a linear subspace of $\mathcal{M}_{\mathrm{loc}}(X, \mathbb{R}^n)$. By means of this identification, we consider, henceforth, $\mathcal{M}(X, \mathbb{R}^n)$ as a linear subspace of $\mathcal{M}_{\mathrm{loc}}(X, \mathbb{R}^n)$.

Definition 4.21 (induced $\mathbb{R}^n$-valued Radon measure set functions). Let μ be an $\mathbb{R}^n$-valued Radon measure on a σ-compact locally compact Hausdorff space X. The $\mathbb{R}^n$-valued Radon measure set function induced by μ is the set function $\hat{\mu} : \mathscr{B}^c_X \to \mathbb{R}^n$ defined, for all $A \in \mathscr{B}^c_X$, by

$$\hat{\mu}(A) := \int \chi_A \,\mathrm{d}\mu \in \mathbb{R}^n.$$

If μ is finite, we define $\hat{\mu} : \mathscr{B}_X \to \mathbb{R}^n$ by the same formula.

Note that the definition above makes sense, since $\chi_A \in \mathsf{L}^1(|\mu|)$ if $A \in \mathscr{B}_X^c$ or if $A \in \mathscr{B}_X$ and μ finite (i.e. if $|\mu|$ is finite, by theorem 4.9).

The fact that $\hat{\mu}$ is actually an $\mathbb{R}^n$-valued Radon measure set function is proved in the proposition below.

PROPOSITION 4.22 (induced $\mathbb{R}^n$-valued Radon measure set functions). *With the notation from the previous definition:*

i) $\hat{\mu}$ is a (finite) $\mathbb{R}^n$-valued Radon measure set function on X if μ is a (finite) $\mathbb{R}^n$-valued Radon measure on X.

ii) The maps $I : \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)^* \to \mathcal{M}_{\mathrm{loc}}(X, \mathbb{R}^n)$ and $I : \mathsf{C}_0(X, \mathbb{R}^n)^* \to \mathcal{M}(X, \mathbb{R}^n)$ defined by $\mu \mapsto \hat{\mu}$ are linear 1-1 and commute with the inclusions, i.e. the following diagram is commutative:

$$\begin{tikzcd} \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)^* \arrow[r, "I"] & \mathcal{M}_{\mathrm{loc}}(X, \mathbb{R}^n) \\ \mathsf{C}_0(X, \mathbb{R}^n)^* \arrow[r, "I"] \arrow[u, hook] & \mathcal{M}(X, \mathbb{R}^n) \arrow[u, hook] \end{tikzcd}$$

PROOF. Let $K \subset X$ compact. We assert that $\hat{\mu} : \mathscr{B}_K \to \mathbb{R}^n$ is an $\mathbb{R}^n$-valued measure on $\mathscr{B}_K$ (in the sense of definition 4.18). It is clear that $\hat{\mu}(\emptyset) = 0$. Let $(A_n)_{n\in\mathbb{N}}$ be a disjoint sequence of Borel measurable subsets of K, and $A = \dot{\bigcup}_{n\in\mathbb{N}} A_n$. For each $n \in \mathbb{N}$, let $\phi_n := \sum_{k=1}^n \chi_{A_k}$. Then $\phi_n\nu \to \chi_A\nu$ pointwise, and the convergence is dominated, since $\|\phi_n\nu\| \leq \chi_A \in \mathsf{L}^1(|\mu|)$ (because $A \Subset X$ and $|\mu|$ is finite on compact sets). Applying the dominated convergence theorem 1.64 componentwise, it follows that $\int \phi_n\nu \,\mathrm{d}|\mu| \to \int \chi_A\nu \,\mathrm{d}|\mu| = \hat{\mu}(A)$. As $\int \phi_n\nu \,\mathrm{d}|\mu| = \sum_{k=1}^n \int \chi_{A_k}\nu \,\mathrm{d}|\mu| = \sum_{k=1}^n \hat{\mu}(A_k)$, the assertion is proved. Hence, $\hat{\mu}$ is an $\mathbb{R}^n$-valued Radon measure set function on X if μ is an $\mathbb{R}^n$-valued Radon measure on X. The same argument may be used to prove that $\hat{\mu} : \mathscr{B}_X \to \mathbb{R}^n$ is an $\mathbb{R}^n$-valued measure on $\mathscr{B}_X$ (i.e. a finite $\mathbb{R}^n$-valued Radon measure set function on X) if μ is finite.

We next prove that I is linear. Let μ and ν be $\mathbb{R}^n$-valued Radon measures on X and $c \in \mathbb{R}$.

To prove that $I(\mu+\nu) = I(\mu) + I(\nu)$, we must compare all three of the polar decompositions $\mu = n_1|\mu|$, $\nu = n_2|\nu|$ and $\mu+\nu = N|\mu+\nu|$. In order to accomplish that, note that $\lambda := |\mu|+|\nu|$ is a Radon measure on X, by exercise 1.30, and all three of the measures $|\mu+\nu|$, $|\mu|$ and $|\nu|$ are absolutely continuous with respect to λ (recall that $|\mu+\nu| \leq |\mu|+|\nu|$, from proposition 4.15). Since the restrictions to $\mathscr{B}_X$ of all measures involved are σ-finite (because they are Radon and X is σ-compact), we may take Radon-Nikodym derivatives of those restrictions (theorem 1.103) and apply the chain rule for such derivatives (proposition 1.107), which yields, for all $f \in \mathsf{C_c}(X, \mathbb{R}^n)$:

$$\begin{aligned}\mu \cdot f &= \int f \cdot n_1 \, \mathrm{d}|\mu| \overset{1.107}{=} \int f \cdot n_1 \frac{d|\mu|}{d\lambda} \, \mathrm{d}\lambda, \\ \nu \cdot f &= \int f \cdot n_2 \, \mathrm{d}|\nu| \overset{1.107}{=} \int f \cdot n_2 \frac{d|\nu|}{d\lambda} \, \mathrm{d}\lambda,\end{aligned}$$

hence

$$\begin{aligned}(\mu+\nu) \cdot f &= \int \langle f, n_1 \frac{d|\mu|}{d\lambda} + n_2 \frac{d|\nu|}{d\lambda} \rangle \, \mathrm{d}\lambda = \\ &= \int \Big\langle f, \frac{n_1 \frac{d|\mu|}{d\lambda} + n_2 \frac{d|\nu|}{d\lambda}}{\big|n_1 \frac{d|\mu|}{d\lambda} + n_2 \frac{d|\nu|}{d\lambda}\big|} \Big\rangle \big|n_1 \frac{d|\mu|}{d\lambda} + n_2 \frac{d|\nu|}{d\lambda}\big| \, \mathrm{d}\lambda\end{aligned}$$

(we define the quotient to be, for instance, 0 where the denominator is 0). Note that, since $|\mu| \le \lambda$ and $|\nu| \le \lambda$, we have $\big|\frac{d|\mu|}{d\lambda}\big| \le 1$ λ-a.e. and $\big|\frac{d|\nu|}{d\lambda}\big| \le 1$ λ-a.e., whence $\big|n_1 \frac{d|\mu|}{d\lambda} + n_2 \frac{d|\nu|}{d\lambda}\big| \le 2$ λ-a.e.; modifying those Borelian functions, if necessary, on a λ-null Borel set, we may assume that they are all bounded. Thus, from lemma 4.11, $\frac{d|\mu|}{d\lambda}\lambda$, $\frac{d|\nu|}{d\lambda}\lambda$ and $\big|n_1 \frac{d|\mu|}{d\lambda} + n_2 \frac{d|\nu|}{d\lambda}\big|\lambda$ are Radon measures on $\mathscr{B}_X$; therefore, from remark 1.29, the extensions (denoted with the same notation) of those measures given by theorem 1.8 are outer Radon measures on X. It then follows from the uniqueness of the polar decompositions of μ, ν and $\mu+\nu$ that

$$\begin{aligned}|\mu| &= \frac{d|\mu|}{d\lambda}\lambda, \\ |\nu| &= \frac{d|\nu|}{d\lambda}\lambda, \\ |\mu+\nu| &= \big|n_1 \frac{d|\mu|}{d\lambda} + n_2 \frac{d|\nu|}{d\lambda}\big|\lambda, \\ N &= \frac{n_1 \frac{d|\mu|}{d\lambda} + n_2 \frac{d|\nu|}{d\lambda}}{\big|n_1 \frac{d|\mu|}{d\lambda} + n_2 \frac{d|\nu|}{d\lambda}\big|} \quad |\mu+\nu| - \text{a.e.}\end{aligned}$$

We thus have, for all $A \in \mathscr{B}^c_X$:

$$\begin{aligned}\widehat{\mu+\nu}(A) &= \int_A N \,\mathrm{d}|\mu+\nu| = \\ &= \int_A \frac{n_1\frac{d|\mu|}{d\lambda} + n_2\frac{d|\nu|}{d\lambda}}{\left|n_1\frac{d|\mu|}{d\lambda} + n_2\frac{d|\nu|}{d\lambda}\right|}\left|n_1\frac{d|\mu|}{d\lambda} + n_2\frac{d|\nu|}{d\lambda}\right| \mathrm{d}\lambda = \\ &= \int_A \left(n_1\frac{d|\mu|}{d\lambda} + n_2\frac{d|\nu|}{d\lambda}\right) \mathrm{d}\lambda = \\ &= \int_A n_1\frac{d|\mu|}{d\lambda}\,\mathrm{d}\lambda + \int_A n_2\frac{d|\nu|}{d\lambda}\,\mathrm{d}\lambda = \\ &= \int_A n_1\,\mathrm{d}|\mu| + \int_A n_2\,\mathrm{d}|\nu| = \hat{\mu}(A) + \hat{\nu}(A)\end{aligned}$$

thus showing that $I(\mu+\nu) = I(\mu) + I(\nu)$. Similarly, if $c \neq 0$ and $\mu = n_1|\mu|$ is the polar decomposition of μ, it follows that, for all $f \in \mathsf{C_c}(X,\mathbb{R}^n)$, $(c\mu)\cdot f = c\int\langle f, n_1\rangle\,\mathrm{d}|\mu| = \int\langle f, \operatorname{sgn}(c) n_1\rangle|c||\mu|$. Thus, if $c\mu = n_2|c\nu|$ is the polar decomposition of $c\mu$, it follows from the uniqueness of such decomposition that $n_2 = \operatorname{sgn}(c)n_1$ and $|c\nu| = |c||\nu|$ (which had already been proved in proposition 4.15.(ii)). Therefore, for all $A \in \mathscr{B}^c_X$,

$$\begin{aligned}\widehat{c\mu}(A) &= \int_A n_2\,\mathrm{d}|c\mu| = \int_A \operatorname{sgn}(c) n_1\,\mathrm{d}(|c||\mu|) = \\ &= \int_A \operatorname{sgn}(c)|c| n_1\,\mathrm{d}|\mu| = c\hat{\mu}(A),\end{aligned}$$

thus proving that $I(c\mu) = cI(\mu)$.

The commutativity of the diagram in part ii) is immediate from the definitions. It remains to prove that $I : \mathsf{C_c}(X,\mathbb{R}^n)^* \to \mathcal{M}_{\mathrm{loc}}(X,\mathbb{R}^n)$ is 1-1 (which then implies that $I : \mathsf{C_0}(X,\mathbb{R}^n)^* \to \mathcal{M}(X,\mathbb{R}^n)$ is 1-1 in view of the asserted commutativity).

Let μ and ν be $\mathbb{R}^n$-valued Radon measures such that $\hat{\mu} = \hat{\nu}$, with respective polar decompositions $\mu = n_1|\mu|$ and $\mu = n_2|\nu|$. We have, for all $A \in \mathscr{B}^c_X$, $\int \chi_A n_1\,\mathrm{d}|\mu| = \int \chi_A n_2\,\mathrm{d}|\nu|$. By linearity of the integrals, the latter equality also holds if we substitute χ_A with a simple function $\phi = \sum_{i=1}^k a_i\chi_{A_i}$ such that $\forall 1 \leq i \leq k$, $a_i \in \mathbb{R}$ and $A_i \in \mathscr{B}^c_X$ if $a_i \neq 0$. For any $f \in \mathsf{C_c}(X,\mathbb{R})$, we may take a sequence $(\phi_m)_{m\in\mathbb{N}}$ of such simple functions which converges pointwise to f and $\forall m \in \mathbb{N}$, $|\phi_m| \leq |f|$; that follows from proposition 1.53.iii) (note that, writing a simple function ϕ in the standard representation, i.e. $\phi = \sum_{i=1}^n a_i\chi_{\phi^{-1}(a_i)}$ for $\operatorname{Im}\phi = \{a_1,\ldots,a_n\}$ with $a_i \neq a_j$ if $i \neq j$, then $|\phi| \leq |f|$ implies $\phi^{-1}(a_i) \subset \operatorname{spt} f$ if $a_i \neq 0$, hence

$\phi^{-1}(a_i) \in \mathscr{B}^c_X$ if $a_i \neq 0$). Hence $\phi_m n_1 \to f n_1$ and $\phi_m n_2 \to f n_2$ pointwise, and the convergence is dominated with respect to both $|\mu|$ and $|\nu|$, since $\|\phi_m n_i\| = |\phi_m| \leq |f| \in \mathsf{L}^1(|\mu|) \cap \mathsf{L}^1(|\nu|)$. Applying the dominated convergence theorem 1.64 componentwise, it follows that $\int f n_1 \,\mathrm{d}|\mu| = \lim_{m\to\infty} \int \phi_m n_1 \,\mathrm{d}|\mu| = \lim_{m\to\infty} \int \phi_m n_2 \,\mathrm{d}|\nu| = \int f n_2 \,\mathrm{d}|\nu|$. Writing the latter equality componentwise, if $n_i = \sum_{j=1}^n n_i^j e_j$, we conclude that, for all $f \in \mathsf{C_c}(X, \mathbb{R})$ and $1 \leq j \leq n$, $\int f n_1^j \,\mathrm{d}|\mu| = \int f n_2^j \,\mathrm{d}|\nu|$. Therefore, for all $f \in \mathsf{C_c}(X, \mathbb{R}^n)$,

$$\begin{aligned}\mu \cdot f = \int f \cdot n_1 \,\mathrm{d}|\mu| = \\ &= \sum_{j=1}^n \Big(\int f_j n_1^j \,\mathrm{d}|\mu|\Big) e_j = \\ &= \sum_{j=1}^n \Big(\int f_j n_2^j \,\mathrm{d}|\nu|\Big) e_j = \\ &= \int f \cdot n_2 \,\mathrm{d}|\nu| = \nu \cdot f,\end{aligned}$$

thus $\mu = \nu$, showing that $I : \mathsf{C_c}(X, \mathbb{R}^n)^* \to \mathcal{M}_{\mathrm{loc}}(X, \mathbb{R}^n)$ is 1-1, as asserted.

□

Conversely, every $\mathbb{R}^n$-valued Radon measure set function on a σ-compact locally compact Hausdorff space X induces an $\mathbb{R}^n$-valued Radon measure on X.

Definition 4.23 (induced $\mathbb{R}^n$-valued Radon measures). Let μ be an $\mathbb{R}^n$-valued Radon measure set function on a σ-compact locally compact Hausdorff space X. The *$\mathbb{R}^n$-valued Radon measure induced by μ* is the map $\check{\mu} : \mathsf{C_c}(X, \mathbb{R}^n) \to \mathbb{R}$ defined, for all $f \in \mathsf{C_c}(X, \mathbb{R}^n)$, by

$$\check{\mu} \cdot f := \sum_{i=1}^n \int_K f_i \,\mathrm{d}\nu_i,$$

where the integrals in the second member are taken with respect to signed measures ν_i, $1 \leq i \leq n$, obtained as the the restrictions of the components μ_i of μ to $\mathscr{B}_K$, where $K \in \mathscr{B}^c_X$ contains spt f.

Note that, by definition 4.19, $\mu|_{\mathscr{B}_K} = (\nu_1, \ldots, \nu_n) : \mathscr{B}_K \to \mathbb{R}^n$ is an $\mathbb{R}^n$-valued measure on $\mathscr{B}_K$; thus, for $1 \leq i \leq n$, ν_i is a finite signed measure on the measure space $(K, \mathscr{B}_K)$ in the sense of definition 1.89. We may then take the integrals in the sense of definition 1.97, since f is bounded (hence $f|_K \in \mathsf{L}^1(|\nu_i|)$ for $1 \leq i \leq n$).

That $\check{\mu}\cdot f$ is well-defined (i.e. the definition does not depend on the relatively compact Borel set $K \supset \operatorname{spt} f$) and linear on f will be proved as part of the next proposition.

PROPOSITION 4.24. *With the notation from the previous definition:*

i) $\check{\mu} : \mathsf{C_c}(X, \mathbb{R}^n) \to \mathbb{R}$ *is well-defined and linear continuous, i.e. it is an* $\mathbb{R}^n$*-valued Radon measure on* X*. Moreover,* $\check{\mu}$ *is finite if so is* μ*, i.e.* $\check{\mu} : \mathsf{C_0}(X, \mathbb{R}^n) \to \mathbb{R}$ *is linear continuous if* μ *is finite.*

ii) The maps $J : \mathcal{M}_{\mathrm{loc}}(X, \mathbb{R}^n) \to \mathsf{C_c}(X, \mathbb{R}^n)^*$ *and* $J : \mathcal{M}(X, \mathbb{R}^n) \to \mathsf{C_0}(X, \mathbb{R}^n)^*$ *defined by* $\mu \mapsto \check{\mu}$ *are linear, commute with inclusions and invert* I *(defined in proposition 4.22) on the left, i.e.* $\forall \mu$ $\mathbb{R}^n$*-valued Radon measure,*

$$\check{\check{\mu}} = \mu.$$

PROOF. We firstly show that, given $f \in \mathsf{C_c}(X, \mathbb{R}^n)$, $\check{\mu} \cdot f$ is well-defined, i.e. the definition does not depend on the relatively compact Borel set $K \supset \operatorname{spt} f$. Indeed, let, $\forall 1 \le i \le n$, $\nu_i : \mathscr{B}_K \to \mathbb{R}$ and $\nu_i^0 : \mathscr{B}_{\operatorname{spt} f} \to \mathbb{R}$ denote the i^{th} components of the restrictions of μ to $\mathscr{B}_K$ and $\mathscr{B}_{\operatorname{spt} f}$, respectively (which are finite signed measures). It follows from the uniqueness of the Jordan decomposition 1.94 that the positive and negative parts of ν_i^0 coincide with the restrictions of the positive and negative parts of ν_i, respectively; thus, for $1 \le i \le n$, $\int_K f \,\mathrm{d}\nu_i^\pm = \int_{\operatorname{spt} f} f \,\mathrm{d}(\nu_i^\pm)|_{\operatorname{spt} f} = \int_{\operatorname{spt} f} f \,\mathrm{d}(\nu_i^0)^\pm$, showing that the definition of $\check{\mu}\cdot f$ does not depend on K, as asserted.

To show that $\check{\mu} : \mathsf{C_c}(X, \mathbb{R}^n) \to \mathbb{R}$ is linear continuous, it suffices to show that, for each $K \subset X$ compact, the restriction $\check{\mu} : \mathsf{C_c^K}(X, \mathbb{R}^n) \to \mathbb{R}$ is linear continuous in the normed space $(\mathsf{C_c^K}(X, \mathbb{R}^n), \|\cdot\|_u)$ (see definition 4.1). That follows from the fact that, if $\mu|_{\mathscr{B}_K} = (\nu_1, \ldots, \nu_n)$, $\check{\mu}|_{\mathsf{C_c^K}(X,\mathbb{R}^n)}$ is given by $f \mapsto \sum_{i=1}^n \int_K f_i \,\mathrm{d}\nu_i$, which is clearly linear in $\mathsf{C_c^K}(X, \mathbb{R}^n)$ (by the linearity of the integrals) and $|\check{\mu}\cdot f| \le \sum_{i=1}^n \int_K |f_i| \,\mathrm{d}|\nu_i| \le (\sum_{i=1}^n |\nu_i|(K))\|f\|_u$ (where we used the triangle inequality from exercise 1.99.(e)), which yields the asserted continuity.

If μ is finite, i.e. if $\mu = (\mu_1, \ldots, \mu_n) : \mathscr{B}_X \to \mathbb{R}^n$ is an $\mathbb{R}^n$-valued measure on $\mathscr{B}_X$, each component μ_i of μ is a finite signed measure on $\mathscr{B}_X$ and the same argument used in the first paragraph of this proof to show that $\check{\mu}\cdot f$ is well-defined may be applied to conclude that $\forall f \in \mathsf{C_c}(X, \mathbb{R}^n)$, $\check{\mu}\cdot f = \sum_{i=1}^n \int_X f_i \,\mathrm{d}\mu_i$. Therefore, applying the triangle inequality once more, it follows that $|\check{\mu}\cdot f| \le (\sum_{i=1}^n |\mu_i|(X))\|f\|_u$, thus proving that $\check{\mu} : \big(\mathsf{C_c}(X, \mathbb{R}^n), \|\cdot\|_u\big) \to \mathbb{R}$ is linear continuous; hence, by the density of $\mathsf{C_c}(X, \mathbb{R}^n)$ in $\mathsf{C_0}(X, \mathbb{R}^n)$, $\check{\mu}$ may be uniquely extended to a continuous linear functional $\check{\mu} : \mathsf{C_0}(X, \mathbb{R}^n) \to \mathbb{R}$ with norm $\le \sum_{i=1}^n |\mu_i|(X)$.

The fact that $J : \mathcal{M}_{\text{loc}}(X, \mathbb{R}^n) \to \mathsf{C}_{\mathsf{c}}(X, \mathbb{R}^n)^*$ is linear follows directly from its definition and from exercise 1.100 parts e) and f). Since $J : \mathcal{M}(X, \mathbb{R}^n) \to \mathsf{C}_0(X, \mathbb{R}^n)^*$ was defined by restriction of $J : \mathcal{M}_{\text{loc}}(X, \mathbb{R}^n) \to \mathsf{C}_{\mathsf{c}}(X, \mathbb{R}^n)^*$ to $\mathcal{M}(X, \mathbb{R}^n)$, it is also linear and the commutativity with the inclusions follows by definition.

It remains to show that $J \circ I$ coincides with the identity of $\mathsf{C}_{\mathsf{c}}(X, \mathbb{R}^n)$ (hence it also coincides with the identity of $\mathsf{C}_0(X, \mathbb{R}^n)$, thanks to the commutativity of I and J with the inclusions).

Let $\mu \in \mathsf{C}_{\mathsf{c}}(X, \mathbb{R}^n)^*$ be an $\mathbb{R}^n$-valued Radon measure with polar decomposition $\mu = N|\mu|$, where $N = (N_1, \ldots, N_n)$. We must show that $\check{\hat{\mu}} = \mu$. It follows from definition 4.21 that, for each $A \in \mathscr{B}^c_X$, $\hat{\mu}(A) = \int_A N \, \mathrm{d}|\mu|$. Thus, for each $K \subset X$ compact, for $1 \le i \le n$, the i^{th} component of the $\mathbb{R}^n$-valued measure $\hat{\mu}|_{\mathscr{B}_K}$ on $\mathscr{B}_K$ is the finite signed measure $N_i|\mu|$ on $(K, \mathscr{B}_K)$. The positive and negative parts of its Jordan decomposition are $N_i^+|\mu|, N_i^-|\mu| : \mathscr{B}_K \to \mathbb{R}$, respectively (since $N_i|\mu| = N_i^+|\mu| - N_i^-|\mu|$ and $N_i^+|\mu| \perp N_i^-|\mu|$ as measures on the measurable space $(K, \mathscr{B}_K)$). It then follows from definition 4.23 that, if $f \in \mathsf{C}_{\mathsf{c}}(X, \mathbb{R}^n)$ and $\operatorname{spt} f \subset K$,

$$
\begin{aligned}
\check{\hat{\mu}} \cdot f &= \sum_{i=1}^{n} \int_K f_i \, \mathrm{d}(N_i|\mu|) = \\
&= \sum_{i=1}^{n} \int_K f_i \, \mathrm{d}(N_i^+|\mu|) - \sum_{i=1}^{n} \int_K f_i \, \mathrm{d}(N_i^-|\mu|) = \\
&= \sum_{i=1}^{n} \int_K f_i N_i^+ \, \mathrm{d}|\mu| - \sum_{i=1}^{n} \int_K f_i N_i^- \, \mathrm{d}|\mu| = \\
&= \sum_{i=1}^{n} \int_K f_i N_i \, \mathrm{d}|\mu| = \\
&= \int \langle f, N \rangle \, \mathrm{d}|\mu| = \mu \cdot f.
\end{aligned}
$$

Therefore, by the arbitrariness of K and f, we conclude that $\check{\hat{\mu}} = \mu$, as asserted. □

We will see next that, if X is a locally compact separable metric space, J defined above also inverts I on the right, i.e. $I \circ J$ is the identity of the corresponding domains. That is, $I : \mathsf{C}_{\mathsf{c}}(X, \mathbb{R}^n)^* \to \mathcal{M}_{\text{loc}}(X, \mathbb{R}^n)$ and $I : \mathsf{C}_0(X, \mathbb{R}^n)^* \to \mathcal{M}(X, \mathbb{R}^n)$ are surjective isomorphisms, so that we may identify $\mathsf{C}_{\mathsf{c}}(X, \mathbb{R}^n)^* \equiv \mathcal{M}_{\text{loc}}(X, \mathbb{R}^n)$ and $\mathsf{C}_0(X, \mathbb{R}^n)^* \equiv \mathcal{M}(X, \mathbb{R}^n)$.

PROPOSITION 4.25. *Let X be a locally compact separable metric space, I and J defined in propositions 4.22 and 4.24, respectively.*

Then $I \circ J$ is the identity of $\mathcal{M}_{\mathrm{loc}}(X, \mathbb{R}^n)$, and so is its restriction to $\mathcal{M}(X, \mathbb{R}^n)$.

PROOF. Take $\mu : \mathscr{B}_X^c \to \mathbb{R}^n$ in $\mathcal{M}_{\mathrm{loc}}(X, \mathbb{R}^n)$. We must show that $\hat{\check{\mu}} = \mu$. It suffices to show that, for an arbitrary relatively compact open set $U \subset X$, the restrictions of μ and $\hat{\check{\mu}}$ coincide on $\mathscr{B}_U \subset \mathscr{B}_X^c$ (for the union of such $\mathscr{B}_U$ coincides with $\mathscr{B}_X^c$, i.e. every element of $\mathscr{B}_X^c$ is contained in some relatively compact open set). Fix such a relatively compact open set U and let $\mu|_{\mathscr{B}_U} = (\mu_1, \ldots, \mu_n)$, $\hat{\check{\mu}}|_{\mathscr{B}_U} = (\hat{\check{\mu}}_1, \ldots, \hat{\check{\mu}}_n)$. We must therefore prove that, for $1 \leq i \leq n$, the finite signed measures μ_i and $\hat{\check{\mu}}_i$ coincide on the measurable space $(U, \mathscr{B}_U)$.

Fix $1 \leq i \leq n$. Note that, since U is a locally compact separable metric space, and since the total variations of both μ_i and $\hat{\check{\mu}}_i$ are finite positive Borel measures on $(U, \mathscr{B}_U)$ (in particular, they are finite on compact subsets of U), it follows from remark 1.33 that $|\mu_i|$ and $|\hat{\check{\mu}}_i|$ are positive Radon measures on $\mathscr{B}_U$. Thus, in order to prove that $\mu_i = \hat{\check{\mu}}_i$, it suffices to show, in view of lemma 4.26 below with U in place of X, that $\forall f \in \mathsf{C_c}(U, \mathbb{R})$, $\int_U f \,\mathrm{d}\mu_i = \int_U f \,\mathrm{d}\hat{\check{\mu}}_i$.

Fix $f \in \mathsf{C_c}(U, \mathbb{R})$ and let $F := f e_i \in \mathsf{C_c}(U, \mathbb{R}^n) \subset \mathsf{C_c}(X, \mathbb{R}^n)$. It follows from definition 4.23 that

$$\check{\mu} \cdot F = \int_U f \,\mathrm{d}\mu_i. \tag{4.5}$$

On the other hand, let the polar decomposition of $\check{\mu} \in \mathsf{C_c}(X, \mathbb{R}^n)^*$ be $\check{\mu} = N|\check{\mu}|$, where $N = (N_1, \ldots, N_n)$. By definition 4.21, for each $A \in \mathscr{B}_X^c$, $\hat{\check{\mu}}(A) = \int_A N \,\mathrm{d}|\check{\mu}|$. Thus, the i^{th} component of the $\mathbb{R}^n$-valued measure $\hat{\check{\mu}}|_{\mathscr{B}_U}$ on $\mathscr{B}_U$ is the finite signed measure $N_i|\check{\mu}|$ on $(U, \mathscr{B}_U)$. It then follows that

$$\begin{aligned} \check{\mu} \cdot F = \int \langle F, N \rangle \,\mathrm{d}|\check{\mu}| = \int_U f N_i \,\mathrm{d}|\check{\mu}| = \\ = \int_U f \,\mathrm{d}\hat{\check{\mu}}_i. \end{aligned} \tag{4.6}$$

Comparing (4.5) and (4.6), we conclude that $\int_U f \,\mathrm{d}\mu_i = \int_U f \,\mathrm{d}\hat{\check{\mu}}_i$, as we wanted to show.

$\square$

LEMMA 4.26. *Let X be a σ-compact locally compact Hausdorff space and μ,ν signed measures on $\mathscr{B}_X$ whose total variations $|\mu|$ and $|\nu|$ are Radon measures on $\mathscr{B}_X$. Then $\mu = \nu$ iff $\forall f \in \mathsf{C_c}(X, \mathbb{R})$, $\int f \,\mathrm{d}\mu = \int f \,\mathrm{d}\nu$.*

Note that both integrals make sense, since both $|\mu|$ and $|\nu|$ are Radon, hence $\mathsf{C_c}(X, \mathbb{R}) \subset \mathsf{L}^1(\mu) \cap \mathsf{L}^1(\nu)$.

PROOF. It suffices to prove ($\Leftarrow$), since the reverse implication is trivial.

Suppose that $\forall f \in \mathsf{C_c}(X, \mathbb{R})$, $\int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\nu$.

1) We contend that there exist Borelian functions $h_\mu, h_\nu : X \to \mathbb{R}$ such that $|h_\mu| = |h_\nu| \equiv 1$ and $\mu = h_\mu |\mu|$, $\nu = h_\nu |\nu|$. Indeed, since $\mu^+ \perp \mu^-$, we may take disjoint Borel sets P and N such that $X = P \dot{\cup} N$, μ^+ is concentrated on P and μ^- is concentrated on N. Thus, $\mu^+ = \chi_P |\mu|$ and $\mu^- = \chi_N |\mu|$, so that $\mu = \mu^+ - \mu^- = (\chi_P - \chi_N)|\mu|$, thus proving the contention for μ with $h_\mu := \chi_P - \chi_N$; we do the same for ν.
2) The linear functional L defined on $\mathsf{C_c}(X, \mathbb{R})$ by $f \mapsto \int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\nu$ is continuous, since, by the fact that $|\mu|$ is finite on compact sets and by the triangle inequality 1.99.e), for every $K \subset X$ compact and for every $f \in \mathsf{C_c^K}(X, \mathbb{R})$, $|\int f \, \mathrm{d}\mu| \leq \int_K |f| \, \mathrm{d}|\mu| \leq |\mu|(K) \|f\|_u$. By the previous item, we have, for all $f \in \mathsf{C_c}(X, \mathbb{R})$,
$$L \cdot f = \int f \, \mathrm{d}\mu = \int f \cdot h_\mu \, \mathrm{d}|\mu|,$$
$$L \cdot f = \int f \, \mathrm{d}\nu = \int f \cdot h_\nu \, \mathrm{d}|\nu|.$$
Since $|\mu|$ and $|\nu|$ are positive Radon measures on $\mathscr{B}_X$ (which, by remark 1.29, correspond to outer Radon measures on X, denoted with the same notation), we conclude that both $h_\mu |\mu|$ and $h_\nu |\nu|$ are polar decompositions for L. Hence, by the uniqueness of the polar decomposition stated in theorem 4.9 (with $n = 1$), it follows that $|\mu| = |\nu|$ and $h_\mu = h_\nu$ $|\mu|$-a.e., which implies $\mu = h_\mu |\mu| = h_\nu |\nu| = \nu$. $\square$

COROLLARY 4.27. *If X is a locally compact separable metric space, then I and J defined in propositions 4.22 and 4.24, respectively, are surjective isomorphisms, inverses of each other.*

PROOF. It is a consequence of propositions 4.22, 4.24 and 4.25. $\square$

REMARK 4.28. If X is a locally compact separable metric space, we may therefore identify $\mathsf{C_c}(X, \mathbb{R}^n)^* \equiv \mathcal{M}_{\mathrm{loc}}(X, \mathbb{R}^n)$ and $\mathsf{C_0}(X, \mathbb{R}^n)^* \equiv \mathcal{M}(X, \mathbb{R}^n)$ by means of the linear isomorphisms of the previous corollary. With these identifications in mind, we will henceforth abandon our provisional nomenclature and drop the hats and checks from the notation, treating an $\mathbb{R}^n$-valued Radon measure μ and the corresponding $\mathbb{R}^n$-valued Radon measure set function $\hat{\mu}$ as being one and the same object.

COROLLARY 4.29. *If X is a locally compact separable metric space, $\mathcal{M}(X, \mathbb{R}^n)$ is a Banach space with the norm $\|\mu\| := |\mu|(X)$.*

PROOF. It is a consequence of the identification $\mathsf{C}_0(X, \mathbb{R}^n)^* \equiv \mathcal{M}(X, \mathbb{R}^n)$ and of theorem 4.9.ii), which asserts that the operator norm of $\mu \in \mathsf{C}_0(X, \mathbb{R}^n)^*$ is $|\mu|(X)$. □

EXERCISE 4.30 (properties of the total variation, part II). Let X be a locally compact separable metric space and μ an $\mathbb{R}^n$-valued Radon measure on X. Define, for all $E \in \mathscr{B}_X$:

i) $\mu_1(E) := \sup\{\sum_{k=1}^m \|\mu(E_k)\| \mid m \in \mathbb{N};\ \forall 1 \le k \le m, E_k \in \mathscr{B}_X, |\mu|(E_k) < \infty;\ \dot{\bigcup}_{k=1}^m E_k \subset E\}$.
ii) $\mu_2(E) := \sup\{\sum_{k\in\mathbb{N}} \|\mu(E_k)\| \mid \forall k \in \mathbb{N}, E_k \in \mathscr{B}_X, |\mu|(E_k) < \infty;\ \dot{\bigcup}_{k\in\mathbb{N}} E_k = E\}$.
iii) $\mu_3(E) := \sup\{\int_E f \cdot \mathrm{d}\mu \mid f \in \mathsf{L}^1(|\mu|, \mathbb{R}^n), \|f\| \le 1\}$.
iv) $\mu_4(E) := \sup\{\int_E f \cdot \mathrm{d}\mu \mid f \in \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n), \|f\| \le 1\}$.

Then $\mu_1(E) = \mu_2(E) = \mu_3(E) = \mu_4(E) = |\mu|(E)$.

4.2. Operations with $\mathbb{R}^n$-valued Radon measures

We generalize to $\mathbb{R}^n$-valued Radon measures the operations for positive measures introduced in definitions 1.13 and 1.14.

DEFINITION 4.31 (restrictions of $\mathbb{R}^n$-valued Radon measures). Let X be a locally compact separable metric space, $\mu \in \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)^*$ an $\mathbb{R}^n$-valued Radon measure and $g : X \to \mathbb{R}$ a bounded Borelian function on X. We define the *restriction of μ to g*, denoted by $\mu \llcorner g$, as the continuous linear functional on $\mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$ given by $\mu \llcorner g \cdot f := \int \langle fg, \nu \rangle \,\mathrm{d}|\mu|$ if $(\nu, |\mu|)$ is the polar decomposition of μ.

NOTATION. If λ is a positive measure on X and $h \in \mathsf{L}^+(\lambda)$, we introduce the alternative notation $\lambda \llcorner h$ to denote the measure on X which has been denoted so far by $h\lambda$, i.e. the extension given by theorem 1.8 of the measure on $\sigma(\lambda)$ given by $A \mapsto \int_A h \,\mathrm{d}\lambda$. This alternative notation is motivated by the following remark.

REMARK 4.32. With the notation from the previous definition:

1) Note that $\mu \llcorner g$ is indeed a well-defined continuous linear functional: $\forall K \subset X$ compact and $\forall f \in \mathsf{C}_\mathsf{c}^\mathsf{K}(X, \mathbb{R}^n)$, $fg \in \mathsf{L}^1(|\mu|)$, $\mu \llcorner g$ is clearly linear in f and, by the triangle inequality (remark 4.17.3), $|\mu \llcorner g \cdot f| \le |\mu|(K)\|g\|_u\|f\|_u$, hence $\mu \llcorner g$ is continuous.
2) The polar decomposition of $\mu \llcorner g$ is $(\frac{g\nu}{|g|}, |g||\mu|)$, where we define $\frac{g\nu}{|g|} := 0$ on the Borel set $\{g = 0\}$. That follows from the fact that $|g||\mu|$ is a positive Radon measure on X (by lemma 4.11) and from

the uniqueness of the polar decomposition. In particular, using the notation above, we have

$$|\mu \llcorner g| = |\mu| \llcorner |g|.$$

3) If μ is a positive Radon measure on X (which we identify with the element of $\mathsf{C_c}(X, \mathbb{R})^*$ whose polar decomposition is $(1, \mu)$) and $A \in \mathscr{B}_X$, then $\mu \llcorner \chi_A$ coincides with the positive Radon measure $\mu \llcorner A$ (that this positive measure is Radon follows from proposition 1.36). We extend this notation for an arbitrary $\mu \in \mathsf{C_c}(X, \mathbb{R}^n)^*$, i.e. we use the notation $\mu \llcorner A$ in place of $\mu \llcorner \chi_A$. It then follows from the previous item that

$$|\mu \llcorner A| = |\mu| \llcorner A.$$

4) We may similarly define $\mu \llcorner g \in \mathsf{C_c}(X, \mathbb{R}^n)^*$ for $\mu \in \mathsf{C_c}(X, \mathbb{R})^*$ and $g : X \to \mathbb{R}^n$ bounded Borelian: that is the continuous linear functional $f \in \mathsf{C_c}(X, \mathbb{R}^n) \mapsto \int \langle f, g\rangle \nu \, \mathrm{d}|\mu|$ if $(\nu, |\mu|)$ is the polar decomposition of μ. Then $(\frac{g\nu}{\|g\|}, \|g\||\mu|)$ is the polar decomposition of $\mu \llcorner g$. In particular,

$$|\mu \llcorner g| = |\mu| \llcorner \|g\|.$$

Note that, with this definition, if $\mu \in \mathsf{C_c}(X, \mathbb{R}^n)^*$ has polar decomposition $(\nu, |\mu|)$, then $\mu = |\mu| \llcorner \nu$.

5) We may also define $\mu \llcorner g$ for $g \in \mathsf{L}^1_{\mathsf{loc}}(|\mu|)$ using the same formula. Note that, for all $K \subset X$ compact and for all $f \in \mathsf{C}^{\mathsf{K}}_{\mathsf{c}}(X, \mathbb{R}^n)$, $|\mu \llcorner g \cdot f| = \left|\int_K \langle fg, \nu\rangle \, \mathrm{d}|\mu|\right| \leq (\int_K |g| \, \mathrm{d}|\mu|)\|f\|_u$, whence the continuity of $\mu \llcorner g$. As before, the polar decomposition of $\mu \llcorner g$ is $(\frac{g\nu}{|g|}, |g||\mu|)$, which follows from the uniqueness of the polar decomposition and from the fact that $|g||\mu|$ is a positive Radon measure on $\mathscr{B}_X$ (by the fact that it is finite on compact subsets of X and by remark 1.33). Thus, as before,

$$|\mu \llcorner g| = |\mu| \llcorner |g|.$$

6) As a final generalization of the restriction operation, we may define $\mu \llcorner T \in \mathsf{C_c}(X, \mathbb{R}^m)^*$ for $\mu \in \mathsf{C_c}(X, \mathbb{R}^n)^*$ and $T \in \mathsf{L}^1_{\mathsf{loc}}\big(|\mu|, \mathsf{L}(\mathbb{R}^m, \mathbb{R}^n)\big)$ by $f \in \mathsf{C_c}(X, \mathbb{R}^m) \mapsto \int \langle T \cdot f, \nu\rangle \, \mathrm{d}|\mu|$, where $(\nu, |\mu|)$ is the polar decomposition of μ. As before, it follows from the triangle inequality that, for all $K \subset X$ compact and for all $f \in \mathsf{C}^{\mathsf{K}}_{\mathsf{c}}(X, \mathbb{R}^m)$, $|\mu \llcorner T \cdot f| \leq (\int_K \|T\| \, \mathrm{d}|\mu|)\|f\|_u$, hence $\mu \llcorner T$ is indeed a continuous linear functional. Note that, defining $T^* : X \to \mathsf{L}(\mathbb{R}^n, \mathbb{R}^m)$ by $x \mapsto T(x)^*$, i.e. the adjoint of T, we have, $\forall f \in \mathsf{C_c}(X, \mathbb{R}^m)$:

$$\mu \llcorner T \cdot f = \int \langle T \cdot f, \nu\rangle \, \mathrm{d}|\mu| = \int \Big\langle f, \frac{T^* \cdot \nu}{\|T^* \cdot \nu\|}\Big\rangle \|T^* \cdot \nu\| \, \mathrm{d}|\mu|.$$

Since $\|T^* \cdot \nu\| \,\mathrm{d}|\mu|$ is a positive Radon measure on X, it follows as before from the uniqueness of the polar decomposition that $\left(\frac{T^* \cdot \nu}{\|T^* \cdot \nu\|}, \|T^* \cdot \nu\| |\mu|\right)$ is the polar decomposition of $\mu \llcorner T$. In particular,

$$|\mu \llcorner T| = |\mu| \llcorner \|T^* \cdot \nu\|.$$

Exercise 4.33 (Lebesgue decomposition and Radon-Nikodym derivative for $\mathbb{R}^n$-valued Radon measures). Let X be a locally compact separable metric space $\mu \in \mathsf{C_c}(X, \mathbb{R}^n)^*$ and λ a positive Radon measure on X. We say that

- $\mu \perp \lambda$ if $|\mu| \perp \lambda$ in the sense of definition 3.34;
- $\mu \ll \lambda$ if $|\mu| \ll \lambda$ in the sense of 3.34.

Then:

a) (Lebesgue decomposition) There exist unique $\mathbb{R}^n$-valued Radon measures μ_a, μ_s on X such that $\mu_s \perp \lambda$, $\mu_a \ll \lambda$, $\mu = \mu_s + \mu_a$.
b) (Radon-Nikodym derivative) If $\mu \ll \lambda$, there exists a unique (up to λ-null sets) Borelian map $f : X \to \mathbb{R}^n$ with $f \in \mathsf{L}^1_{\mathsf{loc}}(\lambda)$ and $\mu = \lambda \llcorner f$. We call f the *Radon-Nikodym derivative* of μ with respect to λ and denote it by $\frac{d\mu}{d\lambda}$.

Exercise 4.34 (fundamental lemma of the Calculus of Variations). Let X be an open set in $\mathbb{R}^m$. If $\mu : \mathsf{C_c}(X, \mathbb{R}^n) \to \mathbb{R}$ is an $\mathbb{R}^n$-valued Radon measure on X such that $\mu \cdot f = 0$ for all $f \in \mathsf{C}^\infty_\mathsf{c}(X, \mathbb{R}^n)$, then $\mu = 0$. In particular, if $g \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^m|_X, \mathbb{R}^n)$ and

$$\int_X \langle f, g\rangle \,\mathrm{d}\mathcal{L}^m = 0$$

for all $f \in \mathsf{C}^\infty_\mathsf{c}(X, \mathbb{R}^n)$, then $g = 0$ $\mathcal{L}^m$-a.e. on X.

Definition 4.35 (trace of $\mathbb{R}^n$-valued Radon measures). Let X be a locally compact separable metric space and $A \subset X$ a locally compact subspace of X (i.e the intersection of an open with a closed subset of X). If μ is an $\mathbb{R}^n$-valued Radon measure on X with polar decomposition $(\nu, |\mu|)$, we define an $\mathbb{R}^n$-valued Radon measure $\mu|_A$ on A by $f \in \mathsf{C_c}(A, \mathbb{R}^n) \mapsto \int \langle \widetilde{f}, \nu\rangle \,\mathrm{d}|\mu|$, where $\widetilde{f} : X \to \mathbb{R}^n$ is the extension of f by 0 in the complement of A.

Proposition 4.36. *Let X be a locally compact separable metric space, $A \subset X$ a locally compact subspace and $\mu \in \mathsf{C_c}(X, \mathbb{R}^n)^*$ with polar decomposition $(\nu, |\mu|)$. Then $\mu|_A$ is a well-defined $\mathbb{R}^n$-valued Radon measure on A and it is finite if so is μ. Moreover, the polar decomposition of $\mu|_A$ is $(\nu|_A, |\mu|\big|_A)$, where $|\mu|\big|_A$ denotes the trace of $|\mu|$ on A in* *the sense of definition 1.13. In particular, if μ is a positive Radon measure on X, the trace of μ on A in the sense of definition 4.35 coincides with the trace in the sense of definition 1.13.*

PROOF. Note that $\widetilde{f}$ is a bounded Borel measurable function and vanishes on the complement of a compact subset of A (which is also a compact subset of X, since the inclusion is continuous and X is Hausdorff), hence $\widetilde{f} \in \mathsf{L}^1(|\mu|)$ and the integral makes sense, i.e. $\mu|_A \cdot f$ is well-defined and clearly linear in f. Moreover, for all $K \subset A$ compact and $f \in \mathsf{C}_\mathsf{c}^\mathsf{K}(A, \mathbb{R}^n)$, we have, by the triangle inequality $\left|\mu|_A \cdot f\right| = \left|\int_K \langle \widetilde{f}, \nu \rangle \,\mathrm{d}|\mu|\right| \leq |\mu|(K)\|f\|_u$, hence $\mu|_A$ is continuous, i.e. $\mu|_A \in \mathsf{C}_\mathsf{c}(A, \mathbb{R}^n)^*$.

For all $f \in \mathsf{C}_\mathsf{c}(A, \mathbb{R}^n)$, it follows from exercise 1.69 that $\mu|_A \cdot f = \int \langle \widetilde{f}, \nu \rangle \,\mathrm{d}|\mu| = \int_A \langle \widetilde{f}, \nu \rangle \,\mathrm{d}|\mu| = \int \langle f, \nu|_A \rangle \,\mathrm{d}|\mu|\big|_A$, where $|\mu|\big|_A$ denotes the trace of $|\mu|$ on A in the sense of definition 1.13. Since $|\mu|\big|_A$ is a positive Radon measure on $\mathscr{B}_A$ (it is a Borelian measure by proposition 1.15.ii and it is finite on compact subsets of A, hence it is Radon by remark 1.33), and since $\nu|_A$ is Borelian with $\|\nu|_A\| = 1$ $|\mu|\big|_A$-a.e., it follows that the polar decomposition of $\mu|_A$ is $(\nu|_A, |\mu|\big|_A)$. In particular, $|\mu|_A| = |\mu|\big|_A$. Hence, if μ is finite (i.e. if $|\mu|$ is a positive finite Radon measure), so is $\mu|_A$ (since its total variation is finite). □

We next introduce the pushforward operation for $\mathbb{R}^n$-valued Radon measures by transposition. For that purpose, and for those who are not acquainted with locally convex spaces and LF topologies, we make an ad hoc definition of continuity which generalizes the notion of continuity introduced in definition 4.1 (and coincides with the notion of continuity with respect to the locally convex topologies of the spaces involved, cf. remark 4.2).

DEFINITION 4.37 (continuity of linear maps on $\mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$). Let X and Y be locally compact separable metric spaces.

i) We say that $A \subset \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$ is *bounded* it there exists $K \subset X$ compact such that $A \subset \mathsf{C}_\mathsf{c}^\mathsf{K}(X, \mathbb{R}^n)$ and A is bounded in the latter space (i.e. it bounded as a subset of the Banach space $\mathsf{C}_\mathsf{c}^\mathsf{K}(X, \mathbb{R}^n)$).
ii) We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$ converges to $x \in \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$ if there exists $K \subset X$ compact such that the image of the sequence is contained in $\mathsf{C}_\mathsf{c}^\mathsf{K}(X, \mathbb{R}^n)$, $x \in \mathsf{C}_\mathsf{c}^\mathsf{K}(X, \mathbb{R}^n)$ and $x_n \to x$ in $\mathsf{C}_\mathsf{c}^\mathsf{K}(X, \mathbb{R}^n)$.
iii) We say that a linear map $T : \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n) \to \mathsf{C}_\mathsf{c}(Y, \mathbb{R}^m)$ is continuous if one of the following equivalent conditions hold:
    - $T(A)$ is bounded whenever $A \subset \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$ is bounded.

- $T(x_n) \to 0$ whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathsf{C_c}(X, \mathbb{R}^n)$ such that $x_n \to 0$.

REMARK 4.38. That the two conditions in part iii) of the above definition are indeed equivalent to continuity in the LF topology of the spaces involved, cf. remark 4.2, is a consequence of the fact that LF spaces are *bornological*. Actually, in the above definition, we could replace the codomain of T by any locally convex space; in particular, those conditions may also be used to characterize continuity of linear functionals $\mathsf{C_c}(X, \mathbb{R}^n) \to \mathbb{R}$.

PROPOSITION 4.39. *Let X and Y be locally compact separable metric spaces and $T : \mathsf{C_c}(X, \mathbb{R}^n) \to \mathsf{C_c}(Y, \mathbb{R}^m)$ a linear map.*

i) If T is continuous and μ is an $\mathbb{R}^m$-valued Radon measure on Y, then $\mu \circ T$ is an $\mathbb{R}^n$-valued Radon measure on X.

ii) If T is continuous with respect to the $\mathsf{C_0}$ topology (i.e. the topology induced by $\|\cdot\|_u$) on both domain and codomain, and μ is a finite $\mathbb{R}^m$-valued Radon measure on Y, then $\mu \circ T$ is a finite $\mathbb{R}^n$-valued Radon measure on X.

PROOF. It is immediate from the definitions. □

DEFINITION 4.40. With the notation from the previous proposition, we define the *transpose* of T, $T^{\mathsf{t}} : \mathsf{C_c}(Y, \mathbb{R}^m)^* \to \mathsf{C_c}(X, \mathbb{R}^n)^*$ in case (i) or $T^{\mathsf{t}} : \mathsf{C_0}(Y, \mathbb{R}^m)^* \to \mathsf{C_0}(X, \mathbb{R}^n)^*$ in case (ii), by $T^{\mathsf{t}} \cdot \mu := \mu \circ T$.

EXAMPLE 4.41. Let X be a locally compact separable metric space.

1) Let $T : X \to \mathsf{L}(\mathbb{R}^m, \mathbb{R}^n)$ be a continuous map. We define $\hat{T} : \mathsf{C_c}(X, \mathbb{R}^m) \to \mathsf{C_c}(X, \mathbb{R}^n)$ by $(\hat{T} \cdot f)(x) := T(x) \cdot f(x)$. Then $\hat{T}$ is clearly linear and, for all $K \subset X$ compact and $f \in \mathsf{C_c^K}(X, \mathbb{R}^m)$, we have $\hat{T} \cdot f \in \mathsf{C_c^K}(X, \mathbb{R}^n)$ and $\|\hat{T} \cdot f\|_u \leq \|T|_K\|_u \|f\|_u$, which implies the continuity of $\hat{T}$. The transpose of $\hat{T}$ is given by $\mu \mapsto \mu \llcorner T$, where $\mu \llcorner T$ was defined in part 6) of remark 4.32 (but the situation in that remark is more general, since it the continuity of T is not required).
2) Let $U \subset X$ open. The inclusion $\mathsf{C_c}(U, \mathbb{R}^n) \subset \mathsf{C_c}(X, \mathbb{R}^n)$ (which maps $f \in \mathsf{C_c}(U, \mathbb{R}^n)$ to its extension by 0 on the complement of U) is clearly continuous; its transpose coincides with $\mu \mapsto \mu|_U$, where $\mu|_U$ is the trace of μ on U in the sense of definition 4.35. For a general locally compact subspace $A \subset X$, we cannot define the trace by means of transposition, since, in general, there is no canonical inclusion $\mathsf{C_c}(A, \mathbb{R}^n) \subset \mathsf{C_c}(X, \mathbb{R}^n)$.

PROPOSITION 4.42. *Let X and Y be locally compact separable metric spaces and $f : X \to Y$ a continuous proper map. Then both*

$(\circ f) : \mathsf{C_c}(Y,\mathbb{R}^n) \to \mathsf{C_c}(X,\mathbb{R}^n)$ *and* $(\circ f) : \mathsf{C_0}(Y,\mathbb{R}^n) \to \mathsf{C_0}(X,\mathbb{R}^n)$ *given by* $g \mapsto g \circ f$ *are well-defined and linear continuous.*

Proof. If $g \in \mathsf{C_c}(Y,\mathbb{R}^n)$, then $\operatorname{spt}(g \circ f) \subset f^{-1}(\operatorname{spt} g)$, which is compact, since f is proper, hence $g \circ f \in \mathsf{C_c}(X,\mathbb{R}^n)$. If $g \in \mathsf{C_0}(Y,\mathbb{R}^n)$ and $\epsilon > 0$, there exists $K \subset Y$ compact such that $K^c \subset \{\|g\| < \epsilon\}$. Since f is proper, $f^{-1}(K)$ is compact and $f^{-1}(K)^c = f^{-1}(K^c) \subset \{\|g \circ f\| < \epsilon\}$, hence $g \circ f \in \mathsf{C_0}(X,\mathbb{R}^n)$. Thus, both $(\circ f) : \mathsf{C_c}(Y,\mathbb{R}^n) \to \mathsf{C_c}(X,\mathbb{R}^n)$ and $(\circ f) : \mathsf{C_0}(Y,\mathbb{R}^n) \to \mathsf{C_0}(X,\mathbb{R}^n)$ are well-defined and clearly linear. It remains to prove their continuity. Indeed, for all $K \subset Y$ compact and for all $g \in \mathsf{C_c^K}(Y,\mathbb{R}^n)$, we have $g \circ f \in \mathsf{C_c^{f^{-1}(K)}}(X,\mathbb{R}^n)$ and $\|g \circ f\|_u \le \|g\|_u$, which implies the continuity of $(\circ f) : \mathsf{C_c}(Y,\mathbb{R}^n) \to \mathsf{C_c}(X,\mathbb{R}^n)$, and the continuity of $(\circ f) : \mathsf{C_0}(Y,\mathbb{R}^n) \to \mathsf{C_0}(X,\mathbb{R}^n)$ follows by the same argument. □

Definition 4.43. With the notation from definition 4.42, the transposes $(\circ f)^{\mathsf{t}} : \mathsf{C_c}(X,\mathbb{R}^n)^* \to \mathsf{C_c}(Y,\mathbb{R}^n)^*$ and $(\circ f)^{\mathsf{t}} : \mathsf{C_0}(X,\mathbb{R}^n)^* \to \mathsf{C_0}(Y,\mathbb{R}^n)^*$ are called *pushforward by* f and denoted by $f_\#{:}\mu \mapsto f_\#\mu$.

Proposition 4.44. *Let* X *and* Y *be locally compact separable metric spaces,* $f : X \to Y$ *a continuous proper map and* $\mu \in \mathsf{C_c}(X,\mathbb{R}^n)^*$ *with polar decomposition* $(\nu_X, |\mu|)$*. Suppose that there exists a Borelian map* $\nu_Y : Y \to \mathbb{R}^n$ *such that* $\nu_Y \circ f = \nu_X$*. Then the polar decomposition of* $f_\#\mu$ *is* $(\nu_Y, f_\#|\mu|)$*, where* $f_\#|\mu|$ *is the pushforward of* $|\mu|$ *by* f *in the sense of the definition 1.14. In particular, if* μ *is a positive Radon measure on* X*, the pushforward of* μ *by* f *in the sense of definition 4.43 coincides with the pushforward in the sense of definition 1.14.*

Proof. For all $g \in \mathsf{C_c}(Y,\mathbb{R}^n)$, we have:

$$\begin{aligned} f_\#\mu \cdot g = \mu \cdot (g \circ f) = \int \langle g \circ f, \nu_X \rangle \, \mathrm{d}|\mu| = \\ = \int \langle g \circ f, \nu_Y \circ f \rangle \, \mathrm{d}|\mu| \overset{ex.1.70}{=} \\ = \int \langle g, \nu_Y \rangle \, \mathrm{d} f_\#|\mu|, \end{aligned}$$

where $f_\#|\mu|$ is the pushforward of $|\mu|$ by f in the sense of the definition 1.14. Since $f_\#|\mu|$ is a positive Radon measure on Y, by proposition 1.37, and since $f^{-1}\big(\{\|\nu_Y\| \neq 1\}\big) \subset \{\|\nu_X\| \neq 1\}$ is $|\mu|$-null (i.e. $\{\|\nu_Y\| \neq 1\}$ is $f_\#|\mu|$-null), we conclude that the polar decomposition of $f_\#\mu$ is $(\nu_Y, f_\#|\mu|)$, as asserted.

In particular, if μ is a positive Radon measure on X, the polar decomposition of $\check{\mu}$ (using the notation of definition 4.23 for clarity) is

$(1, \mu)$, so that we can take $\nu_Y \equiv 1$, hence the polar decomposition of $f_\# \check{\mu}$ is $(1, f_\# \mu)$, whence the thesis. $\square$

EXERCISE 4.45. Let X and Y be locally compact separable metric spaces, $T : X \to \mathsf{L}(\mathbb{R}^m, \mathbb{R}^n)$ continuous, $f : X \to Y$ a continuous proper map and μ an $\mathbb{R}^n$-valued Radon measure on X. Suppose that there exists $S : Y \to \mathsf{L}(\mathbb{R}^m, \mathbb{R}^n)$ such that $S \circ f = T$. Then $f_\#(\mu \llcorner T) = f_\# \mu \llcorner S$. In particular, if f is an homeomorphism, then $f_\#(\mu \llcorner T) = f_\# \mu \llcorner f_\# T$, where $f_\# T := T \circ f^{-1}$.

REMARK 4.46. We may define the pushforward with respect to more general maps. For instance, let X and Y be locally compact separable metric spaces, $\mu \in \mathsf{C_c}(X, \mathbb{R}^n)^*$ with polar decomposition $(\nu, |\mu|)$ and $f : X \to Y$ a Borelian map such that $\forall K \subset Y$ compact, $|\mu|\big(f^{-1}(K)\big) < \infty$. We define $f_\# \mu : \mathsf{C_c}(Y, \mathbb{R}^n) \to \mathbb{R}$ by

$$g \mapsto \int \langle g \circ f, \nu \rangle \, \mathrm{d}|\mu|.$$

Note that the integral makes sense, since $\|g \circ f\| \in \mathsf{L}^1(|\mu|)$ (because it is a bounded Borelian map which vanishes in the complement of the $|\mu|$-finite set $f^{-1}(\mathrm{spt}\ g)$). Moreover, $f_\# \mu$ is clearly linear and, for all $K \subset Y$ compact and for all $g \in \mathsf{C}^{\mathsf{K}}_{\mathsf{c}}(Y, \mathbb{R}^n)$, $|f_\# \mu \cdot g| = \left| \int_{f^{-1}(K)} \langle g \circ f, \nu \rangle \, \mathrm{d}|\mu| \right| \leq |\mu|\big(f^{-1}(K)\big) \|g\|_u$, hence $f_\# \mu \in \mathsf{C_c}(Y, \mathbb{R}^n)^*$, i.e. $f_\# \mu$ is indeed an $\mathbb{R}^n$-valued Radon measure on Y. If f is a continuous proper map, the latter pushforward coincides with the one defined by transposition in definition 4.43. As before, we may find the polar decomposition of $f_\# \mu$ from the polar decomposition $(\nu_X, |\mu|)$ if there exists a Borelian map $\nu_Y : Y \to \mathbb{R}^n$ such that $\nu_Y \circ f = \nu_X$. For that purpose, we take $f_\# |\mu|$ in the sense of remark 1.38, which is a positive Radon measure on Y, and repeat the same computations in the proof of the previous proposition, for $g \in \mathsf{C_c}(Y, \mathbb{R}^n)$:

$$\begin{aligned} f_\# \mu \cdot g &= \int \langle g \circ f, \nu_X \rangle \, \mathrm{d}|\mu| = \\ &= \int \langle g \circ f, \nu_Y \circ f \rangle \, \mathrm{d}|\mu| \overset{\text{remark } 1.71}{=} \\ &= \int \langle g, \nu_Y \rangle \, \mathrm{d} f_\# |\mu|, \end{aligned}$$

where $f_\# |\mu|$ is the pushforward of $|\mu|$ by f in the sense of remark 1.38. Since $f_\# |\mu|$ is a Radon measure on Y and $\|\nu_Y\| = 1$ $f_\# |\mu|$-a.e. on Y, we conclude that the polar decomposition of $f_\# \mu$ is $(\nu_Y, f_\# |\mu|)$.

4.3. Weak-star convergence

DEFINITION 4.47. Let X be a locally compact separable metric space. We say that

i) a sequence $(\mu_k)_{k\in\mathbb{N}}$ in $\mathsf{C_c}(X,\mathbb{R}^n)^*$ is *weakly-star convergent* to $\mu \in \mathsf{C_c}(X,\mathbb{R}^n)^*$ (notation: $\mu_k \overset{*}{\rightharpoonup} \mu$) if, for all $f \in \mathsf{C_c}(X,\mathbb{R}^n)$, $\int f \cdot \mathrm{d}\mu_k \to \int f \cdot \mathrm{d}\mu$;
ii) a sequence $(\mu_k)_{k\in\mathbb{N}}$ in $\mathsf{C_0}(X,\mathbb{R}^n)^*$ is *weakly-star convergent in the sense of finite measures* to $\mu \in \mathsf{C_0}(X,\mathbb{R}^n)^*$ (notation: $\mu_k \overset{*\mathsf{f}}{\rightharpoonup} \mu$) if, for all $f \in \mathsf{C_0}(X,\mathbb{R}^n)$, $\int f \cdot \mathrm{d}\mu_k \to \int f \cdot \mathrm{d}\mu$.

REMARK 4.48.

1) Some authors use the nomenclature *locally weakly star convergent* and *weakly star convergent* for our definitions above in i) and ii), respectively.
2) We have used different names to distinguish one from the other, but both types of convergence above are actually the same notion, i.e. convergence of sequences with respect to weak star topologies: the first type in the weak-star dual of $\mathsf{C_c}(X,\mathbb{R}^n)$ and the second in the weak-star dual of $\mathsf{C_0}(X,\mathbb{R}^n)$. Note that, in general, none of these weak-star topologies is first-countable, so that we may have to use nets or filters to handle general topological problems. However, note that, thanks to the Banach-Alaoglu theorem and to the separability of $\mathsf{C_0}(X,\mathbb{R}^n)$, strongly bounded subsets of $\mathsf{C_0}(X,\mathbb{R}^n)^*$ are relatively compact and metrizable in the weak-star topology of $\mathsf{C_0}(X,\mathbb{R}^n)^*$.

PROPOSITION 4.49 (relation between weak-star convergence and weak-star convergence in the sense of finite measures). *Let X be a locally compact separable metric space, $(\mu_k)_{k\in\mathbb{N}}$ a sequence in $\mathsf{C_c}(X,\mathbb{R}^n)^*$ and $\mu \in \mathsf{C_c}(X,\mathbb{R}^n)^*$. The following conditions are equivalent:*

i) $\mu_k \overset{*}{\rightharpoonup} \mu$ *and* $\sup_{k\in\mathbb{N}}|\mu_k|(X) < \infty$.
ii) $(\mu_k)_{k\in\mathbb{N}}$ *is a sequence in* $\mathsf{C_0}(X,\mathbb{R}^n)^*$, $\mu \in \mathsf{C_0}(X,\mathbb{R}^n)^*$ *and* $\mu_k \overset{*\mathsf{f}}{\rightharpoonup} \mu$.

PROOF.

(i ⇒ ii): For every $f \in \mathsf{C_c}(X,\mathbb{R}^n)$ such that $\|f\|_u \le 1$, $\mu \cdot f = \lim \mu_k \cdot f \le \liminf |\mu_k|(X) \le \sup_{k\in\mathbb{N}}|\mu_k|(X) < \infty$; taking the sup over all such f, we conclude that $|\mu|(X) \le \liminf |\mu_k|(X) < \infty$, i.e. $\mu \in \mathsf{C_0}(X,\mathbb{R}^n)^*$. Moreover, given $g \in \mathsf{C_0}(X,\mathbb{R}^n)$ and $\epsilon > 0$, we may take $f \in \mathsf{C_c}(X,\mathbb{R}^n)$ such that $\|f - g\|_u < \epsilon$;

hence, for all $k \in \mathbb{N}$,

$$\begin{aligned}|\mu_k \cdot g - \mu \cdot g| &\leq |\mu_k \cdot g - \mu_k \cdot f| + |\mu_k \cdot f - \mu \cdot f| + |\mu \cdot f - \mu \cdot g| \leq \\ &\leq |\mu_k|(X)\|f-g\|_u + |\mu_k \cdot f - \mu \cdot f| + |\mu|(X)\|f-g\|_u \leq \\ &\leq 2\epsilon \sup_{k\in\mathbb{N}} |\mu_k|(X) + |\mu_k \cdot f - \mu \cdot f|.\end{aligned}$$

Taking $k \to \infty$, it follows that $\limsup|\mu_k \cdot g - \mu \cdot g| \leq 2\epsilon \sup_{k\in\mathbb{N}}|\mu_k|(X)$; by the arbitrariness of the $\epsilon > 0$ taken, it then follows that $\limsup|\mu_k \cdot g - \mu \cdot g| = 0$, i.e. $\mu_k \cdot g \to \mu \cdot g$, whence $\mu_k \xrightharpoonup{*\mathsf{f}} \mu$.

(ii $\Rightarrow$ i): For all $g \in \mathsf{C}_0(X, \mathbb{R}^n)$, $\mu_k \cdot g \to \mu \cdot g$; in particular, that holds for $g \in \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$ and, by the principle of uniform boundedness, $\sup_{k\in\mathbb{N}}|\mu_k|(X) < \infty$ (recall that the operator norm of $\mu_k \in \mathsf{C}_0(X, \mathbb{R}^n)^*$ is $|\mu_k|(X)$).

□

Proposition 4.50. *Let X and Y be locally compact separable metric spaces and $T : \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n) \to \mathsf{C}_\mathsf{c}(Y, \mathbb{R}^m)$ linear continuous. Then $T^\mathsf{t} : \mathsf{C}_\mathsf{c}(Y, \mathbb{R}^m)^* \to \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)^*$ preserves weak-star convergence of sequences. The same holds for weak-star convergence in the sense of finite measures if T is continuous with respect to the C_0 topologies.*

Proof. It is immediate from the definitions. □

Remark 4.51.

1) Actually, with the same hypothesis from the previous proposition , $T^\mathsf{t} : \mathsf{C}_\mathsf{c}(Y, \mathbb{R}^m)^* \to \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)^*$ and $T^\mathsf{t} : \mathsf{C}_0(Y, \mathbb{R}^m)^* \to \mathsf{C}_0(X, \mathbb{R}^n)^*$ are continuous with respect to the corresponding weak-star topologies.
2) In particular, this proposition applies to the operations with $\mathbb{R}^n$-valued Radon measures which may be defined by transpositions: the restriction $\mu \mapsto \mu \llcorner T$ with $T : X \to \mathsf{L}(\mathbb{R}^m, \mathbb{R}^n)$ continuous (example 4.41.1), the trace on open sets $\mu \mapsto \mu|_U$ with $U \subset X$ open (example 4.41.2) and the pushforward by a continuous proper map (definition 4.43 and proposition 4.44).

Exercise 4.52.

a) Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$ convergent to $x \in \mathbb{R}^n$. Then $\delta_{x_k} \xrightharpoonup{*\mathsf{f}} \delta_x$.
b) (concentration of mass) Let $(\mu_k)_{k\in\mathbb{N}}$ be the sequence of positive Radon measures on $\mathbb{R}^n$ given by $(\forall k \in \mathbb{N})\mu_k := k^n \mathcal{L}^n \llcorner (0, k^{-1})^n$. Then $\mu_k \xrightharpoonup{*\mathsf{f}} \delta$.
c) (spreading of mass) Let $(\mu_k)_{k\in\mathbb{N}}$ be the sequence of positive Radon measures on $\mathbb{R}^n$ given by $(\forall k \in \mathbb{N})\mu_k := \sum_{m=1}^k k^{-1}\delta_{m/k}$. Then $\mu_k \xrightharpoonup{*\mathsf{f}} \mathcal{L}^1 \llcorner (0, 1)$.

d) Let $(\mu_k)_{k\in\mathbb{N}}$ be a sequence of signed Radon measures on $\mathbb{R}^n$ (i.e. signed measures on $\mathscr{B}_{\mathbb{R}^n}$ whose total variation is Radon) and μ a signed Radon measure on $\mathbb{R}^n$ such that $\mu_k \overset{*}{\rightharpoonup} \mu$. It is not necessarily true that $\mu_k^+ \overset{*}{\rightharpoonup} \mu^+$, $\mu_k^- \overset{*}{\rightharpoonup} \mu^-$ or $|\mu_k| \overset{*}{\rightharpoonup} |\mu|$.

PROPOSITION 4.53 (foliations by Borel sets for positive Radon measures). *Let X be a locally compact separable metric space, μ a positive Radon measure on X and $(E_\alpha)_{\alpha\in A}$ a disjoint family of Borel sets in X. Then $\{\alpha \in A \mid \mu(E_\alpha) > 0\}$ is countable.*

PROOF. Since X can be covered by countably many relatively compact open sets, it suffices to show that, for each relatively compact open set U, the thesis holds for the finite Radon measure $\nu := \mu \llcorner U$. For each $A' \subset A$ finite, $\sum_{\alpha\in A'} \nu(E_\alpha) = \nu(\dot{\bigcup}_{\alpha\in A'} E_\alpha) \leq \nu(X) < \infty$. It then follows that $\alpha \in A \mapsto \nu(E_\alpha)$ is summable with respect to the counting measure, hence $\{\alpha \in A \mid \nu(E_\alpha) > 0\}$ is countable, as asserted. $\square$

THEOREM 4.54 (characterization of weak-star convergence for positive Radon measures). *Let X be a locally compact separable metric space, $(\mu_k)_{k\in\mathbb{N}}$ a sequence of positive Radon measures in X and μ a positive Radon measure in X. The following conditions are equivalent:*

i) $\mu_k \overset{*}{\rightharpoonup} \mu$.
ii) For all $K \subset X$ compact and for all $U \subset X$ open,

$$\mu(K) \geq \limsup \mu_k(K) \quad \textit{and} \quad \mu(U) \leq \liminf \mu_k(U).$$

iii) For all $E \in \mathscr{B}_X^c$ such that $\mu(\partial E) = 0$, $\mu_k(E) \to \mu(E)$.

Moreover, if $\mu_k \overset{*}{\rightharpoonup} \mu$ and $x \in \operatorname{spt} \mu$, there exists $n \in \mathbb{N}$ and a sequence $(x_k)_{k\geq n}$ in X such that $\forall k \geq n$, $x_k \in \operatorname{spt} \mu_k$ and $x_k \to x$.

PROOF.
(i⇒ii) For each $f \in \mathsf{C_c}(U,\mathbb{R})$ with $|f| \leq 1$, $\mu \cdot f = \lim \mu_k \cdot f \leq \liminf \mu_k(U)$. Taking the sup over all such f, it follows that $\mu(U) \leq \liminf \mu_k(U)$.

Note that $\mu(K) = \inf\{\mu \cdot f \mid f \in \mathsf{C_c}(X,\mathbb{R}), \chi_K \leq f \leq 1\}$ (the inequality $\leq$ is clear and the reverse inequality follows from the outer regularity of μ in K and from the Urysohn lemma 4.5). For each $f \in \mathsf{C_c}(X,\mathbb{R})$ such that $\chi_K \leq f \leq 1$, $\mu \cdot f = \lim \mu_k \cdot f \geq \limsup \int \chi_K \, \mathrm{d}\mu_k = \limsup \mu_k(K)$. Taking the inf over all such f, we conclude that $\mu(K) \geq \limsup \mu_k(K)$.
(ii⇒iii) Let $E \in \mathscr{B}_X^c$ such that $\mu(\partial E) = 0$. Applying (ii) for the compact set $K = \overline{E}$ and for the open set $U = E^{\mathrm{o}}$, it follows that $\mu(E) = \mu(\overline{E}) \geq \limsup \mu_k(\overline{E}) \geq \limsup \mu_k(E)$ and $\mu(E) = \mu(E^{\mathrm{o}}) \leq \liminf \mu_k(E^{\mathrm{o}}) \leq \liminf \mu_k(E)$, whence $\mu(E) = \lim \mu_k(E)$, as asserted.

(iii⇒i) We must show that, for every $f \in \mathsf{C_c}(X,\mathbb{R})$, $\mu_k \cdot f \to \mu \cdot f$. Since an arbitrary $f \in \mathsf{C_c}(X,\mathbb{R})$ can be written as $f = f^+ - f^-$ with $f^\pm \geq 0$ in $\mathsf{C_c}(X,\mathbb{R})$, it suffices to consider the case $f \geq 0$. So, fix $f \geq 0$ in $\mathsf{C_c}(X,\mathbb{R})$.

Since f is continuous with compact support, for every $t > 0$ the set $\{f > t\}$ is open relatively compact and $\partial\{f > t\} \subset \{f = t\}$. We may apply proposition 4.53 to the disjoint family of Borel sets $(\{f = t\})_{t>0}$ to conclude that there exists a countable set $I \subset (0,\infty)$ such that $\mu(\{f = t\}) = 0$ for $t \in (0,\infty) \setminus I$; hence $\mu(\partial\{f > t\}) = 0$ for $t \in (0,\infty) \setminus I$. It then follows from (iii) that, for every $t \in (0,\infty) \setminus I$, $\mu_k(\{f > t\}) \to \mu(\{f > t\})$.

Define $\forall k \in \mathbb{N}$, $F, F_k : (0,\infty) \to [0,\infty)$ by $F_k(t) := \mu_k(\{f > t\})$ and $F(t) := \mu(\{f > t\})$. Then F, F_k are Borelian (since they are decreasing functions) and, as we saw in the previous paragraph, $F_k \to F$ pointwise in $(0,\infty) \setminus I$, i.e. $F_k \to F$ $\mathcal{L}^1$-a.e. (since countable sets have Lebesgue measure 0). We contend that the convergence is dominated. Indeed, $\forall k \in \mathbb{N}$, $\forall t > 0$, $F_k(t) \leq \mu_k(\mathrm{spt}\ f)\chi_{[0,\|f\|_u]}(t)$; hence, if we show that $C := \sup_{k\in\mathbb{N}} \mu_k(\mathrm{spt}\ f) < \infty$, then the sequence F_k will be dominated by the $\mathcal{L}^1$-summable function $C\chi_{[0,\|f\|_u]}$, thus proving our contention.

In order to show that $\sup_{k\in\mathbb{N}} \mu_k(\mathrm{spt}\ f) < \infty$, take a relatively compact open set $U \supset K := \mathrm{spt}\ f$ and, denoting by d the distance in X, let $\forall 0 < \epsilon < d(K, U^c)$, $U_\epsilon := \{x \in X \mid d(x,K) < \epsilon\}$; applying proposition 4.53 to the Radon measure μ and the disjoint family of Borel sets $(\partial U_\epsilon)_{0<\epsilon<d(K,U^c)}$, we conclude that there exists $0 < \epsilon < d(K,U^c)$ such that $\mu(\partial U_\epsilon) = 0$. Since U_ϵ is open and relatively compact (because it is contained in U), it then follows from (iii) that $\mu_k(U_\epsilon) \to \mu(U_\epsilon)$. Hence $\sup_{k\in\mathbb{N}} \mu_k(\mathrm{spt}\ f) \leq \sup_{k\in\mathbb{N}} \mu_k(U_\epsilon) < \infty$, as asserted.

Finally, by the dominated convergence theorem 1.64 and by the layer-cake formula 1.87 (note that μ, μ_k are σ-finite, since they are Radon and X is σ-compact), we have:

$$\mu_k \cdot f = \int f \, \mathrm{d}\mu_k = \int_0^\infty F_k \, \mathrm{d}\mathcal{L}^1 \to$$
$$\to \int_0^\infty F \, \mathrm{d}\mathcal{L}^1 = \int f \, \mathrm{d}\mu = \mu \cdot f,$$

as we wanted to show.

Finally, suppose that $\mu_k \xrightharpoonup{*} \mu$ and let $x \in \mathrm{spt}\ \mu$. We claim that $\forall \epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\forall k \geq n$, $\mathbb{U}(x,\epsilon) \cap \mathrm{spt}\ \mu_k \neq \emptyset$. Arguing by contradiction, suppose that there exists $\epsilon > 0$ and a sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $k_n \to \infty$ and $\mathbb{U}(x,\epsilon) \cap \mathrm{spt}\ \mu_{k_n} = \emptyset$. It then follows from ii) that $\mu\big(\mathbb{U}(x,\epsilon)\big) \leq \liminf \mu_k\big(\mathbb{U}(x,\epsilon)\big) \leq \lim \mu_{k_n}\big(\mathbb{U}(x,\epsilon)\big) = 0$, hence $\mu\big(\mathbb{U}(x,\epsilon)\big) = 0$, which contradicts the fact that $x \in \operatorname{spt} \mu$ and proves the claim. We now apply the claim for $\epsilon = i \in \mathbb{N}$, yielding a sequence $(n_i)_{\in\mathbb{N}}$, which we may assume to be strictly increasing. Then, for each $i \in \mathbb{N}$, we may choose $x_{n_i}, x_{n_i+1}, \dots, x_{n_{i+1}-1}$ such that $x_j \in \mathbb{U}(x, 1/i) \cap \operatorname{spt} \mu_j$ for $n_i \le j < n_{i+1}$, thus yielding a sequence $(x_j)_{j \ge n_1}$ in X such that $\forall j \ge n_1$, $x_j \in \operatorname{spt} \mu_j$ and $x_j \to x$.

□

EXERCISE 4.55. Let X be a locally compact separable metric space, $(\mu_k)_{k\in\mathbb{N}}$ a sequence of positive Radon measures on X and μ a positive Radon measure on X. Suppose that $\mu_k \overset{*}{\rightharpoonup} \mu$ and for every $r > 0$, $\limsup_{k\to\infty} \inf\{\mu_k\big(\mathbb{U}(x,r)\big) \mid x \in \operatorname{spt} \mu_k\} > 0$. If $(x_k)_{k\in\mathbb{N}}$ is a convergent sequence in X with $\forall k \in \mathbb{N}$, $x_k \in \operatorname{spt} \mu_k$, then $\lim x_k \in \operatorname{spt} \mu$.

EXERCISE 4.56 (narrow convergence). We say that a sequence $(\mu_k)_{k\in\mathbb{N}}$ in $\mathsf{C}_0(X,\mathbb{R}^n)^*$ is *narrowly convergent* to $\mu \in \mathsf{C}_0(X,\mathbb{R}^n)^*$ (notation: $\mu_k \overset{*\,\mathsf{nc}}{\rightharpoonup} \mu$) if, for all $f \in \mathsf{C}_\mathsf{b}(X,\mathbb{R}^n)$, $\int f \cdot \mathrm{d}\mu_k \to \int f \cdot \mathrm{d}\mu$, where $\mathsf{C}_\mathsf{b}(X,\mathbb{R}^n)$ denotes the Banach space of bounded continuous functions $X \to \mathbb{R}^n$ (endowed with the norm of uniform convergence $\|\cdot\|_u$). That is, $\mu_k \overset{*\,\mathsf{nc}}{\rightharpoonup} \mu$ if it converges to μ in the weak-star dual of $\mathsf{C}_\mathsf{b}(X,\mathbb{R}^n)$.

If $(\mu_k)_{k\in\mathbb{N}}$ is a sequence of positive finite Radon measures on X and μ is a positive finite Radon measure on X, then $\mu_k \overset{*\,\mathsf{nc}}{\rightharpoonup} \mu$ iff $\mu_k(X) \to \mu(X)$ and $\forall A \subset X$ open, $\mu(A) \le \liminf \mu_k(A)$.

HINT. To prove that the stated condition implies $\mu_k \overset{*\,\mathsf{nc}}{\rightharpoonup} \mu$, it suffices to show that $\int g \,\mathrm{d}\mu_k \to \int g \,\mathrm{d}\mu$ for $g \in \mathsf{C}_\mathsf{b}(X,\mathbb{R})$ with $0 \le g \le 1$ (since $\mathsf{C}_\mathsf{b}(X,\mathbb{R})$ is the linear span of such g). Prove that $\int g \,\mathrm{d}\mu \le \liminf \int g \,\mathrm{d}\mu_k$, using the layer-cake formula 1.87 to compute the integrals and Fatou's lemma. The same holds for $1 - g$ in the place of g.

PROPOSITION 4.57 (weak convergence and total variation, part I). *Let X be a locally compact separable metric space and $(\mu_k)_{k\in\mathbb{N}}$ a sequence in $\mathsf{C}_\mathsf{c}(X,\mathbb{R}^n)^*$ weakly-star convergent to $\mu \in \mathsf{C}_\mathsf{c}(X,\mathbb{R}^n)^*$. Then, for every $A \subset X$ open, $|\mu|(A) \le \liminf |\mu_k|(A)$.*

PROOF. For every $f \in \mathsf{C}_\mathsf{c}(X,\mathbb{R}^n)$ with $\operatorname{spt} f \subset A$ and $\|f\| \le 1$, we have $\mu \cdot f = \lim \mu_k \cdot f \le \liminf |\mu_k|(A)$. Taking the sup over all such f yields the thesis. □

PROPOSITION 4.58 (weak convergence and total variation, part II). *Let X be a locally compact separable metric space and $(\mu_k)_{k\in\mathbb{N}}$ a sequence in $\mathsf{C}_\mathsf{c}(X,\mathbb{R}^n)^*$ weakly-star convergent to $\mu \in \mathsf{C}_\mathsf{c}(X,\mathbb{R}^n)^*$.*

i) If ν is a positive Radon measure on X and $|\mu_k| \overset{*}{\rightharpoonup} \nu$, then $\forall E \subset X$, $|\mu|(E) \leq \nu(E)$. Moreover, if $E \in \mathscr{B}^c_X$ and $\nu(\partial E) = 0$, then $\mu_k(E) \to \mu(E)$.

ii) If $|\mu_k|(X) \to |\mu|(X) < \infty$, then $|\mu_k| \overset{*\mathsf{f}}{\rightharpoonup} |\mu|$ (actually $|\mu_k| \overset{*\mathsf{nc}}{\rightharpoonup} |\mu|$ in the sense of exercise 4.56).

PROOF.

i) It suffices, by outer regularity, to prove the inequality for $E \subset X$ open. Let $A \subset X$ open with $A \Subset E$, and take $f \in \mathsf{C_c}(X, \mathbb{R})$ such that $\chi_A \leq f \leq 1$ and spt $f \subset E$ (which exists, by Urysohn's lemma 4.5). Then

$$\begin{aligned}|\mu|(A) &\overset{4.57}{\leq} \liminf |\mu_k|(A) \leq \\ &\leq \liminf \int f \,\mathrm{d}|\mu_k| \overset{|\mu_k| \overset{*}{\rightharpoonup} \nu}{=} \lim \int f \,\mathrm{d}\nu \leq \\ &\leq \nu(E).\end{aligned}$$

Since, by inner regularity, $|\mu|(E) = \sup\{|\mu|(K) \mid K \subset E \text{ compact}\} = \sup\{|\mu|(A) \mid A \subset E \text{ open}, A \Subset E\}$, taking the sup in the inequality above yields $|\mu|(E) \leq \nu(E)$, as asserted.

Suppose that $E \in \mathscr{B}^c_X$ with $\nu(\partial E) = 0$. Fix $\epsilon > 0$. It follows from lemma 4.59 below that there exists $K \subset X$ compact and $A \subset X$ open such that $\overline{A} \subset E \subset K^{\mathrm{o}}$ and $\nu(K \setminus A) < \epsilon$. Take $f \in \mathsf{C_c}(X, \mathbb{R})$ such that $\chi_{\overline{A}} \leq f \leq 1$ and spt $f \subset K^{\mathrm{o}}$. We then have, for all $k \in \mathbb{N}$:

$$\Big|\int f \,\mathrm{d}\mu_k - \mu_k(E)\Big| \leq \int |f - \chi_E| \,\mathrm{d}|\mu_k| \leq |\mu_k|(K \setminus A)$$
$$\Big|\int f \,\mathrm{d}\mu - \mu(E)\Big| \leq |\mu|(K \setminus A) \leq \nu(K \setminus A)$$

Thus,

$$\begin{aligned}|\mu_k(E) - \mu(E)| &\leq \Big|\int f \,\mathrm{d}\mu_k - \mu_k(E)\Big| + \Big|\int f \,\mathrm{d}\mu - \mu(E)\Big| + \Big|\int f \,\mathrm{d}\mu_k - \int f \,\mathrm{d}\mu\Big| \leq \\ &\leq |\mu_k|(K \setminus A) + \nu(K \setminus A) + \Big|\int f \,\mathrm{d}\mu_k - \int f \,\mathrm{d}\mu\Big|.\end{aligned}$$

Since $\lim_{k\to\infty}\big|\int f \,\mathrm{d}\mu_k - \int f \,\mathrm{d}\mu\big| = 0$ and, by the fact that $K \setminus A$ is compact, $|\mu_k| \overset{*}{\rightharpoonup} \nu$ and theorem 4.54.ii), $\limsup |\mu_k|(K \setminus A) \leq \nu(K \setminus A)$, it follows that

$$\limsup |\mu_k(E) - \mu(E)| \leq 2\nu(K \setminus A) < 2\epsilon.$$

By the arbitrariness of the $\epsilon > 0$ taken, we conclude that $\mu_k(E) \to \mu(E)$, as asserted.

ii) It follows from proposition 4.57 that, for all $A \subset X$ open, $|\mu|(A) \leq \liminf|\mu_k|(A)$. Since $|\mu_k|(X) \to |\mu|(X) < \infty$, we may assume, excluding the first terms of the sequence if necessary, that $|\mu_k|(X) < \infty$ for all $k \in \mathbb{N}$. It then follows from exercise 4.56 that $|\mu_k| \xrightharpoonup{*\,\mathsf{nc}} |\mu|$; in particular, $|\mu_k| \xrightharpoonup{*\,\mathsf{f}} |\mu|$.

□

Lemma 4.59. *Let X be a locally compact Hausdorff space, μ a positive Radon measure on X and $E \in \mathscr{B}_X^{\mathsf{c}}$ such that $\mu(\partial E) = 0$. Then, for every $\epsilon > 0$, there is a compact set $K \subset X$ and an open set $A \subset X$ such that $\overline{A} \subset E \Subset K^{\mathrm{o}}$ and $\mu(K \setminus A) < \epsilon$.*

Proof. Fix $\epsilon > 0$. Since $E \Subset X$, we may take a compact set $K \subset X$ such that $E \Subset K^{\mathrm{o}}$ and $\nu(K \setminus E) < \epsilon/2$. Indeed, by outer regularity, there exists $U \supset \overline{E}$ open such that $\nu(U \setminus \overline{E}) < \epsilon/2$; take a relatively compact open set V such that $\overline{E} \subset V \Subset U$ (which exists, since $\overline{E}$ is compact) and put $K := \overline{V}$. Then $E \Subset K^{\mathrm{o}}$ and, as $\nu(\partial E) = 0$, we have $\nu(K \setminus E) = \nu(K \setminus \overline{E}) \leq \nu(U \setminus \overline{E}) < \epsilon/2$.

Similarly, by inner regularity there exists a compact set $C \subset E^{\mathrm{o}}$ such that $\mu(E^{\mathrm{o}} \setminus C) < \epsilon/2$. Take a relatively compact open set $A \subset X$ such that $C \subset A \Subset E^{\mathrm{o}}$. Since $\mu(\partial E) = 0$, we have $\mu(E \setminus A) = \mu(E^{\mathrm{o}} \setminus A) \leq \mu(E^{\mathrm{o}} \setminus C) < \epsilon/2$. Finally, since $K \setminus A = (K \setminus E) \cup (E \setminus A)$, the thesis follows. □

Exercise 4.60. Let X be a locally compact separable metric space, $(\mu_j)_{j \in \mathbb{N}}$ a sequence of $\mathbb{R}^n$-valued Radon measures on X weakly-star convergent to an $\mathbb{R}^n$-valued Radon measure μ on X and $(V_m)_{m \in \mathbb{N}}$ an increasing sequence of relatively compact open subsets of X such that $X = \cup_{m \in \mathbb{N}} V_m$. Suppose that $\forall m \in \mathbb{N}$, $\lim_{j \to \infty} |\mu_j|(V_m) = |\mu|(V_m)$. Then $|\mu_j| \xrightharpoonup{*} |\mu|$.

Theorem 4.61 (De La Vallée Poussin). *Let X be a locally compact separable metric space and $(\mu_k)_{k \in \mathbb{N}}$ be a sequence of finite $\mathbb{R}^n$-valued Radon measures on X such that $\sup\{|\mu_k|(X) \mid k \in \mathbb{N}\} < \infty$. Then there exists a finite $\mathbb{R}^n$-valued Radon measure μ on X and a subsequence $(\mu_{k_j})_{j \in \mathbb{N}}$ of $(\mu_k)_{k \in \mathbb{N}}$ such that $\mu_{k_j} \xrightharpoonup{*\,\mathsf{f}} \mu$. Moreover, $|\mu|(X) \leq \liminf|\mu_{k_j}|(X)$.*

Proof. The first assertion is a direct consequence of the fact that strongly closed balls in $\mathsf{C}_0(X, \mathbb{R}^n)^*$ are compact and metrizable in the weak-star topology (hence sequentially compact): the compactness follows from Banach-Alaoglu theorem, and the metrizability follows from the fact that $\mathsf{C}_0(X, \mathbb{R}^n)$ is a separable Banach space.

The second assertion is a consequence of the first and of proposition 4.57. $\square$

REMARK 4.62. The second assertion in the previous proposition is also a consequence of the fact that, for any Banach space Y, the norm $\|\cdot\|$ of Y^* is weakly-star lower semicontinuous, since $\|\cdot\| = \sup\{\langle y, \cdot\rangle \mid y \in Y, \|y\| \leq 1\}$ is the sup of a family of weakly-star continuous functions.

COROLLARY 4.63. *Let X be a locally compact separable metric space and $(\mu_k)_{k\in\mathbb{N}}$ be a sequence of $\mathbb{R}^n$-valued Radon measures on X such that, for any $K \subset X$ compact, $\sup\{|\mu_k|(K) \mid k \in \mathbb{N}\} < \infty$. Then there exists an $\mathbb{R}^n$-valued Radon measure μ on X and a subsequence $(\mu_{k_j})_{j\in\mathbb{N}}$ of $(\mu_k)_{k\in\mathbb{N}}$ such that $\mu_{k_j} \overset{*}{\rightharpoonup} \mu$.*

PROOF. Let $(V_m)_{m\in\mathbb{N}}$ be an increasing sequence of relatively compact open subsets of X such that $\cup_{m\in\mathbb{N}} V_m = X$. We apply De La Vallée Poussin's theorem 4.61 to each of the traces $(\mu_k|_{V_m})_{k\in\mathbb{N}}$, $m \in \mathbb{N}$, and then we use a diagonal argument:

1) Since $\forall k \in \mathbb{N}$, $\left|\mu_k|_{V_1}\right| = |\mu_k|\big|_{V_1}$ (by proposition 4.36) and $\sup_{k\in\mathbb{N}} |\mu_k|(V_1) \leq \sup_{k\in\mathbb{N}} |\mu_k|(\overline{V}_1) < \infty$, there exists $\nu_1 \in \mathsf{C}_0(V_1, \mathbb{R}^n)^*$ and a subsequence $\mu^1 = (\mu^1_k)_{k\in\mathbb{N}}$ of $(\mu_k)_k$ such that $\mu^1_k|_{V_1} \overset{*\mathsf{f}}{\rightharpoonup} \nu_1$.
2) Suppose that we have defined subsequences $\mu^1, \ldots, \mu^i$ of $(\mu_k)_{k\in\mathbb{N}}$ and $\nu_j \in \mathsf{C}_0(V_j, \mathbb{R}^n)^*$ for $1 \leq j \leq i$ such that μ^j is a subsequence of μ^{j-1} for $2 \leq j \leq i$ and $\mu^j_k|_{V_j} \overset{*\mathsf{f}}{\rightharpoonup} \nu_j$ for $1 \leq j \leq i$. We reapply to μ^i the argument of the previous item to find a subsequence μ^{i+1} if μ^i and $\nu_{i+1} \in \mathsf{C}_0(V_{i+1}, \mathbb{R}^n)^*$ such that $\mu^{i+1}_k|_{V_{i+1}} \overset{*\mathsf{f}}{\rightharpoonup} \nu_{i+1}$. Inductively, we have thus defined a sequence $(\mu^i)_{i\in\mathbb{N}}$ of subsequences of the original sequence $(\mu_k)_{k\in\mathbb{N}}$ and a sequence $(\nu_i)_{i\in\mathbb{N}}$ with $\forall i \in \mathbb{N}$, $\nu_i \in \mathsf{C}_0(V_i, \mathbb{R}^n)^*$.
3) Take the subsequence $(\lambda_k)_{k\in\mathbb{N}}$ of $(\mu_k)_{k\in\mathbb{N}}$ given by $\lambda_k := \mu^k_k$. For all $i \in \mathbb{N}$, $(\lambda_k)_{k\in\mathbb{N}}$ is a subsequence of μ^i, hence $\lambda_k|_{V_i} \overset{*\mathsf{f}}{\rightharpoonup} \nu_i$. In particular, given $f \in \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)$ and $i \in \mathbb{N}$ such that spt $f \Subset V_i$, we have
$$\lambda_k \cdot f = \int f \cdot \mathrm{d}\lambda_k = \int f|_{V_i} \cdot \mathrm{d}\lambda_k|_{V_i} \overset{k\to\infty}{\to} \nu_i \cdot f,$$
hence $\nu : \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n) \to \mathbb{R}$ given by $f \mapsto \lim \lambda_k \cdot f$ ($= \nu_i \cdot f$ for any $i \in \mathbb{N}$ such that spt $f \Subset V_i$) is a well-defined linear functional. It is continuous, i.e. $\nu \in \mathsf{C}_\mathsf{c}(X, \mathbb{R}^n)^*$, since, for each $K \subset X$ compact,

we can take $i \in \mathbb{N}$ such that $K \Subset V_i$, hence $\nu|_{\mathsf{C}_\mathsf{c}^\mathsf{K}(X,\mathbb{R}^n)} = \nu_i|_{\mathsf{C}_\mathsf{c}^\mathsf{K}(X,\mathbb{R}^n)}$. Since $\lambda_k \overset{*}{\rightharpoonup} \nu$, the thesis follows.

$\square$

CHAPTER 5

Area and Coarea Formulas

In this chapter we study Lipschitz maps $\mathbb{R}^n \to \mathbb{R}^m$ and two generalizations of the change of variables formula 1.82: the *area formula*, for $n \leq m$, and the *coarea formula* (which is also an extension of Fubini-Tonelli's theorem), for $n \geq m$. Both theorems have the same statement for $n = m$.

5.1. Lipschitz maps on $\mathbb{R}^n$

Recall that, given metric spaces X and Y, a map $f : X \to Y$ is called *Lipschitz* if there exists $C \geq 0$ such that $\forall x, y \in X$, $d_Y\big(f(x), f(y)\big) \leq C d_X(x, y)$. If f is Lipschitz, there exists a smallest such constant C, namely

$$\operatorname{Lip} f := \sup\{\frac{d_Y\big(f(x), f(y)\big)}{d_X(x, y)} \mid x \neq y \in X\},$$

called *Lipschitz constant* of f.

In this section we derive some basic properties of Lipschitz maps $\mathbb{R}^n \to \mathbb{R}^m$.

5.1.1. Extensions of Lipschitz maps. Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ a Lipschitz map. As we will see in subsequent developments, it is useful to be able to extend f to a Lipschitz map with the same Lipschitz constant defined on all of $\mathbb{R}^n$. The theorems stated below ensure the existence of such extensions.

Firstly, we consider the case $m = 1$:

THEOREM 5.1 (McShane's lemma). *Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}$ a Lipschitz map. Define $F : \mathbb{R}^n \to \mathbb{R}$ by:*

$$F(x) := \inf\{f(a) + \operatorname{Lip} f \cdot \|x - a\| \mid a \in A\}. \tag{5.1}$$

Then F extends f and $\operatorname{Lip} F = \operatorname{Lip} f$.

That F is well defined (i.e. the second member in the previous equality is $> -\infty$, so that F indeed takes values in $\mathbb{R}$) will be seen as part of the proof.

The geometric idea behind formula (5.1) is the following: $g : X \to \mathbb{R}$ is a Lipschitz function on the metric space X iff there exists $C \geq 0$

such that $\operatorname{gr} g = \{\big(x, g(x)\big) \mid x \in X\} \subset X \times \mathbb{R}$ lies in the intersection of all cones $\{(x,y) \in X \times \mathbb{R} \mid |y - g(a)| \le C\|x-a\|\}$ for $a \in X$. If that is the case, the least such C is the Lipschitz constant of g.

PROOF.

1) The formula in the statement of the theorem defines $F : \mathbb{R}^n \to [-\infty, \infty)$. We shall prove that $\operatorname{Im} F \subset \mathbb{R}$.
2) If $x \in A$, it is clear that the infimum in (5.1) is attained for $a = x$, since $\forall a \in A$, $f(x) - f(a) \le |f(x) - f(a)| \le \operatorname{Lip} f \|x - a\|$, hence $f(x) \le \inf\{f(a) + \operatorname{Lip} f \cdot \|x-a\| \mid a \in A\}$. Thus, $F(x) = f(x)$, i.e. F extends f.
3) If $x, y \in \mathbb{R}^n$ and $a \in A$, $F(x) \le f(a) + \operatorname{Lip} f \cdot \|x-a\| \le f(a) + \operatorname{Lip} f \cdot \|y-a\| + \operatorname{Lip} f \cdot \|x-y\|$. Taking the infimum of the second member over all $a \in A$, we conclude that $F(x) \le F(y) + \operatorname{Lip} f \cdot \|x-y\|$. In particular, if $x \in A$, we conclude that $F(y) \ge f(x) - \operatorname{Lip} f \cdot \|x-y\| > -\infty$ for all $y \in \mathbb{R}^n$, hence $\operatorname{Im} F \subset \mathbb{R}$.

   Exchanging x and y, we also have $F(y) \le F(x) + \operatorname{Lip} f \cdot \|x-y\|$, so that $|F(x) - F(y)| \le \operatorname{Lip} f \cdot \|x-y\|$. Hence, F is Lipschitz with Lipschitz constant $\le \operatorname{Lip} f$; since it extends f, its Lipschitz constant must be $\operatorname{Lip} f$.

$\square$

For a Lipschitz map $f = (f_1, \ldots, f_m) : A \subset \mathbb{R}^n \to \mathbb{R}^m$, we may apply McShane's lemma to each component of f, yielding a map $F = (F_1, \ldots, F_m) : \mathbb{R}^n \to \mathbb{R}^m$ which extends f with Lipschitz constant $\operatorname{Lip} F \le \sqrt{m} \operatorname{Lip} f$. It is possible, however, to obtain an extension which has the same Lipschitz constant as f:

THEOREM 5.2 (Kirszbraun's theorem). *Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ a Lipschitz map. Then there exists a Lipschitz extension $f : \mathbb{R}^n \to \mathbb{R}^m$ of f such that* $\operatorname{Lip} F = \operatorname{Lip} f$.

PROOF. We refer the reader to [**Mag12**], page 69. $\square$

5.1.2. Rademacher's theorem. We prove in this subsection that every Lipschitz function on $\mathbb{R}^n$ is differentiable in the sense of Fréchet $\mathcal{L}^n$-a.e. on $\mathbb{R}^n$. Besides, its a.e. defined partial derivatives coincide with its *weak* partial derivatives, introduced below.

If Ω is an open subset of $\mathbb{R}^n$ and $X \in \mathsf{C}^1_{\mathsf{c}}(\Omega, \mathbb{R}^n)$, then a direct application of the Fundamental Theorem of Calculus combined with Fubini-Tonelli's theorem yields

$$\int_\Omega \operatorname{div} X \, \mathrm{d}\mathcal{L}^n = 0.$$

If $u \in \mathsf{C}^1(\Omega)$ and $\varphi \in \mathsf{C}^1_\mathsf{c}(\Omega, \mathbb{R}^n)$, the previous equality applied to $X = u\varphi$ yields the elementary Gauss-Green's identity in divergence form:

$$\int_\Omega \langle \nabla u, \varphi \rangle \,\mathrm{d}\mathcal{L}^n = -\int_\Omega u \,\mathrm{div}\, \varphi \,\mathrm{d}\mathcal{L}^n.$$

That motivates, for less regular u, let us say $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$, the introduction of the *distributional gradient* of u (that is, the gradient of u in the sense of the theory of *Schwartz distributions*) as the linear functional $\nabla u : \mathsf{C}^\infty_\mathsf{c}(\Omega, \mathbb{R}^n) \to \mathbb{R}$ given by the second member in the previous equality, i.e.

$$\nabla u \cdot \varphi := -\int_\Omega u \,\mathrm{div}\, \varphi \,\mathrm{d}\mathcal{L}^n.$$

Similarly, for $1 \leq i \leq n$, the *distributional i-th partial derivative* of u is the linear functional $\frac{\partial u}{\partial x_i} : \mathsf{C}^\infty_\mathsf{c}(\Omega) \to \mathbb{R}$ given by

$$\langle \frac{\partial u}{\partial x_i}, \varphi \rangle := -\int_\Omega u \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}\mathcal{L}^n.$$

Whenever those linear functionals are representable as integration of φ against an $\mathsf{L}^1_{\mathsf{loc}}$ function on Ω, we say that u admits *weak gradient* or *weak partial derivatives*:

Definition 5.3 (weak derivatives and gradients). Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$. We say that:

i) For $1 \leq i \leq n$, u has *weak i-th partial derivative* $v_i \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ if $\forall \varphi \in \mathsf{C}^\infty_\mathsf{c}(\Omega)$,

$$\int_\Omega v_i \varphi \,\mathrm{d}\mathcal{L}^n = -\int_\Omega u \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}\mathcal{L}^n.$$

ii) u has *weak gradient* $v \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$ if $\forall \varphi \in \mathsf{C}^\infty_\mathsf{c}(\Omega, \mathbb{R}^n)$,

$$\int_\Omega \langle v, \varphi \rangle \,\mathrm{d}\mathcal{L}^n = -\int_\Omega u \,\mathrm{div}\, \varphi \,\mathrm{d}\mathcal{L}^n. \tag{5.2}$$

We denote the weak derivatives by the same notations used for the classical derivatives, i.e. $\frac{\partial u}{\partial x_i}$ for the i-th weak partial derivative and ∇u for the weak gradient of u, if they exist; if distinction is needed, we use "$\frac{\partial^{\mathsf{w}} u}{\partial x_i}$" or "$\nabla^{\mathsf{w}} u$" for the weak partial derivatives and gradient.

Exercise 5.4 (weak gradients, bis). Weak gradients may be also characterized by means of Gauss-Green identity in gradient form. That is, let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$; then u admits weak gradient $v \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$ iff $\forall \varphi \in \mathsf{C}^\infty_\mathsf{c}(\Omega)$,

$$\int_\Omega \varphi v \,\mathrm{d}\mathcal{L}^n = -\int_\Omega u \nabla\varphi \,\mathrm{d}\mathcal{L}^n. \tag{5.3}$$

EXERCISE 5.5. Let Ω be an open subset of $\mathbb{R}^n$, $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ and $1 \leq i \leq n$. If there exists $\frac{\partial^\mathsf{w} u}{\partial x_i} \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$, then $\forall \varphi \in \mathsf{C}^1_\mathsf{c}(\Omega)$,

$$\int_\Omega \frac{\partial^\mathsf{w} u}{\partial x_i}\,\varphi \,\mathrm{d}\mathcal{L}^n = -\int_\Omega u \frac{\partial \varphi}{\partial x_i}\,\mathrm{d}\mathcal{L}^n.$$

PROPOSITION 5.6. *Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$.*

i) If the weak partial derivatives or weak gradient of u exist, they are unique up to $\mathcal{L}^n$-null sets.
ii) u has weak gradient $v = (v_1, \ldots, v_n) \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$ iff $\forall 1 \leq i \leq n$, u has i-th weak partial derivative $v_i \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$.

PROOF. Part i) follows from the fundamental lemma of the calculus of variations 4.34 and part ii) is immediate from exercise 5.4. □

It is clear that, if $u \in \mathsf{C}^1(\Omega)$, the classical and weak gradients of u coincide. The converse holds in the following sense: if $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ has weak gradient $v \in \mathsf{C}(\Omega, \mathbb{R}^n)$, then $u \in \mathsf{C}^1(\Omega)$. We postpone the proof of this fact to exercise 6.22 in chapter 6.

PROPOSITION 5.7 (vanishing weak gradient). *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ such that $\forall \varphi \in \mathsf{C}^\infty_\mathsf{c}(\Omega)$, $\int_\Omega u \nabla\varphi \,\mathrm{d}\mathcal{L}^n = 0$. Then u coincides $\mathcal{L}^n$-a.e. with a constant function.*

PROOF. 1) For each $\epsilon > 0$, let $\Omega_\epsilon := \{x \in \mathbb{R}^n \mid \mathbb{B}(x, \epsilon) \subset \Omega\} = \{x \in \mathbb{R}^n \mid d(x, \Omega^c) > \epsilon\}$, so that $(\Omega_\epsilon)_{\epsilon > 0}$ is a family of open subsets of Ω which increases to Ω as $\epsilon \downarrow 0$.

Let $(\phi_t)_{t>0}$ be the standard mollifier in $\mathbb{R}^n$. Given $\epsilon > 0$, let $u_\epsilon : \mathbb{R}^n \to \mathbb{R}$ be given by $u_\epsilon = u$ on $\overline{\Omega_{\epsilon/2}}$ and $u_\epsilon = 0$ on $\mathbb{R}^n \setminus \overline{\Omega_{\epsilon/2}}$. Since $\overline{\Omega_{\epsilon/2}} \subset \Omega$, it follows that $u_\epsilon \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n)$ (because, for each compact $K \subset \mathbb{R}^n$, $\int_K |u_\epsilon| \,\mathrm{d}\mathcal{L}^n = \int_{K \cap \overline{\Omega_{\epsilon/2}}} |u| \,\mathrm{d}\mathcal{L}^n < \infty$, since $K \cap \overline{\Omega_{\epsilon/2}}$ is a compact subset of Ω). Take $g_\epsilon := \phi_{\epsilon/2} * u_\epsilon$. Then, by proposition 1.108, $g_\epsilon \in \mathsf{C}^\infty(\mathbb{R}^n)$ and $\nabla g_\epsilon = (\nabla\phi_{\epsilon/2}) * u_\epsilon$. Therefore, since spt $\phi_{\epsilon/2} \subset \mathbb{B}_{\epsilon/2}$, we have, $\forall x \in \Omega_\epsilon$:

$$\begin{aligned}
\nabla g_\epsilon(x) &= \int \nabla\phi_{\epsilon/2}(x-y)u_\epsilon(y)\,\mathrm{d}\mathcal{L}^n(y) = \\
&= -\int \nabla[\phi_{\epsilon/2}(x-\cdot)]u_\epsilon\,\mathrm{d}\mathcal{L}^n = \\
&= -\int_{x+\mathbb{B}_{\epsilon/2}} \nabla[\phi_{\epsilon/2}(x-\cdot)]u_\epsilon\,\mathrm{d}\mathcal{L}^n \overset{x+\mathbb{B}_{\epsilon/2}\subset\overline{\Omega_{\epsilon/2}}}{=} \\
&= -\int_\Omega \nabla[\phi_{\epsilon/2}(x-\cdot)]u\,\mathrm{d}\mathcal{L}^n = 0,
\end{aligned}$$

where the last equality is justified by the fact that spt $\phi_{\epsilon/2}(x-\cdot) \subset x+\mathbb{B}_{\epsilon/2} \subset \Omega$, so that $\phi_{\epsilon/2}(x-\cdot) \in \mathsf{C}^\infty_\mathsf{c}(\Omega)$. That is, $\nabla g_\epsilon \equiv 0$ in the open set Ω_ϵ; by elementary Calculus, it then follows that g_ϵ is a constant function in each connected component of Ω_ϵ.

2) We contend that g_ϵ is convergent to u in $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ as $\epsilon \to 0$, i.e. for each compact $K \subset \Omega$, $\|g_\epsilon - u\|_{\mathsf{L}^1(\mathcal{L}^n|_K)} \to 0$. Indeed, given $K \subset \Omega$ compact, let $\epsilon_0 := \frac{1}{2}d(K,\Omega^c)$. Then, $\forall\, 0 < \epsilon < \epsilon_0$, both $g_\epsilon = \phi_{\epsilon/2} * u_\epsilon$ and $\phi_{\epsilon/2} * u_{\epsilon_0}$ coincide in each $x \in \Omega_{\epsilon_0} \supset K$ with $\int_{x+\mathbb{B}_{\epsilon/2}} \phi_{\epsilon/2}(x-y)u(y)\,\mathrm{d}\mathcal{L}^n(y)$. By exercise 1.115,
$$\phi_{\epsilon/2} * u_{\epsilon_0} \overset{\epsilon\to 0}{\to} u_{\epsilon_0}$$
in $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n)$; therefore, we conclude that $g_\epsilon|_K \to u_{\epsilon_0}|_K = u|_K$ in $\mathsf{L}^1(\mathcal{L}^n|_K)$, thus proving our contention.

3) Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ with $\epsilon_n \downarrow 0$. It follows from the contention in the previous item that $(g_n := g_{\epsilon_n})_{n\in\mathbb{N}}$ is convergent to u in $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$; therefore, for each compact $K \subset \Omega$, there exists a subsequence of $(g_n)_{n\in\mathbb{N}}$ which converges $\mathcal{L}^n$-a.e. on K to $u|_K$. Since Ω is σ-compact, we may take a sequence of compact subsets which increases to Ω and apply a diagonal argument to obtain a subsequence of $(g_n)_{n\in\mathbb{N}}$ which converges $\mathcal{L}^n$-a.e. on Ω to u. We denote such subsequence with the same notation $(g_n)_{n\in\mathbb{N}}$.

4) Let B be an arbitrary open ball with $B \Subset \Omega$. Since $\overline{B}$ is compact and $(\Omega_{\epsilon_n})_{n\in\mathbb{N}}$ increases to Ω as $n \uparrow \infty$, there exists $N \in \mathbb{N}$ such that $\overline{B} \subset \Omega_{\epsilon_n}$ for all $n \geq N$. Since B is connected, it follows from part 1) that g_n is constant on B for every $n \geq N$. We then conclude from part 3) that $(g_n)_{n\in\mathbb{N}}$ is pointwise convergent on B to a constant function. By the arbitrariness of the open ball $B \Subset \Omega$, it follows that $(g_n)_{n\in\mathbb{N}}$ converges pointwise on Ω to a locally constant function $g : \Omega \to \mathbb{R}$; but, since Ω is connected, g must be a constant function. As $(g_n)_{n\in\mathbb{N}}$ converges pointwise to g and converges pointwise $\mathcal{L}^n$-a.e.

to u, it finally follows that $u = g$ $\mathcal{L}^n$-a.e., i.e. u coincides $\mathcal{L}^n$-a.e. with a constant function, as we wanted to show.

$\square$

DEFINITION 5.8 (Sobolev spaces and functions). Let Ω be an open subset of $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}$ and $1 \le p \le \infty$. We say that

i) u is a $(1,p)$-*Sobolev function* if $u \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega)$ and, $\forall 1 \le i \le n$, u has weak partial derivatives $\frac{\partial u}{\partial x_i} \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega)$. We use the notation $\mathsf{W}^{1,\mathsf{p}}(\Omega)$ to denote the space of $(1,p)$-Sobolev functions on Ω.
ii) u is a *local* $(1,p)$-*Sobolev function* if $u \in \mathsf{L}^\mathsf{p}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ and, $\forall 1 \le i \le n$, u has weak partial derivatives $\frac{\partial u}{\partial x_i} \in \mathsf{L}^\mathsf{p}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$. We use the notation $\mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(\Omega)$ to denote the space of local $(1,p)$-Sobolev functions on Ω.

It is immediate from the definitions that $\mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(\Omega)$ and $\mathsf{W}^{1,\mathsf{p}}(\Omega)$ are linear subspaces of $\mathbb{R}^\Omega$, and the weak partial derivatives and weak gradient are linear on these spaces. We further develop the basic theory of weak derivatives and Sobolev spaces in chapter 6. For the moment, we prove that Lipschitz functions on $\mathbb{R}^n$ belong to $\mathsf{W}^{1,\infty}_{\mathsf{loc}}(\mathbb{R}^n)$, but firstly we introduce some notation.

Let $u : \mathbb{R}^n \to \mathbb{R}$ and $\tau \in \mathbb{S}^{n-1}$. For $h \in \mathbb{R} \setminus \{0\}$, we denote by $\tau_h u : \mathbb{R}^n \to \mathbb{R}$ the incremental ratio of u in the direction τ:

$$\tau_h u(x) := \frac{u(x+h\tau) - u(x)}{h}.$$

Note that, by the invariance of the Lebesgue measure under translations, if $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathbb{R}^n)$, $v : \mathbb{R}^n \to \mathbb{R}$ bounded $\mathcal{L}^n$-measurable with compact support and $h \in \mathbb{R} \setminus \{0\}$:

$$\int u(x+h\tau)v(x)\,\mathrm{d}\mathcal{L}^n(x) = \int u(x)v(x-h\tau)\,\mathrm{d}\mathcal{L}^n(x),$$

hence

$$\int \tau_h u(x)v(x)\,\mathrm{d}\mathcal{L}^n(x) = -\int u(x)\tau_{-h}v(x)\,\mathrm{d}\mathcal{L}^n(x). \tag{5.4}$$

PROPOSITION 5.9. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function. Then $f \in \mathsf{W}^{1,\infty}_{\mathsf{loc}}(\mathbb{R}^n)$.*

PROOF. It is clear that $f \in \mathsf{L}^\infty_{\mathsf{loc}}(\mathcal{L}^n)$. We show that f has weak gradient in $\mathsf{L}^\infty(\mathcal{L}^n, \mathbb{R}^n)$.

Let $\tau \in \mathbb{S}^{n-1}$ and $(h_k)_{k\in\mathbb{N}}$ a sequence in $(0,\infty)$ convergent to 0. Then $\forall k \in \mathbb{N}$, $\|\tau_{h_k} f\|_\infty \le \operatorname{Lip} f$. Since $\mathsf{L}^1(\mathcal{L}^n)$ is a separable Banach space with $\mathsf{L}^1(\mathcal{L}^n)^* \equiv \mathsf{L}^\infty(\mathcal{L}^n)$ (in view of Riesz representation theorem 1.79), it follows from Banach-Alaoglu theorem that the closed balls in

$\mathsf{L}^\infty(\mathcal{L}^n)$ are compact and metrizable in the weak-star topology. Hence, passing to a subsequence, if necessary, we may assume that there exists $g_\tau \in \mathsf{L}^\infty(\mathcal{L}^n)$ such that $\tau_{h_k} f \overset{*}{\rightharpoonup} g_\tau$, i.e. for all $v \in \mathsf{L}^1(\mathcal{L}^n)$,

$$\int v\, \tau_{h_k} f \,\mathrm{d}\mathcal{L}^n \overset{k\to\infty}{\to} \int v g_\tau \,\mathrm{d}\mathcal{L}^n. \tag{5.5}$$

Note that, $\forall \varphi \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$, $\forall k \in \mathbb{N}$, $\tau_{-h_k}\varphi$ has support in the compact set $K := \operatorname{spt}\varphi + \mathbb{B}(0, \sup_{k\in\mathbb{N}} h_k)$. Thus, $\forall x \in \mathbb{R}^n$, $f(x)\tau_{-h_k}\varphi(x) \to f(x)\nabla\varphi(x)\cdot\tau$ and the convergence is dominated, since, by the mean value inequality, $\forall k \in \mathbb{N}$, $|f\,\tau_{-h_k}\varphi| \le \|\nabla\varphi\|_\infty\,|f|\,\chi_K \in \mathsf{L}^1(\mathcal{L}^n)$. That justifies the application of the dominated convergence theorem in the last equality below, $\forall \varphi \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$:

$$\begin{aligned}\int g_\tau\,\varphi \,\mathrm{d}\mathcal{L}^n &\overset{(5.5)}{=} \lim_{k\to\infty}\int \tau_{h_k} f\,\varphi\,\mathrm{d}\mathcal{L}^n \overset{(5.4)}{=}\\ &= -\lim_{k\to\infty}\int f\,\tau_{-h_k}\varphi\,\mathrm{d}\mathcal{L}^n \overset{1.64}{=}\\ &= -\int f\,\nabla\varphi\cdot\tau\,\mathrm{d}\mathcal{L}^n.\end{aligned}$$

Taking $\tau = e_i$, $1 \le i \le n$, we conclude that f has weak partial derivatives $g_{e_i} \in \mathsf{L}^\infty(\mathcal{L}^n)$. □

Let $U \subset \mathbb{R}^n$ open and $f : U \to \mathbb{R}$. Recall that f is differentiable at $x_0 \in U$ in the sense of Fréchet if there exists $A \in \mathsf{L}(\mathbb{R}^n, \mathbb{R})$ such that

$$\lim_{h\to 0}\frac{f(x_0+h) - f(x_0) - A\cdot h}{\|h\|} = 0.$$

If that is the case, f has first order partial derivatives at x_0, A satisfying the above condition is unique and coincides with $\langle \nabla f(x_0), \cdot\rangle : \mathbb{R}^n \to \mathbb{R}$; A is called *Fréchet derivative* of f at x_0 and denoted by $\mathsf{D}f(x_0)$.

Equivalently, f is differentiable at x_0 if it satisfies the condition stated in the exercise below. We will use the following two exercises in the proof of Rademacher's theorem.

Exercise 5.10 (characterization of Fréchet differentiability). Let $U \subset \mathbb{R}^n$ open and $f : U \to \mathbb{R}$. Then f is differentiable at $x_0 \in U$ iff there exists $A \in \mathsf{L}(\mathbb{R}^n, \mathbb{R})$ and there exists $r > 0$ such that

$$\lim_{t\to 0^+}\frac{f(x_0+tv) - f(x_0)}{t} = A\cdot v$$

uniformly in $v \in r\mathbb{S}^{n-1}$. If so,

- the above condition holds for all $r > 0$ (i.e. if it holds for some $r > 0$, then it holds for all $r > 0$);

- $A = \mathsf{D}f(x_0)$.

EXERCISE 5.11 (weak gradients under scaling and translations). Let $x \in \mathbb{R}^n$, $h > 0$, $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $\tau \mapsto x + h\tau$ and $u \in \mathsf{W}^{1,1}_{\mathsf{loc}}(\mathbb{R}^n)$. Then $u \circ T \in \mathsf{W}^{1,1}_{\mathsf{loc}}(\mathbb{R}^n)$ and $\nabla^{\mathsf{w}}(u \circ T)(\tau) = h\, \nabla^{\mathsf{w}}\, u(x + h\tau)$.

THEOREM 5.12 (Rademacher's theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. Then f is differentiable in the sense of Fréchet $\mathcal{L}^n$-a.e. and $\nabla f = \nabla^{\mathsf{w}} f$ $\mathcal{L}^n$-a.e.*

PROOF. Recall that $f \in \mathsf{W}^{1,\infty}_{\mathsf{loc}}$, by proposition 5.9, so that $\nabla^{\mathsf{w}} f \in \mathsf{L}^{\infty}_{\mathsf{loc}}(\mathcal{L}^n, \mathbb{R}^n) \subset \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n, \mathbb{R}^n)$. Furthermore, by corollary 3.31 applied to each component of $\nabla^{\mathsf{w}} f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n, \mathbb{R}^n)$, $\mathcal{L}^n$-almost every $x \in \mathbb{R}^n$ is a Lebesgue point of $\nabla^{\mathsf{w}} f$; fix such a Lebesgue point $x \in \mathbb{R}^n$. We will show that f is differentiable at x and $\nabla f(x) = \nabla^{\mathsf{w}} f(x)$.

For each $h > 0$, let $g_h : \mathbb{R}^n \to \mathbb{R}$ be given by

$$g_h(\tau) := \frac{f(x + h\tau) - f(x)}{h}.$$

By exercise 5.10, the thesis follows once we show that $g_h(\tau)$ converges to $\nabla^{\mathsf{w}} f(x) \cdot \tau$ uniformly with respect to τ on $\mathbb{S}^{n-1}$.

Note that $\forall h > 0$, g_h is Lipschitz with $\operatorname{Lip} g_h \le \operatorname{Lip} f$ and $g_h(0) = 0$.

Let $(h_k)_{k\in\mathbb{N}}$ be a sequence in $(0, \infty)$ convergent to 0. We have:

1) By proposition 5.9, $g_h \in \mathsf{W}^{1,\infty}_{\mathsf{loc}}(\mathbb{R}^n)$. Besides, by the linearity of the weak gradient and exercise 5.11, $\forall \tau \in \mathbb{R}^n$:

$$\nabla^{\mathsf{w}}\, g_h(\tau) = \nabla^{\mathsf{w}}\, f(x + h\tau).$$

Hence, the fact that x is a Lebesgue point of $\nabla^{\mathsf{w}} f$ implies that

$$\begin{aligned}\int_{\mathbb{U}(0,1)} |\nabla^{\mathsf{w}}\, g_h(\tau) - \nabla^{\mathsf{w}}\, f(x)|\, \mathrm{d}\tau &= \int_{\mathbb{U}(0,1)} |\nabla^{\mathsf{w}}\, f(x + h\tau) - \nabla^{\mathsf{w}}\, f(x)|\, \mathrm{d}\tau \overset{y=x+h\tau}{=} \\ &= \frac{1}{h^n} \int_{\mathbb{U}(x,h)} |\nabla^{\mathsf{w}}\, f(y) - \nabla^{\mathsf{w}}\, f(x)|\, \mathrm{d}y \overset{h\to 0}{\to} 0,\end{aligned}$$

i.e. $\nabla^{\mathsf{w}}\, g_h$ converges to the constant function $\nabla^{\mathsf{w}}\, f(x)$ in $\mathsf{L}^1(\mathcal{L}^n|_{\mathbb{U}(0,1)})$ as $h \to 0$.

2) $(g_h)_{h>0}$ is equicontinuous and pointwise bounded. It then follows from the Arzelà-Ascoli theorem that there exists $g : \mathbb{R}^n \to \mathbb{R}$ and a subsequence $(h_{k_j})_{j\in\mathbb{N}}$ of $(h_k)_{k\in\mathbb{N}}$ such that $g_j := g_{h_{k_j}} \to g$ uniformly on compact subsets of $\mathbb{R}^n$. In particular, g is Lipschitz with $\operatorname{Lip} g \le \operatorname{Lip} f$ and $g(0) = 0$.

3) $\forall \varphi \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{U}(0,1))$,

$$\begin{aligned}\int \nabla^{\mathsf{w}} g \; \varphi \,\mathrm{d}\mathcal{L}^n &= -\int g \; \nabla\varphi \,\mathrm{d}\mathcal{L}^n = \\ &= -\lim_{j\to\infty} \int g_j \; \nabla\varphi \,\mathrm{d}\mathcal{L}^n = \\ &= \lim_{j\to\infty} \int \nabla^{\mathsf{w}} g_j \; \varphi \,\mathrm{d}\mathcal{L}^n = \\ &= \int \nabla^{\mathsf{w}} f(x) \; \varphi(y) \,\mathrm{d}y.\end{aligned}$$

Thus, from the fundamental lemma of the Calculus of Variations 4.34, it follows that $\nabla^{\mathsf{w}} g(y) = \nabla^{\mathsf{w}} f(x)$ for $\mathcal{L}^n$-a.e. $y \in \mathbb{U}(0,1)$.

4) Define $g_0 : \mathbb{R}^n \to \mathbb{R}$ by $g_0(\tau) := g(\tau) - \nabla^{\mathsf{w}} f(x) \cdot \tau$. Then g_0 is Lipschitz and, by the previous item, $\nabla^{\mathsf{w}} g_0 = 0$ on $\mathbb{U}(0,1)$. Since $\mathbb{U}(0,1)$ is connected, it follows from proposition 5.7 that g_0 coincides $\mathcal{L}^n$-a.e. on $\mathbb{U}(0,1)$ with a constant function. As g_0 is continuous and $g_0(0) = 0$, we conclude that g_0 is identically null on $\mathbb{U}(0,1)$ and, by continuity, identically null on $\mathbb{B}(0,1)$. Thus, $\forall \tau \in \mathbb{B}(0,1)$, $g(\tau) = \nabla^{\mathsf{w}} f(x) \cdot \tau$. Hence, $g_j(\tau) \to \nabla^{\mathsf{w}} f(x) \cdot \tau$ uniformly with respect to $\tau \in \mathbb{B}(0,1)$. Since the sequence $(h_k)_{k\in\mathbb{N}}$ convergent to 0 was arbitrarily taken, we have shown that every such sequence admits a subsequence $(h_{k_j})_{j\in\mathbb{N}}$ such that $g_j = g_{h_{k_j}}$ converges to g in the metric space $\big(\mathsf{C}(\mathbb{B}_n), \|\cdot\|_u\big)$, which implies that $\lim_{h\to 0} g_h = g$ in the same metric space. In particular,

$$\lim_{h\to 0^+} \frac{f(x+h\tau) - f(x)}{h} = \nabla^{\mathsf{w}} f(x) \cdot \tau$$

uniformly with respect to $\tau \in \mathbb{S}^{n-1} \subset \mathbb{B}_n$.

By exercise 5.10, it follows that f is differentiable at x and $\nabla f(x) = \nabla^{\mathsf{w}} f(x)$, as we wanted to show. □

Exercise 5.13. Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. The set D_f of points where f is differentiable in the sense of Fréchet is Borel measurable and $\mathsf{D}f : D_f \to \mathsf{L}(\mathbb{R}^n, \mathbb{R})$ is Borelian.

Corollary 5.14. *If $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}$ is locally Lipschitz, then f is $\mathcal{L}^n|_\Omega$-a.e. differentiable in the sense of Fréchet.*

Proof. We may cover Ω with a countable family $(U_k)_{k\in\mathbb{N}}$ of open subsets of Ω such that $\forall k \in \mathbb{N}$, $f|_{U_k}$ is Lipschitz. For each $k \in \mathbb{N}$, we may extend $f|_{U_k}$ to a Lipschitz function $f_k : \mathbb{R}^n \to \mathbb{R}$, which is differentiable $\mathcal{L}^n$-a.e. on $\mathbb{R}^n$ in view of Rademacher's theorem. As differentiability is a local notion, we conclude that $f|_{U_k}$ is differentiable on the complement of a $\mathcal{L}^n$-null set $S_k \subset U_k$. Then f is differentiable on the complement of the $\mathcal{L}^n$-null set $S = \cup_{k\in\mathbb{N}} S_k$. $\square$

REMARK 5.15. We postpone to 6.16 in chapter 6, after we prove the locality of the weak derivative, the proof that that $f : \Omega \to \mathbb{R}$ locally Lipschitz has weak gradient $\nabla^{\mathsf{w}} f \in \mathsf{L}^\infty_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$, which coincides $\mathcal{L}^n|_\Omega$-a.e. with ∇f.

COROLLARY 5.16. *If $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}^m$ is locally Lipschitz, then f is $\mathcal{L}^n|_\Omega$-a.e. differentiable in the sense of Fréchet.*

PROOF. Apply the previous corollary to each component of f. $\square$

COROLLARY 5.17.

i) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz and $Z_f := \{x \in \mathbb{R}^n \mid f(x) = 0\}$. Then $\mathsf{D}f(x) = 0$ for $\mathcal{L}^n$-a.e. $x \in Z_f$.

ii) Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz and $Y := \{x \in \mathbb{R}^n \mid g\big(f(x)\big) = x\}$. Then $\mathsf{D}g\big(f(x)\big) \circ \mathsf{D}f(x) = \mathrm{id}_{\mathbb{R}^n}$ for $\mathcal{L}^n$-a.e. $x \in Y$.

PROOF.

1) It suffices to prove part i) for $m = 1$ (in the general case, we argue componentwise).
2) Note that $Z_f \in \mathscr{B}_{\mathbb{R}^n}$. Let $x \in Z_f$ such that $\exists \mathsf{D}f(x)$ and

$$\lim_{r\to 0} \frac{\mathcal{L}^n\big(Z \cap \mathbb{B}(x,r)\big)}{\mathcal{L}^n\big(\mathbb{B}(x,r)\big)} = 1.$$

In view of Rademacher's theorem 5.16 and of theorem 3.29 (with $\mathcal{L}^n$ in place of μ and Z_f in place of A), $\mathcal{L}^n$-a.e. $x \in Z_f$ satisfies the above conditions. Therefore, part i) will be proved once we show that $\nabla f(x) = 0$.

Suppose that $\nabla f(x) = a \in \mathbb{R}^n \setminus \{0\}$. Define $S := \{v \in \mathbb{S}^{n-1} \mid \langle a, v\rangle > \frac{1}{2}\|a\|\}$. Note that S is an open neighborhood of $a/\|a\|$ in $\mathbb{S}^{n-1}$. For each $r > 0$, we define $S_r := \{tv \mid 0 < t \le r, v \in S\} \subset \mathbb{B}(0, r)$, so that $S_r = rS_1$.

By exercise 5.10,

$$\lim_{t\to 0} \frac{f(x + tv) - f(x)}{t} = \langle a, v\rangle$$

uniformly on $v \in \mathbb{S}^{n-1}$. It then follows, by the definition of S, that there exists $R > 0$ such that, $\forall 0 < t < R$ and $\forall\, v \in S$,

$$\frac{f(x + tv)}{t} = \frac{f(x + tv) - f(x)}{t} > \frac{1}{2}\|a\| > 0.$$

In particular, $\forall 0 < r < R$, $f > 0$ on $x + S_r$, i.e. $Z_f \cap \mathbb{B}(x,r) \subset \mathbb{B}(x,r) \setminus (x + S_r)$. Consequently, $\forall 0 < r < R$,

$$\frac{\mathcal{L}^n\big(Z_f \cap \mathbb{B}(x,r)\big)}{\mathcal{L}^n\big(\mathbb{B}(x,r)\big)} \le \frac{\mathcal{L}^n\big(\mathbb{B}(x,r) \setminus (x+S_r)\big)}{\mathcal{L}^n\big(\mathbb{B}(x,r)\big)} \overset{1.4}{=}$$
$$= 1 - \frac{\mathcal{L}^n(S_1)}{\alpha(n)},$$

whence

$$\limsup_{r\to 0} \frac{\mathcal{L}^n\big(Z_f \cap \mathbb{B}(x,r)\big)}{\mathcal{L}^n\big(\mathbb{B}(x,r)\big)} \le 1 - \frac{\mathcal{L}^n(S_1)}{\alpha(n)}.$$

In view of our choice of x, the latter inequality implies $1 - \frac{\mathcal{L}^n(S_1)}{\alpha(n)} \ge 1$, hence $\mathcal{L}^n(S_1) = 0$. As S_1 has nonempty interior, we have reached a contradiction, thus showing that $\nabla f(x) \neq 0$ cannot occur.

3) To prove part ii), let $F := g \circ f - \mathrm{id}_{\mathbb{R}^n}$. Then F is locally Lipschitz and $Y = Z_F$; it then follows from part i) that $\mathsf{D}(g \circ f)(x) - \mathrm{id}_{\mathbb{R}^n} = \mathsf{D}F(x) = 0$ for $\mathcal{L}^n$-a.e. $x \in Y$. Therefore, part ii) will be proved once we show that $\mathsf{D}(g\circ f)(x) = \mathsf{D}g\big(f(x)\big)\circ \mathsf{D}g(x)$ for $\mathcal{L}^n$-a.e. $x \in Y$.

   Let $D_f := \{x \in \mathbb{R}^n \mid \exists \mathsf{D}f(x)\}$, $D_g := \{x \in \mathbb{R}^n \mid \exists \mathsf{D}g(x)\}$, and $X := Y \cap D_f \cap f^{-1}(D_g)$. Then $Y \setminus X = (Y \setminus D_f) \cup \big(Y \setminus f^{-1}(D_g)\big)$. If $x \in Y \setminus f^{-1}(D_g)$, then $f(x) \in \mathbb{R}^n \setminus D_g$, hence $x = g\big(f(x)\big) \in g(\mathbb{R}^n \setminus D_g)$. Therefore, $Y \setminus f^{-1}(D_g) \subset g(\mathbb{R}^n \setminus D_g)$, so that

$$Y \setminus X \subset (\mathbb{R}^n \setminus D_f) \cup g(\mathbb{R}^n \setminus D_g).$$

   Since both $\mathbb{R}^n \setminus D_f$ and $\mathbb{R}^n \setminus D_g$ are $\mathcal{L}^n$-null sets (in view of Rademacher's theorem 5.17), and since the image of a $\mathcal{L}^n$-null set by a locally Lipschitz map is $\mathcal{L}^n$-null, it follows that $Y \setminus X$ is $\mathcal{L}^n$-null. On the other hand, $\forall x \in X$, $\exists \mathsf{D}f(x)$ and $\exists \mathsf{D}g\big(f(x)\big)$, hence the chain rule ensures that $\exists \mathsf{D}(g \circ f)(x) = \mathsf{D}g\big(f(x)\big) \circ \mathsf{D}g(x)$.

□

5.1.3. Linear maps and Jacobians. In this subsection we recall some linear algebra and introduce pertinent notations that will be used in the two main theorems which name this chapter.

Definition 5.18. Let V and W be finite-dimensional Hilbert spaces.

i) A linear map $O : \mathsf{V} \to \mathsf{W}$ is called an *orthogonal injection* if $\forall x, y \in \mathsf{V}$, $\langle O \cdot x, O \cdot y\rangle = \langle x, y\rangle$. We denote the set of orthogonal injections $\mathsf{V} \to \mathsf{W}$ by $\mathrm{O}(\mathsf{V}, \mathsf{W})$; we abbreviate $\mathrm{O}(n,m) := \mathrm{O}(\mathbb{R}^n, \mathbb{R}^m)$ and $\mathrm{O}(n) := \mathrm{O}(n,n)$.
ii) Let $T : \mathsf{V} \to \mathsf{W}$ be a linear map. We denote by T^* the *adjoint* of T with respect to the inner products on V and W, i.e. the unique

linear map such that $\forall x \in \mathsf{V}, \forall y \in \mathsf{W}$, $\langle x, T^* \cdot y\rangle = \langle T \cdot x, y\rangle$. If $\mathsf{V} = \mathsf{W}$ and $T = T^*$, we call T *self-adjoint* or *symmetric*. We denote by $\mathrm{Sym}(\mathsf{V})$ the set of symmetric linear maps in $\mathsf{L}(\mathsf{V})$; we abbreviate $\mathrm{Sym}(n) := \mathrm{Sym}(\mathbb{R}^n)$.

iii) We say that a linear map $T : \mathsf{V} \to \mathsf{V}$ is *positive* if it is symmetric and $\forall x \in \mathsf{V}$, $\langle T \cdot x, x\rangle \geq 0$.

Note that $\mathrm{O}(\mathsf{V}, \mathsf{W}) = \emptyset$ if $\dim \mathsf{V} > \dim \mathsf{W}$.

Recall that, for any symmetric linear map T on a finite-dimensional Hilbert space V, there exists an orthonormal basis of V formed by eigenvectors of T. Equivalently, there exist unique $c_1, \ldots, c_k \in \mathbb{R}$ pairwise distinct and unique $P_1, \ldots, P_k \in \mathsf{L}(\mathsf{V})$ such that $\forall 1 \leq i, j \leq k$, $P_i = P_i^*$, $P_i^2 = P_i$, $P_i P_j = 0$ if $i \neq j$, $\sum_{i=1}^k P_i = \mathrm{id}_{\mathsf{V}}$ and $T = \sum_{i=1}^k c_i P_i$; the c_i's are the eigenvalues of T and the P_i's are the orthogonal projections on the corresponding eigenspaces. The decomposition $T = \sum_{i=1}^k c_i P_i$ is called the *spectral resolution* of T.

THEOREM 5.19 (existence of square roots). *If V is a finite-dimensional Hilbert space and $P \in \mathsf{L}(\mathsf{V})$ is a positive operator, there exists a unique positive operator $N \in \mathsf{L}(\mathsf{V})$ such that $N^2 = P$.*

NOTATION. We denote N by $\sqrt{P}$.

PROOF. Let $P = \sum_{i=1}^k c_i E_i$ be the spectral resolution of P. The positiveness of P implies $c_i \geq 0$ for $1 \leq i \leq k$. Define $N := \sum_{i=1}^k \sqrt{c_i} E_i$; then N is positive and $N^2 = P$, thus proving the existence. On the other hand, suppose that M is another positive operator such that $M^2 = P$. Let the spectral resolution of M be $M = \sum_{i=1}^j d_i F_i$. Then $P = M^2 = \sum_{i=1}^j d_i^2 F_i$. By the uniqueness of the spectral resolution of P, it then follows that $j = k$ and, reordering the d_i's if necessary, $c_i = d_i^2$ for $1 \leq i \leq k$, thus proving the uniqueness. □

THEOREM 5.20 (polar decomposition). *Let V and W be finite-dimensional Hilbert spaces and $L : \mathsf{V} \to \mathsf{W}$ be a linear map.*

i) If $\dim \mathsf{V} \leq \dim \mathsf{W}$, there exists a positive $S \in \mathrm{Sym}(\mathsf{V})$ and $O \in \mathrm{O}(\mathsf{V}, \mathsf{W})$ such that
$$L = O \circ S.$$
Moreover, in the above decomposition, $S \in \mathrm{Sym}(\mathsf{V})$ positive is unique, and so is $O \in \mathrm{O}(\mathsf{V}, \mathsf{W})$ if L is injective.

ii) If $\dim \mathsf{V} \geq \dim \mathsf{W}$, there exists a positive $S \in \mathrm{Sym}(\mathsf{W})$ and $O \in \mathrm{O}(\mathsf{W}, \mathsf{V})$ such that
$$L = S \circ O^*.$$
Moreover, in the above decomposition, $S \in \mathrm{Sym}(\mathsf{W})$ positive is unique, and so is $O \in \mathrm{O}(\mathsf{W}, \mathsf{V})$ if L is surjective.

Proof. Part ii) follows from part i) applied to $L^* : \mathsf{W} \to \mathsf{V}$, so it is enough to prove part i).

1) (uniqueness) Suppose that there exists $S \in \mathrm{Sym}(\mathsf{V})$ and $O \in \mathsf{O}(\mathsf{V}, \mathsf{W})$ such that $L = O \circ S$. Then, since $O^*O = \mathrm{id}_{\mathsf{V}}$, it follows that $L^*L = S^2$. As L^*L is positive, we conclude that S is the (unique) positive square root of L^*L given by theorem 5.19. Moreover, if L is injective, so is S, hence S is invertible and we must have $O = L \circ S^{-1}$.
2) (existence) Let $S := \sqrt{L^*L}$. For each $v \in \mathsf{V}$, we must have $O \cdot S \cdot v = L \cdot v$. Thus, define O on the range of S by $O \cdot w := L \cdot v$ if $v \in \mathsf{V}$ is such that $S \cdot v = w$. If $v' \in \mathsf{V}$ is another vector such that $S \cdot v' = w$, we must have $\|L \cdot (v - v')\|^2 = \langle L^*L \cdot (v - v'), v - v' \rangle = \langle S^2 \cdot (v - v'), v - v' \rangle = 0$, hence $L \cdot v = L \cdot v'$, which shows that O is well-defined on the range of S. Besides, it is clearly linear and satisfies $\forall v \in \mathsf{V}$, $L \cdot v = O \cdot S \cdot v$. If $w, w' \in \mathrm{Im}\, S$ and $v, v' \in \mathsf{V}$ are such that $S \cdot v = w$, $S \cdot v' = w'$, we have:
$$\begin{aligned}\langle O \cdot w, O \cdot w' \rangle &= \langle L \cdot v, L \cdot v' \rangle = \\ &= \langle L^*L \cdot v, v' \rangle = \langle S^2 \cdot v, v' \rangle = \\ &= \langle S \cdot v, S \cdot v' \rangle = \langle w, w' \rangle,\end{aligned}$$
hence $O : \mathrm{Im}\, S \to \mathsf{W}$ is orthogonal. Finally, since $\dim \mathsf{V} \leq \dim \mathsf{W}$, we have $\dim(\mathrm{Im}\, S)^\perp \leq \dim(O \cdot \mathrm{Im}\, S)^\perp$, hence we may extend O to an orthogonal injection on V (take any orthonormal set in $(\mathrm{Im}\, S)^\perp$ and map it to an orthonormal set on $(O \cdot \mathrm{Im}\, S)^\perp$), thus yielding $O \in \mathsf{O}(\mathsf{V}, \mathsf{W})$ such that $O \circ S = L$.

□

Definition 5.21 (Jacobian of a linear map). Let V and W be finite-dimensional Hilbert spaces and $L \in \mathsf{L}(\mathsf{V}, \mathsf{W})$, with polar decomposition $O \circ S$ if $\dim \mathsf{V} \leq \dim \mathsf{W}$ or $S \circ O^*$ if $\dim \mathsf{V} > \dim \mathsf{W}$, cf. theorem 5.20. We define the *Jacobian* $[\![L]\!]$ of L by:
$$[\![L]\!] := |\det S|.$$

Remark 5.22.

1) Note that $[\![L]\!]$ is well-defined, by the uniqueness of S in the polar decomposition.
2) It is clear that
$$[\![L]\!] = [\![L^*]\!] = \begin{cases} \sqrt{\det L^*L} & \text{if } \dim \mathsf{V} \leq \dim \mathsf{W} \\ \sqrt{\det LL^*} & \text{if } \dim \mathsf{V} \geq \dim \mathsf{W}. \end{cases}$$

The next theorem provides a useful formula for computing the Jacobian of a linear map $L : \mathsf{V} \to \mathsf{W}$ between finite-dimensional Hilbert spaces.

THEOREM 5.23 (Binet-Cauchy formula). *Let* V *and* W *be finite-dimensional Hilbert spaces with* $n = \dim \mathsf{V} \leq \dim \mathsf{W} = m$. *If* $L \in \mathsf{L}(\mathsf{V}, \mathsf{W})$, *then*

$$[\![L]\!] = \sqrt{\sum_{B \in \mu(m,n)} (\det B)^2},$$

where $\mu(m,n)$ *is the set of* $n \times n$ *minors in some matrix representation of* L *with respect to orthonormal bases on* V *and* W.

Choosing orthonormal bases on V and W, we identify $\mathsf{V} \equiv \mathbb{R}^n$ and $\mathsf{W} \equiv \mathbb{R}^m$. We will use the following notation:

NOTATION. Let $n \leq m$.

1) We denote by:
   - $\Phi(m,n)$ the set of all maps $\{1, \dots, n\} \to \{1, \dots, m\}$.
   - $\Sigma(m,n) := \{\lambda \in \Phi(m,n) \mid \lambda \text{ 1-1}\}$. We abbreviate $\Sigma_n := \Sigma(n,n)$ (i.e the set of permutaions of $\{1, \dots, n\}$).
   - $\Lambda(m,n) := \{\lambda \in \Sigma(m,n) \mid \lambda \text{ strictly increasing}\}$.
2) For $\lambda \in \Lambda(m,n)$, let $S_\lambda := \langle e_{\lambda(i)} \mid 1 \leq i \leq n \rangle \subset \mathbb{R}^m$ and $P_\lambda \in \mathsf{L}(\mathbb{R}^m, S_\lambda)$ the orthogonal projection onto S_λ, i.e. $P_\lambda(x_1, \dots, x_m) := (x_{\lambda(1)}, \dots, x_{\lambda(n)})$.

PROOF. With the above notation in force, we must prove that

$$[\![L]\!]^2 = \sum_{\lambda \in \Lambda(m,n)} (\det P_\lambda \circ L)^2.$$

Let $(L_{ij})_{m \times n}$ be the matrix of L with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. Then the matrix $(A_{ij})_{n \times n}$ of $A := L^* L \in \mathsf{L}(\mathbb{R}^n)$ with respect to the standard basis is given by $A_{ij} = \sum_{k=1}^{m} L_{ki}L_{kj}$. Therefore:

$$
\begin{aligned}
[\![L]\!]^2 = \det A &= \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} = \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \sum_{k=1}^{m} L_{ki}L_{k\sigma(i)} = \\
&= \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) \sum_{\varphi\in\Phi(m,n)} \prod_{i=1}^{n} L_{\varphi(i)i}L_{\varphi(i)\sigma(i)} \overset{(*)}{=} \\
&= \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) \sum_{\varphi\in\Sigma(m,n)} \prod_{i=1}^{n} L_{\varphi(i)i}L_{\varphi(i)\sigma(i)} \overset{\Sigma(m,n)=\dot\cup_{\lambda\in\Lambda(m,n)}\cup_{\theta\in\Sigma_n}\lambda\circ\theta}{=} \\
&= \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) \sum_{\lambda\in\Lambda(m,n)} \sum_{\theta\in\Sigma_n} \prod_{i=1}^{n} L_{\lambda\circ\theta(i),i}L_{\lambda\circ\theta(i),\sigma(i)} = \\
&= \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) \sum_{\lambda\in\Lambda(m,n)} \sum_{\theta\in\Sigma_n} \prod_{\{i=\theta^{-1}(j)|1\le j\le n\}} L_{\lambda\circ\theta(i),i}L_{\lambda\circ\theta(i),\sigma(i)} = \\
&= \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) \sum_{\lambda\in\Lambda(m,n)} \sum_{\theta\in\Sigma_n} \prod_{j=1}^{n} L_{\lambda(j),\theta^{-1}(j)}L_{\lambda(j),\sigma\circ\theta^{-1}(j)} = \\
&= \sum_{\lambda\in\Lambda(m,n)} \sum_{\theta\in\Sigma_n} \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} L_{\lambda(i),\theta(i)}L_{\lambda(i),\sigma\circ\theta(i)} = \\
&= \sum_{\lambda\in\Lambda(m,n)} \sum_{\theta\in\Sigma_n} \sum_{\{\sigma=\rho\circ\theta^{-1}|\rho\in\Sigma_n\}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} L_{\lambda(i),\theta(i)}L_{\lambda(i),\sigma\circ\theta(i)} \overset{\operatorname{sgn}(\sigma)=\operatorname{sgn}(\rho)\cdot\operatorname{sgn}(\theta)}{=} \\
&= \sum_{\lambda\in\Lambda(m,n)} \sum_{\theta\in\Sigma_n} \sum_{\rho\in\Sigma_n} \operatorname{sgn}(\rho)\cdot\operatorname{sgn}(\theta) \prod_{i=1}^{n} L_{\lambda(i),\theta(i)}L_{\lambda(i),\rho(i)} = \\
&= \sum_{\lambda\in\Lambda(m,n)} \Big(\sum_{\theta\in\Sigma_n} \operatorname{sgn}(\theta) \sum_{i=1}^{n} L_{\lambda(i),\theta(i)}\Big)^2 = \\
&= \sum_{\lambda\in\Lambda(m,n)} (\det P_\lambda\circ L)^2,
\end{aligned}
$$

where the equality $(*)$ is justified by the fact that, if $\varphi\in\Phi(m,n)$ is not injective, then

$$\sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} L_{\varphi(i)i}L_{\varphi(i)\sigma(i)} = 0.$$

□

REMARK 5.24. Theorem 5.23 may also be obtained as a corollary of the Pythagorean theorem. Indeed, with the notation preceding the above proof in force, let $\forall 1 \le i \le n$, $v_i := L \cdot e_i = \sum_{k=1}^m L_{ki} e_k \in \mathbb{R}^m$. Consider the n-vector

$$v = v_1 \wedge \cdots \wedge v_n = \sum_{\lambda \in \Lambda(m,n)} (\det P_\lambda \circ L) e_\lambda \in \bigwedge^n \mathbb{R}^m, \tag{5.6}$$

where $e_\lambda := e_{\lambda(1)} \wedge \cdots \wedge e_{\lambda(n)} \in \bigwedge^n \mathbb{R}^m$.

The Euclidean inner product on $\mathbb{R}^m$ induces an inner product on $\bigwedge^n \mathbb{R}^m$ for which $\{e_\lambda \mid \lambda \in \Lambda(m,n)\}$ is an orthonormal basis (cf. [**Fed69**], page 32, or [**dL65**], page 113). For decomposable n-vectors $w = w_1 \wedge \cdots w_n$, $z = z_1 \wedge \cdots \wedge z_n \in \bigwedge^n \mathbb{R}^m$, we have

$$\langle w, z \rangle = \det\big(\langle w_i, z_j \rangle\big)_{1 \le i,j \le n}.$$

Therefore, computing $\|v\|^2$ by the Pythagorean theorem:

$$\begin{aligned} [\![L]\!]^2 &= \det L^* L = \det\big(\langle L^* L \cdot e_i, e_j \rangle\big)_{1 \le i,j \le n} = \\ &= \det\big(\langle L \cdot e_i, L \cdot e_j \rangle\big)_{1 \le i,j \le n} = \\ &= \langle v, v \rangle \overset{\text{Pythagoras}+(5.6)}{=} \\ &= \sum_{\lambda \in \Lambda(m,n)} (\det P_\lambda \circ L)^2. \end{aligned}$$

Definition 5.25 (Jacobian of Lipschitz maps). Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. It follows from Rademacher's theorem 5.12 (applied componentwise) and from exercise 5.13 that f is differentiable in the complement of a Borel set of $\mathcal{L}^n$-null measure and $x \mapsto \mathsf{D} f(x)$ is Borelian $\mathcal{L}^n$-a.e. defined.

We define, for each point x where f is differentiable, the *Jacobian* of f at x,

$$\mathsf{J} f(x) := [\![\mathsf{D} f(x)]\!],$$

so that $\mathsf{J} f$ is a Borelian function defined on the complement of a Borel subset of $\mathbb{R}^n$ of $\mathcal{L}^n$-null measure.

Exercise 5.26. With the notation above, check that $\mathsf{J} f$ is indeed Borelian.

Notation. For a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$, we will use henceforth the following notation:

- $D_f := \{x \in \mathbb{R}^n \mid \exists \mathsf{D} f(x)\}$;
- $J_f^+ := \{x \in D_f \mid \mathsf{J} f(x) > 0\}$;
- $J_f^0 := \{x \in D_f \mid \mathsf{J} f(x) = 0\}$.

5.2. The area formula

In this section we assume $n \leq m$.

LEMMA 5.27 (Area Formula, linear case). *If $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear and $n \leq m$, then $\forall A \subset \mathbb{R}^n$,*

$$\mathcal{H}^n\big(L(A)\big) = [\![L]\!]\mathcal{L}^n(A). \tag{5.7}$$

PROOF. Let $L = O \circ S$ be a polar decomposition of L, cf. theorem 5.20, where $S \in \mathrm{Sym}(n)$ positive and $O \in \mathrm{O}(n,m)$. Then $[\![L]\!] = |\det S|$. We have:

1) If $[\![L]\!] = 0$, then $\det S = 0$, so that $\dim \operatorname{Im} L = \dim \operatorname{Im} S \leq n - 1$. It then follows from exercise 2.23 that $\mathcal{H}$-dim $\operatorname{Im} L \leq n-1$, hence $\mathcal{H}^n\big(L(\mathbb{R}^n)\big) = 0$, whence $\forall A \subset \mathbb{R}^n$, $\mathcal{H}^n\big(L(A)\big) = 0$ and both members of (5.7) are zero.
2) If $[\![L]\!] > 0$, then $\det S > 0$ and $O : \mathbb{R}^n \to \mathbb{R}^m$ is a linear isometry into $\mathbb{R}^n$. It then follows from corollary 2.5 that, for each closed ball $\mathbb{B}(x,r) \subset \mathbb{R}^n$:

$$\begin{aligned}\frac{\mathcal{H}^n\Big(L\big(\mathbb{B}(x,r)\big)\Big)}{\mathcal{L}^n\big(\mathbb{B}(x,r)\big)} &= \frac{\mathcal{H}^n\Big(O \circ S\big(\mathbb{B}(x,r)\big)\Big)}{\mathcal{L}^n\big(\mathbb{B}(x,r)\big)} = \\ &= \frac{\mathcal{H}^n\Big(S\big(\mathbb{B}(x,r)\big)\Big)}{\mathcal{L}^n\big(\mathbb{B}(x,r)\big)} \overset{2.21}{=} \\ &= \frac{\mathcal{L}^n\Big(S\big(\mathbb{B}(x,r)\big)\Big)}{\mathcal{L}^n\big(\mathbb{B}(x,r)\big)} \overset{1.81.i)}{=} \\ &= |\det S| = [\![L]\!].\end{aligned} \tag{5.8}$$

Define $\forall A \subset \mathbb{R}^n$, $\nu(A) := \mathcal{H}^n\big(L(A)\big)$. We contend that ν is a Radon measure on $\mathbb{R}^n$ and $\nu \ll \mathcal{L}^n$. Indeed,

- $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear isomorphism onto $\operatorname{Im} L$. In particular, $L : \mathbb{R}^n \to \operatorname{Im} L$ is a homeomorphism (endowing $\operatorname{Im} L$ with the relative topology), hence the pushforward operation $L_\#$ defines a bijection between Borel measures on $\mathbb{R}^n$ and Borel measures on $\operatorname{Im} L$, with inverse ${L^{-1}}_\#$; moreover, it is clear that this bijection restricts to a bijection between Borel regular measures. Since $\mathcal{H}^n|_{\operatorname{Im} L}$ is a Borel regular measure (which can be checked directly in view of the Borel regularity of $\mathcal{H}^n$ on $\mathbb{R}^m$, or from the fact that the trace $\mathcal{H}^n|_{\operatorname{Im} L}$ coincides with the m-dimensional Hausdorff measure of $\operatorname{Im} L$ as a metric subspace of $\mathbb{R}^m$, by proposition 2.4.1), it follows that $\nu = {L^{-1}}_\#(\mathcal{H}^n|_{\operatorname{Im} L})$ is a Borel regular measure on $\mathbb{R}^n$. Besides, $\forall K \subset \mathbb{R}^n$ compact, it follows from proposition 2.4.3) that $\nu(K) = \mathcal{H}^n\big(L(K)\big) \leq (\operatorname{Lip} L)^n \mathcal{H}^n(K) = (\operatorname{Lip} L)^n \mathcal{L}^n(K) < \infty$. That is, ν is a locally finite Borel regular measure on $\mathbb{R}^n$, hence it is Radon by exercise 1.32.

- If $A \subset \mathbb{R}^n$ is $\mathcal{L}^n$-null, then it follows from proposition 2.4.3) that $\nu(A) = \mathcal{H}^n\big(L(A)\big) \leq (\operatorname{Lip} L)^n \mathcal{H}^n(A) = (\operatorname{Lip} L)^n \mathcal{L}^n(A) = 0$, hence $\nu \ll \mathcal{L}^n$, thus proving our contention.

It follows from (5.8) that $\forall x \in \mathbb{R}^n$, $\Theta^\nu(\mathcal{L}^n, x) = [\![L]\!]$. Recall that, from proposition 3.23, every Borel measure on $\mathbb{R}^n$ has the symmetric Vitaly property, so that theorem 3.40 applies to Radon measures on $\mathbb{R}^n$, from which we conclude that, $\forall A \in \mathscr{B}_{\mathbb{R}^n}$,

$$\nu(A) = \int_A \Theta^\nu(\mathcal{L}^n, x)\, \mathrm{d}\mathcal{L}^n(x) = [\![L]\!]\mathcal{L}^n(A).$$

By Borel regularity, both members must coincide for all $A \subset \mathbb{R}^n$, i.e. $\mathcal{H}^n\big(L(A)\big) = [\![L]\!]\mathcal{L}^n(A)$, as we wanted to show.

□

EXERCISE 5.28. Let $T \in \mathsf{L}(\mathbb{R}^n, \mathbb{R}^m)$, $n \leq m$.

a) If $R \in \mathsf{L}(\mathbb{R}^n)$, then $[\![T \circ R]\!] = [\![T]\!][\![R]\!]$.
b) $[\![T]\!] \leq \|T\|^n$. If T is 1-1, then $\|T^{-1}\|^{-n} \leq [\![T]\!] \leq \|T\|^n$.
c) If $m \leq k$ and $R \in \mathsf{L}(\mathbb{R}^m, \mathbb{R}^k)$, then $[\![R \circ T]\!] \leq \|R\|^n [\![T]\!]$. If R is 1-1, then $\|R^{-1}\|^{-n}[\![T]\!] \leq [\![R \circ T]\!] \leq \|R\|^n [\![T]\!]$.

LEMMA 5.29. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, with $n \leq m$, and $A \subset \mathbb{R}^n$ $\mathcal{L}^n$-measurable. Then:*

i) $f(A)$ is $\mathcal{H}^n$-measurable.
ii) The function $N(f|_A) : \mathbb{R}^m \to [0, \infty]$ given by $y \mapsto \mathcal{H}^0(A \cap f^{-1}\{y\})$ is $\mathcal{H}^n$-measurable.
iii) $\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\, \mathrm{d}\mathcal{H}^n(y) \leq (\operatorname{Lip} f)^n \mathcal{L}^n(A)$.

DEFINITION 5.30 (multiplicity function). With the notation from the previous lemma, $N(f|_A) : y \mapsto \mathcal{H}^0(A \cap f^{-1}\{y\})$ is called the *multiplicity function* of $f|_A$.

REMARK 5.31. Concerning part a) of the previous lemma, for continuous images of Borel sets we have the following theorem. If X is a complete, separable metric space, Y a Hausdorff topological space, μ a Borel measure on Y and $f : X \to Y$ continuous, then $\forall A \in \mathscr{B}_X$, $f(A)$ is μ-measurable — see [**Fed69**], paragraph 2.2.13. Actually, $\forall A \in \mathscr{B}_X$, $f(A)$ is a *Suslin* set. This result is pertinent to the so-called *descriptive set theory*, for which we refer the interested reader to [**Sri98**] or [**Mos09**].

PROOF.

i) Since $\mathcal{L}^n$ is σ-finite, we may take a sequence $(A_k)_{k\in\mathbb{N}}$ in $\sigma(\mathcal{L}^n)$ such that $\forall k \in \mathbb{N}$, $\mathcal{L}^n(A_k) < \infty$ and $\cup_{k\in\mathbb{N}} A_k = A$. Then $f(A) = \cup_{k\in\mathbb{N}} f(A_k)$, so that $f(A) \in \sigma(\mathcal{H}^n)$ once we show that $\forall k \in \mathbb{N}$, $f(A_k) \in \sigma(\mathcal{H}^n)$. It is therefore enough to prove the case in which $\mathcal{L}^n(A) < \infty$. Since $\mathcal{L}^n$ is a Radon measure and $\mathbb{R}^n$ is σ-compact, we may take (by exercise 1.31) an increasing sequence $(K_i)_{i\in\mathbb{N}}$ of compact subsets of A such that $\mathcal{L}^n(K_i) \to \mathcal{L}^n(A)$. Since $A \in \sigma(\mathcal{L}^n)$ and $\mathcal{L}^n(A) < \infty$, it follows that $\mathcal{L}^n(A \setminus K_i) \to 0$, hence $\mathcal{L}^n(A \setminus \cup_{i\in\mathbb{N}} K_i) = 0$. Therefore, by proposition 2.4.3), we conclude that $\mathcal{H}^n\big(f(A \setminus \cup_{i\in\mathbb{N}} K_i)\big) \le (\operatorname{Lip} f)^n \mathcal{L}^n(A \setminus \cup_{i\in\mathbb{N}} K_i) = 0$. Since $\forall i \in \mathbb{N}$, $f(K_i)$ is compact, it follows that $\cup_{i\in\mathbb{N}} f(K_i) \in \mathscr{B}_{\mathbb{R}^m} \subset \sigma(\mathcal{H}^n)$. As $f(A) \setminus \cup_{i\in\mathbb{N}} f(K_i) \subset f(A \setminus \cup_{i\in\mathbb{N}} K_i)$, we conclude that $f(A) \setminus \cup_{i\in\mathbb{N}} f(K_i)$ is $\mathcal{H}^n$-null, i.e. $f(A)$ is the union of a Borel set with an $\mathcal{H}^n$-null set, thus $f(A) \in \sigma(\mathcal{H}^n)$.

ii) We may take a sequence $(\mathcal{F}_i)_{i\in\mathbb{N}}$ such that
   - $\forall i \in \mathbb{N}$, $\mathcal{F}_i = (F^i_j)_{j\in\mathbb{N}}$ is a disjoint family of Borel subsets of $\mathbb{R}^n$ with $\forall j \in \mathbb{N}$, diam $F^i_j \le 1/i$ and $\dot\cup_{j\in\mathbb{N}} F^i_j = \mathbb{R}^n$;
   - $\forall i \in \mathbb{N}$, each F^{i+1}_j is a subset of some F^i_k (so that each F^i_k is a disjoint union of some of the terms of $\mathcal{F}_{i+1}$).

   Let $(g_i)_{i\in\mathbb{N}}$ be the sequence of functions $\mathbb{R}^m \to [0, \infty]$ defined by, $\forall i \in \mathbb{N}$,

$$g_i := \sum_{j\in\mathbb{N}} \chi_{f(A\cap F^i_j)}$$

   (the idea is that, for each $i \in \mathbb{N}$ and $y \in \mathbb{R}^m$, $g_i(y)$ is the number of terms of $\mathcal{F}_i$ which intersect $A \cap f^{-1}\{y\}$; intuitively, g_i increases pointwise to the multiplicity function). The thesis then follows once we show that each g_i is $\mathcal{H}^n$-measurable and $(g_i)_{i\in\mathbb{N}}$ increases pointwise to the multiplicity function $N(f|_A)$ (which implies the $\mathcal{H}^n$-measurability of the latter function in view of theorem 1.41.iv). That is done along the following steps:

   1) $\forall i, j \in \mathbb{N}$, $A \cap F^i_j \in \sigma(\mathcal{L}^n)$, hence $\chi_{f(A\cap F^i_j)}$ is $\mathcal{H}^n$-measurable by part i). Thus, $\forall i \in \mathbb{N}$, $g_i := \sum_{j\in\mathbb{N}} \chi_{f(A\cap F^i_j)}$ is $\mathcal{H}^n$-measurable by theorem 1.41.iv).
   2) $(g_i)_{i\in\mathbb{N}}$ is pointwise increasing. Indeed, $\forall y \in \mathbb{R}^m$ and $i \in \mathbb{N}$, for each $j \in \mathbb{N}$ such that $A \cap f^{-1}\{y\}$ cuts F^i_j, i.e. such that $\chi_{f(A\cap F^i_j)}(y) = 1$, the fact that F^i_j is a union of terms of $\mathcal{F}_{i+1}$ implies the existence of $k = k_i(j) \in \mathbb{N}$ such that $F^{i+1}_k \subset F^i_j$ (thus $k_i(j) \ne k_i(j')$ if $j \ne j'$, i.e. k_i is 1-1) and $A \cap f^{-1}\{y\}$ cuts F^{i+1}_k, i.e. $\chi_{f(A\cap F^{i+1}_k)}(y) = 1$. Then, defining $N_i := \{j \in \mathbb{N} \mid$

$A \cap f^{-1}\{y\} \cap F^i_j \neq \emptyset\}$, we have

$$\begin{aligned} g_i(y) &= \sum_{j\in\mathbb{N}} \chi_{f(A\cap F^i_j)}(y) = \sum_{j\in N_i} \chi_{f(A\cap F^i_j)}(y) = \\ &= \sum_{j\in N_i} \chi_{f(A\cap F^{i+1}_{k_i(j)})}(y) \overset{k_i \text{ is 1-1}}{\leq} \\ &\leq \sum_{j\in\mathbb{N}} \chi_{f(A\cap F^{i+1}_j)}(y) = g_{i+1}(y). \end{aligned}$$

3) $\forall i \in \mathbb{N}$, $g_i \leq N(f|_A)$. Indeed, since $\mathcal{F}_i$ is a disjoint family, for all $y \in \mathbb{R}^m$, $A \cap f^{-1}\{y\} = \dot\bigcup_{j\in\mathbb{N}} A \cap f^{-1}\{y\} \cap F^i_j$. As $\forall j \in \mathbb{N}$, $\mathcal{H}^0(A \cap f^{-1}\{y\} \cap F^i_j) \geq \chi_{f(A\cap F^i_j)}(y)$, it then follows that $N(f|_A)(y) = \mathcal{H}^0(A \cap f^{-1}\{y\}) = \sum_{j\in\mathbb{N}} \mathcal{H}^0(A \cap f^{-1}\{y\} \cap F^i_j) \geq \sum_{j\in\mathbb{N}} \chi_{f(A\cap F^i_j)}(y) = g_i(y)$.

4) $\forall y \in \mathbb{R}^m$, $\forall k \in \mathbb{N}$ such that $k \leq N(f|_A)(y)$, there exists $i \in \mathbb{N}$ such that $g_i(y) \geq k$. Indeed, since $N(f|_A)(y) = \mathcal{H}^0(A \cap f^{-1}\{y\}) \geq k$, we may choose k distinct points $x_1, \dots, x_k \in A \cap f^{-1}\{y\}$. Take $i \in \mathbb{N}$ such that $\|x_p - x_q\| > 1/i$ for $1 \leq p < q \leq k$. Since the terms of $\mathcal{F}_i$ are disjoint with diameters $\leq 1/i$, it follows that $\forall 1 \leq p \leq k$, x_p belong to exactly one of the terms of $\mathcal{F}_i$, say $F^i_{j(p)}$, with $p \mapsto j(p)$ 1-1. Then $g_i(y) = \sum_{j\in\mathbb{N}} \chi_{f(A\cap F^i_j)}(y) \geq \sum_{1\leq p\leq k} \chi_{f(A\cap F^i_{j(p)})}(y) = k$, as asserted.

iii) Let $(g_i)_{i\in\mathbb{N}}$ be the same sequence of functions $\mathbb{R} \to [0, \infty]$ from the previous item, so that $\forall y \in \mathbb{R}^m$, $g_i(y) \uparrow N(f|_A)(y)$. It follows from the monotone convergence theorem 1.62 that:

$$\begin{aligned} \int_{\mathbb{R}^m} N(f|_A)(y)\,\mathrm{d}\mathcal{H}^n(y) &= \lim_{i\to\infty} \int_{\mathbb{R}^m} g_i(y)\,\mathrm{d}\mathcal{H}^n(y) = \\ &= \lim_{i\to\infty} \sum_{j\in\mathbb{N}} \mathcal{H}^n\big(f(A\cap F^i_j)\big) \overset{2.4.3)}{\leq} \\ &\leq \liminf_{i\to\infty} \sum_{j\in\mathbb{N}} (\mathrm{Lip}\, f)^n \mathcal{L}^n(A \cap F^i_j) = \\ &= (\mathrm{Lip}\, f)^n \mathcal{L}^n(A). \end{aligned}$$

□

Definition 5.32. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map with $n \leq m$ and $t > 1$. We say that (E, S) is a *t-linearization* for f if $E \in \mathscr{B}_{\mathbb{R}^n}$ and $S \in \mathrm{Sym}(n) \cap \mathrm{GL}(\mathbb{R}^n)$ satisfy:

i) $\forall x \in E$, f is differentiable at x and $\mathsf{J}f(x) > 0$;
ii) $\forall x, y \in E$, $t^{-1}\|S \cdot x - S \cdot y\| \leq \|f(x) - f(y)\| \leq t\|S \cdot x - S \cdot y\|$;

iii) $\forall x \in E$, $\forall v \in \mathbb{R}^n$, $t^{-1}\|S\cdot v\| \leq \|\mathsf{D}f(x)\cdot v\| \leq t\|S\cdot v\|$.

PROPOSITION 5.33. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map with $n \leq m$, $t > 1$, $E \in \mathscr{B}_{\mathbb{R}^n}$ such that condition i) in definition 5.32 holds and $S \in \mathrm{Sym}(n) \cap \mathrm{GL}(\mathbb{R}^n)$. Then (E, S) is a t-linearization for f* iff *$f|_E$ is 1-1 with Lipschitz inverse and satisfies:*

ii') $\mathrm{Lip}\, f|_E \circ S^{-1} \leq t$ and $\mathrm{Lip}\, S \circ (f|_E)^{-1} \leq t$;

iii') $\forall x \in E$, $\|\mathsf{D}f(x) \circ S^{-1}\| \leq t$ and $\|S \circ \mathsf{D}f(x)^{-1}\| \leq t$.

PROOF. If (E, S) is a t-linearization for f, then:

1) $f|_E$ is 1-1 in view of the first inequality in ii);
2) $f|_E \circ S^{-1}$ is Lipschitz, with $\mathrm{Lip}\, f|_E \circ S^{-1} \leq t$, in view of the second inequality in ii) with $S^{-1}(x')$ in place of x and $S^{-1}(y')$ in place of y;
3) $S \circ (f|_E)^{-1}$ is Lipschitz, with $\mathrm{Lip}\, S \circ (f|_E)^{-1} \leq t$, in view of the first inequality in ii) with $f^{-1}(x')$ in place of x and $f^{-1}(y')$ in place of y;
4) similarly, the second inequality in iii) implies $\|\mathsf{D}f(x) \circ S^{-1}\| \leq t$ and the first inequality in iii) implies $\|S \circ \mathsf{D}f(x)^{-1}\| \leq t$;
5) since S^{-1} and $S \circ (f|_E)^{-1}$ are both Lipschitz, so is $(f|_E)^{-1} = S^{-1} \circ \big(S \circ (f|_E)^{-1}\big)$.

Thus we have proved that $f|_E$ is 1-1 with Lipschitz inverse and satisfies conditions ii') and iii').

With a similar argument, if $f|_E$ is 1-1 with Lipschitz inverse, then conditions ii') and iii') imply ii) and iii), respectively, in definition 5.32, thus proving the converse implication. □

COROLLARY 5.34. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map with $n \leq m$, $t > 1$ and (E, S) a t-linearization for f. Then $\forall x \in E$,*

$$t^{-n}|\det S| \leq \mathsf{J}f(x) \leq t^n|\det S|. \tag{5.9}$$

PROOF.

$$\begin{aligned}\mathsf{J}f(x) &= [\![\mathsf{D}f(x)]\!]|\det S^{-1}||\det S| \overset{5.28.a)}{=} \\ &= [\![\mathsf{D}f(x) \circ S^{-1}]\!]|\det S|.\end{aligned}$$

Hence, from exercise 5.28.b) with $\mathsf{D}f(x) \circ S^{-1}$ in place of T and from proposition 5.33.iii'), the thesis follows. □

THEOREM 5.35 (Lipschitz linearization, [**Fed69**]). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map with $n \leq m$, $t > 1$ and $J_f^+ = \{x \in \mathbb{R}^n \mid \exists \mathsf{D}f(x)$ and $\mathsf{J}f(x) > 0\}$ (which is a Borel set, by exercises 5.13 and 5.26). Then there exists a countable disjoint family $(E_k)_{k\in\mathbb{N}}$ in $\mathscr{B}_{\mathbb{R}^n}$ such that $J_f^+ = \dot\bigcup_{k\in\mathbb{N}} E_k$ and, $\forall k \in \mathbb{N}$, there exists $S_k \in \mathrm{Sym}(n) \cap \mathrm{GL}(\mathbb{R}^n)$ such that (E_k, S_k) is a t-linearization for f.*

Proof. Fix $\epsilon > 0$ such that $t^{-1} + \epsilon < 1 < t - \epsilon$. Let $\mathcal{S}$ be a countable dense subset of $\mathrm{Sym}(n) \cap \mathrm{GL}(\mathbb{R}^n)$ and $\mathcal{G}$ a countable dense subset of J_f^+. For all $S \in \mathcal{S}$, $k \in \mathbb{N}$ and $c \in \mathcal{G}$, we define $E(S,k,c) := \mathbb{B}(c, \frac{1}{2k}) \cap F(S,k)$, where $F(S,k) \subset J_f^+$ is the set of all $x \in J_f^+$ such that[1]:

F1) $\forall v \in \mathbb{R}^n$, $(t^{-1}+\epsilon)\|S \cdot v\| \leq \|\mathsf{D}f(x) \cdot v\| \leq (t-\epsilon)\|S \cdot v\|$;

F2) $\forall v \in \mathbb{R}^n$ such that $\|v\| \leq k^{-1}$, $\|f(x+v) - f(x) - \mathsf{D}f(x) \cdot v\| \leq \epsilon\|S \cdot v\|$.

Since $\mathsf{D}f$ is Borelian (by exercise 5.13), it is clear that $F(S,k) \in \mathscr{B}_{\mathbb{R}^n}$, hence $E(S,k,c) \in \mathscr{B}_{\mathbb{R}^n}$. Furthermore,

1) For all $S \in \mathcal{S}$, $k \in \mathbb{N}$ and $c \in \mathcal{G}$, $\forall x, y \in E(S,k,c)$,

$$\|f(y) - f(x)\| \overset{F2)}{\leq} \|\mathsf{D}f(x) \cdot (y-x)\| + \epsilon\|S \cdot (y-x)\| \overset{F1)}{\leq} t\|S \cdot (y-x)\|,$$

$$\|f(y) - f(x)\| \overset{F2)}{\geq} \|\mathsf{D}f(x) \cdot (y-x)\| - \epsilon\|S \cdot (y-x)\| \overset{F1)}{\geq} t^{-1}\|S \cdot (y-x)\|.$$

Therefore, the condition ii) in definition 5.32 is satisfied for $\big(E(S,k,c), S\big)$. Besides, in view of F1), condition iii) in the same definition is trivially satisfied, so that $\big(E(S,k,c), S\big)$ is a t-linearization for f.

2) We contend that J_f^+ is the union of the countable family $\{E(S,k,c) \mid S \in \mathcal{S}, k \in \mathbb{N}, c \in \mathcal{G}\}$. Once we prove this contention, we enumerate this family as $(\hat{E}_k)_{k\in\mathbb{N}}$ and we take the disjoint sequence $(E_k)_{k\in\mathbb{N}}$ given by $E_k := \hat{E}_k \setminus \cup_{i=1}^{k-1} \hat{E}_i$, thus reaching the thesis in view of the previous item.

   To prove the contention, fix $x \in J_f^+$ and let the polar decomposition of $\mathsf{D}f(x)$ be $\mathsf{D}f(x) = P_x \circ S_x$, with $P_x \in \mathrm{O}(n,m)$ and $S_x \in \mathrm{Sym}(n)$. Note that, since $\mathsf{D}f(x)$ is 1-1, so is S_x, i.e. $S_x \in$

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[1] The idea is the following:

- We want to ensure ii) and iii) in definition 5.32. In order to ensure iii) we might take F1) with t instead of $t - \epsilon$; however, with $t - \epsilon$ it will work as well and, as we shall see, we need a little "space" for the estimate in the next step.
- To ensure ii), we need to use somehow the differentiability of f. Assume that $x, y \in E \subset J_f^+$ with $\mathrm{diam}\,(E) \leq 1/k$ sufficiently small (to be chosen). We then have

$$f(y) - f(x) = \mathsf{D}f(x) \cdot (y-x) + R_x(y-x).$$

  Thus, in order to obtain the desired inequalities in ii) to $\|f(y) - f(x)\|$, say the second one, we must control the norms of both terms in the second member. As $\|\mathsf{D}f(x) \cdot (y-x)\| \leq (t-\epsilon)\|S \cdot x - S \cdot y\|$ if F1) holds, we must ensure that $\|R_x(y-x)\| \leq \epsilon\|S \cdot x - S \cdot y\| = \epsilon\|S \cdot (y-x)\|$ with $1/k$ sufficiently small.

$\mathrm{GL}(\mathbb{R}^n)$. Moreover, since $\mathcal{S}$ is dense in $\mathrm{Sym}(n) \cap \mathrm{GL}(\mathbb{R}^n)$, we may take a sequence $(S_i)_{i\in\mathbb{N}}$ in $\mathcal{S}$ convergent to S_x; hence, by continuity, $S_x \circ S_i^{-1} \to \mathrm{id}_{\mathbb{R}^n}$ and $S_i \circ S_x^{-1} \to \mathrm{id}_{\mathbb{R}^n}$. Taking i sufficiently large, we then conclude that there exists $S \in \mathcal{S}$ such that

$$\|S_x \circ S^{-1}\| \le t - \epsilon \quad \text{and} \quad \|S \circ S_x^{-1}\| \le (t^{-1} + \epsilon)^{-1}.$$

It then follows that, $\forall v \in \mathbb{R}^n$,

$$\begin{aligned}
\|\mathsf{D}f(x)\cdot v\| &= \|P_x \cdot S_x \cdot v\| = \|S_x \cdot v\| = \|(S_x S^{-1})S\cdot v\| \le \\
&\le \|S_x S^{-1}\|\|S\cdot v\| \le (t-\epsilon)\|S\cdot v\|, \\
(t^{-1}+\epsilon)\|S\cdot v\| &= (t^{-1}+\epsilon)\|(SS_x^{-1})S_x\cdot v\| \le \\
&\le (t^{-1}+\epsilon)\|SS_x^{-1}\|\|S_x\cdot v\| \le \|S_x\cdot v\| = \|\mathsf{D}f(x)\cdot v\|.
\end{aligned}$$

That is, F1) is satisfied. Moreover, by the differentiability of f at x, there exists $R(x,\cdot) : \mathbb{R}^n \to [0,\infty)$ continuous and null at $v = 0$, such that $\forall v \in \mathbb{R}^n$:

$$\begin{aligned}
\|f(x+v) - f(x) - \mathsf{D}f(x)\cdot v\| = R(x,v)\|v\| = R(x,v)\|S^{-1}S\cdot v\| \le \\
\le R(x,v)\|S^{-1}\|\|S\cdot v\|.
\end{aligned}$$

Since $\lim_{v\to 0} R(x,v) = R(x,0) = 0$, we may take $k \in \mathbb{N}$ sufficiently large so that $R(x,v)\|S^{-1}\| \le \epsilon$ for $\|v\| \le k^{-1}$, hence F2) is satisfied for this choice of k. We then conclude that $x \in F(S,k)$. Finally, since $\mathcal{G}$ is dense in J_f^+, there exists $c \in \mathcal{G}$ such that $c \in \mathbb{U}(x, \frac{1}{2k}) \Leftrightarrow x \in \mathbb{U}(c, \frac{1}{2k})$, so that $x \in E(S,k,c) = \mathbb{B}(c, \frac{1}{2k}) \cap F(S,k)$, thus proving our contention.

□

THEOREM 5.36 (Area Formula). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \le m$. Then, for all $A \in \sigma(\mathcal{L}^n)$,*

$$\int_A \mathsf{J}f \,\mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y).$$

PROOF. If $\mathcal{L}^n(A) = 0$, the first member is trivially null, and so is the second member in view of lemma 5.29.iii). Therefore, in view of Rademacher's theorem 5.12, we may assume that $A \subset D_f = \{x \in \mathbb{R}^n \mid \exists \mathsf{D}f(x)\}$. Let $J_f^+ = \{x \in D_f \mid \mathsf{J}f(x) > 0\}$ and $J_f^0 = \{x \in D_f \mid \mathsf{J}f(x) = 0\}$, so that $D_f = J_f^+ \,\dot\cup\, J_f^0$.

1) Case 1: $A \subset J_f^+$. Fix $t > 1$. Let $(E_k)_{k\in\mathbb{N}}$ be a sequence in $\mathscr{B}_{\mathbb{R}^n}$ given by the Lipschitz linearization theorem 5.35, i.e. such that $J_f^+ = \dot\bigcup_{k\in\mathbb{N}} E_k$ and, for each $k \in \mathbb{N}$, there exists $S_k \in \mathrm{Sym}(n) \cap \mathrm{GL}(\mathbb{R}^n)$

such that (E_k, S_k) is a t-linearization for f. Then, $\forall k \in \mathbb{N}$,

$$\begin{aligned}\mathcal{H}^n\big(f(A \cap E_k)\big) &= \mathcal{H}^n\big(f|_{E_k} \circ S_k^{-1} \circ S_k(A \cap E_k)\big) \overset{2.4}{\leq} \\ &\leq (\operatorname{Lip} f|_{E_k} \circ S_k^{-1})^n \mathcal{H}^n\big(S_k(A \cap E_k)\big) \overset{5.33.ii')}{\leq} \\ &\leq t^n \mathcal{H}^n\big(S_k(A \cap E_k)\big),\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\mathcal{H}^n\big(S_k(A \cap E_k)\big) &= \mathcal{H}^n\big(S_k \circ (f|_{E_k})^{-1} \circ f(A \cap E_k)\big) \overset{2.4}{\leq} \\ &\leq (\operatorname{Lip} S_k \circ (f|_{E_k})^{-1})^n \mathcal{H}^n\big(f(A \cap E_k)\big) \overset{5.33.ii')}{\leq} \\ &\leq t^n \mathcal{H}^n\big(f(A \cap E_k)\big),\end{aligned} \tag{5.11}$$

On the other hand, it follows from corollary 5.34 that, $\forall k \in \mathbb{N}$, $\forall x \in E_k$,

$$t^{-n}[\![S_k]\!] \leq \mathsf{J}f(x) \leq t^n [\![S_k]\!]. \tag{5.12}$$

Therefore, $\forall k \in \mathbb{N}$:

$$\begin{aligned}t^{-2n}\mathcal{H}^n\big(f(A \cap E_k)\big) &\overset{(5.10)}{\leq} t^{-n}\mathcal{H}^n\big(S_k(A \cap E_k)\big) \overset{5.27}{=} \\ &= t^{-n}[\![S_k]\!]\mathcal{L}^n(A \cap E_k) \overset{(5.12)}{\leq} \\ &\leq \int_{A \cap E_k} \mathsf{J}f(x)\, \mathrm{d}\mathcal{L}^n(x) \overset{(5.12)}{\leq} \\ &\leq t^n [\![S_k]\!]\mathcal{L}^n(A \cap E_k) \overset{5.27}{=} \\ &= t^n \mathcal{H}^n\big(S_k(A \cap E_k)\big) \overset{(5.11)}{\leq} \\ &\leq t^{2n}\mathcal{H}^n\big(f(A \cap E_k)\big)\end{aligned} \tag{5.13}$$

Since, $\forall k \in \mathbb{N}$, $f|_{E_k}$ is 1-1 (by proposition 5.33), we have $\forall y \in \mathbb{R}^m$, $\mathcal{H}^0(A \cap E_k \cap f^{-1}\{y\}) = \chi_{f(A \cap E_k)}(y)$, so that $\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap E_k \cap f^{-1}\{y\})\, \mathrm{d}\mathcal{H}^n(y) = \mathcal{H}^n\big(f(A \cap E_k)\big)$. Therefore, from (5.13) and from the monotone convergence theorem 1.62, we conclude that:

$$\begin{aligned}
t^{-2n}\int_{\mathbb{R}^m}\mathcal{H}^0(A\cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y) &\overset{1.62}{=} \sum_{k\in\mathbb{N}} t^{-2n}\int_{\mathbb{R}^m}\mathcal{H}^0(A\cap E_k\cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y) = \\
&= \sum_{k\in\mathbb{N}} t^{-2n}\mathcal{H}^n\big(f(A\cap E_k)\big) \overset{(5.13)}{\le} \\
&\le \sum_{k\in\mathbb{N}}\int_{A\cap E_k}\mathsf{J}f(x)\,\mathrm{d}\mathcal{L}^n(x) \overset{1.62}{=} \int_A \mathsf{J}f(x)\,\mathrm{d}\mathcal{L}^n(x) \overset{(5.13)}{\le} \\
&\le \sum_{k\in\mathbb{N}} t^{2n}\mathcal{H}^n\big(f(A\cap E_k)\big) = \\
&= \sum_{k\in\mathbb{N}} t^{2n}\int_{\mathbb{R}^m}\mathcal{H}^0(A\cap E_k\cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y) \overset{1.62}{=} \\
&= t^{2n}\int_{\mathbb{R}^m}\mathcal{H}^0(A\cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y),
\end{aligned}$$

thus $t^{-2n}\int_{\mathbb{R}^m}\mathcal{H}^0(A\cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y) \le \int_A \mathsf{J}f\,\mathrm{d}\mathcal{L}^n \le t^{2n}\int_{\mathbb{R}^m}\mathcal{H}^0(A\cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y)$. Taking $t\downarrow 1$, it follows that $\int_{\mathbb{R}^m}\mathcal{H}^0(A\cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y) = \int_A \mathsf{J}f\,\mathrm{d}\mathcal{L}^n$, as asserted.

2) Case 2: $A\subset J_f^0$. Then $\int_A \mathsf{J}f\,\mathrm{d}\mathcal{L}^n = 0$; we must show that $\int \mathcal{H}^0(A\cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y) = 0$. We may assume that $\mathcal{L}^n(A)<\infty$ (since the general case is obtained from this and from the monotone convergence theorem, writing $A=\dot\bigcup_{n\in\mathbb{N}}A_n$, with $\forall n\in\mathbb{N}$, $A_n\in\sigma(\mathcal{L}^n)$ and $\mathcal{L}^n(A_n)<\infty$, which is possible thanks to the σ-finiteness of the Lebesgue measure).

Fix $0<\epsilon<1$. Define $g:\mathbb{R}^n\to\mathbb{R}^{m+n}\equiv\mathbb{R}^m\times\mathbb{R}^n$ by $g(x):=\big(f(x),\epsilon x\big)$. Then g is Lipschitz 1-1 and $\forall x\in D_g=D_f$, $\mathsf{D}g(x)=\big(\mathsf{D}f(x),\epsilon\,\mathrm{id}_{\mathbb{R}^n}\big)\in\mathsf{L}(\mathbb{R}^n,\mathbb{R}^m\times\mathbb{R}^n)$.

We contend that there exists $C=C(n,m,\operatorname{Lip} f)>0$ (in particular, C does not depend on ϵ) such that $\forall x\in A$, $0<\mathsf{J}g(x)\le C\epsilon$. Assuming this contention, we have, denoting by $\mathsf{pr}_1:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^m$

the projection on the first factor:

$$\begin{aligned}\mathcal{H}^n\big(f(A)\big) &= \mathcal{H}^n\big(\mathsf{pr}_1 \circ g(A)\big) \overset{2.4.3)}{\leq} \\ &\leq (\operatorname{Lip}\mathsf{pr}_1)^n \mathcal{H}^n\big(g(A)\big) \overset{g \text{ 1-1 and } \operatorname{Lip}\mathsf{pr}_1 = 1}{=} \\ &= \int_{\mathbb{R}^{m+n}} \mathcal{H}^0(A \cap g^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y) \overset{\text{case 1}}{=} \\ &= \int_A \mathsf{J}g\,\mathrm{d}\mathcal{L}^n \overset{\text{contention}}{\leq} \\ &\leq C\epsilon\mathcal{L}^n(A).\end{aligned}$$

Thus, since $0 < \epsilon < 1$ was arbitrarily taken and we assumed $\mathcal{L}^n(A) < \infty$, it follows that $\mathcal{H}^n\big(f(A)\big) = 0$. As the multiplicity function $N(f|_A) : \mathbb{R}^n \to \mathbb{R}$, $y \mapsto \mathcal{H}^0(A \cap f^{-1}\{y\})$, is supported on $f(A)$, it then follows that $\int \mathcal{H}^0(A \cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y) = \int_{f(A)} \mathcal{H}^0(A \cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y) = 0$, as asserted.

It remains to prove the contention. Since $\forall x \in D_g = D_f$, $\mathsf{D}g(x) = \big(\mathsf{D}f(x), \epsilon\,\mathrm{id}_{\mathbb{R}^n}\big) \in \mathsf{L}(\mathbb{R}^n, \mathbb{R}^m \times \mathbb{R}^n)$, the Jacobian matrix of $\mathsf{D}g(x)$ is the $(m+n) \times n$ matrix written in block form:

$$[\mathsf{D}g(x)] = \begin{pmatrix} [\mathsf{D}f(x)] \\ \epsilon I_n \end{pmatrix}. \tag{5.14}$$

By the Binet-Cauchy formula 5.23, $\big(\mathsf{J}g(x)\big)^2$ is the sum of the squares of the $n \times n$-minors of the above matrix. In particular taking the minor corresponding the the last n rows, we conclude that $\forall x \in D_g = D_f$, $\mathsf{J}g(x) \geq \epsilon^n > 0$. On the other hand, to obtain an upper bound for that sum:

- Note that the i-th row of the matrix $[\mathsf{D}f(x)]$ is $\nabla f^i(x)$, where f^i is the i-th component of f in the standard basis of $\mathbb{R}^m$; the norm of this row is therefore $\|\nabla f^i(x)\| = \|\mathsf{D}f^i(x)\| \leq \operatorname{Lip} f^i \leq \operatorname{Lip} f$.
- The sum of the squares of the $n \times n$ minors of $[\mathsf{D}g(x)]$ may be written as $M_1 + M_2$, where the terms in M_1 are the squares of the $n \times n$ minors with rows in $[\mathsf{D}f(x)]$, i.e. $M_1 = \big(\mathsf{J}f(x)\big)^2$, and the terms in M_2 are the squares of the other minors, i.e. $n \times n$ minors which have at least one row in ϵI_n. Since $\epsilon < 1$ and the rows in $[\mathsf{D}f(x)]$ are bounded in norm by $\operatorname{Lip} f$, each minor of the latter type is bounded by $\epsilon \cdot \max\{1, (\operatorname{Lip} f)^{n-1}\}$. Since there are $\binom{m+n}{n} - \binom{m}{n}$ summands in M_2, $M_2 \leq \big(\binom{m+n}{n} - \binom{m}{n}\big)\epsilon^2 \cdot \max\{1, (\operatorname{Lip} f)^{n-1}\}^2$. Hence, $\forall x \in D_g = D_f$:

$$\big(\mathsf{J}g(x)\big)^2 \leq \big(\mathsf{J}f(x)\big)^2 + \Big(\binom{m+n}{n} - \binom{m}{n}\Big)\epsilon^2 \cdot \max\{1, (\operatorname{Lip} f)^{n-1}\}^2.$$

In particular, if $x \in A \subset J_f^0$, we conclude that $\mathsf{J}g(x) \leq C\epsilon$, where

$$C := \sqrt{\binom{m+n}{n} - \binom{m}{n}} \max\{1, (\operatorname{Lip} f)^{n-1}\},$$

thus proving our contention.

3) General case: $A \subset D_f$. It is a direct consequence of cases 1 and 2:

$$\begin{aligned}\int_A \mathsf{J}f \, \mathrm{d}\mathcal{L}^n = \int_{A\cap J_f^+} \mathsf{J}f \, \mathrm{d}\mathcal{L}^n + \int_{A\cap J_f^0} \mathsf{J}f \, \mathrm{d}\mathcal{L}^n = \\ = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap J_f^+ \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{H}^n(y) + \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap J_f^0 \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{H}^n(y) = \\ = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{H}^n(y).\end{aligned}$$

$\square$

Corollary 5.37. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, $n \leq m$, then for $\mathcal{H}^n$-a.e. $y \in \mathbb{R}^m$, $f^{-1}\{y\}$ is countable.*

Proof. Since $\forall x \in D_f$, $\mathsf{J}f(x) \overset{5.28.b)}{\leq} \|\mathsf{D}f(x)\|^n \leq (\operatorname{Lip} f)^n$, it follows from the area formula 5.36 that, $\forall K \subset \mathbb{R}^n$ compact, $\int_{\mathbb{R}^m} \mathcal{H}^0(K \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{H}^n(y) = \int_K \mathsf{J}f \, \mathrm{d}\mathcal{L}^n < \infty$. Then, $\forall K \subset \mathbb{R}^n$ compact, for $\mathcal{H}^n$-a.e. $y \in \mathbb{R}^m$, $\mathcal{H}^0(K \cap f^{-1}\{y\}) < \infty$. Since $\mathbb{R}^n$ is σ-compact, it then follows that for $\mathcal{H}^n$-a.e. $y \in \mathbb{R}^n$, $\forall K \subset \mathbb{R}^n$ compact, $\mathcal{H}^0(K \cap f^{-1}\{y\}) < \infty$. For such y, $f^{-1}\{y\} \cap K$ is finite for each compact $K \subset \mathbb{R}^n$, hence $f^{-1}\{y\}$ is countable. $\square$

Corollary 5.38 (Change of variables formula). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \leq m$. Then for all $g : \mathbb{R}^n \to \mathbb{R}$ $\mathcal{L}^n$-measurable with $g \geq 0$ or g summable,*

$$\int_{\mathbb{R}^n} g \, \mathsf{J}f \, \mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^m} \big(\sum_{x \in f^{-1}\{y\}} g(x) \big) \, \mathrm{d}\mathcal{H}^n(y).$$

Proof. Suppose that $g \geq 0$. By exercise 1.54, there exists a sequence $(A_i)_{i \in \mathbb{N}}$ in $\sigma(\mathcal{L}^n)$ such that

$$g = \sum_{i=1}^{\infty} \frac{1}{i} \chi_{A_i}.$$

Let $\psi : \mathbb{R}^m \to [0, \infty]$ be given by $\psi(y) := \sum_{x \in f^{-1}\{y\}} g(x)$. Given $y \in \mathbb{R}^m$, we may compute $\sum_{x \in f^{-1}\{y\}} g(x)$ by means of the monotone convergence theorem (with respect to the counting measure on $f^{-1}\{y\}$):

$$\begin{aligned}\psi(y) = \sum_{x\in f^{-1}\{y\}} g(x) = \sum_{x\in f^{-1}\{y\}} \sum_{i\in\mathbb{N}} \frac{1}{i}\chi_{A_i}(x) &\overset{\text{MCT } 1.62}{=} \\ &= \sum_{i\in\mathbb{N}} \frac{1}{i} \sum_{x\in f^{-1}\{y\}} \chi_{A_i}(x) = \\ &= \sum_{i\in\mathbb{N}} \frac{1}{i}\mathcal{H}^0(A_i \cap f^{-1}\{y\}).\end{aligned}$$

Since, for each $i \in \mathbb{N}$, the multiplicity function $N(f|_{A_i}) : y \mapsto \mathcal{H}^0(A_i \cap f^{-1}\{y\})$ is $\mathcal{H}^n$-measurable, by lemma 5.29.ii), we therefore conclude that ψ is $\mathcal{H}^n$-measurable and ≥ 0. Besides, using the monotone convergence theorem once more and the area formula, we have:

$$\begin{aligned}\int_{\mathbb{R}^m} \psi(y)\,\mathrm{d}\mathcal{H}^n(y) &= \int_{\mathbb{R}^m} \sum_{i\in\mathbb{N}} \frac{1}{i}\mathcal{H}^0(A_i\cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y) \overset{\text{MCT } 1.62}{=} \\ &= \sum_{i\in\mathbb{N}} \frac{1}{i} \int_{\mathbb{R}^m} \mathcal{H}^0(A_i\cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y) \overset{\text{AF } 5.36}{=} \\ &= \sum_{i\in\mathbb{N}} \frac{1}{i} \int_{A_i} \mathsf{J}f\,\mathrm{d}\mathcal{L}^n \overset{\text{MCT } 1.62}{=} \\ &= \int_{\mathbb{R}^n} \sum_{i\in\mathbb{N}} \frac{1}{i}\chi_{A_i}\,\mathsf{J}f\,\mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^n} g\,\mathsf{J}f\,\mathrm{d}\mathcal{L}^n,\end{aligned}$$

thus proving the case in which $g \geq 0$.

If $g : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{L}^n$-summable, we write $g = g^+ - g^-$ and apply the case already proved to g^+ and g^-, from which the thesis follows.

□

Corollary 5.39. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz 1-1, $n \leq m$.*

i) $\forall A \in \sigma(\mathcal{L}^n)$, $\mathcal{H}^n\big(f(A)\big) = \int_A \mathsf{J}f\,\mathrm{d}\mathcal{L}^n$. In particular, we have

$$f_\#(\mathcal{L}^n \llcorner \mathsf{J}f) = \mathcal{H}^n \llcorner \operatorname{Im} f \tag{5.15}$$

(equality as Borel regular outer measures on $\mathbb{R}^m$).

ii) If $g : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{L}^n$-measurable with $g \geq 0$ or $g \in \mathsf{L}^1(\mathcal{L}^n)$, then $\int_{\operatorname{Im} f} g\circ f^{-1}\,\mathrm{d}\mathcal{H}^n = \int_{\mathbb{R}^n} g\,\mathsf{J}f\,\mathrm{d}\mathcal{L}^n$. In particular, if $g : \operatorname{Im} f \to [0,\infty]$ is Borelian, then

$$\int_{\operatorname{Im} f} g\,\mathrm{d}\mathcal{H}^n = \int g\circ f\,\mathsf{J}f\,\mathrm{d}\mathcal{L}^n. \tag{5.16}$$

Proof. If f is 1-1,

i) $\forall A \in \sigma(\mathcal{L}^n)$, $\forall y \in \mathbb{R}^m$, $\mathcal{H}^0(A \cap f^{-1}\{y\}) = \chi_{f(A)}(y)$. Hence, $\mathcal{H}^n\big(f(A)\big) = \int_A \mathsf{J}f \,\mathrm{d}\mathcal{L}^n$ as a direct consequence of the area formula 5.36. The proof of (5.15) is done along the following steps:
   - $\mathcal{H}^n \llcorner \mathrm{Im}\, f$ is a Borel regular outer measure. Indeed, since $\mathrm{Im}\, f = \cup_{N\in\mathbb{N}} f(\mathbb{B}_N)$ is σ-compact, hence Borelian, we may apply proposition 1.36.i).
   - $f_\#(\mathcal{L}^n \llcorner \mathsf{J}f)$ is a Borel regular outer measure. Indeed, $\mathcal{L}^n \llcorner \mathsf{J}f$ is a Radon measure on $\mathbb{R}^n$, in view of lemma 4.11; hence $f_\#(\mathcal{L}^n \llcorner \mathsf{J}f)$ is a Borel outer measure on $\mathbb{R}^m$, since $\forall U \subset \mathbb{R}^m$ open, $f^{-1}(U)$ is open by the continuity of f, thus $\mathcal{L}^n \llcorner \mathsf{J}f$-measurable, so that U is $f_\#(\mathcal{L}^n \llcorner \mathsf{J}f)$-measurable in view of proposition 1.15.iii). It remains to prove the Borel regularity of $f_\#(\mathcal{L}^n \llcorner \mathsf{J}f)$. Given $T \subset \mathbb{R}^m$, the fact that $\mathcal{L}^n \llcorner \mathsf{J}f$ is Radon ensures the existence of a sequence of open sets $(U_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $\forall k \in \mathbb{N}, U_k \supset f^{-1}(T)$ and $\inf\{\mathcal{L}^n \llcorner \mathsf{J}f(U_k) \mid k \in \mathbb{N}\} = \mathcal{L}^n \llcorner \mathsf{J}f\big(f^{-1}(T)\big)$. Since $\forall k \in \mathbb{N}$, U_k is σ-compact, so is $f(U_k)$ (because f is continuous, hence it maps compact sets to compact sets), thus $f(U_k) \in \mathscr{B}_{\mathbb{R}^m}$. Take $\forall k \in \mathbb{N}$, $B_k := f(U_k) \cup (\mathbb{R}^m \setminus \mathrm{Im}\, f) \in \mathscr{B}_{\mathbb{R}^m}$. Then $\forall k \in \mathbb{N}$, $B_k \supset T$ and, as $f^{-1}(B_k) = U_k$, $\inf\{f_\#(\mathcal{L}^n \llcorner \mathsf{J}f)(B_k) \mid k \in \mathbb{N}\} = \mathcal{L}^n \llcorner \mathsf{J}f\big(f^{-1}(T)\big) = f_\#(\mathcal{L}^n \llcorner \mathsf{J}f)(T)$, which implies the Borel regularity of $f_\#(\mathcal{L}^n \llcorner \mathsf{J}f)$, as asserted.
   - In view of the two previous items, it suffices to show that $\mathcal{H}^n \llcorner \mathrm{Im}\, f$ and $f_\#(\mathcal{L}^n \llcorner \mathsf{J}f)$ coincide in each $B \in \mathscr{B}_{\mathbb{R}^m}$. Indeed,

$$\mathcal{H}^n \llcorner \mathrm{Im}\, f(B) = \mathcal{H}^n(\mathrm{Im}\, f \cap B) = \mathcal{H}^n\big(f[f^{-1}(B)]\big) \overset{(*)}{=}$$

$$= \int_{f^{-1}(B)} \mathsf{J}f \,\mathrm{d}\mathcal{L}^n = f_\#(\mathcal{L}^n \llcorner \mathsf{J}f)(B),$$

     where the equality $(*)$ is due to the area formula applied to $(\forall y \in \mathbb{R}^m)\chi_{f[f^{-1}(B)]}(y) = \mathcal{H}^0\big(f^{-1}(B) \cap f^{-1}\{y\}\big)$ (because f is 1-1).

ii) $\forall y \in \mathbb{R}^m$,

$$\sum_{x\in f^{-1}\{y\}} g(x) = \begin{cases} g \circ f^{-1}(y) & y \in \mathrm{Im}\, f \\ 0 & y \in \mathbb{R}^m \setminus \mathrm{Im}\, f. \end{cases}$$

It then follows from corollary 5.38 that $\int_{\mathrm{Im}\, f} g\circ f^{-1} \,\mathrm{d}\mathcal{H}^n = \int_{\mathbb{R}^n} g \,\mathsf{J}f \,\mathrm{d}\mathcal{L}^n$. If $g : \mathrm{Im}\, f \to [0,\infty]$ is Borelian, we may apply the latter equality to $g \circ f$ in place of g, thus yielding (5.16).

□

EXAMPLE 5.40 (applications of the area formula).

1) (length of a curve) Let $-\infty < a < b < \infty$ and $\gamma : [a,b] \to \mathbb{R}^m$ be Lipschitz 1-1. We may extend γ to a Lipschitz function on $\mathbb{R}$, which we still denote by γ. Note that, for all t in the set D_γ of the points of differentiability of γ,
$$\mathsf{J}\gamma(t) = \|\gamma'(t)\|.$$
It then follows from the change of variables formula 5.38 with $g = \chi_{[a,b]}$ that
$$\begin{aligned}\int_a^b \|\gamma'(t)\|\,\mathrm{d}t &= \int_{\mathbb{R}} \chi_{[a,b]}\,\mathsf{J}\gamma\,\mathrm{d}\mathcal{L}^1 \overset{5.38}{=} \\ &= \int_{\mathbb{R}^m} \Big(\sum_{x\in\gamma^{-1}\{y\}} \chi_{[a,b]}(x)\Big)\,\mathrm{d}\mathcal{H}^1(y) = \\ &= \int_{\mathbb{R}^m} \chi_{\gamma([a,b])}(y)\,\mathrm{d}\mathcal{H}^1(y) = \\ &= \mathcal{H}^1\big(\gamma([a,b])\big).\end{aligned}$$

2) (area of a graph) Let $g : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz and $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be given by $f(x) := \big(x, g(x)\big)$. Then f is Lipschitz 1-1 and, computing by means of the Binet-Cauchy formula 5.23, $\forall x \in D_f = D_g$,
$$\mathsf{J}f(x) = \sqrt{1 + \|\nabla g(x)\|^2}.$$
For each $U \subset \mathbb{R}^n$ open, it follows from corollary 5.39.i) that the "surface area" of the graph of g over U, $\Gamma = \Gamma(g;U) := \{\big(x, g(x)\big) \mid x \in U\} = f(U)$, is given by:
$$\mathcal{H}^n(\Gamma) = \int_U \mathsf{J}f\,\mathrm{d}\mathcal{L}^n = \int_U \sqrt{1 + \|\nabla g(x)\|^2}\,\mathrm{d}x.$$

EXERCISE 5.41 (Area Formula for locally Lipschitz maps). The area formula and its corollaries remain valid for locally Lipschitz maps defined on open subsets of $\mathbb{R}^n$. That is, let $n \le m$, $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}^m$ locally Lipschitz.

a) (area formula) For all $\mathcal{L}^n$-measurable $A \subset \Omega$, the multiplicity function $N(f|_A) : \mathbb{R}^m \to [0,\infty]$, $y \mapsto \mathcal{H}^0(A \cap f^{-1}\{y\})$, is $\mathcal{H}^n$-measurable and
$$\int_A \mathsf{J}f\,\mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^n(y).$$

b) For $\mathcal{H}^n$-a.e. $y \in \mathbb{R}^m$, $f^{-1}\{y\}$ is countable.

c) (change of variables formula) If $g : \Omega \to \mathbb{R}$ is $\mathcal{L}^n|_\Omega$-measurable and $g \geq 0$ or $g \in \mathsf{L}^1(\mathcal{L}^n|_\Omega)$, then

$$\int_\Omega g \, \mathsf{J} f \, \mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^m} \Big(\sum_{x \in f^{-1}\{y\}} g(x) \Big) \, \mathrm{d}\mathcal{H}^n(y),$$

meaning that the integral in the second member makes sense and the equality holds. In particular, if f is 1-1, it follows that $\int_\Omega g \, \mathsf{J} f \, \mathrm{d}\mathcal{L}^n = \int_{\mathrm{Im}\, f} g \circ f^{-1} \, \mathrm{d}\mathcal{H}^n$.

The classical C^1 change of variables formula 1.82 may be obtained as a corollary of part c) of the previous exercise. The classical formula actually holds with much weaker hypotheses on the change of variables $\phi : U \to \mathbb{R}^n$ with $U \subset \mathbb{R}^n$ open; it suffices, for instance, that ϕ be a 1-1 C^1-map (it need not be a diffeomorphism).

Exercise 5.42 (Hausdorff dimension and Lebesgue measure of a k-dimensional Riemannian submanifold of $\mathbb{R}^n$). For any smooth embedded k-Riemannian submanifold $\mathsf{M} \subset \mathbb{R}^n$, the measure induced by the Riemannian metric on M (i.e. the Lebesgue measure of M) coincides with the trace $\mathcal{H}^k|_{\mathsf{M}}$. Conclude that $\mathcal{H}$-dim $M = k$ and, if M is closed (i.e. topologically closed), $\mathcal{H}^k \llcorner \mathsf{M}$ is a Radon measure on $\mathbb{R}^n$.

5.3. The coarea formula

In this section we assume $n \geq m$. The coarea formula is a powerful generalization of Fubini-Tonelli's theorem 1.84.

Lemma 5.43 (Coarea formula, linear case). *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be linear, $n \geq m$, $A \in \sigma(\mathcal{L}^n)$. Then:*

i) $N(L|_A) : \mathbb{R}^m \to [0, \infty]$ given by $N(L|_A)(y) := \mathcal{H}^{n-m}(A \cap L^{-1}\{y\})$ is $\mathcal{L}^m$-measurable.

ii)

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap L^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y) = [\![L]\!] \mathcal{L}^n(A) \tag{5.17}$$

Proof. Let $O \in \mathrm{O}(m, n)$ and $S \in \mathrm{Sym}(m)$ be given by theorem 5.20, i.e. such that $L = S \circ O^*$ is a polar decomposition of L.

(1) Case 1: $\dim \mathrm{Im}\, L < m$. Then, for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$, $L^{-1}\{y\} = \emptyset$, thus $N(L|_A)(y) = \mathcal{H}^{n-m}(A \cap L^{-1}\{y\}) = 0$. That is, $N(L|_A)$ is null $\mathcal{L}^m$-a.e., hence it is $\mathcal{L}^m$-measurable. On the other hand, since $\mathrm{Im}\, L = \mathrm{Im}\, S$, we have $[\![L]\!] = |\det S| = 0$. Therefore, both members of (5.17) are null.

(2) Case 2: $L = P : \mathbb{R}^n \equiv \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ is the projection on the first factor (hence $O = P^*$ and $S = \mathrm{id}_{\mathbb{R}^m}$). Fix $y \in \mathbb{R}^m$ and let $\mathsf{pr}_2 : \mathbb{R}^n \equiv \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ be the projection on the second factor. Then the restriction $\mathsf{pr}_2 : P^{-1}\{y\} \to \mathbb{R}^{n-m}$ is an isometry which maps $A \cap P^{-1}\{y\}$ to the y-section $A_y \subset \mathbb{R}^{n-m}$ (see notation preceding Fubini-Tonelli's theorem 1.84). Therefore, from proposition 2.4 parts i) and ii) and from theorem 2.21, we conclude that $N(P|_A)(y) = \mathcal{H}^{n-m}(A \cap P^{-1}\{y\}) = \mathcal{H}^{n-m}(A_y) = \mathcal{L}^{n-m}(A_y)$. Hence, from Fubini-Tonelli's theorem 1.84.ii) applied to the product measure $\mathcal{L}^m \times \mathcal{L}^{m-n}$ (which coincides with $\mathcal{L}^n$, in view of example 1.86), we conclude that $N(P|_A)$ is $\mathcal{L}^m$-measurable and $\int_{\mathbb{R}^m} N(P|_A)\,\mathrm{d}\mathcal{L}^m = \mathcal{L}^n(A)$, thus proving (5.17) (since $[\![P]\!] = 1$).

(3) Case 3: $L : \mathbb{R}^n \to \mathbb{R}^m$ surjective. Note that, since $\operatorname{Im} L = \operatorname{Im} S$, we have $S \in \mathrm{Sym}(m) \cap \mathrm{GL}(\mathbb{R}^m)$.

We contend that there exists $Q \in \mathrm{O}(n)$ such that $O^* = P \circ Q$, where $P : \mathbb{R}^n \equiv \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ is the projection on the first factor, as in the previous item. Indeed, extend $O \in \mathrm{O}(m,n)$ to a linear isometry $S : \mathbb{R}^n \equiv \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^n$ and define $Q := S^*$. Since $P^* : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ is the inclusion on the first factor, we have $S \circ P^* = O$, hence $O^* = P \circ S^* = P \circ Q$, as we wanted.

With $Q \in \mathrm{O}(n)$ given by the contention proved above, we have, $\forall y \in \mathbb{R}^m$,

$$\begin{aligned} N(L|_A)(y) = \mathcal{H}^{n-m}(A \cap L^{-1}\{y\}) &= \\ &= \mathcal{H}^{n-m}\big(A \cap (S \circ P \circ Q)^{-1}\{y\}\big) = \\ &= \mathcal{H}^{n-m}\big(Q^{-1}\big[Q(A) \cap P^{-1}\{S^{-1}(y)\}\big]\big) \overset{2.4.2)}{=} \\ &= \mathcal{H}^{n-m}\big(Q(A) \cap P^{-1}\{S^{-1}(y)\}\big) = \\ &= N(P|_{Q(A)}) \circ S^{-1}(y). \end{aligned}$$

That is, $N(L|_A) = N(P|_{Q(A)}) \circ S^{-1}$. By the previous item, $N(P|_{Q(A)})$ is $\mathcal{L}^m$-measurable, and S^{-1} is continuous, hence Borelian; it then follows that the composition $N(L|_A) = N(P|_{Q(A)}) \circ S^{-1}$ is $\mathcal{L}^m$-measurable and ≥ 0. Moreover,

$$\begin{aligned} \int_{\mathbb{R}^m} N(f|A)(y)\,\mathrm{d}\mathcal{L}^m(y) &= \int_{\mathbb{R}^m} N(P|_{Q(A)}) \circ S^{-1}(y)\,\mathrm{d}\mathcal{L}^m(y) \overset{1.81.ii)}{=} \\ &= |\det S| \int_{\mathbb{R}^m} N(P|_{Q(A)})\,\mathrm{d}\mathcal{L}^m \overset{\text{Case 2}}{=} \\ &= |\det S| \mathcal{L}^n\big(Q(A)\big) \overset{Q \in \mathrm{O}(n)}{=} [\![L]\!] \mathcal{L}^n(A), \end{aligned}$$

thus proving (5.17).

∎

In the next lemma we make computations with the *upper integral* introduced in exercise 1.68.

LEMMA 5.44. *Let $n, m \in \mathbb{N}$, $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. Then, $\forall k, l \in [0, \infty)$ and $\forall A \subset \mathbb{R}^n$,*

$$\int_{\mathbb{R}^m}^{*} \mathcal{H}^k(A \cap f^{-1}\{y\})\,\mathrm{d}\mathcal{H}^l(y) \le \frac{\alpha(k)\alpha(l)}{\alpha(k+l)}(\mathrm{Lip}\, f)^l \mathcal{H}^{k+l}(A).$$

Note that we neither assume $n \ge m$ nor the measurability of A in the statement of the lemma above. This is a particular case from Federer's theorem 2.10.25 in [**Fed69**]; the theorem actually holds for any Lipschitz map $f : X \to Y$ between metric spaces X and Y. We will prove only the case $l = m$, for which it is possible to make a simpler argument, adapted from [**EG91**], thanks to the isodiametric inequality 2.19. Only this case will be needed in the proof of the coarea formula.

PROOF FOR THE CASE $l = m$. For each $j \in \mathbb{N}$, by proposition 2.4.4) there exists $(B_i^j)_{i \in \mathbb{N}}$ cover of A by closed sets with diameters $\le 1/j$ such that

$$\sum_{i \in \mathbb{N}} \alpha(k+m)\Big(\frac{\mathrm{diam}\ B_i^j}{2}\Big)^{k+m} \le \mathcal{H}_{1/j}^{k+m}(A) + \frac{1}{j}. \tag{5.18}$$

$\forall i, j \in \mathbb{N}$, define

$$g_i^j := \alpha(k)\Big(\frac{\mathrm{diam}\ B_i^j}{2}\Big)^k \chi_{f(B_i^j)}.$$

Since $f(B_i^j)$ is σ-compact (because B_i^j is closed, hence σ-compact, and f is continuous), hence Borel measurable, g_i^j is Borelian and ≥ 0, and so is $\sum_{i \in \mathbb{N}} g_i^j : \mathbb{R}^m \to [0, \infty]$. Moreover, for each $y \in \mathbb{R}^m$ and $j \in \mathbb{N}$, $A \cap f^{-1}\{y\}$ is contained in the union of the balls of $(B_i^j)_{i \in \mathbb{N}}$ which cut $f^{-1}\{y\}$, i.e. such that $y \in f(B_i^j)$. It then follows that, $\forall y \in \mathbb{R}^m$,

$$\mathcal{H}_{1/j}^k(A \cap f^{-1}\{y\}) \le \sum_{i \in \mathbb{N}} g_i^j(y).$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^m}^* \mathcal{H}^k(A \cap f^{-1}\{y\})\,\mathrm{d}\mathcal{L}^m(y) = \\
&= \int_{\mathbb{R}^m}^* \lim_{j\to\infty} \mathcal{H}^k_{1/j}(A \cap f^{-1}\{y\})\,\mathrm{d}\mathcal{L}^m(y) \overset{\text{monotonicity of } \int^*}{\leq} \\
&\leq \int_{\mathbb{R}^m}^* \liminf_{j\to\infty} \sum_{i\in\mathbb{N}} g_i^j(y)\,\mathrm{d}\mathcal{L}^m(y) \overset{1.68.b)}{=} \\
&= \int_{\mathbb{R}^m} \liminf_{j\to\infty} \sum_{i\in\mathbb{N}} g_i^j(y)\,\mathrm{d}\mathcal{L}^m(y) \overset{\text{Fatou } 1.63}{\leq} \\
&\leq \liminf_{j\to\infty} \int_{\mathbb{R}^m} \sum_{i\in\mathbb{N}} g_i^j(y)\,\mathrm{d}\mathcal{L}^m(y) \overset{\text{MCT } 1.62}{=} \\
&= \liminf_{j\to\infty} \sum_{i\in\mathbb{N}} \int_{\mathbb{R}^m} g_i^j(y)\,\mathrm{d}\mathcal{L}^m(y) = \\
&= \liminf_{j\to\infty} \sum_{i\in\mathbb{N}} \alpha(k)\Big(\frac{\operatorname{diam} B_i^j}{2}\Big)^k \mathcal{L}^m\big(f(B_i^j)\big) \overset{\text{isodiametric } 2.19}{\leq} \\
&\leq \liminf_{j\to\infty} \sum_{i\in\mathbb{N}} \alpha(k)\Big(\frac{\operatorname{diam} B_i^j}{2}\Big)^k \alpha(m)\Big(\frac{\operatorname{diam} f(B_i^j)}{2}\Big)^m \leq \\
&\leq \frac{\alpha(k)\alpha(m)}{\alpha(k+m)}(\operatorname{Lip} f)^m \liminf_{j\to\infty} \sum_{i\in\mathbb{N}} \alpha(k+m)\Big(\frac{\operatorname{diam} B_i^j}{2}\Big)^{k+m} \overset{(5.18)}{\leq} \\
&\leq \frac{\alpha(k)\alpha(m)}{\alpha(k+m)}(\operatorname{Lip} f)^m \liminf_{j\to\infty} \Big(\mathcal{H}^{k+m}_{1/j}(A) + \frac{1}{j}\Big) = \\
&= \frac{\alpha(k)\alpha(m)}{\alpha(k+m)}(\operatorname{Lip} f)^m \mathcal{H}^{k+m}(A).
\end{aligned}$$

□

LEMMA 5.45. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \geq m$, $A \in \sigma(\mathcal{L}^n)$. Then:*

i) For $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$, $A \cap f^{-1}\{y\}$ is $\mathcal{H}^{n-m}$-measurable.
ii) $N(f|_A) : \mathbb{R}^m \to [0,\infty]$ given by $N(f|_A)(y) := \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})$ is $\mathcal{L}^m$-measurable.

PROOF.

1) Case 1: A is compact. Fix $t \geq 0$. For each $i \in \mathbb{N}$, let U_i be defined as the set of points $y \in \mathbb{R}^m$ such that there exist finitely many open

sets $(S_j)_{1\leq j\leq k}$ satisfying the following conditions:

$$\left\{\begin{array}{l} A\cap f^{-1}\{y\}\subset \cup_{j=1}^k S_j,\\ \text{diam } S_j\leq \frac{1}{i}, \forall 1\leq j\leq k,\\ \sum_{j=1}^k \alpha(n-m)\left(\frac{\text{diam } S_j}{2}\right)^{n-m}\leq t+\frac{1}{i}.\end{array}\right. \tag{5.19}$$

2) Claim #1: $\forall i\in\mathbb{N}$, U_i is open. Indeed, let $y\in U_i$. Take $(S_j)_{1\leq j\leq k}$ satisfying the conditions (5.19). We contend that there exists $r>0$ such that $A\cap f^{-1}\big(\mathbb{U}(y,r)\big)\subset\cup_{j=1}^k S_j$, from which we conclude that $\mathbb{U}(y,r)\subset U_i$, thus proving the claim. The contention is a direct consequence of the fact that A is compact and f is continuous: if there were no such $r>0$, we could take a sequence $(y_h)_{h\in\mathbb{N}}$ in $\mathbb{R}^m$ convergent to y such that $\forall h\in\mathbb{N}$, there exists $x_h\in f^{-1}\{y_h\}\cap A\setminus\cup_{1\leq j\leq k}S_j$. Since $A\setminus\cup_{1\leq j\leq k}S_j$ is compact, there would be a subsequence of $(x_h)_{h\in\mathbb{N}}$, which we assume to be $(x_h)_{h\in\mathbb{N}}$ itself up to changing the notation, such that $x_h\to x\in A\setminus\cup_{1\leq j\leq k}S_j$. By continuity, we conclude that $f(x)=\lim f(x_h)=\lim y_h=y$, hence $x\in f^{-1}\{y\}\setminus\cup_{1\leq j\leq k}S_j$, thus yielding a contradiction which proves our contention.
3) Claim #2: $\{y\in\mathbb{R}^m \mid \mathcal{H}^{n-m}(A\cap f^{-1}\{y\})\leq t\}=\cap_{i\in\mathbb{N}}U_i$, hence it is a Borel set. Since $\{y\in\mathbb{R}^m\mid\mathcal{H}^{n-m}(A\cap f^{-1}\{y\})\leq t\}=\emptyset$ for $t<0$, the claim then implies that $N(f|_A)$ is Borelian if A is compact. Since $\forall y\in\mathbb{R}^m$, $A\cap f^{-1}\{y\}$ is compact, hence Borelian, we achieve the proof of case 1 once we show the claim.

   Proof of claim #2:
   - Assume that $\mathcal{H}^{n-m}(A\cap f^{-1}\{y\})\leq t$. Then, $\forall\delta>0$, $\mathcal{H}_\delta^{n-m}(A\cap f^{-1}\{y\})\leq t$. Given $i\in\mathbb{N}$, choose $\delta\in(0,\frac{1}{i})$. In view of proposition 2.4.4), there exists a countable cover $\mathcal{G}$ of $A\cap f^{-1}\{y\}$ by open subsets of $\mathbb{R}^n$ with diameters $\leq\delta$ such that $\sum_{S\in\mathcal{G}}\alpha(n-m)\left(\frac{\text{diam } S}{2}\right)^{n-m}<t+\frac{1}{i}$. Since $A\cap f^{-1}\{y\}$ is compact, we may take a finite subcover $(S_j)_{1\leq j\leq k}$ of $\mathcal{G}$ satisfying (5.19), so that $y\in U_i$. As $i\in\mathbb{N}$ is arbitrary, it then follows that $y\in\cap_{i\in\mathbb{N}}U_i$.
   - Conversely, if $\forall\, i\in\mathbb{N}$, $y\in U_i$, then (5.19) ensures that $\mathcal{H}_{1/i}^{n-m}(A\cap f^{-1}\{y\})\leq t+\frac{1}{i}$; hence, taking $i\to\infty$, we conclude that $\mathcal{H}^{n-m}(A\cap f^{-1}\{y\})\leq t$.
4) Case 2: A is σ-compact (in particular, that holds if A is open). Then $\forall y\in\mathbb{R}^m$, $A\cap f^{-1}\{y\}$ is σ-compact, hence Borelian. Moreover, we may take an increasing sequence $(K_i)_{i\in\mathbb{N}}$ of compact subsets of A whose union is A, so that $\forall y\in\mathbb{R}^m$, the sequence of Borel sets $(K_i\cap f^{-1}\{y\})_{i\in\mathbb{N}}$ increases to $A\cap f^{-1}\{y\}$. Then, applying the continuity from below 1.11 for $\mathcal{H}^{n-m}$, it follows that $N(f|_{K_i})$ increases

pointwise to $N(f|_A)$; from case 1 and from theorem 1.41.iv) we therefore conclude that $N(f|_A)$ is Borelian.

5) Case 3: $\mathcal{L}^n(A) = 0$. It follows from lemma 5.44 with $k = n - m$ and $l = m$, and from theorem 2.21, that $\int^*_{\mathbb{R}^m} N(f|_A)\,\mathrm{d}\mathcal{L}^m = 0$. Hence, from exercise 1.68.a), $N(f|_A)$ is $\mathcal{L}^m$-measurable and $\int_{\mathbb{R}^m} N(f|_A)\,\mathrm{d}\mathcal{L}^m = 0$, so that $N(f|_A)$ is $\mathcal{L}^m$-a.e. null. That is, for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$, $\mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) = 0$, which implies that $A \cap f^{-1}\{y\}$ is $\mathcal{H}^{n-m}$-measurable.
6) Case 4: $\mathcal{L}^n(A) < \infty$. Since $\mathcal{L}^n$ is a Radon measure, we may take a decreasing sequence $(U_k)_{k\in\mathbb{N}}$ of open sets containing A such that $\inf\{\mathcal{L}^n(U_k) \mid k \in \mathbb{N}\} = \mathcal{L}^n(A)$. Hence, taking $B := \cap_{k\in\mathbb{N}} U_k \in \mathscr{B}_{\mathbb{R}^n}$, we have $\mathcal{L}^n(B) = \mathcal{L}^n(A) < \infty$; as A is $\mathcal{L}^n$-measurable, we conclude that $\mathcal{L}^n(B \setminus A) = 0$. In particular, it follows from case 3 that, for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$, $(B \setminus A) \cap f^{-1}\{y\}$ is $\mathcal{H}^{n-m}$-null. For such y, $A \cap f^{-1}\{y\} = (B \cap f^{-1}\{y\}) \setminus \big((B \setminus A) \cap f^{-1}\{y\}\big)$ is $\mathcal{H}^{n-m}$-measurable and $\mathcal{H}^{n-m}(B \cap f^{-1}\{y\}) = \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})$, thus showing that $N(f|_A) = N(f|_B)$ $\mathcal{L}^m$-a.e., so that case 4 will be done once we prove that $N(f|_B)$ is $\mathcal{L}^m$-measurable. Indeed,
   - for each $y \in \mathbb{R}^m$ and $k \in \mathbb{N}$, $U_k \cap f^{-1}\{y\}$ is Borelian and the sequence $(U_k \cap f^{-1}\{y\})_{k\in\mathbb{N}}$ decreases to $B \cap f^{-1}\{y\}$;
   - since $\mathcal{L}^n(A) < \infty$ and $\mathcal{L}^n(U_k) \downarrow \mathcal{L}^n(A)$, we may assume that $\mathcal{L}^n(U_1) < \infty$ (discarding the first terms of the sequence $(U_k)_{k\in\mathbb{N}}$, if necessary). It then follows from lemma 5.44 with $k = n - m$ and $l = m$ that $\int_{\mathbb{R}^m} N(f|_{U_1})\,\mathrm{d}\mathcal{L}^m = \int^*_{\mathbb{R}^m} N(f|_{U_1})\,\mathrm{d}\mathcal{L}^m < \infty$. Hence, for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$, $N(f|_{U_1})(y) = \mathcal{H}^{n-m}(U_1 \cap f^{-1}\{y\}) < \infty$; for such y, we may apply the continuity from above 1.11 to conclude that $N(f|_{U_k})(y) \downarrow N(f|_B)(y)$. That is $N(f|_{U_k})$ decreases $\mathcal{L}^m$-a.e. to $N(f|_B)$. It then follows that $N(f|_B)$ is $\mathcal{L}^m$-measurable (in view of case 2 and of theorem 1.41.iv), as asserted.
7) General case. By the σ-finiteness of $\mathcal{L}^n$, we may write $A = \dot\cup_{k\in\mathbb{N}} A_k$, where $\forall k \in \mathbb{N}$, $A_k \in \sigma(\mathcal{L}^n)$ and $\mathcal{L}^n(A_k) < \infty$. Then $A \cap f^{-1}\{y\} = \dot\cup_{k\in\mathbb{N}}(A_k \cap f^{-1}\{y\})$. It follows from case 4 that, for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$, $\forall k \in \mathbb{N}$, $A_k \cap f^{-1}\{y\}$ is $\mathcal{H}^{n-m}$-measurable; hence, for such y, $A \cap f^{-1}\{y\}$ is $\mathcal{H}^{n-m}$-measurable and $N(f|_A)(y) = \sum_{k\in\mathbb{N}} N(f|_{A_k})(y)$ by the σ-additivity of $\mathcal{H}^{n-m}$. Then $N(f|_A)$ is $\mathcal{L}^m$-measurable, in view of case 4 and theorem 1.41.iv).

$\square$

Lemma 5.46 (Lipschitz linearization, part II). *Let $t > 1$, $h : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz and $J_h^+ = \{x \in D_h \mid \mathsf{J}h(x) > 0\}$. Then there exists a countable disjoint family $(E_k)_{k\in\mathbb{N}}$ in $\mathscr{B}_{J_h^+}$ such that $\mathcal{L}^n(J_h^+ \setminus \cup_{k\in\mathbb{N}} E_k) =$*

0 *and,* $\forall k \in \mathbb{N}$, $h|_{E_k}$ *is 1-1 and there exists* $S_k \in \mathrm{Sym}(n) \cap \mathrm{GL}(\mathbb{R}^n)$ *satisfying*

i) $\mathrm{Lip}\, S_k^{-1} \circ h|_{E_k} \le t$ *and* $\mathrm{Lip}(h|_{E_k})^{-1} \circ S_k \le t$*;*
ii) $\forall x \in E_k$*,* $\|S_k^{-1} \circ \mathsf{D}h(x)\| \le t$ *and* $\|\mathsf{D}h(x)^{-1} \circ S_k\| \le t$*.*

Remark 5.47. With the notation from the previous lemma:

1) Conditions i) and ii) are equivalent to, respectively:
   i') $\forall x, y \in h(E_k)$,
$$t^{-1}\|S_k^{-1} \cdot (x-y)\| \le \|(h|_{E_k})^{-1}(x) - (h|_{E_k})^{-1}(y)\| \le t\|S_k^{-1} \cdot (x-y)\|;$$
   ii') $\forall x \in E_k$, $\forall v \in \mathbb{R}^n$, $t^{-1}\|S_k^{-1} \cdot v\| \le \|\mathsf{D}h(x)^{-1} \cdot v\| \le t\|S_k^{-1} \cdot v\|$.
   The proof is immediate and similar to the argument used in proposition 5.33.
2) Condition i) implies that $h|_{E_k}$ has Lipschitz inverse, since $(h|_{E_k})^{-1} = [(h|_{E_k})^{-1} \circ S_k] \circ S_k^{-1}$, whence $\mathrm{Lip}(h|_{E_k})^{-1} \le t\|S_k^{-1}\|$.
3) Condition ii) implies that, $\forall x \in E_k$:
$$t^{-n}|\det S_k| \le \mathsf{J}h(x) \le t^n|\det S_k|. \tag{5.20}$$
   Indeed,
$$\begin{aligned}\mathsf{J}h(x) &= |\det S_k||\det S_k^{-1}|[\![\mathsf{D}h(x)]\!] \overset{5.28.a)}{=} \\ &= |\det S_k|[\![S_k^{-1} \circ \mathsf{D}h(x)]\!],\end{aligned}$$
   hence (5.20) follows from ii) and from exercise 5.28.b) with $S_k^{-1} \circ \mathsf{D}h(x)$ in place of T.

Proof.

1) Let $(F_k)_{k\in\mathbb{N}}$ be a countable disjoint family in $\mathscr{B}_{\mathbb{R}^n}$ such that $J_h^+ = \dot\bigcup_{k\in\mathbb{N}} F_k$ and $\forall k \in \mathbb{N}$, $h|_{F_k}$ is 1-1 with Lipschitz inverse. The existence of such a family follows from theorem 5.35 and proposition 5.33 with h in place of f.
2) Fix $k \in \mathbb{N}$. As $(h|_{F_k})^{-1} : h(F_k) \to \mathbb{R}^n$ is Lipschitz, by theorem 5.1 (or theorem 5.2) it may be extended to a Lipschitz map $h_k : \mathbb{R}^n \to \mathbb{R}^n$. Since $h(F_k) \subset \{x \in \mathbb{R}^n \mid h \circ h_k(x) = x\}$, it follows from corollary 5.17 with h_k in place of f and h in place of g that $\mathsf{D}h\big(h_k(x)\big) \circ \mathsf{D}h_k(x) = \mathrm{id}_{\mathbb{R}^n}$ for $\mathcal{L}^n$-a.e. $x \in h(F_k)$. Thus, defining
$$Y_k := \{x \in \mathbb{R}^n \mid \exists \mathsf{D}h_k(x), \exists \mathsf{D}h\big(h_k(x)\big), \mathsf{D}h\big(h_k(x)\big) \circ \mathsf{D}h_k(x) = \mathrm{id}_{\mathbb{R}^n}\}$$
   then $Y_k \in \mathscr{B}_{\mathbb{R}^n}$ (in view of exercise 5.13) and $h(F_k) \setminus Y_k$ is $\mathcal{L}^n$-null. Besides, $\forall x \in Y_k$, $\mathsf{J}h\big(h_k(x)\big) \cdot \mathsf{J}h_k(x) = 1$, so that $\mathsf{J}h_k(x) > 0$, hence $Y_k \subset J_{h_k}^+$.

3) Applying the Lipschitz linearization theorem 5.35 to h_k, there exists a countable disjoint family $(G_j^k)_{j\in\mathbb{N}}$ in $\mathscr{B}_{\mathbb{R}^n}$ and a sequence $(R_j^k)_{j\in\mathbb{N}}$ in $\operatorname{Sym}(n)\cap\mathrm{GL}(\mathbb{R}^n)$ such that $J_{h_k}^+ = \dot\cup_{j\in\mathbb{N}} G_j^k$ and $\forall j\in\mathbb{N}$, (G_j^k, R_j^k) is a t-linearization for h_k. Define, for each $j\in\mathbb{N}$,

$$E_j^k := F_k \cap h^{-1}(G_j^k \cap Y_k) \in \mathscr{B}_{\mathbb{R}^n} \quad \text{and} \quad S_j^k := (R_j^k)^{-1} \in \operatorname{Sym}(n)\cap\mathrm{GL}(\mathbb{R}^n).$$

We will prove that the countable family $(E_j^k, S_j^k)_{k,j\in\mathbb{N}}$ satisfies conditions stated in the theorem.

4) It is clear that $(E_j^k)_{k,j\in\mathbb{N}}$ is a disjoint family in $\mathscr{B}_{J_f^+}$ and $\forall k,j\in\mathbb{N}$, $h|_{E_j^k}$ is 1-1 with Lipschitz inverse, since $E_j^k\subset F^k$. Namely, $(h|_{E_j^k})^{-1}$ is the restriction of h_k to $h(E_j^k) = h(F_k)\cap G_j^k\cap Y_k$.

5) We contend that $J_h^+\setminus \cup_{k,n\in\mathbb{N}} E_{k,j}$ is $\mathcal{L}^n$-null. Indeed, since $J_h^+ = \dot\cup_{k\in\mathbb{N}} F_k$, it suffices to show that, for each $k\in\mathbb{N}$, $F_k\setminus\cup_{j\in\mathbb{N}}E_j^k$ is $\mathcal{L}^n$-null. Since $h|_{F_k}$ is bi-Lipschitz onto $h(F_k)$, the latter condition is equivalent to $h(F_k\setminus\cup_{j\in\mathbb{N}}E_j^k)$ being $\mathcal{L}^n$-null. As

$$\begin{aligned} h(F_k\setminus\cup_{j\in\mathbb{N}}E_j^k) &= h\big(F_k\setminus[F_k\cap h^{-1}(Y_k\cap\cup_{j\in\mathbb{N}}G_j^k)]\big) = \\ &= h\big(F_k\setminus h^{-1}(\underbrace{Y_k\cap J_{h_k}^+}_{=Y_k})\big) = \\ &= h(F_k)\setminus Y_k, \end{aligned}$$

the contention follows from part 2).

6) $\forall k,j\in\mathbb{N}$, $h_k|_{G_j^k}$ extends $(h|_{E_j^k})^{-1}$ (by part 4), hence $(h_k|_{G_j^k})^{-1}$ extends $h|_{E_j^k}$. Therefore, $\forall k,j\in\mathbb{N}$,

$$\begin{aligned} &\operatorname{Lip}(S_j^k)^{-1}\circ h|_{E_j^k} = \operatorname{Lip} R_j^k\circ h|_{E_j^k} \le \operatorname{Lip} R_j^k\circ(h_k|_{G_j^k})^{-1}\le t,\\ &\operatorname{Lip}(h|_{E_j^k})^{-1}\circ S_j^k = \operatorname{Lip}(h|_{E_j^k})^{-1}\circ(R_j^k)^{-1}\le \operatorname{Lip} h_k|_{G_j^k}\circ(R_j^k)^{-1}\le t, \end{aligned}$$

where the last inequalities in both lines are justified by the fact that (G_j^k, R_j^k) is a t-linearization for h_k and by proposition 5.33. Thus, $\forall k,j\in\mathbb{N}$, condition i) in the statement of the lemma is fulfilled by (E_j^k, R_j^k).

7) $\forall k,j\in\mathbb{N}$, $\forall x\in E_j^k$, we have $h(x)\in G_j^k\cap Y_k$ and $x = h_k\big(h(x)\big)$; in particular, by the definition of Y_k, $\mathsf{D}h(x)\circ\mathsf{D}h_k\big(h(x)\big) = \mathrm{id}_{\mathbb{R}^n}$, i.e. $\mathsf{D}h(x) = \mathsf{D}h_k\big(h(x)\big)^{-1}$. It then follows that, $\forall k,j\in\mathbb{N}$, $\forall x\in E_j^k$:

$$\begin{aligned} \|(S_j^k)^{-1}\circ\mathsf{D}h(x)\| &= \|R_j^k\circ\mathsf{D}h_k\big(h(x)\big)^{-1}\|\le t,\\ \|\mathsf{D}h(x)^{-1}\circ S_j^k\| &= \|\mathsf{D}h_k\big(h(x)\big)\circ(R_j^k)^{-1}\|\le t, \end{aligned}$$

where the last inequalities in both lines are justified by the fact that (G_j^k, R_j^k) is a t-linearization for h_k and by proposition 5.33. Thus,

$\forall k, j \in \mathbb{N}$, condition ii) in the statement of the lemma is fulfilled by (E_j^k, R_j^k), which concludes the proof.

□

THEOREM 5.48 (Coarea formula). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \geq m$. Then, for each $\mathcal{L}^n$-measurable $A \subset \mathbb{R}^n$,*

$$\int_A \mathsf{J}f \, \mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y). \tag{5.21}$$

REMARK 5.49.

1) Recall that $N(f|_A) : y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})$ is $\mathcal{L}^m$-measurable, by lemma 5.45, so that the integral in the second member of the coarea formula makes sense.
2) If $f : \mathbb{R}^n \equiv \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ is the projection on the first factor, we have $\mathsf{J}f \equiv 1$ and the coarea formula reduces to Fubini-Tonelli's theorem 1.84. The general case may be interpreted, therefore, as a "curvilinear" generalization of Fubini-Tonelli's theorem.
3) If $n = m$, the coarea formula coincides with the area formula 5.36.
4) If we take the Borel set $A := (\mathbb{R}^n \setminus D_f) \cup J_f^0 = \{x \in \mathbb{R}^n \mid \nexists \mathsf{D}f(x) \text{ or } \mathsf{J}f(x) = 0\}$ in the coarea formula, we conclude that $\mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) = 0$ for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$. That may be interpreted as a measure theoretic version of Morse-Sard's theorem: $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$ is a measure theoretic "regular value" of f, in the sense that, up to $\mathcal{H}^{n-m}$ null sets, $f^{-1}\{y\}$ lies in the set J_f^+ of points where $\mathsf{D}f$ has maximal rank.

PROOF.

1) If $A \subset \mathbb{R}^n \setminus D_f$, then $\mathcal{L}^n(A) = 0$ by Rademacher's theorem 5.12, hence the first member in (5.21) is null, and so is the second in view of lemma 5.44 with $k = n - m$ and $l = m$. Therefore, it suffices to show (5.21) for $A \subset D_f = J_f^+ \,\dot\cup\, J_f^0$. Since both members are additive on $\sigma(\mathcal{L}^n)$, it suffices to consider the cases $A \subset J_f^+$ and $A \subset J_f^0$.
2) Case 1: $A \subset J_f^+$. For each $\lambda \in \Lambda(n, n-m)$, define $h_\lambda : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ by
$$h_\lambda(x) := \big(f(x), P_\lambda(x)\big),$$
where P_λ is the orthogonal projection onto $\langle e_{\lambda(1)}, \ldots, e_{\lambda(n-m)} \rangle \equiv \mathbb{R}^{n-m}$. For all $x \in D_{h_\lambda} = D_f$, we have $\mathsf{D}h_\lambda(x) = \big(\mathsf{D}f(x), P_\lambda\big) \in \mathsf{L}(\mathbb{R}^n)$; therefore, $\mathsf{J}h_\lambda(x) > 0$ iff $x \in J_f^\lambda$, where
$$J_f^\lambda := \{x \in J_f^+ \mid P_\lambda|_{\ker \mathsf{D}f(x)} \text{ is 1-1 }\}.$$
For each $x \in J_f^+$, there exists $\lambda \in \Lambda(n, n-m)$ such that $\ker \mathsf{D}f(x)$ is transversal to $\langle e_{\lambda(1)}, \ldots, e_{\lambda(n-m)} \rangle \equiv \mathbb{R}^{n-m}$ (hence $x \in J_f^\lambda$). That

is, $J_f^+ = \cup_{\lambda\in\Lambda(n,n-m)} J_f^\lambda$. Therefore, we may decompose A as a disjoint union $A = \dot\cup_{\lambda\in\Lambda(n,n-m)} A_\lambda$, with $A_\lambda \subset J_f^\lambda$ $\mathcal{L}^n$-measurable. By the additivity of both members of (5.21) on $\mathcal{L}^n$, it then suffices to show the equality for each A_λ. We may therefore assume that $A \subset J_f^\lambda = J_{h_\lambda}^+$ for a given $\lambda \in \Lambda(n, n-m)$.

3) For simplicity of notation, we put $h := h_\lambda : \mathbb{R}^n \to \mathbb{R}^n$. Let $q : \mathbb{R}^n \equiv \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ be the projection on the first factor, so that $f = q \circ h$.

   Fix $t > 1$. Apply lemma 5.46 to h in order to obtain a disjoint countable family $(E_k)_{k\in\mathbb{N}}$ in $\mathscr{B}_{J_h^+}$ and a sequence $(S_k)_{k\in\mathbb{N}} \in \mathrm{Sym}(n)\cap \mathrm{GL}(\mathbb{R}^n)$ such that $\mathcal{L}^n(J_h^+ \setminus \dot\cup_{k\in\mathbb{N}} E_k) = 0$ and $\forall k \in \mathbb{N}$, $h|_{E_k}$ is 1-1 and conditions i), ii) in the statement of the lemma are fulfilled. Let $\forall k \in \mathbb{N}$, $A_k := A \cap E_k \in \sigma(\mathcal{L}^n)$, so that $A \setminus \dot\cup_{k\in\mathbb{N}} A_k$ is $\mathcal{L}^n$-null (since $A \subset J_h^+$).

   We contend that, $\forall k \in \mathbb{N}$, $\forall x \in A_k$,

$$t^{-m}[\![q\circ S_k]\!] \le \mathsf{J}f(x) \le t^m [\![q\circ S_k]\!]. \tag{5.22}$$

Indeed, since $f = q\circ h$, $\mathsf{D}f(x) = q\circ \mathsf{D}h(x) = q\circ S_k \circ \big(S_k^{-1}\circ \mathsf{D}h(x)\big)$. Thus, defining $C := S_k^{-1}\circ \mathsf{D}h(x)$, we have $q\circ S_k = \mathsf{D}f(x)\circ C^{-1}$, whence $(q\circ S_k)^* = (C^{-1})^* \circ \mathsf{D}f(x)^*$. Therefore, applying exercise 5.28.c) with $\mathsf{D}f(x)^* : \mathbb{R}^m \to \mathbb{R}^n$ in place of T and $(C^{-1})^* \in \mathrm{GL}(\mathbb{R}^n)$ in place of R, and noting that $(C^{-1})^* = (C^*)^{-1}$ and that the transposition $(\cdot)^*$ preserves Jacobians and is a linear isometry, we conclude that

$$\|C\|^{-m}\,\mathsf{J}f(x) \le [\![q\circ S_k]\!] \le \|C^{-1}\|^m\,\mathsf{J}f(x),$$

hence

$$\mathsf{J}f(x) \le \|C\|^m [\![q\circ S_k]\!] \overset{ii)\text{ from lemma 5.46}}{\le} t^m[\![q\circ S_k]\!],$$
$$\mathsf{J}f(x) \ge \|C^{-1}\|^{-m} [\![q\circ S_k]\!] \overset{ii)\text{ from lemma 5.46}}{\ge} t^{-m}[\![q\circ S_k]\!],$$

thus proving our contention.

4) $\forall k \in \mathbb{N}$,

$$
\begin{aligned}
&t^{-2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A_k \cap f^{-1}\{y\}) \,\mathrm{d}\mathcal{L}^m(y) = \\
&= t^{-2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big((S_k^{-1} \circ h|_{A_k})^{-1} \circ S_k^{-1} \circ h|_{A_k}(A_k \cap h^{-1}q^{-1}\{y\})\big) \,\mathrm{d}\mathcal{L}^m(y) = \\
&= t^{-2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big((S_k^{-1} \circ h|_{A_k})^{-1}[S_k^{-1} \circ h(A_k) \cap (q \circ S_k)^{-1}\{y\}]\big) \,\mathrm{d}\mathcal{L}^m(y) \overset{2.4.3)}{\leq} \\
&\leq [\mathrm{Lip}(h|_{A_k})^{-1} \circ S_k]^{n-m}\, t^{-2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big(S_k^{-1} \circ h(A_k) \cap (q \circ S_k)^{-1}\{y\}\big) \,\mathrm{d}\mathcal{L}^m(y) \overset{5.46.i)}{\leq} \\
&\leq t^{-n-m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big(S_k^{-1} \circ h(A_k) \cap (q \circ S_k)^{-1}\{y\}\big) \,\mathrm{d}\mathcal{L}^m(y) \overset{5.43}{=} \\
&= t^{-n-m}[\![q \circ S_k]\!]\mathcal{L}^n\big(S_k^{-1} \circ h(A_k)\big) \overset{2.4.3)}{\leq} \\
&\leq t^{-n-m}[\![q \circ S_k]\!](\mathrm{Lip}\, S_k^{-1} \circ h|_{A_k})^n \mathcal{L}^n(A_k) \overset{5.46.i)}{\leq} \\
&\leq t^{-m}[\![q \circ S_k]\!]\mathcal{L}^n(A_k) \overset{(5.22)}{\leq} \\
&\leq \int_{A_k} Jf \,\mathrm{d}\mathcal{L}^n \overset{(5.22)}{\leq} \\
&\leq t^m[\![q \circ S_k]\!]\mathcal{L}^n(A_k) = t^m[\![q \circ S_k]\!]\mathcal{L}^n\big((h|_{A_k}^{-1} \circ S_k) \circ (S_k^{-1} \circ h|_{A_k})(A_k)\big) \overset{2.4.3)}{\leq} \\
&\leq t^m[\![q \circ S_k]\!](\mathrm{Lip}\, h|_{A_k}^{-1} \circ S_k)^n \mathcal{L}^n\big(S_k^{-1} \circ h|_{A_k}(A_k)\big) \overset{5.46.i)}{\leq} \\
&\leq t^{m+n}[\![q \circ S_k]\!]\mathcal{L}^n\big(S_k^{-1} \circ h|_{A_k}(A_k)\big) \overset{5.43}{=} \\
&= t^{m+n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big(S_k^{-1} \circ h(A_k) \cap (q \circ S_k)^{-1}\{y\}\big) \,\mathrm{d}\mathcal{L}^m(y) = \\
&= t^{m+n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big[(S_k^{-1} \circ h|_{A_k}) \circ (h|_{A_k}^{-1} \circ S_k)\big(S_k^{-1} \circ h(A_k) \cap (q \circ S_k)^{-1}\{y\}\big)\big] \,\mathrm{d}\mathcal{L}^m(y) = \\
&= t^{m+n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big[S_k^{-1} \circ h|_{A_k}\big(A_k \cap h|_{A_k}^{-1} q^{-1}\{y\}\big)\big] \,\mathrm{d}\mathcal{L}^m(y) = \\
&= t^{m+n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big[S_k^{-1} \circ h|_{A_k}(A_k \cap f^{-1}\{y\})\big] \,\mathrm{d}\mathcal{L}^m(y) \overset{2.4.3)}{\leq} \\
&\leq t^{m+n}(\mathrm{Lip}\, S_k^{-1} \circ h|_{A_k})^{n-m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A_k \cap f^{-1}\{y\}) \,\mathrm{d}\mathcal{L}^m(y) \overset{5.46.i)}{\leq} \\
&\leq t^{2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A_k \cap f^{-1}\{y\}) \,\mathrm{d}\mathcal{L}^m(y).
\end{aligned}
$$

In particular, $\forall k \in \mathbb{N}$,

$$
\begin{aligned}
t^{-2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A_k \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y) &\leq \int_{A_k} Jf \, \mathrm{d}\mathcal{L}^n \leq \\
&\leq t^{2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A_k \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y).
\end{aligned} \tag{5.23}
$$

5) It follows from lemma 5.45 that, for $\mathcal{L}^n$-a.e. $y \in \mathbb{R}^m$, $\forall k \in \mathbb{N}$, $A_k \cap f^{-1}\{y\}$ is $\mathcal{H}^{n-m}$-measurable, so that $\mathcal{H}^{n-m}(\dot\bigcup_{k\in\mathbb{N}} A_k \cap f^{-1}\{y\}) = \sum_{k\in\mathbb{N}} \mathcal{H}^{n-m}(A_k \cap f^{-1}\{y\})$. It then follows from the monotone convergence theorem 1.62 that

$$
\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(\dot\bigcup_{k\in\mathbb{N}} A_k \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y) = \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A_k \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y). \tag{5.24}
$$

6) We contend that

$$
\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(\dot\bigcup_{k\in\mathbb{N}} A_k \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y). \tag{5.25}
$$

Indeed, since $\dot\bigcup_{k\in\mathbb{N}} A_k \subset A$, the inequality "$\leq$" trivially holds in (5.25). On the other hand, by subadditivity we have, $\forall y \in \mathbb{R}^m$, $\mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \leq \mathcal{H}^{n-m}(\dot\bigcup_{k\in\mathbb{N}} A_k \cap f^{-1}\{y\}) + \mathcal{H}^{n-m}\big((A \setminus \dot\bigcup_{k\in\mathbb{N}} A_k) \cap f^{-1}\{y\}\big)$, whence $\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y) \leq \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(\dot\bigcup_{k\in\mathbb{N}} A_k \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y) + \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big((A \setminus \dot\bigcup_{k\in\mathbb{N}} A_k) \cap f^{-1}\{y\}\big) \, \mathrm{d}\mathcal{L}^m(y)$. As $A \setminus \dot\bigcup_{k\in\mathbb{N}} A_k$ is $\mathcal{L}^n$-null (by part 3), it follows from lemma 5.44 with $k = n-m$ and $l = m$ that $\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big((A \setminus \dot\bigcup_{k\in\mathbb{N}} A_k) \cap f^{-1}\{y\}\big) \, \mathrm{d}\mathcal{L}^m(y) = 0$, thus proving the reverse inequality and our contention follows.

It then follows from (5.24) and (5.25) that

$$
\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y) = \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A_k \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y). \tag{5.26}
$$

7) Since $A \setminus \dot\bigcup_{k\in\mathbb{N}} A_k$ is $\mathcal{L}^n$-null, we have

$$
\int_A \mathrm{J}f \, \mathrm{d}\mathcal{L}^n = \int_{\dot\bigcup_{k\in\mathbb{N}} A_k} \mathrm{J}f \, \mathrm{d}\mathcal{L}^n \overset{\text{MCT } 1.62}{=} \sum_{k\in\mathbb{N}} \int_{A_k} \mathrm{J}f \, \mathrm{d}\mathcal{L}^n. \tag{5.27}
$$

Thus, from (5.27), (5.26) and (5.23), we finally conclude that

$$
\begin{aligned}
t^{-2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y) &\leq \int_A Jf \, \mathrm{d}\mathcal{L}^n \leq \\
&\leq t^{2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y).
\end{aligned}
$$

Since $t > 1$ was arbitrarily taken, making $t \downarrow 1$ it follows that

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})\,\mathrm{d}\mathcal{L}^m(y) = \int_A Jf\,\mathrm{d}\mathcal{L}^n,$$

which concludes the proof of case 1.

8) Case 2: $A \subset J_f^0$.

Note that both members in (5.21) are σ-additive on $\sigma(\mathcal{L}^n)$ (for the second member, that is a consequence of lemma 5.45 and of the monotone convergence theorem 1.62). By the σ-finiteness of $\mathcal{L}^n$, we may therefore assume that $\mathcal{L}^n(A) < \infty$.

Fix $\epsilon > 0$ and define $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ by $g(x, y) := f(x) + \epsilon y$, $q : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ the projections on the first and second factors, respectively. Then g is Lipschitz and $\forall\,(x, y) \in D_g = D_f \times \mathbb{R}^m$, $\mathsf{D}g(x, y) = \mathsf{D}f(x) \circ q + \epsilon p$. Hence, the transpose of the Jacobian matrix $[\mathsf{D}g(x, y)]$ is the $(n+m)\times m$ matrix written in block form as

$$[\mathsf{D}g(x, y)^*] = \begin{pmatrix} [\mathsf{D}f(x)^*] \\ \epsilon I_m \end{pmatrix},$$

i.e. it is of the same form of (5.14), exchanging m with n. Since the i-th row of the matrix $[\mathsf{D}f(x)^*]$ is the i-th partial derivative of f at x, i.e. $\mathsf{D}f(x) \cdot e_i$, the norm of such row is $\leq \operatorname{Lip} f$. Therefore, with exactly the same argument used in page 146 (case 2 of the area formula), i.e. using Binet-Cauchy formula 5.23, we conclude that, $\forall\,(x, y) \in D_g = D_f \times \mathbb{R}^m$,

$$\mathsf{J}g(x, y) \geq \epsilon^m > 0,$$

$$\big(\mathsf{J}g(x, y)\big)^2 \leq \big(\mathsf{J}f(x)\big)^2 + \Big(\binom{n+m}{m} - \binom{n}{m}\Big)\epsilon^2 \cdot \max\{1, (\operatorname{Lip} f)^{m-1}\}^2.$$

In particular, if $(x, y) \in A \times \mathbb{R}^m \subset J_f^0 \times \mathbb{R}^m$, we conclude that $\mathsf{J}g(x, y) \leq C\epsilon$, where

$$C := \sqrt{\binom{n+m}{m} - \binom{n}{m}}\,\max\{1, (\operatorname{Lip} f)^{m-1}\}.$$

Hence, $\forall\,(x, y) \in A\times\mathbb{R}^m$, $0 < \mathsf{J}g(x, y) \leq C\epsilon$; in particular, $A\times\mathbb{R}^m \subset J_g^+$, so that we may apply case 1 with g in place of f and any $\mathcal{L}^{n+m}$-measurable subset of $A \times \mathbb{R}^m$ in place of A.

9) Recall that, by lemma 5.45, the map $N(f|_A) : \mathbb{R}^m \to [0, \infty]$ given by $N(f|_A)(y) = \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})$ is $\mathcal{L}^m$-measurable. Since $\eta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ given by $\eta(y, w) := y - \epsilon w$ is linear and surjective, it is measurable as a map $(\mathbb{R}^{2m} \equiv \mathbb{R}^m \times \mathbb{R}^m, \mathscr{L}_{\mathbb{R}^{2m}}) \to (\mathbb{R}^m, \mathscr{L}_{\mathbb{R}^m})$ (because it may be factored as $\eta = q \circ \psi$, where $\psi \in \mathrm{GL}(\mathbb{R}^{2m})$

and $q : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is the projection on the first factor, and linear isomorphisms preserve the Lebesgue σ-algebra). Therefore, the composite map $N(f|_A) \circ \eta$ is ≥ 0 and $\mathcal{L}^{2m}$-measurable, and so is the map $\mathbb{R}^m \times \mathbb{R}^m \to [0, \infty]$ given by $(y, w) \mapsto \chi_{\mathbb{B}(0,1)}(w) \cdot N(f|_A)(y - \epsilon w)$. As $\mathcal{L}^{2m} = \mathcal{L}^m \times \mathcal{L}^m$ (by example 1.86), that justifies the application of Fubini-Tonelli's theorem 1.84 in the computation below:

(5.28)

$$
\begin{aligned}
\forall w \in \mathbb{R}^m, & \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \,\mathrm{d}\mathcal{L}^m(y) \overset{1.4}{=} \\
&= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y - \epsilon w\}) \,\mathrm{d}\mathcal{L}^m(y) = \\
&= \frac{1}{\alpha(m)} \int_{\mathbb{B}(0,1)} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y - \epsilon w\}) \,\mathrm{d}\mathcal{L}^m(y) \,\mathrm{d}\mathcal{L}^m(w) = \\
&= \frac{1}{\alpha(m)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \chi_{\mathbb{B}(0,1)}(w) \cdot \mathcal{H}^{n-m}(A \cap f^{-1}\{y - \epsilon w\}) \,\mathrm{d}\mathcal{L}^m(y) \,\mathrm{d}\mathcal{L}^m(w) \overset{\text{Fubini } 1.84}{=} \\
&= \frac{1}{\alpha(m)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \chi_{\mathbb{B}(0,1)}(w) \cdot \mathcal{H}^{n-m}(A \cap f^{-1}\{y - \epsilon w\}) \,\mathrm{d}\mathcal{L}^m(w) \,\mathrm{d}\mathcal{L}^m(y) = \\
&= \frac{1}{\alpha(m)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \chi_{\mathbb{B}(0,1)}(w) \cdot \mathcal{H}^{n-m}\big((A \cap f^{-1}\{y - \epsilon w\}) \times \{w\}\big) \,\mathrm{d}\mathcal{L}^m(w) \,\mathrm{d}\mathcal{L}^m(y),
\end{aligned}
$$

where the last equality is due to corollary 2.5 with the isometry $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ given by $x \mapsto (x, w)$.

10) Note that, if $x \in \mathbb{R}^n$ and $w, y \in \mathbb{R}^m$, we have ($x \in A$ and $g(x, w) = y$) iff ($x \in A$ and $f(x) = y - \epsilon w$) iff $x \in A \cap f^{-1}\{y - \epsilon w\}$. Therefore, defining $B := A \times \mathbb{B}(0, 1) \subset \mathbb{R}^n \times \mathbb{R}^m$, the following equality holds:

$$
B \cap g^{-1}\{y\} \cap p^{-1}\{w\} = \begin{cases} \emptyset & \text{if } w \notin \mathbb{B}(0,1) \\ (A \cap f^{-1}\{y - \epsilon w\}) \times \{w\} & \text{if } w \in \mathbb{B}(0,1). \end{cases}
$$

Thus, $\forall (y, w) \in \mathbb{R}^m \times \mathbb{R}^m$,

$$
\chi_{\mathbb{B}(0,1)}(w) \cdot \mathcal{H}^{n-m}\big((A \cap f^{-1}\{y - \epsilon w\}) \times \{w\}\big) = \mathcal{H}^{n-m}(B \cap g^{-1}\{y\} \cap p^{-1}\{w\}).
$$

It then follows from (5.28) that

$$
\begin{aligned}
&\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \,\mathrm{d}\mathcal{L}^m(y) = \\
&\quad = \frac{1}{\alpha(m)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(B \cap g^{-1}\{y\} \cap p^{-1}\{w\}) \,\mathrm{d}\mathcal{L}^m(w) \,\mathrm{d}\mathcal{L}^m(y).
\end{aligned}
$$

To continue this computation, we apply lemma 5.44 to the inner integral, with $B \cap g^{-1}\{y\} \in \sigma(\mathcal{L}^{n+m})$ in place of A, $p : \mathbb{R}^n \times \mathbb{R}^m \to$

$\mathbb{R}^m$ in place of f, $k = n - m$ and $l = m$, which yields

$$\begin{aligned}
&\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y) \leq \\
&= \frac{1}{\alpha(m)} \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} \int_{\mathbb{R}^m} \mathcal{H}^n(B \cap g^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y) \overset{\text{by case 1}}{=} \\
&= \frac{\alpha(n-m)}{\alpha(n)} \int_B \mathsf{J}g \, \mathrm{d}\mathcal{L}^{n+m} \overset{\text{part 8}}{\leq} \\
&\leq \frac{\alpha(n-m)}{\alpha(n)} C\epsilon \cdot \mathcal{L}^{n+m}(B) = \\
&= \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} C\epsilon \cdot \mathcal{L}^n(A).
\end{aligned}$$

Since $\mathcal{L}^n(A) < \infty$ (by the reduction in the first part of step 8) and $\epsilon > 0$ was arbitrarily taken, making $\epsilon \downarrow 0$ yields

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \, \mathrm{d}\mathcal{L}^m(y) = 0 = \int_A \mathsf{J}f \, \mathrm{d}\mathcal{L}^n,$$

which concludes the proof of case 2 and the thesis follows.

$\square$

Corollary 5.50 (curvilinear Fubini-Tonelli's theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \geq m$. Then for all $g : \mathbb{R}^n \to \mathbb{R}$ $\mathcal{L}^n$-measurable with $g \geq 0$ or g summable,*

$$\int_{\mathbb{R}^n} g \, \mathsf{J}f \, \mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^m} \Big(\int_{f^{-1}\{y\}} g(x) \, \mathrm{d}\mathcal{H}^{n-m}(x) \Big) \, \mathrm{d}\mathcal{L}^m(y), \tag{5.29}$$

meaning that the iterated integrals in second member make sense and the equality holds.

Proof. Suppose that $g \geq 0$. By exercise 1.54, there exists a sequence $(A_i)_{i \in \mathbb{N}}$ in $\sigma(\mathcal{L}^n)$ such that

$$g = \sum_{i=1}^{\infty} \frac{1}{i} \chi_{A_i}.$$

Hence, for all $y \in \mathbb{R}^m$,

$$g \cdot \chi_{f^{-1}\{y\}} = \sum_{i=1}^{\infty} \frac{1}{i} \chi_{A_i \cap f^{-1}\{y\}}.$$

It follows from lemma 5.45 that, for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$, $\forall i \in \mathbb{N}$, $\chi_{A_i \cap f^{-1}\{y\}}$ is $\mathcal{H}^{n-m}$-measurable; for such y, theorem 1.41 ensures that $g \cdot \chi_{f^{-1}\{y\}}$ is $\mathcal{H}^{n-m}$-measurable and ≥ 0, so that the inner integral in the second member of (5.29) makes sense. Moreover, still for y satisfying the above condition, it follows from the monotone convergence theorem 1.62 that

$$\int g(x) \cdot \chi_{f^{-1}\{y\}}(x) \,\mathrm{d}\mathcal{H}^{n-m}(x) = \sum_{i=1}^{\infty} \frac{1}{i} \mathcal{H}^{n-m}(A_i \cap f^{-1}\{y\}).$$

Since the second member of the above equality defines a $\mathcal{L}^m$-measurable function $\mathbb{R}^m \to [0,\infty]$ (in view of lemma 5.45 and theorem 1.41), we conclude that the $\mathcal{L}^m$-a.e. defined positive function $y \mapsto \int g(x) \cdot \chi_{f^{-1}\{y\}}(x) \,\mathrm{d}\mathcal{H}^{n-m}(x)$ is $\mathcal{L}^m$-measurable and

$$\begin{aligned}
&\int_{\mathbb{R}^m} \Big(\int_{f^{-1}\{y\}} g(x) \,\mathrm{d}\mathcal{H}^{n-m}(x) \Big) \,\mathrm{d}\mathcal{L}^m(y) = \\
&\quad = \int_{\mathbb{R}^m} \sum_{i=1}^{\infty} \frac{1}{i} \mathcal{H}^{n-m}(A_i \cap f^{-1}\{y\}) \,\mathrm{d}\mathcal{L}^m(y) \overset{\text{MCT } 1.62}{=} \\
&\quad = \sum_{i=1}^{\infty} \frac{1}{i} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A_i \cap f^{-1}\{y\}) \,\mathrm{d}\mathcal{L}^m(y) \overset{\text{CAF } 5.48}{=} \\
&\quad = \sum_{i=1}^{\infty} \frac{1}{i} \int \chi_{A_i} \,\mathsf{J}f \,\mathrm{d}\mathcal{L}^n \overset{\text{MCT } 1.62}{=} \\
&\quad = \int \sum_{i=1}^{\infty} \frac{1}{i} \chi_{A_i} \,\mathsf{J}f \,\mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^n} g \,\mathsf{J}f \,\mathrm{d}\mathcal{L}^n,
\end{aligned}$$

thus proving the case in which $g \geq 0$. For $g : \mathbb{R}^n \to \mathbb{R}$ $\mathcal{L}^n$-summable, we apply the case just proved to the positive and negative parts of g. □

Exercise 5.51 (Coarea Formula for locally Lipschitz maps). The coarea formula and its corollary remain valid for locally Lipschitz maps defined on open subsets of $\mathbb{R}^n$. That is, let $n \geq m$, $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}^m$ locally Lipschitz.

a) (coarea formula) For all $\mathcal{L}^n$-measurable $A \subset \Omega$,
  - for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$, $f^{-1}\{y\} \cap A$ is $\mathcal{H}^{n-m}$-measurable;
  - the function $N(f|_A) : \mathbb{R}^m \to [0,\infty]$, $y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})$, is $\mathcal{L}^m$-measurable and
  $$\int_A \mathsf{J}f \,\mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \,\mathrm{d}\mathcal{L}^m(y).$$

b) (curvilinear Fubini-Tonelli's theorem) If $g : \Omega \to \mathbb{R}$ is $\mathcal{L}^n|_\Omega$-measurable and $g \geq 0$ or $g \in \mathsf{L}^1(\mathcal{L}^n|_\Omega)$, then
$$\int_\Omega g \,\mathsf{J}f \,\mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^m} \Big(\int_{f^{-1}\{y\}} g(x) \,\mathrm{d}\mathcal{H}^{n-m}(x) \Big) \,\mathrm{d}\mathcal{L}^n(y),$$

meaning that the iterated integrals in the second member make sense and the equality holds.

5.3.1. Applications of the coarea formula.

PROPOSITION 5.52 (polar coordinates). *If $g : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{L}^n$-measurable and $g \geq 0$ or $g \in \mathsf{L}^1(\mathcal{L}^n)$, then*

$$\int_{\mathbb{R}^n} g \,\mathrm{d}\mathcal{L}^n = \int_0^\infty \Big(\int_{\partial\mathbb{B}(0,r)} g \,\mathrm{d}\mathcal{H}^{n-1} \Big) \,\mathrm{d}r. \tag{5.30}$$

PROOF. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = \|x\|$. Then f is Lipschitz and $\forall\, x \in D_f = \mathbb{R}^n \setminus \{0\}$, $\nabla f(x) = x/\|x\|$, hence $\mathsf{J}f(x) = \|\nabla f(x)\| = 1$. Since $\forall r \in \mathbb{R}$, $f^{-1}\{r\} = \partial\mathbb{B}(0,r)$ (in particular, $= \emptyset$ for $r < 0$), (5.30) is a direct consequence of corollary 5.50. $\square$

PROPOSITION 5.53. *Let $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}$ be locally Lipschitz. Then*

$$\int_\Omega \|\nabla f\| \,\mathrm{d}\mathcal{L}^n = \int_{-\infty}^\infty \mathcal{H}^{n-1}(\{f = t\}) \,\mathrm{d}t.$$

PROOF. It is a direct consequence of exercise 5.51.a), with $A = \Omega$, taking into account that $\mathsf{J}f = \|\nabla f\|$. $\square$

CHAPTER 6

Sobolev Spaces

In this chapter we study some basic theory of Sobolev spaces $\mathsf{W}^{1,\mathsf{p}}(\Omega)$ and $\mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(\Omega)$ on open sets $\Omega \subset \mathbb{R}^n$. Our primary purpose is to develop some background for the study of *functions of bounded variation* and *Cacciopoli sets* in the subsequent chapters. For a more extensive treatment on this subject, we refer the reader to standard textbooks — for instance, [**Maz11**] or [**AF03**].

Recall the definitions of *weak derivatives* and *sobolev functions* introduced in the previous chapter in 5.3 and 5.8. In order to introduce vector space topologies on the spaces $\mathsf{W}^{1,\mathsf{p}}(\Omega)$ and $\mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(\Omega)$, we make the following definition:

Definition 6.1. Let $\Omega \subset \mathbb{R}^n$ open and $f \in \mathsf{W}^{1,1}_{\mathsf{loc}}(\Omega)$, i.e. $f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ admits weak partial derivatives of first order. We define

- for $1 \leq p < \infty$, $\|f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)} := (\int_\Omega |f|^p + \|\nabla f\|^p \, \mathrm{d}\mathcal{L}^n)^{1/p} \in [0, \infty]$;
- for $p = \infty$, $\|f\|_{\mathsf{W}^{1,\infty}(\Omega)} := \big\| |f| + \|\nabla f\| \big\|_{\mathsf{L}^\infty(\Omega)} \in [0, \infty]$.

Thus, with the notation above, $\forall 1 \leq p \leq \infty$, $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$ iff $\|f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)} < \infty$, and it is clear that $\|\cdot\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)}$ is a seminorm on $\mathsf{W}^{1,\mathsf{p}}(\Omega)$. Similarly to the discussion on L^p spaces, the linear subspace $N := \{f \in \mathsf{W}^{1,\mathsf{p}}(\Omega) \mid \|f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)} = 0\}$ of $\mathsf{W}^{1,\mathsf{p}}(\Omega)$ consists of the measurable functions on Ω which are null almost everywhere. Therefore, the quotient $\mathsf{W}^{1,\mathsf{p}}(\Omega)/N$ consists of classes of equivalence of functions in $\mathsf{W}^{1,\mathsf{p}}(\Omega)$ which coincide almost everywhere, and $\|\cdot\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)}$ is a norm on this quotient, which is complete by the following proposition. As in remark 1.61, we shall henceforth overload the notation "$\mathsf{W}^{1,\mathsf{p}}(\Omega)$", which will be used both with its original meaning and also to denote the aforementioned quotient space.

Proposition 6.2. *Let $\Omega \subset \mathbb{R}^n$ open. For $1 \leq p \leq \infty$, $\mathsf{W}^{1,\mathsf{p}}(\Omega)$ is a Banach space (for $p = 2$, it is a Hilbert space). It is reflexive for $1 < p < \infty$ and it is separable for $1 \leq p < \infty$.*

Proof. Let $H := \mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega) \times \mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$, which is a Banach space with the norm

$$\|(f, F)\| := \begin{cases} \left(\int_\Omega |f|^p + \|F\|^p \,\mathrm{d}\mathcal{L}^n\right)^{1/p} & \text{for } 1 \le p < \infty, \\ \big\| |f| + \|F\| \big\|_{\mathsf{L}^\infty(\Omega)} & \text{for } p = \infty. \end{cases}$$

The fact that H is indeed a Banach space, reflexive for $1 < p < \infty$ and separable for $1 \le p < \infty$ follows from the corresponding properties of the spaces $\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega)$ and $\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega, \mathbb{R}^n) \equiv \mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega)^n$ and from the fact that the topology of H is the product topology. Moreover, for $p = 2$, the norm defined above is induced by the inner product $\langle (f, F), (g, G)\rangle := \langle f, g\rangle_{\mathsf{L}^2(\mathcal{L}^n|_\Omega)} + \langle F, G\rangle_{\mathsf{L}^2(\mathcal{L}^n|_\Omega, \mathbb{R}^n)}$, hence it is a Hilbert space in that case.

We contend that the graph of the weak gradient operator $\nabla^\mathsf{w} : \mathsf{W}^{1,\mathsf{p}}(\Omega) \to \mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$ is a closed subspace of H. Indeed, let $(u_k, v_k)_{k\in\mathbb{N}}$ be a sequence in gr ∇^w such that $(u_k, v_k) \to (u, v) \in H$. We must show that u is weakly differentiable and $\nabla^\mathsf{w} u = v$. Indeed, $\forall \varphi \in \mathsf{C}^\infty_\mathsf{c}(\Omega, \mathbb{R}^n)$, $\forall k \in \mathbb{N}$,

$$\int_\Omega u_k \operatorname{div} \varphi \,\mathrm{d}\mathcal{L}^n = -\int_\Omega \langle v_k, \varphi\rangle \,\mathrm{d}\mathcal{L}^n.$$

Since $u_k \to u$ in $\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega)$ and $v_k \to v$ in in $\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$, the above equality holds with u in place of u_k and v in place of v_k, thus proving our contention.

As a closed subspace of H, gr ∇^w is also a Banach space (Hilbert for $p = 2$), reflexive for $1 < p < \infty$ and separable for $1 \le p < \infty$. Since the projection on the first factor gr $\nabla^\mathsf{w} \to \mathsf{W}^{1,\mathsf{p}}(\Omega)$ is a linear isometry onto $\mathsf{W}^{1,\mathsf{p}}(\Omega)$ endowed with the norm defined in 6.1 (in other words, that norm is the "graph norm"), the thesis follows. □

Proposition 6.3. *Let $1 \le p \le \infty$, $\Omega \subset \mathbb{R}^n$ open and $(u_k)_{k\in\mathbb{N}}$ a sequence in $\mathsf{W}^{1,\mathsf{p}}(\Omega)$.*

i) If $(u_k)_{k\in\mathbb{N}}$ is L^p-convergent to $u \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega)$ and $(\nabla u_k)_{k\in\mathbb{N}}$ is L^p-convergent to $v \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$, then $u \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$ and $\nabla^\mathsf{w} u = v$.

ii) If $1 < p \le \infty$, $(u_k)_{k\in\mathbb{N}}$ is L^p-convergent to $u \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega)$ and the sequence $(\nabla u_k)_{k\in\mathbb{N}}$ is bounded in $\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$, then $u \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$.

Proof. With the notation from the previous proof, the first assertion is a direct consequence of the fact that gr ∇^w is closed in H.

As to the second assertion, let $1 < p \le \infty$ and q the conjugate exponent of p. It follows from the Riesz representation theorem 1.79 that $\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$ is the dual of $\mathsf{L}^\mathsf{q}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$; hence, by the fact that $\mathsf{L}^\mathsf{q}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$ is separable (since $1 \le q < \infty$) and by the Banach-Alaoglu theorem, strongly closed balls in $\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$ are sequentially compact in the weak-star topology. Thus, passing to a subsequence if necessary, we may assume that $(\nabla u_k)_{k\in\mathbb{N}}$ is weakly-star convergent to $v \in \mathsf{L}^{\mathsf{p}}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$. In particular, for every $\varphi \in \mathsf{C}^\infty_{\mathsf{c}}(\Omega)$,

$$\int_\Omega u \operatorname{div} \varphi \, d\mathcal{L}^n = \lim_{k\to\infty} \int_\Omega u_k \operatorname{div} \varphi \, d\mathcal{L}^n = \\ = - \lim_{k\to\infty} \int_\Omega \langle \nabla u_k, \varphi \rangle \, d\mathcal{L}^n = \\ = - \int_\Omega \langle v, \varphi \rangle \, d\mathcal{L}^n$$

thus showing that u is weakly differentiable and $\nabla^{\mathsf{w}} u = v \in \mathsf{L}^{\mathsf{p}}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$, hence $u \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$. □

In order to establish the *locality* of the weak derivative, we recall from *Advanced Calculus* the construction of smooth partitions of unity (skip to theorem 6.13 if you don't need to recall that stuff).

Definition 6.4 (point-finite and locally finite families). Let X be a topological space. We say that a family $(F_\alpha)_{\alpha\in A}$ of subsets of X is

- *point-finite* if $\forall x \in X$, $\{\alpha \in A \mid x \in F_\alpha\}$ is finite;
- *locally finite* if $\forall x \in X$, there exists a neighborhood V of x such that $\{\alpha \in A \mid V \cap F_\alpha \neq \emptyset\}$ is finite.

Remark 6.5. Let X be a topological space.

1) If X is second countable and $(F_\alpha)_{\alpha\in A}$ is a locally finite family of subsets of X, then A is countable, since we may cover X by countably many open sets, each of which intersects subsets in the family for at most finitely many indices.
2) It is clear that every locally finite family of subsets of X is point-finite.
3) It is also clear that, if $K \subset X$ is compact and $(F_\alpha)_{\alpha\in A}$ is a locally finite family of subsets of X, then $\{\alpha \in A \mid K \cap F_\alpha \neq \emptyset\}$ is finite.

Definition 6.6 (smooth partitions of unity on open sets of $\mathbb{R}^n$). Let $\Omega \subset \mathbb{R}^n$ open. A *smooth partition of unity* of Ω is a family $(\xi_\alpha)_{\alpha\in A}$ such that:

PU1) $\forall \alpha \in A$, $\xi_\alpha \in \mathsf{C}^\infty(\Omega, [0,1])$ and $(\operatorname{spt} \xi_\alpha)_{\alpha\in A}$ is a locally finite family of subsets of Ω;

PU2) $\forall x \in \Omega$, $\sum_{\alpha\in A} \xi_\alpha(x) = 1$.

If $\mathcal{F}$ is an open cover of Ω, we say that a smooth partition of unity $(\xi_\alpha)_{\alpha\in A}$ of Ω is *subordinate* to $\mathcal{F}$ if $\forall \alpha \in A$, there exists $U \in \mathcal{F}$ such that $\operatorname{spt} \xi_\alpha \subset U$.

With the notation above, note that, $\forall x \in \Omega$, the sum in PU2) is finite. Actually, thanks to PU1), there exists a neighborhood $V \subset \Omega$

of x such that $A_V := \{\alpha \in A \mid \operatorname{spt} \xi_\alpha \cap V \neq \emptyset\}$ is finite; hence, $\forall y \in V$, $\sum_{\alpha\in A} \xi_\alpha(y) = \sum_{\alpha\in A_V} \xi_\alpha(y)$ is a finite sum.

We will need the following theorem from *General Topology*:

THEOREM 6.7 (shrinking lemma). *Let X be a normal topological space and $(U_\alpha)_{\alpha\in A}$ a point-finite open cover of X. Then there exists an open cover $(V_\alpha)_{\alpha\in A}$ of X such that, $\forall \alpha \in A$, $\overline{V_\alpha} \subset U_\alpha$.*

PROOF. We refer the reader to [**Eng89**], theorem 1.5.18, page 44. ☐

THEOREM 6.8 (existence of partitions of unity on open sets of $\mathbb{R}^n$). *Let $\Omega \subset \mathbb{R}^n$ open and $(U_\alpha)_{\alpha\in A}$ a locally finite open cover of Ω with $\forall \alpha \in A$, $U_\alpha \Subset \Omega$. Then there exists a smooth partition of unity $(\xi_\alpha)_{\alpha\in A}$ of Ω such that, $\forall \alpha \in A$, $\operatorname{spt} \xi_\alpha \Subset U_\alpha$.*

PROOF. Apply the shrinking lemma 6.7 to the locally finite (hence point-finite) open cover $(U_\alpha)_{\alpha\in A}$ of Ω endowed with the relative topology. We obtain an open cover $(V_\alpha)_{\alpha\in A}$ of Ω such that, $\forall \alpha \in A$, $\overline{V_\alpha}^{\Omega} = \overline{V_\alpha} \cap \Omega \subset U_\alpha$; in particular, since $U_\alpha \Subset \Omega$, $\overline{V_\alpha} = \overline{V_\alpha}^{\Omega}$ is a compact subset of U_α. Then, for each $\alpha \in A$, we may apply the differentiable Urysohn's lemma 1.114 to obtain $\psi_\alpha \in \mathsf{C}^\infty_\mathsf{c}(U_\alpha, [0, 1])$ such that $\psi_\alpha \equiv 1$ on $\overline{V_\alpha}$. Since $(\operatorname{spt} \psi_\alpha)_{\alpha\in A}$ is a locally finite family of subsets of Ω, $\psi := \sum_{\alpha\in A} \psi_\alpha$ is a real-valued smooth function on Ω; it is strictly positive, because $(V_\alpha)_{\alpha\in A}$ cover Ω. Define, $\forall \alpha \in A$,

$$\xi_\alpha := \frac{\psi_\alpha}{\psi}.$$

Then $\forall \alpha \in A$, $\xi_\alpha \in \mathsf{C}^\infty_\mathsf{c}(\Omega)$, $\operatorname{spt} \xi_\alpha \Subset U_\alpha$, $(\operatorname{spt} \xi_\alpha)_{\alpha\in A}$ is locally finite family of subsets of Ω and $\forall x \in \Omega$, $\sum_{\alpha\in A} \xi_\alpha(x) = 1$. ☐

COROLLARY 6.9. *Let $\Omega \subset \mathbb{R}^n$ be open and $\mathcal{F}$ an open cover of Ω. Then there exists a partition of unity $(\xi_\alpha)_{\alpha\in A}$ of Ω subordinate to $\mathcal{F}$ such that $\forall \alpha \in A$, $\operatorname{spt} \xi_\alpha$ is compact.*

PROOF. Take a refinement of $\mathcal{F}$ formed by open sets with compact closures in Ω and then a locally finite open refinement $\mathcal{G}$ of the latter cover, which exists, due the paracompactness of Ω. Then apply theorem 6.8 to Ω with the cover $\mathcal{G}$. ☐

COROLLARY 6.10. *Let $K \subset \mathbb{R}^n$ be compact and $(U_i)_{1\le i\le N}$ a cover of K by open subsets of $\mathbb{R}^n$. Then there exists $(\xi_i)_{1\le i\le N}$ such that $\forall 1 \le i \le N$, $\xi_i \in \mathsf{C}^\infty_\mathsf{c}(U_i, [0, 1])$ and $\sum_{i=1}^N \xi_i \equiv 1$ on K.*

PROOF. Let $\Omega := \cup_{1\le i\le N} U_i$ and apply the previous corollary to obtain a smooth partition of unity $(\eta_\alpha)_{\alpha\in A}$ of Ω subordinate to the cover $(U_i)_{1\leq i\leq N}$ such that $\forall \alpha \in A$, spt η_α is compact. Since (spt $\eta_\alpha)_{\alpha\in A}$ is a locally finite family of subsets of Ω and K is a compact subset of Ω, the set $A_K := \{\alpha \in A \mid \text{spt } \eta_\alpha \cap K \neq \emptyset\}$ is finite. Since the partition of unity is subordinate to $(U_i)_{1\leq i\leq N}$, for each $\alpha \in A_K$ we may choose $i(\alpha) \in \{1, \dots, N\}$ such spt $\eta_\alpha \subset U_{i(\alpha)}$. Define, for $1 \leq i \leq N$,

$$\xi_i := \sum_{\{\alpha\in A_K | i(\alpha)=i\}} \eta_\alpha,$$

recalling that the sum over the empty family is 0.

□

COROLLARY 6.11. *Let $\Omega \subset \mathbb{R}^n$ be open and $\mathcal{F}$ an open cover of Ω. Then there exists a partition of unity $(\xi_V)_{V\in\mathcal{F}}$ of Ω* strictly subordinate *to $\mathcal{F}$, i.e. such that for all $V \in \mathcal{F}$,* spt *$\xi_V \subset V$.*

In general, it is not possible to choose such a strictly subordinate partition of unit with compact supports, i.e. for each $V \in \mathcal{F}$, the support of ξ_V may be not compact.

PROOF. We may apply corollary 6.9 to obtain a smooth partition of unity $(\eta_\alpha)_{\alpha\in A}$ of Ω subordinate to the cover $\mathcal{F}$ such that $\forall\alpha \in A$, spt η_α is compact. For each $\alpha \in A$ we may choose $V(\alpha) \in \mathcal{F}$ such spt $\eta_\alpha \subset V(\alpha)$. Define, for $V \in \mathcal{F}$,

$$\xi_V := \sum_{\{\alpha\in A | V(\alpha)=V\}} \eta_\alpha.$$

Since the above sum is locally finite, for each $V \in \mathcal{F}$, ξ_V is smooth with $0 \leq \xi_V \leq 1$ and $\sum_{V\in\mathcal{F}} \xi_V \equiv 1$ on Ω. Moreover, as the family (spt $\eta_\alpha)_{\alpha\in A}$ is locally finite, for all $V \in \mathcal{F}$, $\cup_{V(\alpha)=V}$ spt $\eta_{V(\alpha)} \subset V$ is closed in Ω (because it is the union of a locally finite family of closed subsets of Ω), hence spt $\xi_V = \cup_{V(\alpha)=V}$ spt $\eta_{V(\alpha)} \subset V$. That is, $(\xi_V)_{V\in\mathcal{F}}$ is a smooth partition of unity of Ω with spt $\xi_V \subset V$ for all $V \in \mathcal{F}$. □

EXERCISE 6.12 (differentiable Urysohn's lemma, part II). Let F_0 and F_1 be disjoint closed subsets of $\mathbb{R}^n$. There exists a smooth function $\xi \in \mathsf{C}^\infty(\mathbb{R}^n)$ such that $\xi \equiv 0$ on F_0 and $\xi \equiv 1$ on F_1.

THEOREM 6.13 (locality of the weak derivative). *Let $\Omega \subset \mathbb{R}^n$, $f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ and $\mathcal{F} \subset 2^\Omega$ an open cover of Ω. Then f admits weak partial derivatives of first order on Ω* iff *$\forall U \in \mathcal{F}$, $f|_U$ admits weak partial derivatives of first order on U. Moreover, weak derivatives commute with restrictions.*

LEMMA 6.14. *Let $U \subset \mathbb{R}^n$ open, $1 \leq p \leq \infty$ and $\xi \in \mathsf{C}^\infty_\mathsf{c}(U)$.*

i) If $f \in \mathsf{L}^{\mathsf{p}}_{\mathsf{loc}}(\mathcal{L}^n|_U)$*, then* $\xi \cdot f$ *(defined as* 0 *on* $\mathbb{R}^n \setminus U$*) belongs to* $\mathsf{L}^{\mathsf{p}}(\mathcal{L}^n)$*.*

ii) If $f \in \mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(U)$*, then* $\xi \cdot f \in \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ *and,* $\forall 1 \le i \le n$*,*

$$\frac{\partial^{\mathsf{w}}(\xi \cdot f)}{\partial x_i} = \frac{\partial \xi}{\partial x_i} \cdot f + \xi \cdot \frac{\partial^{\mathsf{w}} f}{\partial x_i}.$$

Proof.

i) $\xi \cdot f$ is clearly $\mathcal{L}^n$-measurable (for instance, by proposition 1.50). If $1 \le p < \infty$, $\int_{\mathbb{R}^n} |\xi \cdot f|^p \, \mathrm{d}\mathcal{L}^n \le \|\xi\|_u^p \int_{\mathrm{spt}\ \xi} |f|^p \, \mathrm{d}\mathcal{L}^n < \infty$; if $p = \infty$, $\|\xi \cdot f\|_\infty \le \|\xi\|_u \|f\|_{\mathsf{L}^\infty(\mathcal{L}^n|_{\mathrm{spt}\ \xi})} < \infty$.

ii) For all $1 \le i \le n$, it follows from the previous item that both $\xi \cdot f$ and $g := \frac{\partial \xi}{\partial x_i} \cdot f + \xi \cdot \frac{\partial^{\mathsf{w}} f}{\partial x_i}$ belong to $\mathsf{L}^{\mathsf{p}}(\mathcal{L}^n)$. It therefore suffices to show that $\xi \cdot f$ admits weak i-th partial derivative equal to g. Indeed, $\forall \varphi \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\mathbb{R}^n} (\xi \cdot f) \cdot \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}\mathcal{L}^n &= \int_\Omega f \cdot \Big(\frac{\partial(\xi \cdot \varphi)}{\partial x_i} - \frac{\partial \xi}{\partial x_i} \cdot \varphi \Big) \, \mathrm{d}\mathcal{L}^n = \\ &= -\int_\Omega \Big(\xi \cdot \frac{\partial^{\mathsf{w}} f}{\partial x_i} \cdot \varphi + f \cdot \frac{\partial \xi}{\partial x_i} \cdot \varphi \Big) \, \mathrm{d}\mathcal{L}^n = \\ &= -\int_{\mathbb{R}^n} g \cdot \varphi \, \mathrm{d}\mathcal{L}^n,\end{aligned}$$

as asserted.

$\square$

Proof of theorem 6.13. The implication "$\Rightarrow$" and the fact that weak derivatives commute with restrictions are clear. We must prove the converse implication, i.e. if $1 \le i \le n$ and $\forall U \in \mathcal{F}$, $f|_U$ admits weak i-th partial derivative $\partial_i(f|_U) \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_U)$, then f admits weak i-th partial derivative on Ω.

1) We may assume that $\mathcal{F}$ is locally finite and $\forall U \in \mathcal{F}$, $U \Subset \Omega$. Indeed, in the general case, take a locally finite open refinement $\mathcal{G}$ of $\mathcal{F}$ such that $\forall U \in \mathcal{G}$, $U \Subset \Omega$. For each $V \in \mathcal{G}$, there exists $U \in \mathcal{F}$ such that $V \subset U$; since $f|_U$ admits weak i-th partial derivative $\partial_i(f|_U) \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_U)$, it follows that $f|_V = (f|_U)|_V$ admits weak i-th partial derivative, so that we may replace $\mathcal{F}$ by $\mathcal{G}$.

2) Take a smooth partition of unity $(\xi_V)_{V \in \mathcal{F}}$ of Ω, given by theorem 6.8, such that $\forall V \in \mathcal{F}$, $\xi_V \in \mathsf{C}^\infty_{\mathsf{c}}(V)$. We contend that $g := \sum_{V \in \mathcal{F}} \xi_V \partial_i(f|_V) \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$. Indeed, g is clearly $\mathcal{L}^n$-measurable and, for each compact $K \subset \Omega$, there are finitely many $V_1, \ldots, V_N \in \mathcal{F}$ which intersect K, so that $|g|\chi_K \le \sum_{j=1}^N \xi_{V_j} |\partial_i(f|_{V_j})| \in \mathsf{L}^1(\mathcal{L}^n)$ by lemma 6.14.i), thus proving our contention.

3) Let $\varphi \in \mathsf{C}^\infty_\mathsf{c}(\Omega)$. Since spt φ is compact, there are finitely many $V_1, \dots, V_N \in \mathcal{F}$ which intersect K. We then have

$$\begin{aligned}\int_\Omega g\,\varphi\,\mathrm{d}\mathcal{L}^n &= \sum_{j=1}^N \int_\Omega \xi_j \partial_i(f|_{V_j})\varphi\,\mathrm{d}\mathcal{L}^n = \\ &= \sum_{j=1}^N \int_{V_j} \partial_i(f|_{V_j}) \cdot (\xi_j\varphi)\,\mathrm{d}\mathcal{L}^n = \\ &= -\sum_{j=1}^N \int_{V_j} f|_{V_j} \cdot \partial_i(\xi_j\varphi)\,\mathrm{d}\mathcal{L}^n = \\ &= -\sum_{j=1}^N \int_\Omega f \cdot \partial_i(\xi_j\varphi)\,\mathrm{d}\mathcal{L}^n = \\ &= -\int_\Omega f \cdot \partial_i(\sum_{j=1}^N \xi_j\varphi)\,\mathrm{d}\mathcal{L}^n = \\ &= -\int_\Omega f \cdot \partial_i(\varphi)\,\mathrm{d}\mathcal{L}^n,\end{aligned}$$

thus proving that $\partial_i^{\mathsf{w}} f = g$ on Ω.

□

Corollary 6.15. *Let $\Omega \subset \mathbb{R}^n$ open, $1 \le p \le \infty$ and $f : \Omega \to \mathbb{R}$ Lebesgue measurable. Then $f \in \mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(\Omega)$* iff *for all open $V \Subset \Omega$, $f|_V \in \mathsf{W}^{1,\mathsf{p}}(V)$.*

Proof. The implication "⇒" is clear, in view of the fact that weak derivatives commute with restrictions. Conversely, assume that for all open $V \Subset \Omega$, $f|_V \in \mathsf{W}^{1,\mathsf{p}}(V)$. In particular, for all open $V \Subset \Omega$, $f|_V \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n|_V)$, hence $f \in \mathsf{L}^\mathsf{p}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega) \subset \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$. It then follows from theorem 6.13 that $\exists\, \nabla^{\mathsf{w}} f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$; besides, for all open $V \Subset \Omega$, $(\nabla^{\mathsf{w}} f)|_V = \nabla^{\mathsf{w}}(f|_V) \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n|_V, \mathbb{R}^n)$. That is, $f \in \mathsf{L}^\mathsf{p}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ and $\nabla^{\mathsf{w}} f \in \mathsf{L}^\mathsf{p}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$, i.e. $f \in \mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(\Omega)$. □

Corollary 6.16. *Let $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}^n$ locally Lipschitz. Then $f \in \mathsf{W}^{1,\infty}_{\mathsf{loc}}(\Omega)$, f is differentiable in the sense of Fréchet $\mathcal{L}^n$-a.e. on Ω and $\nabla^{\mathsf{w}} f = \nabla f$ $\mathcal{L}^n$-a.e. on Ω.*

Proof. We have already proved in corollary 5.14 that f is differentiable in the sense of Fréchet $\mathcal{L}^n$-a.e. on Ω. It therefore suffices to show, in view of the locality of both the weak derivative 6.13 and of the classical Fréchet derivative, that for each open $V \Subset \Omega$ on which f has Lipschitz restriction, $f|_V \in \mathsf{W}^{1,\infty}(V)$ and $\nabla^{\mathsf{w}} f = \nabla f$ $\mathcal{L}^n$-a.e. on V.

Indeed, by McShane's theorem 5.1 we may extend $f|_V$ to a Lipschitz map $\mathbb{R}^n \to \mathbb{R}$, for which proposition 5.9 and theorem 5.12 apply. $\square$

6.1. Approximation by smooth functions, part I

In this section we fix an open set $\Omega \subset \mathbb{R}^n$. Our goal is to derive theorems on approximation of Sobolev functions on Ω by smooth functions. In order to accomplish that, it will be convenient to generalize the operation of convolution with the standard mollifier $(\phi_t)_{t>0}$ to functions $f : \Omega \to \mathbb{R}$. One possible approach is to proceed like we did in the proof of proposition 5.7. Another approach, which we adopt here, is to define the regularized functions $f_t = \phi_t * f$ on smaller subsets Ω_t of Ω, cf. definition 6.17 below. We call the reader's attention to the fact that, in general, it is not possible to simply extend f by zero on $\mathbb{R}^n \setminus \Omega$ and then regularize the extension $\bar{f}$: it might be the case that $\bar{f} \notin \mathsf{L}^1_{\mathsf{loc}}(\mathbb{R}^n)$, so that "$\phi_t * \bar{f}$ " would not be defined.

Definition 6.17. For each $t > 0$, let

$$\Omega_t := \{x \in \mathbb{R}^n \mid \mathbb{B}(x,t) \subset \Omega\} = \{x \in \mathbb{R}^n \mid d(x, \Omega^c) > t\},$$

so that $(\Omega_t)_{t>0}$ is a family of open subsets of Ω which increases to Ω as $t \downarrow 0$.

Let $(\phi_t)_{t>0}$ be the standard mollifier in $\mathbb{R}^n$. For each $t > 0$ and $f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$, we define $f_t : \Omega_t \to \mathbb{R}$ by, $\forall x \in \Omega_t$,

$$f_t(x) := (\phi_t * f)(x) = \int_{\mathbb{B}(x,t)} f(y)\phi_t(x-y)\, \mathrm{d}\mathcal{L}^n(y).$$

We call f_t the *t-approximation or t-regularization of f*.

Note that: 1) $f_t(x)$ is well-defined since, for $x \in \Omega_t$, $\mathbb{B}(x,t) \subset \Omega$; 2) if $\Omega = \mathbb{R}^n$, then $\Omega_t = \mathbb{R}^n$ for all $t > 0$.

Notation. If $f : \Omega \to \mathbb{R}$, we denote by $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ the extension of f by zero on $\mathbb{R}^n \setminus \Omega$.

In order to simplify the notation, sometimes we omit the bar from the extension, whenever no confusion arises.

Definition 6.18 (convergence in the sense of $\mathsf{L}^{\mathsf{p}}_{\mathsf{loc}}$ and $\mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}$). Let $1 \le p \le \infty$, $f : \Omega \to \mathbb{R}$ $\mathcal{L}^n$-measurable and, for each $k \in \mathbb{N}$, let $f_k : \operatorname{dom} f_k \subset \Omega \to \mathbb{R}$ be $\mathcal{L}^n$-measurable.

- We say that $(f_k)_{k\in\mathbb{N}}$ *converges to f in the sense of* $\mathsf{L}^{\mathsf{p}}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ (notation: "$f_k \to f$ in $\mathsf{L}^{\mathsf{p}}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$") if, for all open $V \Subset \Omega$, there exists $k_0 \in \mathbb{N}$ (possibly depending on V) such that $\forall k \ge k_0$, $V \subset \operatorname{dom} f_k$ and $\|f_k - f\|_{\mathsf{L}^{\mathsf{p}}(\mathcal{L}^n|_V)} \to 0$.

- If $\forall k \in \mathbb{N}$, dom f_k is open, f and f_k belong to $\mathsf{L}^1_{\mathsf{loc}}$ on their domains and admit weak partial derivatives of first order, we say that $(f_k)_{k\in\mathbb{N}}$ *converges to f in the sense of* $\mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(\Omega)$ (notation: "$f_k \to f$ in $\mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(\Omega)$") if, for all open $V \Subset \Omega$, there exists $k_0 \in \mathbb{N}$ (possibly depending on V) such that $\forall k \geq k_0$, $V \subset \operatorname{dom} f_k$ and $\|f_k - f\|_{\mathsf{W}^{1,\mathsf{p}}(V)} \to 0$.

We make similar definitions of convergence in the sense of $\mathsf{L}^{\mathsf{p}}_{\mathsf{loc}}$ or in the sense of $\mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}$ for a family $(f_\epsilon)_{\epsilon>0}$ in place of $(f_k)_{k\in\mathbb{N}}$.

Remark 6.19. Concerning the previous definition:

(1) What we have in mind is the family $(f_t)_{t>0}$ of the regularized functions of some $f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$, cf. definition 6.18.
(2) For a sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathsf{L}^{\mathsf{p}}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ and $f \in \mathsf{L}^{\mathsf{p}}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$, the convergence defined above coincides with the convergence in the natural topology of $\mathsf{L}^{\mathsf{p}}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$, which is a Fréchet space topology induced by the family of seminorms $\{\|\cdot\|_{\mathsf{L}^{\mathsf{p}}(\mathcal{L}^n|_V)} \mid V \Subset \Omega \text{ open}\}$.
(3) Similarly, for a sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(\Omega)$ and $f \in \mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(\Omega)$, the convergence defined above coincides with the convergence in the natural topology of $\mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(\Omega)$, which is a Fréchet space topology induced by the family of seminorms $\{\|\cdot\|_{\mathsf{W}^{1,\mathsf{p}}(V)} \mid V \Subset \Omega \text{ open}\}$.

Theorem 6.20 (mollifiers, part II). *Let $f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$, $(\phi_\epsilon)_{\epsilon>0}$ the standard mollifier and $f_\epsilon = \phi_\epsilon * f : \Omega_\epsilon \to \mathbb{R}$ the ϵ-approximation of f, cf. definition 6.17.*

i) $\forall\epsilon > 0$, $f_\epsilon \in \mathsf{C}^\infty(\Omega_\epsilon)$.
ii) $\forall\epsilon > 0$, $\forall\varphi \in \mathsf{C}^0_{\mathsf{c}}(\Omega_\epsilon)$, $\int f_\epsilon\varphi \,\mathrm{d}\mathcal{L}^n = \int f\varphi_\epsilon \,\mathrm{d}\mathcal{L}^n$.
iii) $\lim_{\epsilon\to0} f_\epsilon(x) = f(x)$ *if $x \in \Omega$ is a Lebesgue point of f; in particular, $f_\epsilon \to f$ $\mathcal{L}^n$-a.e. on Ω.*
iv) If $f \in \mathsf{C}(\Omega)$, $f_\epsilon \to f$ uniformly on compact subsets of Ω.
v) If $f \in \mathsf{L}^{\mathsf{p}}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ for some $1 \leq p < \infty$, then $f_\epsilon \to f$ in the sense of $\mathsf{L}^{\mathsf{p}}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$.
vi) If $f \in \mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(\Omega)$ for some $1 \leq p \leq \infty$, then $\forall\epsilon > 0, \forall 1 \leq i \leq n$,

$$\frac{\partial f_\epsilon}{\partial x_i} = \phi_\epsilon * \frac{\partial^{\mathsf{w}} f}{\partial x_i} = \left(\frac{\partial^{\mathsf{w}} f}{\partial x_i}\right)_\epsilon$$

on Ω_ϵ.
vii) In particular, if $f \in \mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(\Omega)$ for some $1 \leq p < \infty$, then $f_\epsilon \to f$ in the sense of $\mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}(\Omega)$.

We interpret the theorem above by saying that the ϵ-regularized functions of f approximate f in the "natural topology" of its class of regularity.

PROOF.

i) That is a direct consequence of the dominated convergence theorem, like we argued in the proof of proposition 1.108.j). Indeed, let $x_0 \in \Omega_\epsilon$ and $r > 0$ such that $\mathbb{U}(x_0, r) \subset \Omega_\epsilon$. Then, $\forall x \in \mathbb{U}(x_0, r)$,

$$f_\epsilon(x) = \int_K \phi_\epsilon(x-y) f(y) \, \mathrm{d}\mathcal{L}^n(y),$$

where $K := \mathbb{B}(x_0, r+\epsilon) \Subset \Omega$. It then follows that, for each multi-index $\alpha \in \mathbb{Z}_+^n$, the integral which results from the second member above by derivation under the integral sign (to be justified) is

$$\int_K \partial^\alpha(\phi_\epsilon)(x-y) f(y) \, \mathrm{d}\mathcal{L}^n(y),$$

whose integrand is dominated in absolute value by $\|\partial^\alpha \phi_\epsilon\|_u |f|\big|_K \in \mathsf{L}^1(\mathcal{L}^n|_K)$. Therefore, we may differentiate successively under the integral sign by means of the dominated convergence theorem, cf. proposition 1.67, to conclude that $\forall x \in \mathbb{U}(x_0, r)$, $\forall \alpha \in \mathbb{Z}_+^n$,

$$\exists \partial^\alpha(f_\epsilon)(x) = \int_K \partial^\alpha(\phi_\epsilon)(x-y) f(y) \, \mathrm{d}\mathcal{L}^n(y).$$

Since $x_0 \in \Omega_\epsilon$ was arbitrarily taken, we have proved that f_ϵ has partial derivatives of all orders on all points of Ω_ϵ.

ii) If $\epsilon > 0$ and $\varphi \in \mathsf{C}_\mathsf{c}^0(\Omega_\epsilon)$, we have:

$$\begin{aligned}
\int f_\epsilon \varphi \, \mathrm{d}\mathcal{L}^n &= \int_{\mathrm{spt}\ \varphi} \varphi(x) \int_\Omega f(y)\phi_\epsilon(x-y) \, \mathrm{d}y \, \mathrm{d}x \overset{\text{Fubini } 1.84}{=} \\
&= \int_\Omega f(y) \int_{\mathrm{spt}\ \varphi} \phi_\epsilon(x-y)\varphi(x) \, \mathrm{d}x \, \mathrm{d}y \overset{\phi(z)=\phi(-z)}{=} \\
&= \int_\Omega f(y) \int_{\mathrm{spt}\ \varphi} \phi_\epsilon(y-x)\varphi(x) \, \mathrm{d}x \, \mathrm{d}y = \\
&= \int f(y)\varphi_\epsilon(y) \, \mathrm{d}y,
\end{aligned}$$

where the application of Fubini's theorem is justified by the fact that

$$\int_{\text{spt }\varphi\times\Omega} |f(y)|\cdot\phi_\epsilon(x-y)\cdot|\varphi(x)|\,\mathrm{d}(\mathcal{L}^n\times\mathcal{L}^n)(x,y) \overset{\text{Tonelli 1.84}}{=}$$
$$= \int_\Omega |f(y)| \int_{\text{spt }\varphi} |\varphi(x)|\phi_\epsilon(y-x)\,\mathrm{d}x\,\mathrm{d}y \leq$$
$$\leq \|\varphi\|_u\|\phi_\epsilon\|_u \int_{\text{spt }\varphi+\mathbb{B}(0,\epsilon)} |f(y)|\,\mathrm{d}y < \infty.$$

iii) Let $x\in\Omega$ be a Lebesgue point of f. Take $\delta>0$ such that $x\in\Omega_\delta$. Then, $\forall\, 0<\epsilon\leq\delta$:

$$f_\epsilon(x)-f(x) = \int_{\mathbb{B}(x,\epsilon)} \phi_\epsilon(x-y)[f(y)-f(x)]\,\mathrm{d}\mathcal{L}^n(y) =$$
$$= \int_{\mathbb{B}(x,\epsilon)} \frac{1}{\epsilon^n}\phi(\frac{x-y}{\epsilon})[f(y)-f(x)]\,\mathrm{d}\mathcal{L}^n(y),$$

so that

$$|f_\epsilon(x)-f(x)| \leq \|\phi\|_u\,\alpha(n)\cdot\frac{1}{\mathcal{L}^n\big(\mathbb{B}(x,\epsilon)\big)}\int_{\mathbb{B}(x,\epsilon)} |f(y)-f(x)|\,\mathrm{d}\mathcal{L}^n(y) \overset{\epsilon\to 0}{\to} 0.$$

iv) Let K be a compact subset of Ω. Take $\delta>0$ such that $K_\delta := K+\mathbb{B}(0,\delta)\Subset\Omega$. Let $F := \overline{f\cdot\chi_{K_\delta}} : \mathbb{R}^n\to\mathbb{R}$. Since f is bounded on K_δ (because f is continuous and $K_\delta\subset\Omega$ is compact), $F\in\mathsf{L}^\infty(\mathcal{L}^n)$; moreover, since f coincides with F on K_δ, F is continuous on $(K_\delta)^{\text{o}}$. It then follows from theorem 1.111.iii) that $\phi_\epsilon * F\to F$ uniformly on compact subsets of $(K_\delta)^{\text{o}}$. On the other hand, as $F|_{K_\delta} = f|_{K_\delta}$, we conclude that, $\forall 0<\epsilon\leq\delta$, $\phi_\epsilon * F = f_\epsilon$ on K, whence $f_\epsilon\to f$ uniformly on K.

v) Let K be a compact subset of Ω. Take $\delta>0$ such that $K_\delta := K+\mathbb{B}(0,\delta)\Subset\Omega$. Let $F := \overline{f\cdot\chi_{K_\delta}} : \mathbb{R}^n\to\mathbb{R}$. Since K_δ is a compact subset of Ω, $f|_{K_\delta}\in\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_{K_\delta})$, which implies $F\in\mathsf{L}^\mathsf{p}(\mathcal{L}^n)$. It then follows from theorem 1.111.i) that $\phi_\epsilon * F\to F$ in $\mathsf{L}^\mathsf{p}(\mathcal{L}^n)$. On the other hand, as $F|_{K_\delta} = f|_{K_\delta}$, we conclude that, $\forall 0<\epsilon\leq\delta$, $\phi_\epsilon * F = f_\epsilon$ on K, whence $\|f_\epsilon - f\|_{\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_K)}\to 0$.

vi) $\forall \epsilon > 0, \forall 1 \leq i \leq n, \forall \varphi \in \mathsf{C}^\infty_\mathsf{c}(\Omega_\epsilon)$,

$$\begin{aligned}\int f_\epsilon \frac{\partial \varphi}{\partial x_i}\, \mathrm{d}\mathcal{L}^n &\overset{\text{(ii)}}{=} \int f \phi_\epsilon * \frac{\partial \varphi}{\partial x_i}\, \mathrm{d}\mathcal{L}^n \overset{1.108}{=} \\ &= \int f \frac{\partial(\phi_\epsilon * \varphi)}{\partial x_i}\, \mathrm{d}\mathcal{L}^n = \\ &= -\int \frac{\partial^\mathsf{w} f}{\partial x_i}\, \varphi_\epsilon \,\mathrm{d}\mathcal{L}^n \overset{\text{(ii)}}{=} \\ &= -\int \left(\frac{\partial^\mathsf{w} f}{\partial x_i}\right)_\epsilon \varphi \,\mathrm{d}\mathcal{L}^n,\end{aligned}$$

thus showing that $\frac{\partial f_\epsilon}{\partial x_i} = \left(\frac{\partial^\mathsf{w} f}{\partial x_i}\right)_\epsilon$.

One alternative to the above proof is to use the formula obtained in (i) for $\frac{\partial f_\epsilon}{\partial x_i}$, i.e. $\forall x \in \Omega_\epsilon$,

$$\begin{aligned}\frac{\partial f_\epsilon}{\partial x_i}(x) &= \int_\Omega \frac{\partial(\phi_\epsilon)}{\partial x_i}(x-y) f(y)\,\mathrm{d}y = \\ &= -\int_\Omega \frac{\partial}{\partial x_i}\bigg|_y \Big[\phi_\epsilon(x-\cdot)\Big] f(y)\,\mathrm{d}y \overset{\phi_\epsilon(x-\cdot)\in \mathsf{C}^\infty_\mathsf{c}(\Omega)}{=} \\ &= \int_\Omega \phi_\epsilon(x-y) \frac{\partial^\mathsf{w} f}{\partial x_i}(y)\,\mathrm{d}y = \\ &= \phi_\epsilon * \frac{\partial^\mathsf{w} f}{\partial x_i}(x).\end{aligned}$$

vii) $f_\epsilon \to f$ in $\mathsf{L}^\mathsf{p}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ by part v) and $\forall 1 \leq i \leq n$, $\frac{\partial f_\epsilon}{\partial x_i} = \left(\frac{\partial^\mathsf{w} f}{\partial x_i}\right)_\epsilon \to \frac{\partial^\mathsf{w} f}{\partial x_i}$ in $\mathsf{L}^\mathsf{p}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ by parts vi) and v).

□

Corollary 6.21. *Let $1 \leq p < \infty$, $(\phi_t)_{t>0}$ the standard mollifier and $f \in \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$. Then:*

i) $\forall \epsilon > 0$, $f_\epsilon = \phi_\epsilon * f \in \mathsf{C}^\infty(\mathbb{R}^n) \cap \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ and $f_\epsilon \to f$ in $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ as $\epsilon \to 0$.

ii) There exists a sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ such that $f_k \to f$ in $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$.

Proof. It follows from theorem 6.20 with $\Omega = \mathbb{R}^n$ that $f_\epsilon \in \mathsf{C}^\infty(\mathbb{R}^n)$ and $\nabla(f_\epsilon) = (\nabla^\mathsf{w} f)_\epsilon$. Since $f \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n)$ and $\nabla^\mathsf{w} f \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n, \mathbb{R}^n)$, it follows from Young's inequality 1.108.g) that $f_\epsilon \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n)$ and $(\nabla^\mathsf{w} f)_\epsilon \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n, \mathbb{R}^n)$, hence $f_\epsilon \in \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$. Finally, from theorem 1.111.i), $f_\epsilon \to f$ in $\mathsf{L}^\mathsf{p}(\mathbb{R}^n)$ and $\nabla(f_\epsilon) = (\nabla^\mathsf{w} f)_\epsilon \to \nabla^\mathsf{w} f$ in $\mathsf{L}^\mathsf{p}(\mathbb{R}^n)$, thus showing that $f_\epsilon \to f$ in $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$.

It remains to prove the second assertion. Let $\forall k \in \mathbb{N}$, $g_k := \phi_{1/k} * f \in \mathsf{C}^\infty(\mathbb{R}^n) \cap \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$, so that $g_k \to f$ in $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ by part i). Choose $\zeta \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n, [0,1])$ such that $\zeta \equiv 1$ on $\mathbb{B}(0,1)$ and spt $\zeta \subset \mathbb{U}(0,2)$ (which exists, thanks to exercise 1.114). Define, $\forall k \in \mathbb{N}$, $\zeta_k \in \mathsf{C}^\infty_{\mathsf{c}}\big(\mathbb{U}(0,2k)\big)$ by $\zeta_k(x) := \zeta(x/k)$, and $f_k := \zeta_k \cdot g_k$. Then $\forall k \in \mathbb{N}$, $f_k \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n)$. We will prove that $f_k \to f$ in $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$.

1) For all $u \in \mathsf{L}^{\mathsf{p}}(\mathcal{L}^n)$, $\zeta_k \cdot u \to u$ in $\mathsf{L}^{\mathsf{p}}(\mathcal{L}^n)$. Indeed, $|u - \zeta_k \cdot u|^p \to 0$ pointwise and $|u - \zeta_k \cdot u|^p \le 2^p |u|^p \in \mathsf{L}^1(\mathcal{L}^n)$, hence the dominated convergence theorem 1.64 yields the assertion.
2) Since $f - f_k = (f - \zeta_k \cdot f) + \zeta_k \cdot (f - g_k)$, we have $\|f - f_k\|_p \le \|f - \zeta_k \cdot f\|_p + \|\zeta_k\|_u \|f - g_k\|_p \to 0$, since $|\zeta_k| \le 1$, $\|f - \zeta_k \cdot f\|_p \to 0$ by the previous item and $\|f - g_k\|_p \le \|f - g_k\|_{\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)} \to 0$.
3) $\forall x \in \mathbb{R}^n$,
$$\begin{aligned}\nabla f_k(x) &= \nabla \zeta_k(x) \cdot g_k(x) + \zeta_k(x) \cdot \nabla g_k(x) = \\ &= \frac{1}{k} \cdot \nabla \zeta(x/k) \cdot g_k(x) + \zeta_k(x) \cdot \nabla g_k(x).\end{aligned}$$
Hence,
$$\begin{aligned}\nabla^{\mathsf{w}} f(x) - \nabla f_k(x) = \nabla^{\mathsf{w}} f(x) - \zeta_k(x)\, \nabla^{\mathsf{w}} f(x) + \\ + \zeta_k(x)[\nabla^{\mathsf{w}} f(x) - \nabla g_k(x)] - \frac{1}{k} \cdot \nabla \zeta(x/k) \cdot g_k(x),\end{aligned}$$
so that
$$\|\nabla^{\mathsf{w}} f - \nabla f_k\|_p \le \|\nabla^{\mathsf{w}} f - \zeta_k \nabla^{\mathsf{w}} f\|_p + \|\zeta_k\|_u \|\nabla^{\mathsf{w}} f - \nabla g_k\|_p + \frac{1}{k}\|\nabla \zeta\|_u \|g_k\|_p.$$
Since $\|\nabla^{\mathsf{w}} f - \zeta_k \nabla^{\mathsf{w}} f\|_p \to 0$ by item 1), $\|\nabla^{\mathsf{w}} f - \nabla g_k\|_p \le \|f - g_k\|_{\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)} \to 0$ and, as $\|g_k\|_p = \|\phi_{1/k} * f\|_p \le \|f\|_p$ by Young's inequality 1.108.g), $\frac{1}{k}\|\nabla \zeta\|_u \|g_k\|_p \to 0$, it follows that $\|\nabla^{\mathsf{w}} f - \nabla f_k\|_p \to 0$. We have thus proved that $f_k \to f$ in $\mathsf{L}^{\mathsf{p}}(\mathbb{R}^n)$ (by the previous item) and $\nabla f_k \to \nabla^{\mathsf{w}} f$ in $\mathsf{L}^{\mathsf{p}}(\mathcal{L}^n, \mathbb{R}^n)$; that is, $f_k \to f$ in $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$.

□

EXERCISE 6.22. If $u \in \mathsf{C}^1(\Omega)$, the classical and weak gradients of u coincide. The converse holds in the following sense: if $u \in \mathsf{L}^1_{\mathsf{loc}}(\Omega)$ has weak gradient $v \in \mathsf{C}(\Omega, \mathbb{R}^n)$, then u coincides $\mathcal{L}^n$-a.e. in Ω with a function $\tilde{u} \in \mathsf{C}^1(\Omega)$.

EXERCISE 6.23 (Friedrichs's theorem). Let $1 \le p < \infty$. If $u \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$, there exists a sequence $(u_k)_{k \in \mathbb{N}}$ in $\mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n)$ such that $u_k|_\Omega \to u$ in $\mathsf{L}^{\mathsf{p}}(\mathcal{L}^n|_\Omega)$ and $\nabla u_k|_\Omega \to \nabla^{\mathsf{w}} u$ in $\mathsf{L}^{\mathsf{p}}_{\mathsf{loc}}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$.

THEOREM 6.24 (Meyers-Serrin's theorem). *Let* $1 \leq p < \infty$ *and* $u \in \mathsf{W}^{1,p}(\Omega)$. *There exists a sequence* $(u_k)_{k \in \mathbb{N}}$ *in* $\mathsf{C}^\infty(\Omega) \cap \mathsf{W}^{1,p}(\Omega)$ *such that* $u_k \to u$ *in* $\mathsf{W}^{1,p}(\Omega)$.

PROOF.

1) Let $(U_k)_{k \in \mathbb{N}}$ be a locally finite open cover of Ω such that $\forall k \in \mathbb{N}$, $U_k \Subset \Omega$. Take a smooth partition of unity $(\xi_k)_{k \in \mathbb{N}}$ of Ω, with $\forall k \in \mathbb{N}$, $\operatorname{spt} \xi_k \Subset U_k$, given by theorem 6.8.
2) Fix $\epsilon > 0$ and $k \in \mathbb{N}$. Let $(\phi_t)_{t>0}$ be the standard mollifier in $\mathbb{R}^n$. It follows from lemma 6.14.ii) that $\xi_k \cdot u \in \mathsf{W}^{1,p}(\mathbb{R}^n)$, with $\operatorname{spt} \xi_k \cdot u \subset \operatorname{spt} \xi_k \Subset U_k$. We may therefore apply proposition 1.108.d) and corollary 6.21.i) to obtain t_k sufficiently small so that $\phi_{t_k} * (\xi_k \cdot u) \in \mathsf{C}^\infty(\mathbb{R}^n) \cap \mathsf{W}^{1,p}(\mathbb{R}^n)$ has compact support in U_k and $\|\phi_{t_k} * (\xi_k \cdot u) - \xi_k \cdot u\|_{\mathsf{W}^{1,p}(\mathbb{R}^n)} < 2^{-k}\epsilon$. Define $u_\epsilon : \Omega \to \mathbb{R}$ by

$$u_\epsilon := \sum_{k=1}^{\infty} \big[\phi_{t_k} * (\xi_k \cdot u)\big]|_\Omega.$$

Note that the above sum is locally finite, since $\big(\operatorname{spt} \phi_{t_k} * (\xi_k \cdot u)\big)_{k \in \mathbb{N}}$ is a locally finite family of subsets of Ω (because $\forall k \in \mathbb{N}$, $\operatorname{spt} \phi_{t_k} * (\xi_k \cdot u) \Subset U_k$). Hence, $u_\epsilon \in \mathsf{C}^\infty(\Omega)$[1]. Similarly, we have locally finite sums

$$\begin{aligned} u_\epsilon - u &= \sum_{k=1}^{\infty} \big(\phi_{t_k} * (\xi_k \cdot u) - \xi_k \cdot u\big) \\ \nabla^{\mathsf{w}}(u_\epsilon - u) &= \sum_{k=1}^{\infty} \nabla^{\mathsf{w}}\big(\phi_{t_k} * (\xi_k \cdot u) - \xi_k \cdot u\big), \end{aligned} \tag{6.1}$$

where the second equality holds because both members coincide on each relatively compact open subset $V \Subset \Omega$ (because on V the sum is finite and the weak gradient is linear).
3) It follows from (6.1), definition 6.1 and from Minkowski's inequality for integrals 1.88 (with $\mu = \mathcal{L}^n|_\Omega$ and ν the counting measure on $\mathbb{N}$)

[1] note that $\big(\operatorname{spt} \phi_{t_k} * (\xi_k \cdot u)\big)_{k \in \mathbb{N}}$ is a locally finite family of subsets of Ω (with the relative topology), but not, in general, a locally finite family of subsets of $\mathbb{R}^n$, hence we cannot define a smooth function on $\mathbb{R}^n$ using the same formula.

that

$$\begin{aligned}
\|u_\epsilon - u\|_{\mathsf{L^p}(\mathcal{L}^n|_\Omega)} &\le \sum_{k=1}^{\infty} \|\phi_{t_k} * (\xi_k \cdot u) - \xi_k \cdot u\|_{\mathsf{L^p}(\mathcal{L}^n|_\Omega)} \le \\
&\le \sum_{k=1}^{\infty} \|\phi_{t_k} * (\xi_k \cdot u) - \xi_k \cdot u\|_{\mathsf{W^{1,p}}(\Omega)} < \epsilon, \\
\|\nabla^{\mathsf{w}}(u_\epsilon - u)\|_{\mathsf{L^p}(\mathcal{L}^n|_\Omega)} &\le \sum_{k=1}^{\infty} \|\nabla^{\mathsf{w}}\big(\phi_{t_k} * (\xi_k \cdot u) - \xi_k \cdot u\big)\|_{\mathsf{L^p}(\mathcal{L}^n|_\Omega)} \le \\
&\le \sum_{k=1}^{\infty} \|\phi_{t_k} * (\xi_k \cdot u) - \xi_k \cdot u\|_{\mathsf{W^{1,p}}(\Omega)} < \epsilon.
\end{aligned}$$

We therefore conclude that $u_\epsilon \in \mathsf{W^{1,p}}(\Omega)$ and $u_\epsilon \to u$ in $\mathsf{W^{1,p}}(\Omega)$ as $\epsilon \to 0$.

□

EXERCISE 6.25. Let Ω be an open subset of $\mathbb{R}^n$, $1 < p \le \infty$ and $f \in \mathsf{L^p}(\mathcal{L}^n|_\Omega)$. Then the following conditions are equivalent:

i) $f \in \mathsf{W^{1,p}}(\Omega)$.

ii) There exists a constant $C \ge 0$ such that, for all $\varphi \in \mathsf{C}_\mathsf{c}^\infty(\Omega)$ and all $1 \le i \le n$,

$$\Big|\int_\Omega f \frac{\partial \varphi}{\partial x_i}\, \mathrm{d}\mathcal{L}^n\Big| \le C\|\varphi\|_{\mathsf{L^{p'}}(\Omega)},$$

where p' is the conjugate exponent of p.

iii) There exists a constant $C \ge 0$ such that, for all relatively compact open $\omega \Subset \Omega$ and all $h \in \mathbb{R}^n$ with $\|h\| < d(\omega, \Omega^c)$,

$$\|\tau_h f - f\|_{\mathsf{L^p}(\mathcal{L}^n|_\omega)} \le C\|h\|.$$

Moreover, we can take $C = \|\nabla^{\mathsf{w}} f\|_{\mathsf{L^p}(\mathcal{L}^n|_\Omega)}$ in (ii) and (iii), and if $\Omega = \mathbb{R}^n$, we have

$$\|\tau_h f - f\|_{\mathsf{L^p}(\mathcal{L}^n)} \le \|\nabla^{\mathsf{w}} f\|_{\mathsf{L^p}(\mathcal{L}^n)}\|h\|$$

for all $h \in \mathbb{R}^n$.

6.2. Product and Chain Rules

THEOREM 6.26 (Product rule). *Let Ω be an open set in $\mathbb{R}^n$ and u, v be real functions on Ω satisfying one of the following conditions:*

i) $u \in \mathsf{W^{1,1}_{loc}}(\Omega)$ and $v \in \mathsf{C^1}(\Omega)$;

ii) $u, v \in \mathsf{W^{1,1}_{loc}}(\Omega) \cap \mathsf{L^\infty_{loc}}(\mathcal{L}^n|_\Omega)$.

Then $uv \in \mathsf{W}^{1,1}_{\mathsf{loc}}(\Omega)$ *and*

$$\nabla^{\mathsf{w}}(uv) = u\,\nabla^{\mathsf{w}} v + v\,\nabla^{\mathsf{w}} u. \tag{6.2}$$

Proof. Note that, assuming either i) or ii), the second member in (6.2) belongs to $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$. In view of the locality of the weak derivative 6.13, it therefore suffices to show that, for each open $V \Subset \Omega$, uv has weak gradient on V given by (6.2).

1) Suppose that i) holds. For each $0 < \epsilon < d(V, \Omega^c)$, we denote by $u_\epsilon \in \mathsf{C}^\infty(\Omega_\epsilon)$ the ϵ-approximation of u, cf. definition 6.17. Note that both v and ∇v are bounded on V; it then follows from theorem 6.20 that:
   - $u_\epsilon v \in \mathsf{L}^1(\mathcal{L}^n|_V)$ and $u_\epsilon v \to uv$ in $\mathsf{L}^1(\mathcal{L}^n|_V)$ as $\epsilon \to 0$;
   - $\nabla(u_\epsilon v) = u_\epsilon \nabla v + v \nabla u_\epsilon \in \mathsf{L}^1(\mathcal{L}^n|_V, \mathbb{R}^n)$ and $\nabla(u_\epsilon v) \to u\nabla v + v\nabla u$ in $\mathsf{L}^1(\mathcal{L}^n|_V, \mathbb{R}^n)$.

   Hence, applying proposition 6.3.i), we conclude that uv has weak gradient on V given by (6.2), as asserted.
2) Suppose that ii) holds. For each $0 < \epsilon < d(V, \Omega^c)$, we denote by $v_\epsilon \in \mathsf{C}^\infty(\Omega_\epsilon)$ the ϵ-approximation of v. It follows from part i) with v_ϵ in place of v that $uv_\epsilon \in \mathsf{W}^{1,1}(V)$ and $\nabla^{\mathsf{w}}(uv_\epsilon) = u\nabla v_\epsilon + v_\epsilon\,\nabla^{\mathsf{w}} u$. Besides:
   - since $u \in \mathsf{L}^\infty(\mathcal{L}^n|_V)$ and $v_\epsilon \to v$ in $\mathsf{L}^1(\mathcal{L}^n|_V)$ (by theorem 6.20.v), it follows that $uv_\epsilon \to uv$ in $\mathsf{L}^1(\mathcal{L}^n|_V)$;
   - since $\nabla v_\epsilon = (\nabla^{\mathsf{w}} v)_\epsilon \to \nabla^{\mathsf{w}} v$ in $\mathsf{L}^1(\mathcal{L}^n|_V, \mathbb{R}^n)$ (by theorem 6.20.vi and 6.20.v, respectively) and $u \in \mathsf{L}^\infty(\mathcal{L}^n|_V)$, we also have $u\nabla v_\epsilon \to u\,\nabla^{\mathsf{w}} v$ in $\mathsf{L}^1(\mathcal{L}^n|_V, \mathbb{R}^n)$;
   - $v_\epsilon \to v$ $\mathcal{L}^n$-a.e. on V (by 6.20.iii), hence $v_\epsilon\,\nabla^{\mathsf{w}} u - v\,\nabla^{\mathsf{w}} u \to 0$ $\mathcal{L}^n$-a.e. on V. The latter convergence is dominated, since, fixing $0 < \epsilon_0 < d(V, \Omega^c)$, for all $0 < \epsilon < \epsilon_0$:

$$\begin{aligned}\|v_\epsilon\,\nabla^{\mathsf{w}} u - v\,\nabla^{\mathsf{w}} u\| &\le \big(\|v_\epsilon\|_{\mathsf{L}^\infty(V)} + \|v\|_{\mathsf{L}^\infty(V)}\big)\|\nabla^{\mathsf{w}} u\| \overset{v_\epsilon = \phi_\epsilon * v}{\le} \\ &\le 2\|v\|_{\mathsf{L}^\infty\left(V + \mathbb{B}(0,\epsilon_0)\right)}\|\nabla^{\mathsf{w}} u\| \in \mathsf{L}^1(\mathcal{L}^n|_V).\end{aligned}$$

     It then follows from the dominated convergence theorem 1.64 (taking $\epsilon \to 0$ along an arbitrary sequence) that $v_\epsilon\,\nabla^{\mathsf{w}} u \to v\,\nabla^{\mathsf{w}} u$ in $\mathsf{L}^1(\mathcal{L}^n|_V, \mathbb{R}^n)$.

   Therefore, $uv_\epsilon \to uv$ in $\mathsf{L}^1(\mathcal{L}^n|_V)$ and $\nabla^{\mathsf{w}}(uv_\epsilon) \to u\,\nabla^{\mathsf{w}} v + v\,\nabla^{\mathsf{w}} u$ in $\mathsf{L}^1(\mathcal{L}^n|_V, \mathbb{R}^n)$; the thesis then follows from proposition 6.3.i).

□

Corollary 6.27. *Let* Ω *be an open set in* $\mathbb{R}^n$, $1 \le p \le \infty$ *and* $u, v \in \mathsf{W}^{1,\mathsf{p}}(\Omega) \cap \mathsf{L}^\infty(\mathcal{L}^n|_\Omega)$. *Then* $uv \in \mathsf{W}^{1,\mathsf{p}}(\Omega) \cap \mathsf{L}^\infty(\mathcal{L}^n|_\Omega)$.

Recall that, for a map F defined on an open set of $\mathbb{R}^n$, we use the notation D_F for the set of points in which F is Fréchet-differentiable.

THEOREM 6.28 (Chain rule). *Let Ω be an open set in $\mathbb{R}^n$, $f \in \mathsf{W}^{1,1}_{\mathsf{loc}}(\Omega)$ and $F : \mathbb{R} \to \mathbb{R}$ Lipschitz with $\mathbb{R} \setminus D_F$ countable. Define $F' \equiv 0$ on $\mathbb{R} \setminus D_F$. Then $F \circ f \in \mathsf{W}^{1,1}_{\mathsf{loc}}(\Omega)$ and*

$$\nabla^{\mathsf{w}}(F \circ f) = (F' \circ f) \cdot \nabla^{\mathsf{w}} f.$$

PROOF.

1) The thesis holds if $f \in \mathsf{C}^1(\Omega)$.

Since f is locally Lipschitz, so is $F \circ f$. It then follows from corollary 6.16 that $F \circ f$ is differentiable in the sense of Fréchet $\mathcal{L}^n$-a.e. on Ω, $F \circ f \in \mathsf{W}^{1,\infty}_{\mathsf{loc}}(\Omega) \subset \mathsf{W}^{1,1}_{\mathsf{loc}}(\Omega)$ and $\nabla^{\mathsf{w}}(F \circ f) = \nabla(F \circ f)$ $\mathcal{L}^n$-a.e. on Ω. Hence, it suffices to show that

$$\nabla(F \circ f) = (F' \circ f) \cdot \nabla f \tag{6.3}$$

$\mathcal{L}^n$-a.e. on Ω. The latter equality holds on $f^{-1}(D_F)$ by the classical chain rule; we must show that it holds $\mathcal{L}^n$-a.e. on $f^{-1}(\mathbb{R} \setminus D_F)$. Indeed, for each $t \in \mathbb{R}^n \setminus D_F$, the second member of (6.3) is null on $f^{-1}\{t\}$ (since we defined $F' \equiv 0$ on $\mathbb{R} \setminus D_F$) and the first member is null $\mathcal{L}^n$-a.e. on $f^{-1}\{t\}$ by corollary 5.17, thus showing that (6.3) holds $\mathcal{L}^n$-a.e. on $f^{-1}\{t\}$. Since $\mathbb{R}^n \setminus D_F$ is countable, we conclude that (6.3) holds $\mathcal{L}^n$-a.e. on $f^{-1}(\mathbb{R}^n \setminus D_F)$, as asserted.

2) General case: let $f \in \mathsf{W}^{1,1}_{\mathsf{loc}}(\Omega)$. Note that
   - $\forall x \in \Omega$, $|F \circ f(x)| \le |F \circ f(x) - F(0)| + |F(0)| \le (\operatorname{Lip} F) \cdot |f(x)| + |F(0)|$, hence $F \circ f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|\Omega)$;
   - $\|(F' \circ f) \cdot \nabla^{\mathsf{w}} f\| \le (\operatorname{Lip} F) \cdot \|\nabla^{\mathsf{w}} f\| \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|\Omega)$, hence $(F' \circ f) \cdot \nabla^{\mathsf{w}} f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|\Omega)$.

In view of the locality of the weak derivative, it therefore suffices to show that, for each open $V \Subset \Omega$, $F \circ f$ has weak derivative on V given by $(F' \circ f) \cdot \nabla^{\mathsf{w}} f$.

Let $(\phi_t)_{t>0}$ be the standard mollifier on $\mathbb{R}^n$. Let $0 < \epsilon_0 < d(V, \Omega^c)$ and $(\epsilon_k)_{k \in \mathbb{N}}$ a sequence in $]0, \epsilon_0[$ with $\epsilon_k \downarrow 0$. With the notation from definition 6.17 in force, let $\forall k \in \mathbb{N}$, $f_k := f_{\epsilon_k} = \phi_{\epsilon_k} * f \in \mathsf{C}^\infty(\Omega_{\epsilon_k})$. It follows from theorem 6.20 that $f_k \to f$ in $\mathsf{W}^{1,1}(V)$, $f_k \to f$ $\mathcal{L}^n$-a.e. on V and $\nabla f_k = (\nabla^{\mathsf{w}} f)_{\epsilon_k} \to \nabla^{\mathsf{w}} f$ $\mathcal{L}^n$-a.e. on V.

Fix $\varphi \in \mathsf{C}^\infty_{\mathsf{c}}(V)$. Since $\forall k \in \mathbb{N}$, $f_k \in \mathsf{C}^1(V)$, it follows from part 1) of the proof that, $\forall k \in \mathbb{N}$:

$$-\int_V F \circ f_k \, \nabla\varphi \, \mathrm{d}\mathcal{L}^n = \int_V (F' \circ f_k) \cdot \nabla f_k \, \varphi \, \mathrm{d}\mathcal{L}^n. \tag{6.4}$$

The thesis then follows once we prove the following claims:

- Claim 1: $\int_V F \circ f_k \nabla\varphi \, \mathrm{d}\mathcal{L}^n \to \int_V F \circ f \nabla\varphi \, \mathrm{d}\mathcal{L}^n$.
- Claim 2: $\int_V (F' \circ f_k) \cdot \nabla f_k \, \varphi \, \mathrm{d}\mathcal{L}^n \to \int_V (F' \circ f) \cdot \nabla f \, \varphi \, \mathrm{d}\mathcal{L}^n$.

Proof of claim 1: We have:

i) $\forall k \in \mathbb{N}$, $\|F \circ f_k \nabla\varphi\| \le \underbrace{[(\mathrm{Lip}\, F)|f_k| + |F(0)|]\|\nabla\varphi\|_u}_{g_k :=}$;

ii) $F \circ f_k \nabla\varphi \to F \circ f \nabla\varphi$ $\mathcal{L}^n$-a.e. on V;

iii) $g_k \to g := [(\mathrm{Lip}\, F)|f| + |F(0)|]\|\nabla\varphi\|_u$ $\mathcal{L}^n$-a.e. on V;

iv) $\int_V g_k \, \mathrm{d}\mathcal{L}^n \to \int_V g \, \mathrm{d}\mathcal{L}^n < \infty$ (because $f_k \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_V)$).

An application of the generalized dominated convergence theorem 1.66 concludes the proof. $\square$

Proof of claim 2: We have:

i) $\forall k \in \mathbb{N}$, $\|(F' \circ f_k) \cdot \nabla f_k \, \varphi\| \le \underbrace{(\mathrm{Lip}\, F)\|\varphi\|_u \|\nabla f_k\|}_{g_k :=}$;

ii) $(F' \circ f_k) \cdot \nabla f_k \, \varphi \to (F' \circ f) \cdot \nabla f \, \varphi$ $\mathcal{L}^n$-a.e. on V;

iii) $g_k \to g := (\mathrm{Lip}\, F)\|\varphi\|_u \|\nabla^{\mathsf{w}} f\|$ $\mathcal{L}^n$-a.e. on V;

iv) $\int_V g_k \, \mathrm{d}\mathcal{L}^n \to \int_V g \, \mathrm{d}\mathcal{L}^n < \infty$ (because $\nabla f_k \to \nabla^{\mathsf{w}} f$ in $\mathsf{L}^1(\mathcal{L}^n|_V, \mathbb{R}^n)$).

An application of the generalized dominated convergence theorem 1.66 concludes the proof. $\square$

$\square$

Corollary 6.29. *Let Ω be an open set in $\mathbb{R}^n$, $f \in \mathsf{W}^{1,1}_{\mathsf{loc}}(\Omega)$ and $F : \mathbb{R} \to \mathbb{R}$ sectionally C^1 on each compact subinterval of $\mathbb{R}$, with $F' \in \mathsf{L}^\infty(\mathbb{R})$. Define $F' \equiv 0$ on $\mathbb{R} \setminus D_F$. Then $F \circ f \in \mathsf{W}^{1,1}_{\mathsf{loc}}(\Omega)$ and*

$$\nabla^{\mathsf{w}}(F \circ f) = (F' \circ f) \cdot \nabla^{\mathsf{w}} f.$$

Proof. The hypothesis on F implies F Lipschitz with $\mathbb{R} \setminus D_F$ countable. $\square$

Corollary 6.30. *Let Ω be an open set in $\mathbb{R}^n$, $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$ and $F : \mathbb{R} \to \mathbb{R}$ Lipschitz with $\mathbb{R} \setminus D_F$ countable. Define $F' \equiv 0$ on $\mathbb{R} \setminus D_F$. If $\mathcal{L}^n(\Omega) = \infty$, assume that $F(0) = 0$. Then $F \circ f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$ and*

$$\nabla^{\mathsf{w}}(F \circ f) = (F' \circ f) \cdot \nabla^{\mathsf{w}} f.$$

Proof. Since $\|(F' \circ f) \cdot \nabla^{\mathsf{w}} f\| \le (\mathrm{Lip}\, f)\|\nabla^{\mathsf{w}} f\| \in \mathsf{L}^{\mathsf{p}}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$ and $|F \circ f| \le (\mathrm{Lip}\, F)|f| + |F(0)| \in \mathsf{L}^{\mathsf{p}}(\Omega)$, it follows that $F \circ f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$. $\square$

COROLLARY 6.31. *Let Ω be an open set in $\mathbb{R}^n$ and $f \in \mathsf{W}^{1,1}_{\mathsf{loc}}(\Omega)$. Then $f^+, f^-, |f| \in \mathsf{W}^{1,1}_{\mathsf{loc}}(\Omega)$ and*

$$\nabla^{\mathsf{w}} f^+ = \begin{cases} \nabla^{\mathsf{w}} f & \mathcal{L}^n\text{-a.e. on } \{f > 0\} \\ 0 & \mathcal{L}^n\text{-a.e. on } \{f \leq 0\}, \end{cases}$$

$$\nabla^{\mathsf{w}} f^- = \begin{cases} -\nabla^{\mathsf{w}} f & \mathcal{L}^n\text{-a.e. on } \{f < 0\} \\ 0 & \mathcal{L}^n\text{-a.e. on } \{f \geq 0\}, \end{cases}$$

$$\nabla^{\mathsf{w}} |f| = \begin{cases} \nabla^{\mathsf{w}} f & \mathcal{L}^n\text{-a.e. on } \{f > 0\} \\ 0 & \mathcal{L}^n\text{-a.e. on } \{f = 0\} \\ -\nabla^{\mathsf{w}} f & \mathcal{L}^n\text{-a.e. on } \{f < 0\}. \end{cases}$$

PROOF. Apply theorem 6.28 to $F \circ f$, where F is given by, respectively, $\mathrm{id}_{\mathbb{R}} \cdot \chi_{[0,\infty)}$, $-\mathrm{id}_{\mathbb{R}} \cdot \chi_{(-\infty,0]}$ and $|\cdot|$. ☐

COROLLARY 6.32. *Let Ω be an open set in $\mathbb{R}^n$ and $f \in \mathsf{W}^{1,1}_{\mathsf{loc}}(\Omega)$. Then $\nabla^{\mathsf{w}} f = 0$ $\mathcal{L}^n$-a.e. on $\{f = 0\}$.*

PROOF. Apply the previous corollary and the linearity of the weak derivative to $f = f^+ - f^-$. ☐

6.3. Approximation by smooth functions, part II

We resume the discussion started on section 6.1 on the approximation of Sobolev functions. With regard to Meyers-Serrin's theorem 6.24, for instance, we may obtain better approximation results if we impose some regularity on $\partial\Omega$. For instance, if Ω is a Lipschitz domain, in the sense of following definition, we will prove that Sobolev functions on Ω may be approximated by functions in $\mathsf{C}^\infty(\overline{\Omega})$.

NOTATION. We will use the following notation for cylinders on products of Euclidean spaces $\mathbb{R}^n \equiv \mathbb{R}^k \times \mathbb{R}^{n-k}$. Let $p : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ and $q : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be the projections on the first and second factors, respectively. Given $x \in \mathbb{R}^n$, $0 < r, h \leq \infty$, we define the open and closed cylinders with center x, radius r and half-height h:

- $\mathbb{C}(x, r, h) := \mathbb{U}(p \cdot x, r) \times \mathbb{U}(q \cdot x, h) \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$.
- $\overline{\mathbb{C}}(x, r, h) := \mathbb{B}(p \cdot x, r) \times \mathbb{B}(q \cdot x, h) = \overline{\mathbb{C}(x, r, h)}$.

We use abbreviated notations $\mathbb{C}(x, r) := \mathbb{C}(x, r, r)$ and $\overline{\mathbb{C}}(x, r) := \overline{\mathbb{C}}(x, r, r)$.

DEFINITION 6.33 (Lipschitz domains[2]). Let $n \geq 2$, $U \subset \mathbb{R}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R}$ open and $\Omega \subset U$ an open subset of U. We say that Ω is

[2] there is a weaker notion of "Lipschitz domain" which we do not consider here; our definition corresponds to what sometimes is called a *strong Lipschitz domain*.

a *Lipschitz domain*[3] *in* U if for all $x \in \partial^U \Omega = \partial\Omega \cap U$ (i.e. x in the topological boundary of Ω in U), there exist:

1) a rigid motion $\Phi \in \mathrm{SE}(n)$ with $\Phi(0) = x$;
2) $f : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz with $f(0) = 0$;
3) $\mathbb{C}(0, r, h) \subset \mathbb{R}^{n-1} \times \mathbb{R}$ open cylinder

satisfying the following conditions (see figure 1):

- $C := \Phi\big(\mathbb{C}(0, r, h)\big) \subset U$;
- $\Phi\big(\mathrm{gr}\ f \cap \mathbb{C}(0, r, h)\big) = C \cap \partial\Omega$
- $\Phi\big(\mathrm{epi}_{\mathsf{S}}\ f \cap \mathbb{C}(0, r, h)\big) = C \cap \Omega$,

where $\mathrm{epi}_{\mathsf{S}}\ f = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y > f(x)\}$ is the *strict epigraph of* f.

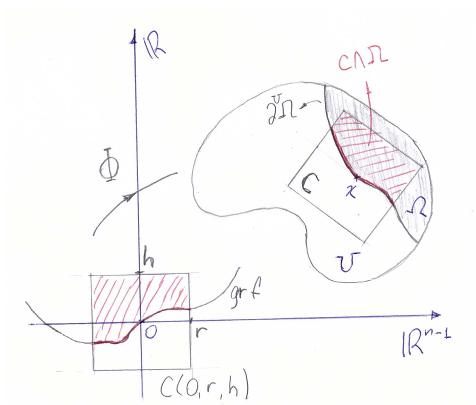

FIGURE 1. Lipschitz Domain

THEOREM 6.34 (Global approximation by smooth functions). *Let* $\Omega \subset \mathbb{R}^n$ *be a Lipschitz domain. If* $1 \le p < \infty$ *and* $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$, *there exists* $(f_k)_{k\in\mathbb{N}}$ *in* $\mathsf{W}^{1,\mathsf{p}}(\Omega) \cap \mathsf{C}^\infty(\overline{\Omega})$ *such that* $f_k \to f$ *in* $\mathsf{W}^{1,\mathsf{p}}(\Omega)$. *Moreover, if* $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega) \cap \mathsf{C}(\overline{\Omega})$, *the sequence* $(f_k)_{k\in\mathbb{N}}$ *may be chosen so that it also converges to* f *uniformly on* $\overline{\Omega}$.

We devote the remaining of this section to the proof of the theorem above, which will be done along the following lemmas.

LEMMA 6.35. *Let* $\Omega \subset \mathbb{R}^n$ *open,* $1 \le p \le \infty$, $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$ *and* $\xi \in \mathsf{C}^\infty(\mathbb{R}^n)$ *with* $\mathrm{spt}\ \xi \subset \Omega$ *and* $\|\xi\|_{\mathsf{W}^{1,\infty}(\mathbb{R}^n)} < \infty$. *Then* $\xi \cdot f$ *(defined as* 0 *on* $\mathbb{R}^n \setminus \Omega$*) belongs to* $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ *and* $\nabla^{\mathsf{w}}(\xi \cdot f) = (\nabla\xi) \cdot f + \xi \cdot \nabla^{\mathsf{w}} f$.

[3] our Lipschitz domains need not be connected, despite the usual meaning of the term "domain", i.e. an open connected set.

Note that it is not required that spt ξ be compact.

PROOF. It is clear that $\xi \cdot f \in \mathsf{L}^{\mathsf{p}}(\mathcal{L}^n)$ and $g := (\nabla\xi)\cdot f + \xi \cdot \nabla^{\mathsf{w}} f \in \mathsf{L}^{\mathsf{p}}(\mathcal{L}^n, \mathbb{R}^n)$, thanks to the hypothesis $\|\xi\|_{\mathsf{W}^{1,\infty}(\mathbb{R}^n)} < \infty$. Then it suffices to show that $\nabla^{\mathsf{w}}(\xi \cdot f)$ exists and coincides with g. Indeed, it follows from theorem 6.26 that $\xi \cdot f \in \mathsf{W}^{1,1}_{\mathsf{loc}}(\Omega)$ and that its weak gradient on Ω coincides with g. The same holds for the restriction of $\xi \cdot f$ to the open set $\mathbb{R}^n \backslash \mathrm{spt}\ \xi$ (because $\xi \equiv 0$ on this open set). Since $\Omega \cup \mathbb{R}^n \backslash \mathrm{spt}\ \xi = \mathbb{R}^n$, the thesis follows from the locality of the weak derivative 6.13. $\square$

LEMMA 6.36. *Let* $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ *Lipschitz,* $h > 0$ *and* $\alpha := (\mathrm{Lip}\,\Gamma) + 2$*. Then* $d\big(\mathrm{gr}\ \Gamma, \mathrm{gr}\,(\Gamma - h)\big) \geq \frac{h}{\alpha}$.

PROOF. Let $P = \big(x, \Gamma(x)\big) \in \mathrm{gr}\ \Gamma$ and $Q = \big(y, \Gamma(y) - h\big) \in \mathrm{gr}\,(\Gamma - h)$. If $\|P - Q\| = \epsilon$, then $\|x - y\| \leq \epsilon$, hence $|\Gamma(x) - \Gamma(y)| \leq (\mathrm{Lip}\,\Gamma)\epsilon$. Thus, putting $R = \big(y, \Gamma(y)\big)$, $\|R - P\| \leq \|x - y\| + |\Gamma(x) - \Gamma(y)| \leq (1 + \mathrm{Lip}\,\Gamma)\epsilon$. That implies, by the triangle inequality, $h = \|R - Q\| \leq \|R - P\| + \|P - Q\| \leq (2 + \mathrm{Lip}\,\Gamma)\epsilon$. We therefore conclude that

$$\|P - Q\| = \epsilon \geq \frac{h}{\alpha},$$

which implies, by the arbitrariness of $P \in \mathrm{gr}\ \Gamma$ and $Q \in \mathrm{gr}\,(\Gamma - h)$, that $d\big(\mathrm{gr}\ \Gamma, \mathrm{gr}\,(\Gamma - h)\big) \geq \frac{h}{\alpha}$. $\square$

NOTATION. Given $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$, $\Omega := \mathrm{epi}_{\mathsf{S}}\ \Gamma = \{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y_n > \Gamma(y')\}$ and $h > 0$, we denote by Ω_{-h} the strict epigraph of $\Gamma - h$, i.e. $\Omega_{-h} = \mathrm{epi}_{\mathsf{S}}\,(\Gamma - h) = \{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y_n > \Gamma(y') - h\}$.

NOTATION. If $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}$, we denote by spt f the closure in A of $\{x \in A \mid f(x) \neq 0\}$ (i.e the usual notation for the support of f) and by $\overline{\mathrm{spt}\ f}$ its closure in $\mathbb{R}^n$.

LEMMA 6.37. *With the notation above in force, let* $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ *Lipschitz. For each* $h > 0$*, there exists* $\Psi \in \mathsf{C}^\infty(\mathbb{R}^n) \cap \mathsf{W}^{1,\infty}(\mathbb{R}^n)$ *such that* $0 \leq \Psi \leq 1$*,* $\Psi \equiv 1$ *on* $\overline{\Omega}$ *and* spt $\Psi \subset \Omega_{-h}$.

PROOF. Let $(\phi_\epsilon)_{\epsilon>0}$ be the standard mollifier. Fix $0 < \epsilon < \frac{h}{2\alpha}$ and define $\Psi := \phi_\epsilon * \chi_{\Omega_{-h/2}}$. It is clear that $\Psi \in \mathsf{C}^\infty(\mathbb{R}^n)$ and $0 \leq \Psi \leq 1$. Moreover:

1) For all $x \in \overline{\Omega}$, $\mathbb{B}(x, \epsilon) \subset \Omega_{-h/2}$ by lemma 6.36, which implies $\Psi \equiv 1$ on $\overline{\Omega}$;
2) spt $\Psi \subset$ spt $\chi_{\Omega_{-h/2} + \mathbb{B}(0,\epsilon)} = \overline{\Omega_{-h/2}} + \mathbb{B}(0, \epsilon) \subset \Omega_{-h}$ by lemma 6.36 applied to $\Gamma - \frac{h}{2}$ in place of Γ;
3) It follows from proposition 1.108 parts g) and j) that $\|\nabla\Psi\|_\infty = \|(\nabla\phi_\epsilon) * \chi_{\Omega_{-h/2}}\| \leq \|\nabla\phi_\epsilon\|_1 < \infty$, hence $\Psi \in \mathsf{W}^{1,\infty}(\mathbb{R}^n)$.

□

LEMMA 6.38. *With the notation above in force, let* $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ *Lipschitz,* $\Omega = \mathrm{epi}_{\mathsf{S}}\,\Gamma$ *and* $1 \le p < \infty$. *Let* $f : \Omega \to \mathbb{R}$ *and, for each* $h > 0$, $\tau_{-h} f : \Omega_{-h} \to \mathbb{R}$ *be given by* $x \mapsto f(x+h)$. *Then:*

i) if $f \in \mathsf{L^p}(\mathcal{L}^n|\Omega)$, *then* $\tau_{-h} f \in \mathsf{L^p}(\mathcal{L}^n|_{\Omega_{-h}})$ *and* $(\tau_{-h} f)|_\Omega \to f$ *in* $\mathsf{L^p}(\mathcal{L}^n|_\Omega)$ *as* $h \to 0$.

ii) if $f \in \mathsf{W^{1,p}}(\Omega)$, *then* $\tau_{-h} f \in \mathsf{W^{1,p}}(\Omega_{-h})$ *and* $(\tau_{-h} f)|_\Omega \to f$ *in* $\mathsf{W^{1,p}}(\Omega)$ *as* $h \to 0$.

iii) if f *is uniformly continuous, so is* $\tau_{-h} f$ *and* $(\tau_{-h} f)|_\Omega \to f$ *uniformly as* $h \to 0$.

PROOF.

(1) It is clear that $\tau_{-h} f \in \mathsf{L^p}(\mathcal{L}^n|_{\Omega_{-h}})$. Moreover, using a bar to denote extensions by zero, $\tau_{-h}\overline{f} = \overline{\tau_{-h} f}$ converges to $\overline{f}$ in $\mathsf{L^p}(\mathcal{L}^n)$ by lemma 1.110, which implies $(\tau_{-h} f)|_\Omega \to f$ in $\mathsf{L^p}(\mathcal{L}^n|_\Omega)$ as $h \to 0$.

(2) It is clear that τ_{-h} maps $\mathsf{W^{1,1}_{loc}}(\Omega)$ to $\mathsf{W^{1,1}_{loc}}(\Omega_{-h})$ and that τ_{-h} commutes with weak derivatives. Hence, by the previous item, if $f \in \mathsf{W^{1,p}}(\Omega)$, then $\nabla^{\mathsf{w}} \tau_{-h} f = \tau_{-h} \nabla^{\mathsf{w}} f \to \nabla^{\mathsf{w}} f$ in $\mathsf{L^p}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$ as $h \to 0$, from which we conclude that $\tau_{-h} f \in \mathsf{W^{1,p}}(\Omega_{-h})$ and $(\tau_{-h} f)|_\Omega \to f$ in $\mathsf{W^{1,p}}(\Omega)$.

(3) It is immediate from the definition of uniform continuity.

□

LEMMA 6.39. *With the notation above in force, let* $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ *Lipschitz and* $\Omega = \mathrm{epi}_{\mathsf{S}}\,\Gamma$. *Then,* $\forall \epsilon > 0$, $\forall 1 \le p < \infty$*:*

i) If $f \in \mathsf{W^{1,p}}(\Omega)$, *there exists* $g \in \mathsf{C^\infty_c}(\mathbb{R}^n)$ *such that* $\|g|_\Omega - f\|_{\mathsf{W^{1,p}}(\Omega)} < \epsilon$.

ii) If $f \in \mathsf{W^{1,p}}(\Omega)$ *and* f *is bounded and uniformly continuous, there exists* $g \in \mathsf{C^\infty}(\mathbb{R}^n)$ *satisfying both* $\|g|_\Omega - f\|_{\mathsf{W^{1,p}}(\Omega)} < \epsilon$ *and* $\|g|_\Omega - f\|_u < \epsilon$.

In both cases, if $U \subset \mathbb{R}^n$ *is open and the closure in* $\mathbb{R}^n$ *of* spt f *is a compact subset of* U, *we can take* $g \in \mathsf{C^\infty_c}(U)$ *satisfying the conditions stated above.*

PROOF. Let $f \in \mathsf{W^{1,p}}(\Omega)$. Fix $\epsilon > 0$. By lemma 6.38, we may take $h > 0$ such that $\|\tau_{-h} f - f\|_{\mathsf{W^{1,p}}(\Omega)} < \epsilon/2$. Besides, if f is uniformly continuous, by the same lemma we may choose $h > 0$ so that $\|\tau_{-h} f - f\|_u < \epsilon/2$ also holds on Ω.

Apply lemma 6.37 to obtain $\Psi \in \mathsf{C^\infty}(\mathbb{R}^n) \cap \mathsf{W^{1,\infty}}(\mathbb{R}^n)$ such that $0 \le \Psi \le 1$, $\Psi \equiv 1$ on $\overline{\Omega}$ and spt $\Psi \subset \Omega_{-h}$. It follows from lemma 6.35 with Ω_{-h} in place of Ω and $\tau_{-h} f$ in place of f that $\Psi \cdot \tau_{-h} f$ belongs to $\mathsf{W^{1,p}}(\mathbb{R}^n)$. Since $\Psi \equiv 1$ on $\overline{\Omega}$, $\Psi \cdot \tau_{-h} f$ and $\tau_{-h} f$ have the same restrictions to Ω; therefore, $\|\Psi\cdot\tau_{-h}f - f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)} < \epsilon/2$ and, if f uniformly continuous, we also have $\|\Psi\cdot\tau_{-h}f - f\|_u < \epsilon/2$ on Ω.

To prove part i), apply corollary 6.21.ii) to obtain $g \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ such that $\|g - \Psi\cdot\tau_{-h}f\|_{\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)} < \epsilon/2$, which yields $\|g|_\Omega - f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)} < \epsilon$.

To prove part ii), let $(\phi_t)_{t>0}$ be the standard mollifier. We contend that $\Psi\cdot\tau_{-h}f : \mathbb{R}^n \to \mathbb{R}$ is bounded and uniformly continuous. Assuming this contention, to be proved below, we may apply theorem 1.111.ii) to obtain $t > 0$ sufficiently small so that $\|\phi_t * \Psi\cdot\tau_{-h}f - \Psi\cdot\tau_{-h}f\|_u < \epsilon/2$; besides, taking a smaller t if necessary, corollary 6.21.i) yields $\|\phi_t * \Psi\cdot\tau_{-h}f - \Psi\cdot\tau_{-h}f\|_{\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)} < \epsilon/2$. We therefore reach the thesis with $g := \phi_t * \Psi\cdot\tau_{-h}f \in \mathsf{C}^\infty(\mathbb{R}^n)$.

Proof of the contention in the previous paragraph: as f is bounded and uniformly continuous, so is $\tau_{-h}f : \Omega_{-h} \to \mathbb{R}$. Then $\Psi\cdot\tau_{-h}f$ is clearly bounded; it remains to show that it is uniformly continuous. Note that Ψ is uniformly continuous, since it is smooth with bounded derivative, hence it is Lipschitz. As $\Psi\cdot\tau_{-h}f$ is continuous (because so are its restrictions to the open sets Ω and $\mathbb{R}^n \setminus \operatorname{spt}\Psi$) with support contained in $\operatorname{spt}\Psi$, and $\forall x, y \in \operatorname{spt}\Psi$,

$$\begin{aligned}
&|\Psi(x)\cdot\tau_{-h}f(x) - \Psi(y)\cdot\tau_{-h}f(y)| \leq \\
&\quad \leq |\Psi(x)||\tau_{-h}f(x) - \tau_{-h}f(y)| + |\tau_{-h}f(y)||\Psi(x) - \Psi(y)| \leq \\
&\quad \leq \|\Psi\|_u|\tau_{-h}f(x) - \tau_{-h}f(y)| + \|f\|_u|\Psi(x) - \Psi(y)|,
\end{aligned}$$

we conclude that $\Psi\cdot\tau_{-h}f$ is uniformly continuous, as asserted.

Finally, if $U \subset \mathbb{R}^n$ is open and the closure in $\mathbb{R}^n$ of $\operatorname{spt} f$ is a compact subset of U, take $\widetilde{g} \in \mathsf{C}^\infty(\mathbb{R})$ satisfying i) or ii) for a given $\epsilon > 0$ and $\zeta \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ with $0 \leq \zeta \leq 1$, $\operatorname{spt}\zeta \subset U$ and $\zeta \equiv 1$ on the closure in $\mathbb{R}^n$ of $\operatorname{spt} f$. Define $g := \zeta\widetilde{g} \in \mathsf{C}^\infty_\mathsf{c}(U)$. Since $\zeta f = f$, we have $g - f = \zeta(\widetilde{g} - f)$, hence (by theorem 6.26) $\nabla^\mathsf{w}(g - f) = \nabla\zeta\cdot(\widetilde{g} - f) + \zeta\cdot\nabla^\mathsf{w}(\widetilde{g} - f)$ on Ω, which implies

- $\|g - f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)} \leq \|\zeta\|_{\mathsf{W}^{1,\infty}(\mathbb{R}^n)}\|\widetilde{g} - f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)} < \|\zeta\|_{\mathsf{W}^{1,\infty}(\mathbb{R}^n)}\epsilon$;
- in case ii), $\|g|_\Omega - f\|_u \leq \|\zeta\|_u\|\widetilde{g}|_\Omega - f\|_u < \epsilon$.

Since $\epsilon > 0$ was arbitrarily taken, the statements in i) and ii) are fulfilled with g in place of $\widetilde{g}$. □

Lemma 6.40. *Let $1 \leq p \leq \infty$, $U \subset \mathbb{R}^n$ open, $\Phi \in \mathrm{SE}(n)$ a rigid motion and $V = \Phi(U)$. The map $(\circ\Phi) : f \mapsto f \circ \Phi$ is:*

1) a linear isometry $\mathsf{L}^\mathsf{p}(V) \to \mathsf{L}^\mathsf{p}(U)$ with inverse $(\circ\Phi^{-1})$;
2) a linear isometry $\mathsf{W}^{1,\mathsf{p}}(V) \to \mathsf{W}^{1,\mathsf{p}}(U)$ with inverse $(\circ\Phi^{-1})$;
3) a linear isometry $\mathsf{C}_\mathsf{b}(V) \to \mathsf{C}_\mathsf{b}(U)$ with inverse $(\circ\Phi^{-1})$.

Proof. Part 3) is immediate and part 1) is an immediate consequence of $\Phi_\#\mathcal{L}^n = \mathcal{L}^n$ (since the Lebesgue measure is invariant by translations and rotations). Part 2) follows from part 1) and from the fact that weak derivatives commute with $(\circ\Phi)$, as it can be directly checked.

$\square$

LEMMA 6.41. *Let $\Omega \subset \mathbb{R}^n$ open, $f \in \mathsf{C}(\overline{\Omega})$ and $\xi \in \mathsf{C_c}(\mathbb{R}^n)$. Then $\xi \cdot f : \Omega \to \mathbb{R}$ is uniformly continuous.*

PROOF. We may extend f to a continuous function $\overline{\Omega} \to \mathbb{R}$, which on its turn may be extended, in view of Tietze's extension theorem, to a continuous function $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$. Then $\xi \cdot \widetilde{f} \in \mathsf{C_c}(\mathbb{R}^n) \subset \mathsf{C_0}(\mathbb{R}^n)$ is uniformly continuous by lemma 1.109, and so is its restriction $(\xi \cdot \widetilde{f})|_\Omega = \xi \cdot f : \Omega \to \mathbb{R}$. $\square$

PROOF OF THEOREM 6.34. Let $(U_i)_{i \geq 0}$ be a countable open cover of $\mathbb{R}^n$, where $U_0 = \mathbb{R}^n \setminus \overline{\Omega}$, $U_1 = \Omega$ and, for each $i \geq 2$, U_i is obtained by rigid motion of a cylinder centered at $0 \in \mathbb{R}^n$ as in definition 6.33, i.e. there exists a rigid motion $\Phi_i \in \mathrm{SE}(n)$ with $\Phi_i(0) = x_i \in \partial\Omega$ and there exists $r_i, h_i > 0$ and $\Gamma_i : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz with $\Gamma_i(0) = 0$ such that $U_i = \Phi_i\big(\mathbb{C}(0, r_i, h_i)\big)$, $\Phi_i\big(\mathrm{gr}\ \Gamma_i \cap \mathbb{C}(0, r_i, h_i)\big) = U_i \cap \partial\Omega$ and $\Phi_i\big(\mathrm{epi}_{\mathsf{S}}\ \Gamma_i \cap \mathbb{C}(0, r_i, h_i)\big) = U_i \cap \Omega$.

Let $(V_k)_{k \in \mathbb{N}}$ be a locally finite refinement of $(U_i)_{i \in \mathbb{N}}$ formed by relatively compact open subsets of $\mathbb{R}^n$, and $(\xi_k)_{k \in \mathbb{N}}$ a smooth partition of unity of $\mathbb{R}^n$ such that for each $k \in \mathbb{N}$, $\mathrm{spt}\ \xi_k \Subset V_k$, given by theorem 6.8. Note that, for each $k \in \mathbb{N}$, the fact that $\xi_k \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ and the product rule 6.26 imply that:

- $\xi_k \cdot f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$ and $\mathrm{spt}\ \xi_k \cdot f \subset \mathrm{spt}\ \xi_k \Subset V_k$;
- if f belongs to $\mathsf{C}(\overline{\Omega})$, it follows from lemma 6.41 that $\xi_k \cdot f : \Omega \to \mathbb{R}$ is uniformly continuous.

Fix $\epsilon > 0$.

1) Claim: for each $k \in \mathbb{N}$, there exists $g_k \in \mathsf{C}^\infty_\mathsf{c}(V_k)$ such that $\|g_k|_\Omega - \xi_k \cdot f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)} < 2^{-k}\epsilon$. Moreover, if $f \in \mathsf{C}(\overline{\Omega})$, we may take $g_k \in \mathsf{C}^\infty_\mathsf{c}(V_k)$ so that $\|g_k|_\Omega - \xi_k \cdot f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)} < 2^{-k}\epsilon$ and $\|g_k|_\Omega - f\|_u < 2^{-k}\epsilon$.

   Indeed, there exists $i_k \in \mathbb{N}$ such that $V_k \subset U_{i_k}$.
   - If $i_k = 0$, $\xi_k \cdot f$ is the null function on Ω and we may take $g_k \equiv 0$.
   - If $i_k = 1$, $\mathrm{spt}\ \xi_k \Subset V_k \subset U_1 = \Omega$, hence $\xi_k \cdot f \in \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ by lemma 6.35 and $\mathrm{spt}\ \xi_k \cdot f \Subset V_k$. Besides, if $f \in \mathsf{C}(\overline{\Omega})$, $\xi_k \cdot f \in \mathsf{C_c}(\mathbb{R}^n)$. Then, if $(\phi_t)_{t>0}$ is the standard mollifier, by corollary 6.21 there exists $t > 0$ sufficiently small so that $\phi_t * (\xi_k \cdot f) \in \mathsf{C}^\infty_\mathsf{c}(V_k)$ and $\|\phi_t * (\xi_k \cdot f) - \xi_k \cdot f\|_{\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)} < 2^{-k}\epsilon$. If $f \in \mathsf{C}(\overline{\Omega})$, the fact that $\xi_k \cdot f \in \mathsf{C_c}(\mathbb{R}^n)$ and theorem 1.111.ii)

ensure the existence of a smaller $t > 0$ such that we also have $\|\phi_t *(\xi_k \cdot f) - \xi_k \cdot f\|_{\mathsf{L}^\infty(\mathbb{R}^n)} < 2^{-k}\epsilon$. Put $g_k := \phi_t *(\xi_k \cdot f)$.

- If $i_k \geq 2$, $(\xi_k \cdot f) \circ \Phi_{i_k} \in \mathsf{W}^{1,\mathsf{p}}\big(\mathbb{C}(0, r_{i_k}, h_{i_k}) \cap \text{epi}_{\mathsf{S}}\ \Gamma_{i_k}\big)$ (by lemma 6.40) and $\overline{\text{spt}\,(\xi_k \cdot f) \circ \Phi_{i_k}} \subset \Phi_{i_k}^{-1}(\text{spt}\ \xi_k) \Subset \Phi_{i_k}^{-1}(V_k) \subset \mathbb{C}(0, r_{i_k}, h_{i_k})$. Hence $(\xi_k \cdot f) \circ \Phi_{i_k} \in \mathsf{W}^{1,\mathsf{p}}(\text{epi}_{\mathsf{S}}\ \Gamma_{i_k})$. Furthermore, if $f \in \mathsf{C}(\overline{\Omega})$, as we saw above $\xi_k \cdot f$ is uniformly continuous on Ω, hence its restriction to $U_{i_k} \cap \Omega$ is bounded (because it may be continuously extended to the compact set $\overline{U_{i_k} \cap \Omega}$, on which it is therefore bounded) and uniformly continuous, and so is $(\xi_k \cdot f) \circ \Phi_{i_k} : \mathbb{C}(0, r_{i_k}, h_{i_k}) \cap \text{epi}_{\mathsf{S}}\ \Gamma_{i_k} \to \mathbb{R}$.
  We contend that $(\xi_k \cdot f) \circ \Phi_{i_k} : \text{epi}_{\mathsf{S}}\ \Gamma_{i_k} \to \mathbb{R}$ is bounded and uniformly continuous. Indeed, it is clearly bounded, since it is null on the complement of the cylinder $\mathbb{C}(0, r_{i_k}, h_{i_k})$ and its restriction to $\mathbb{C}(0, r_{i_k}, h_{i_k}) \cap \text{epi}_{\mathsf{S}}\ \Gamma_{i_k}$ is bounded. Moreover, since $\overline{\text{spt}\,(\xi_k \cdot f) \circ \Phi_{i_k}} \Subset \mathbb{C}(0, r_{i_k}, h_{i_k})$ and the restriction $(\xi_k \cdot f) \circ \Phi_{i_k} : \mathbb{C}(0, r_{i_k}, h_{i_k}) \cap \text{epi}_{\mathsf{S}}\ \Gamma_{i_k} \to \mathbb{R}$ is uniformly continuous, given $\epsilon > 0$, we may take $0 < \delta < \delta_0 := d(\overline{\text{spt}\,(\xi_k \cdot f) \circ \Phi_{i_k}}, \mathbb{R}^n \setminus \mathbb{C}(0, r_{i_k}, h_{i_k}))$ such that, putting $F := (\xi_k \cdot f) \circ \Phi_{i_k}$, $\forall x, y \in \text{epi}_{\mathsf{S}}\ \Gamma_{i_k} \cap \mathbb{C}(0, r_{i_k}, h_{i_k})$ with $\|x - y\| < \delta$, we have $|F(x) - F(y)| < \epsilon$. The same holds for all $x, y \in \text{epi}_{\mathsf{S}}\ \Gamma_{i_k}$ with $\|x-y\| < \delta$ because, if $x \in \mathbb{R}^n \setminus \mathbb{C}(0, r_{i_k}, h_{i_k})$ and $\|x - y\| < \delta$, then $y \in \mathbb{R}^n \setminus \overline{\text{spt}\,(\xi_k \cdot f) \circ \Phi_{i_k}}$, hence $F(x) = F(y) = 0$. Thus, the contention is proved.
  Applying lemma 6.39 with $(\xi_k \cdot f) \circ \Phi_{i_k}$ in place of f and $\Phi_{i_k}^{-1}(V_k)$ in place of U, we obtain $\widetilde{g}_k \in \mathsf{C}_{\mathsf{c}}^\infty\big(\Phi_{i_k}^{-1}(V_k)\big)$ such that $\|\widetilde{g}_k|_{\text{epi}_{\mathsf{S}}\ \Gamma_{i_k}} - (\xi_k \cdot f) \circ \Phi_{i_k}\|_{\mathsf{W}^{1,\mathsf{p}}(\text{epi}_{\mathsf{S}}\ \Gamma_{i_k})} < 2^{-k}\epsilon$ and, if $f \in \mathsf{C}(\overline{\Omega})$, $\|\widetilde{g}_k|_{\text{epi}_{\mathsf{S}}\ \Gamma_{i_k}} - (\xi_k \cdot f) \circ \Phi_{i_k}\|_u < 2^{-k}\epsilon$. Thus, in view of lemma 6.40, $g_k := \widetilde{g}_k \circ \Phi_{i_k}^{-1}$ proves the claim.

2) Let $g := \sum_{k \geq 0} g_k$. Since $\forall k \in \mathbb{N}$, $g_k \in \mathsf{C}_{\mathsf{c}}^\infty(V_k)$ and $(V_k)_{k \in \mathbb{N}}$ is a locally finite open cover of $\mathbb{R}^n$, the sum which defines g is locally finite and $g \in \mathsf{C}^\infty(\mathbb{R}^n)$. Besides, since $f = \sum_{k \geq 0} \xi_k \cdot f$, we have $\|g|_\Omega - f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)} \leq \sum_{k \geq 0} \|g_k|_\Omega - \xi_k \cdot f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)} < 2\epsilon$ and, if $f \in \mathsf{C}(\overline{\Omega})$, $\|g|_\Omega - f\|_u \leq \sum_{k \geq 0} \|g_k|_\Omega - \xi_k \cdot f\|_u < 2\epsilon$.

□

Remark 6.42. With the same hypothesis and notation from theorem 6.34, we have actually proved that there exists $(f_k)_{k \in \mathbb{N}}$ in $\mathsf{W}^{1,\mathsf{p}}(\Omega) \cap \mathsf{C}^\infty(\mathbb{R}^n)$ such that $f_k \to f$ in $\mathsf{W}^{1,\mathsf{p}}(\Omega)$, which also converges to f uniformly on $\overline{\Omega}$ if $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega) \cap \mathsf{C}(\overline{\Omega})$.

Corollary 6.43. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. If $1 \le p < \infty$ and $f \in \mathsf{W}^{1,p}(\Omega)$, there exists $(f_k)_{k\in\mathbb{N}}$ in $\mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ such that $f_k|_\Omega \to f$ in $\mathsf{W}^{1,p}(\Omega)$. Moreover, if $f \in \mathsf{W}^{1,p}(\Omega) \cap \mathsf{C}(\overline{\Omega})$, the sequence $(f_k)_{k\in\mathbb{N}}$ may be chosen so that it also converges to f uniformly on compact subsets of $\overline{\Omega}$.*

Proof. As it was noted in remark 6.42, there exists a sequence $(g_k)_{k\in\mathbb{N}}$ in $\mathsf{C}^\infty(\mathbb{R}^n)$ such that $(\forall k) g_k|_\Omega \in \mathsf{W}^{1,p}(\Omega)$ and $g_k|_\Omega \to f$ in $\mathsf{W}^{1,p}(\Omega)$, and such that $(g_k|_{\overline{\Omega}})$ also converges uniformly to f if $f \in \mathsf{W}^{1,p}(\Omega) \cap \mathsf{C}(\overline{\Omega})$.

We now adapt the argument from part ii) of corollary 6.21. Choose $\zeta \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n, [0,1])$ such that $\zeta \equiv 1$ on $\mathbb{B}(0,1)$ and $\operatorname{spt} \zeta \subset \mathbb{U}(0,2)$. Define, $\forall k \in \mathbb{N}$, $\zeta_k \in \mathsf{C}^\infty_\mathsf{c}\big(\mathbb{U}(0,2k)\big)$ by $\zeta_k(x) := \zeta(x/k)$, and $f_k := \zeta_k \cdot g_k$. Then $\forall k \in \mathbb{N}$, $f_k \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$. We will prove that $f_k|_\Omega \to f$ in $\mathsf{W}^{1,p}(\Omega)$. We omit restrictions for simplicity of notation; the p-norms are taken with respect to $\mathcal{L}^n|_\Omega$.

1) For all $u \in \mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega)$, $\zeta_k \cdot u \to u$ in $\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega)$. Indeed, $|u - \zeta_k \cdot u|^p \to 0$ pointwise and $|u - \zeta_k \cdot u|^p \le 2^p |u|^p \in \mathsf{L}^1(\mathcal{L}^n|_\Omega)$, hence the dominated convergence theorem 1.64 yields the assertion.
2) Since $f - f_k = (f - \zeta_k \cdot f) + \zeta_k \cdot (f - g_k)$, we have $\|f - f_k\|_p \le \|f - \zeta_k \cdot f\|_p + \|\zeta_k\|_u \|f - g_k\|_p \to 0$, since $|\zeta_k| \le 1$, $\|f - \zeta_k \cdot f\|_p \to 0$ by the previous item and $\|f - g_k\|_p \le \|f - g_k\|_{\mathsf{W}^{1,p}(\Omega)} \to 0$.
3) $\forall x \in \mathbb{R}^n$,

$$\begin{aligned}\nabla f_k(x) &= \nabla \zeta_k(x) \cdot g_k(x) + \zeta_k(x) \cdot \nabla g_k(x) = \\ &= \frac{1}{k} \cdot \nabla \zeta(x/k) \cdot g_k(x) + \zeta_k(x) \cdot \nabla g_k(x).\end{aligned}$$

Hence, $\forall x \in \Omega$,

$$\begin{aligned}\nabla^\mathsf{w} f(x) - \nabla f_k(x) = \nabla^\mathsf{w} f(x) &- \zeta_k(x)\, \nabla^\mathsf{w} f(x) + \\ &+ \zeta_k(x)[\nabla^\mathsf{w} f(x) - \nabla g_k(x)] - \frac{1}{k} \cdot \nabla \zeta(x/k) \cdot g_k(x),\end{aligned}$$

so that

$$\|\nabla^\mathsf{w} f - \nabla f_k\|_p \le \|\nabla^\mathsf{w} f - \zeta_k \nabla^\mathsf{w} f\|_p + \|\zeta_k\|_u \|\nabla^\mathsf{w} f - \nabla g_k\|_p + \frac{1}{k} \|\nabla \zeta\|_u \|g_k\|_p.$$

Since $\|\nabla^\mathsf{w} f - \zeta_k \nabla^\mathsf{w} f\|_p \to 0$ by item 1), $\|\nabla^\mathsf{w} f - \nabla g_k\|_p \le \|f - g_k\|_{\mathsf{W}^{1,p}(\Omega)} \to 0$ and, as $(g_k)_{k\in\mathbb{N}}$ is bounded in $\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega)$ (because it is convergent in $\mathsf{W}^{1,p}(\Omega)$, hence in in $\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega)$), $\frac{1}{k} \|\nabla \zeta_k\|_u \|g_k\|_p \to 0$, it follows that $\|\nabla^\mathsf{w} f - \nabla f_k\|_p \to 0$. We have thus proved that $f_k \to f$ in $\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega)$ (by the previous item) and $\nabla f_k \to \nabla^\mathsf{w} f$ in $\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$; that is, $f_k \to f$ in $\mathsf{W}^{1,p}(\Omega)$, as asserted.

Finally, if $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega) \cap \mathsf{C}(\overline{\Omega})$, $(f_k|_{\overline{\Omega}})_{k\in\mathbb{N}}$ converges to f uniformly on compact subsets of $\overline{\Omega}$, bacause so does $(g_k)_{k\in\mathbb{N}}$, and for each $k \in \mathbb{N}$, $f_k \equiv g_k$ on $\mathbb{B}(0,k)$. □

6.4. Lipschitz functions and $\mathsf{W}^{1,\infty}$

THEOREM 6.44. *Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}$. Then $f \in \mathsf{W}^{1,\infty}_{\mathsf{loc}}(\Omega)$ if, and only if, f coincides $\mathcal{L}^n$-a.e. on Ω with a locally Lipschitz function.*

PROOF. We have already proved in corollary 6.16 that, if f is locally Lipschitz, than $f \in \mathsf{W}^{1,\infty}_{\mathsf{loc}}(\Omega)$ (moreover, it is Fréchet-differentiable $\mathcal{L}^n$-a.e. and its Fréchet derivative coincides $\mathcal{L}^n$-a.e. with its weak gradient).

It remains to prove the converse. Suppose that $f \in \mathsf{W}^{1,\infty}_{\mathsf{loc}}(\Omega)$. It suffices to show that, for each relatively compact convex open subset $V \Subset \Omega$, $f|_V$ coincides $\mathcal{L}^n$-a.e. with a Lipschitz function, since Ω may be covered by countably many such convex open sets. Let $W \Subset \Omega$ open such that $V \Subset W$. Let $\epsilon_0 := d(\overline{V}, W^c)$ and $(\phi_t)_{t>0}$ the standard mollifier. Then, for every $0 < \epsilon < \epsilon_0$, $f_\epsilon = \phi_\epsilon * f \in \mathsf{C}^\infty(V)$ and for each $x \in V$,

$$\begin{aligned}\|\nabla f_\epsilon(x)\| &\overset{6.20.vi)}{=} \Big\|\int_{\mathbb{B}(x,\epsilon)} \nabla^{\mathsf{w}} f(y)\phi_\epsilon(x-y)\,\mathrm{d}\mathcal{L}^n(y)\Big\| \le \\ &\le \|\nabla^{\mathsf{w}} f\|_{\mathsf{L}^\infty(\mathcal{L}^n|_W)}.\end{aligned}$$

Thus, putting $C := \|\nabla^{\mathsf{w}} f\|_{\mathsf{L}^\infty(\mathcal{L}^n|_W)}$, we have $\sup\{\|\nabla f_\epsilon\|_{\mathsf{L}^\infty(\mathcal{L}^n|_V)} \mid 0 < \epsilon < \epsilon_0\} \le C < \infty$. Therefore, for all $x, y \in V$ and $0 < \epsilon < \epsilon_0$:

$$\begin{aligned}|f_\epsilon(y) - f_\epsilon(x)| &= \Big|\int_0^1 \nabla f_\epsilon\big(x + t(y-x)\big)\cdot(y-x)\,\mathrm{d}\mathcal{L}^n(t)\Big| \le \\ &\le C\|y-x\|.\end{aligned}$$

Hence, denoting by L_f the set of Lebesgue points of f, it follows from theorem 6.20.iii) that, taking $\epsilon \downarrow 0$, for all $x, y \in V \cap L_f$,

$$|f(y) - f(x)| \le C\|y-x\|,$$

i.e. $f|_{L_f \cap V}$ is Lipschitz. We may therefore extend this restriction to a Lipschitz function on V (even on $\mathbb{R}^n$). Since $\mathcal{L}^n(V \setminus L_f) = 0$, we have proved that f coincides $\mathcal{L}^n$-a.e. on V with a Lipschitz function, as asserted. □

6.5. Traces and Extensions

We prove below a version of the Gauss-Green theorem for epigraphs of Lipschitz functions which will be needed to prove theorems on traces and extensions of Sobolev functions. This theorem will be generalized in chapter 7, theorem 7.18; it is essentially a consequence of the area formula.

THEOREM 6.45 (Gauss-Green theorem for Lipschitz epigraphs). *Let $n \geq 2$, $f : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz and $\Omega := \mathrm{epi}_{\mathsf{S}}\, f$ (hence $\partial\Omega = \mathrm{gr}\, f$). Then*

i) $\mathcal{H}^{n-1} \llcorner \partial\Omega$ is a Radon measure on $\mathbb{R}^n$;
ii) there exists a Borel measurable unit vector field $\nu : \partial\Omega \to \mathbb{R}^n$, unique up to $\mathcal{H}^{n-1} \llcorner \partial\Omega$-null sets, such that, for all $\varphi \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)$,

$$\int_\Omega \nabla\varphi \, \mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} \varphi\nu \, \mathrm{d}\mathcal{H}^{n-1}, \tag{6.5}$$

or, equivalently, such that, for all $\varphi \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int_\Omega \mathrm{div}\, \varphi \, \mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} \varphi \cdot \nu \, \mathrm{d}\mathcal{H}^{n-1}. \tag{6.6}$$

PROOF. Let $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R}$ be given by $x \mapsto \big(x, f(x)\big)$. Then Γ is Lipschitz 1-1 and $\mathrm{Im}\,\Gamma = \mathrm{gr}\, f = \partial\Omega$. Thus, from corollary 5.39.i), it follows that $\mathcal{H}^{n-1} \llcorner \partial\Omega = \Gamma_\#(\mathcal{L}^{n-1} \llcorner \mathrm{J}\Gamma)$, whence $\mathcal{H}^{n-1} \llcorner \partial\Omega$ is a Radon measure on $\mathbb{R}^n$ (because it is Borel regular and finite on compact sets, as one can see directly from the above formula).

It remains to prove the existence and uniqueness up to $\mathcal{H}^{n-1} \llcorner \partial\Omega$-null sets of $\nu : \partial\Omega \to \mathbb{R}^n$ Borel measurable with $\|\nu\| \equiv 1$ such that (6.6) or, equivalently, (6.5) holds. Indeed, for each $x = \big(x', f(x')\big) \in \Gamma(D_f) \subset \partial\Omega$, where $D_f \subset \mathbb{R}^{n-1}$ is the differentiability set of f, let

$$\nu(x) = \frac{\big(\nabla f(x'), -1\big)}{\sqrt{1 + \|\nabla f(x')\|^2}}, \tag{6.7}$$

and let ν be any constant unit vector field on $\partial\Omega \setminus \Gamma(D_f)$. Since $D_f \in \mathscr{B}_{\mathbb{R}^{n-1}}$, ∇f is Borelian on D_f and $\mathcal{L}^n(\mathbb{R}^{n-1} \setminus D_f) = 0$ (by exercise 5.13 and by Rademacher's theorem), it follows that ν is Borelian, $\|\nu\| \equiv 1$ and $\partial\Omega \setminus \Gamma(D_f)$ is $\mathcal{H}^{n-1} \llcorner \partial\Omega$-null, so that ν is given $\mathcal{H}^{n-1} \llcorner \partial\Omega$-a.e. by (6.7). We will prove that (6.5) holds with such ν. If ν' is another such Borel unit vector field, then $(\nu, \mathcal{H}^{n-1} \llcorner \partial\Omega)$ and $(\nu', \mathcal{H}^{n-1} \llcorner \partial\Omega)$ are polar decompositions of the same $\mathbb{R}^n$-valued Radon measure, so that $\nu = \nu'$ $\mathcal{H}^{n-1} \llcorner \partial\Omega$-a.e. by the uniqueness of the polar decomposition.

Given $\delta > 0$, let F_δ be the open strip of amplitude 2δ along the graph of f (see figure 2), i.e. $F_\delta := \{x = (x', x_n) \in \mathbb{R}^n \mid |x_n - f(x')| < \delta\}$. We approximate the characteristic function χ_Ω of $\Omega = \mathrm{epi}_{\mathsf{S}}\, f$ by a Lipschitz function f_δ defined as 1 on $\mathrm{epi}\,(f + \delta)$, 0 on $\mathrm{hyp}\,(f_\delta)$, and by linear interpolation on F_δ, i.e. for all $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

$$f_\delta(x) := \begin{cases} 1 & x_n \geq f(x') + \delta, \\ 0 & x_n \leq f(x') - \delta, \\ (2\delta)^{-1}\big[x_n - \big(f(x') - \delta\big)\big] & x \in F_\delta. \end{cases}$$

It is then clear that $0 \leq f_\delta \leq 1$, $\operatorname{Lip} f_\delta = (2\delta)^{-1}(1 + \operatorname{Lip} f)$ and $f_\delta \to \chi_\Omega$ pointwise on $\mathbb{R}^{n-1} \setminus \operatorname{gr} f$ as $\delta \downarrow 0$; in particular, $f_\delta \to \chi_\Omega$ $\mathcal{L}^n$-a.e. on $\mathbb{R}^n$, since $\mathcal{L}^n(\operatorname{gr} f) = 0$ (for instance, as an immediate consequence of Fubini's theorem). Furthermore, by a direct computation, the classical gradient of f_δ exists on $\mathcal{L}^n$-a.e. $x = (x', x_n) \in \mathbb{R}^n$ and is given by:

$$\nabla f_\delta(x) = \begin{cases} 0 & \text{if } x_n > f(x') + \delta \text{ or } x_n < f(x') - \delta \\ (2\delta)^{-1}\big(-\nabla f(x'), 1\big) & \text{if } x \in F_\delta \text{ and } x' \in D_f. \end{cases}$$

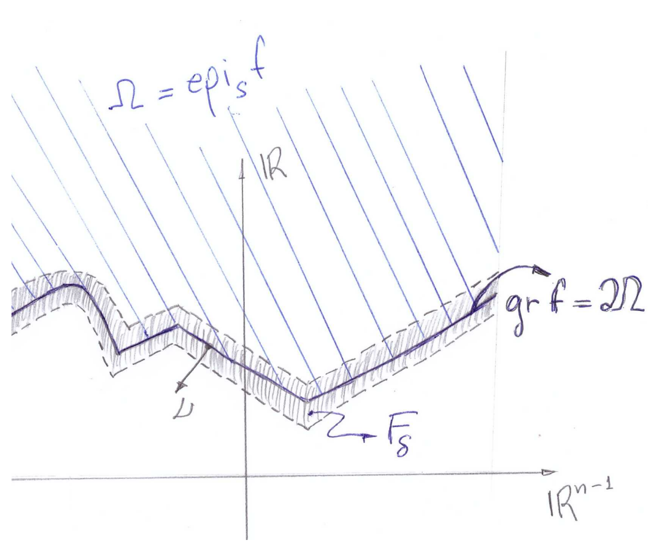

FIGURE 2. Gauss-Green theorem for epigraphs

We now compute, given $\varphi \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)$ and taking a sequence $(\delta_k)_{k\in\mathbb{N}}$ in $(0,\infty)$ with $\delta_k \downarrow 0$:

$$\begin{aligned}
&\int_\Omega \nabla\varphi \,\mathrm{d}\mathcal{L}^n \overset{\text{DCT } 1.64}{=} \lim_{k\to\infty}\int f_{\delta_k}\nabla\varphi\,\mathrm{d}\mathcal{L}^n \overset{5.5, f_{\delta_k}\text{ Lipschitz}}{=}\\
&= -\lim_{k\to\infty}\int \varphi\,\nabla^{\mathsf{w}} f_{\delta_k}\,\mathrm{d}\mathcal{L}^n \overset{\nabla^{\mathsf{w}} f_{\delta_k}=\nabla f_{\delta_k}\text{ by } 5.12}{=}\\
&= -\lim_{k\to\infty}\int_{F_{\delta_k}} \varphi(x)\,\frac{1}{2\delta_k}\big(-\nabla f(x'),1\big)\,\mathrm{d}\mathcal{L}^n(x) \overset{\text{Fubini } 1.84}{=}\\
&= \lim_{k\to\infty}\int_{\mathbb{R}^{n-1}}\big(\nabla f(x'),-1\big)\,\frac{1}{2\delta_k}\int_{f(x')-\delta_k}^{f(x')+\delta_k}\varphi(x',t)\,\mathrm{d}\mathcal{L}^n(t)\,\mathrm{d}\mathcal{L}^{n-1}(x') =\\
&= \lim_{k\to\infty}\int_{\underbrace{\mathsf{pr}_{\mathbb{R}^{n-1}}\,\mathrm{spt}\,\varphi}_{\text{compact}}} \overbrace{\big(\nabla f(x'),-1\big)}^{\text{bounded}}\,\underbrace{\frac{1}{2\delta_k}\int_{f(x')-\delta_k}^{f(x')+\delta_k}\varphi(x',t)\,\mathrm{d}\mathcal{L}^n(t)}_{\le\|\varphi\|_u}\,\mathrm{d}\mathcal{L}^{n-1}(x') \overset{\text{DCT } 1.64}{=}\\
&= \int_{\mathbb{R}^{n-1}}\big(\nabla f(x'),-1\big)\varphi\big(x',f(x')\big)\,\mathrm{d}\mathcal{L}^{n-1}(x') \overset{\mathsf{J}\Gamma(x')=\sqrt{1+\|\nabla f(x')\|^2}}{=}\\
&= \int_{\mathbb{R}^{n-1}}\varphi\big(\Gamma(y)\big)\nu\big(\Gamma(y)\big)\,\mathsf{J}\Gamma(y)\,\mathrm{d}\mathcal{L}^{n-1}(y) \overset{\text{area formula } 5.39.ii)}{=}\\
&= \int_{\partial\Omega}\varphi\nu\,\mathrm{d}\mathcal{H}^{n-1},
\end{aligned}$$

thus proving (6.5). □

DEFINITION 6.46. With the notation from the previous theorem, ν is called *outer unit normal* to $\partial\Omega$.

REMARK 6.47. With the notation from the previous theorem, we have actually proved that, up to $\mathcal{H}^n \llcorner \partial\Omega$-null sets, on each point point $x = \big(x', f(x')\big)$ in $\partial\Omega = \mathrm{gr}\ f$ whose abscissa x' is a differentiability point of f,

$$\nu(x) = \frac{\big(\nabla f(x'),-1\big)}{\sqrt{1+\|\nabla f(x')\|^2}}. \tag{6.8}$$

In particular, if f is C^1, ν coincides with the usual outer unit normal from Differential Geometry.

THEOREM 6.48 (Trace theorem for Sobolev functions on Lipschitz epigraphs). *Let $n \ge 2$, $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz, $\Omega := \mathrm{epi}_{\mathsf{S}}\,\Gamma$ and $1 \le p < \infty$. Then:*

i) There exists a unique bounded linear operator $T : \mathsf{W}^{1,\mathsf{p}}(\Omega) \to \mathsf{L}^{\mathsf{p}}(\mathcal{H}^{n-1}|_{\partial\Omega})$ such that, for all $f \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)$, $T\cdot(f|_\Omega) = f|_{\partial\Omega}$.

ii) The Gauss-Green formula holds for all $f \in \mathsf{W}^{1,1}(\Omega)$, *i.e. denoting by* ν *the unit outer normal to* $\partial\Omega$,

$$\int_\Omega \nabla^{\mathsf{w}} f \,\mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} T \cdot f \;\nu \,\mathrm{d}\mathcal{H}^{n-1}, \tag{6.9}$$

with a similar equality in divergence form. Furthermore, for all $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$ *and* $\varphi \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int_\Omega f \operatorname{div} \varphi \,\mathrm{d}\mathcal{L}^n = -\int_\Omega \langle \nabla^{\mathsf{w}} f, \varphi\rangle \,\mathrm{d}\mathcal{L}^n + \int_{\partial\Omega} T \cdot f \;\langle \varphi, \nu\rangle \,\mathrm{d}\mathcal{H}^{n-1}. \tag{6.10}$$

PROOF. 1) Let $(e_1, \ldots, e_n)$ be the standard basis of $\mathbb{R}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R}$. Since, for all $x \in D_\Gamma$, $\|\nabla\Gamma(x)\| \le \operatorname{Lip}\Gamma$, it follows from (6.8) that, for all $x \in D_\Gamma$,

$$-e_n \cdot \nu \ge \frac{1}{\sqrt{1 + (\operatorname{Lip}\Gamma)^2}}.$$

In particular, putting $C := \sqrt{1 + (\operatorname{Lip}\Gamma)^2}$, we conclude that

$$1 \le C\,(-e_n \cdot \nu) \tag{6.11}$$

$\mathcal{H}^{n-1} \llcorner \partial\Omega$-a.e. on $\partial\Omega$.

2) Given $\epsilon > 0$, let $\beta_\epsilon : \mathbb{R} \to \mathbb{R}$ be given by $\beta_\epsilon(t) := (t^2 + \epsilon^2)^{1/2} - \epsilon$. Note that $\beta_\epsilon \in \mathsf{C}^\infty(\mathbb{R})$, $\beta_\epsilon \ge 0$, $\beta_\epsilon(t)$ increases to $|t|$ as $\epsilon \downarrow 0$ and $|\beta'_\epsilon| \le 1$.

3) Fix $f \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)$. Then $\beta_\epsilon \circ f \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)$ (since $\beta_\epsilon(0) = 0$, hence $\operatorname{spt} \beta_\epsilon \circ f \subset \operatorname{spt} f$). We compute:

$$\begin{aligned}\int_{\partial\Omega} \beta_\epsilon \circ f \,\mathrm{d}\mathcal{H}^{n-1} &\overset{(6.11)}{\le} -C \int_{\partial\Omega} \langle \underbrace{\beta_\epsilon \circ f\, e_n}_{\in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n,\mathbb{R}^n)}, \nu\rangle \,\mathrm{d}\mathcal{H}^{n-1} \overset{(6.6)}{=} \\ &= -C \int_\Omega \frac{\partial}{\partial x_n} [\ \underbrace{\beta_\epsilon \circ f}_{=\beta'_\epsilon \circ f \cdot \frac{\partial f}{\partial x_n}}\ ] \,\mathrm{d}\mathcal{L}^n \le \\ &\le C \int_\Omega |\beta'_\epsilon\big(f(x)\big)| \|\nabla f(x)\| \,\mathrm{d}\mathcal{L}^n(x) \\ &\le C \int_\Omega \|\nabla f\| \,\mathrm{d}\mathcal{L}^n.\end{aligned}$$

Since $\beta_\epsilon \circ f$ increases pointwise to $|f|$ as $\epsilon \downarrow 0$, we may therefore apply the monotone convergence theorem 1.62 to conclude that

$$\int_{\partial\Omega} |f| \,\mathrm{d}\mathcal{H}^{n-1} \le C \int_\Omega \|\nabla f\| \,\mathrm{d}\mathcal{L}^n. \tag{6.12}$$

In particular, since $\mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)$ is dense in $\mathsf{W}^{1,1}(\Omega)$ by lemma 6.39.i), $f \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n) \mapsto f|_{\partial\Omega}$ may be uniquely extended to a bounded linear function $\mathsf{W}^{1,1}(\Omega) \to \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$, thus proving part i) for $p = 1$.

4) For $1 < p < \infty$, given $f \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)$, note that $|f|^p \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)$ (since $|\cdot|^p \in \mathsf{C}^1(\mathbb{R})$ for $p > 1$ and it is null on 0) and $\nabla(|f|^p) = p|f|^{p-1} \cdot \operatorname{sgn} f \cdot \nabla f$ by the chain rule. We may therefore apply (6.12) with $|f|^p$ in place of f; denoting by $q \in (1, \infty)$ the conjugate exponent to p, we compute:

$$\begin{aligned}\int_{\partial\Omega} |f|^p \,\mathrm{d}\mathcal{H}^{n-1} &\leq C \int_\Omega \|\nabla(|f|^p)\| \,\mathrm{d}\mathcal{L}^n = \\ &= pC \int_\Omega \underbrace{|f|^{p-1}\|\nabla f\|}_{=\left[\|\nabla f\|^p\right]^{1/p}\left[|f|^{q(p-1)}\right]^{1/q} \leq \frac{\|\nabla f\|^p}{p} + \frac{|f|^p}{q}} \,\mathrm{d}\mathcal{L}^n \leq \\ &\leq C \int_\Omega \|\nabla f\|^p \,\mathrm{d}\mathcal{L}^n + \frac{pC}{q} \int_\Omega |f|^p \,\mathrm{d}\mathcal{L}^n.\end{aligned}$$

We therefore conclude that $\|f\|_{\mathsf{L}^\mathsf{p}(\mathcal{H}^{n-1}|_{\partial\Omega})} \leq C(n, p, \operatorname{Lip}\Gamma)\|f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)}$; since $\mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)$ is dense in $\mathsf{W}^{1,\mathsf{p}}(\Omega)$, by lemma 6.39.i), the linear map $f \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n) \mapsto f|_{\partial\Omega}$ may be uniquely extended to a bounded linear function $\mathsf{W}^{1,\mathsf{p}}(\Omega) \to \mathsf{L}^\mathsf{p}(\mathcal{H}^{n-1}|_{\partial\Omega})$, thus proving part i) for $1 < p < \infty$.

5) It remains to prove part ii). Let $f \in \mathsf{W}^{1,1}(\Omega)$. By lemma 6.39.i), we may take a sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ such that $f_k \to f$ in $\mathsf{W}^{1,1}(\Omega)$. It then follows, by the continuity of the trace operator, that $f_k|_{\partial\Omega} = T \cdot f_k \to T \cdot f$ in $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$. On the other hand, for each $k \in \mathbb{N}$, it follows from the Gauss-Green theorem (6.5) that

$$\int_\Omega \nabla f_k \,\mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} f_k \,\nu \,\mathrm{d}\mathcal{H}^{n-1}.$$

Hence, taking $k \to \infty$, we obtain (6.9).

Similarly, if $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$ and $\varphi \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n, \mathbb{R}^n)$, we may apply once more lemma 6.39.i) to obtain a sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ such that $f_k \to f$ in $\mathsf{W}^{1,\mathsf{p}}(\Omega)$. It then follows, by the continuity of the trace operator, that $f_k|_{\partial\Omega} = T \cdot f_k \to T \cdot f$ in $\mathsf{L}^\mathsf{p}(\mathcal{H}^{n-1}|_{\partial\Omega})$. On the other hand, for each $k \in \mathbb{N}$ an application of (6.6) to $f_k \cdot \varphi \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n, \mathbb{R}^n)$ yields

$$\int_\Omega f_k \operatorname{div} \varphi \,\mathrm{d}\mathcal{L}^n = -\int_\Omega \langle \nabla f_k, \varphi\rangle \,\mathrm{d}\mathcal{L}^n + \int_{\partial\Omega} f_k \langle \varphi, \nu\rangle \,\mathrm{d}\mathcal{H}^{n-1}.$$

Since $\operatorname{spt}\varphi$ is compact, we have $f_k \operatorname{div}\varphi \to f \operatorname{div}\varphi$, $\langle \nabla f_k, \varphi\rangle \to \langle \nabla^\mathsf{w} f, \varphi\rangle$ and $f_k\langle\varphi,\nu\rangle \to T \cdot f\langle\varphi,\nu\rangle$ in L^1; therefore, taking $k \to \infty$ in the last equality yields (6.10).

□

REMARK 6.49. With the notation from the previous theorem, for $p = 1$ it follows from 6.12 and from the density of $\mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)$ in $\mathsf{W}^{1,1}(\Omega)$ that the inequality

$$\int_{\partial\Omega} |T \cdot f| \, \mathrm{d}\mathcal{H}^{n-1} \leq C \int_\Omega \|\nabla^\mathsf{w} f\| \, \mathrm{d}\mathcal{L}^n.$$

holds for all $f \in \mathsf{W}^{1,1}(\Omega)$, where $C := \sqrt{1 + (\operatorname{Lip}\Gamma)^2}$.

COROLLARY 6.50 (Trace theorem for Sobolev functions on Lipschitz epigraphs). *With the hypothesis from the previous theorem, if $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega) \cap \mathsf{C}(\overline{\Omega})$, then $T \cdot f = f|_{\partial\Omega}$.*

PROOF. It follows from corollary 6.43 that there exists $(f_k)_{k\in\mathbb{N}}$ in $\mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ such that $f_k|_\Omega \to f$ in $\mathsf{W}^{1,\mathsf{p}}(\Omega)$ and $(f_k)_{k\in\mathbb{N}}$ converges to f uniformly on compact subsets of $\overline{\Omega}$. Then $f_k|_{\partial\Omega} = T \cdot f_k \to T \cdot f$ in $\mathsf{L}^\mathsf{p}(\mathcal{H}^{n-1}|_{\partial\Omega})$ and $f_k|_{\partial\Omega} \to f|_{\partial\Omega}$ uniformly on compact subsets of $\partial\Omega$, which implies $T \cdot f = f|_{\partial\Omega}$. □

The theorem below, which generalizes theorem 6.48 for Lipschitz domains on $\mathbb{R}^n$ with bounded frontier, may be skipped on a first reading. We shall need theorem 7.18, which ensures that every Lipschitz domain $\Omega \subset \mathbb{R}^n$ is a set of locally finite perimeter. The material covered in chapter 7 up to its first section 7.1 is independent of the remaining parts of this chapter, so that the reader may study it now if he wishes to better understand the following theorem.

THEOREM 6.51 (Trace theorem for Sobolev functions on Lipschitz domains). *Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ a Lipschitz domain with $\partial\Omega$ bounded, and $1 \leq p < \infty$. Then:*

i) There exists a unique bounded linear operator $T : \mathsf{W}^{1,\mathsf{p}}(\Omega) \to \mathsf{L}^\mathsf{p}(\mathcal{H}^{n-1}|_{\partial\Omega})$ such that, for all $f \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)$, $T \cdot (f|_\Omega) = f|_{\partial\Omega}$.

ii) The Gauss-Green formula holds for all $f \in \mathsf{W}^{1,1}(\Omega)$, i.e. denoting by ν the unit outer normal to $\partial\Omega$, cf. definition 7.12,

$$\int_\Omega \nabla^\mathsf{w} f \, \mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} T \cdot f \; \nu \, \mathrm{d}\mathcal{H}^{n-1}, \tag{6.13}$$

with a similar equality in divergence form. Furthermore, for all $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$ and $\varphi \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int_\Omega f \operatorname{div} \varphi \, \mathrm{d}\mathcal{L}^n = -\int_\Omega \langle \nabla^\mathsf{w} f, \varphi\rangle \, \mathrm{d}\mathcal{L}^n + \int_{\partial\Omega} T \cdot f \; \langle \varphi, \nu\rangle \, \mathrm{d}\mathcal{H}^{n-1}. \tag{6.14}$$

PROOF.

1) For each $x \in \partial\Omega$, there exists an open set $U_x \subset \mathbb{R}^n$ such that $x \in U_x$ and U_x is obtained by rigid motion of a cylinder centered at $0 \in \mathbb{R}^n$ as in definition 6.33, i.e. there exists a rigid motion $\Phi \in \mathrm{SE}(n)$ with $\Phi(0) = x$ and there exists $r, h > 0$ and $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz with $\Gamma(0) = 0$ such that $U_x = \Phi\big(\mathbb{C}(0, r, h)\big)$, $\Phi\big(\mathrm{gr}\ \Gamma \cap \mathbb{C}(0, r, h)\big) = U_x \cap \partial\Omega$ and $\Phi\big(\mathrm{epi}_{\mathsf{S}}\ \Gamma \cap \mathbb{C}(0, r, h)\big) = U_x \cap \Omega$.
2) From the open cover $(U_x)_{x\in\partial\Omega}$ of the compact set $\partial\Omega \subset \mathbb{R}^n$, we may extract a finite subcover $(U_i)_{1\le i\le N}$. For each $1 \le i \le N$, let the corresponding objects defined in the previous item be denoted with a subscript i, so that $\Phi_i\big(\mathbb{C}(0, r_i, h_i)\big) = U_i$.

   Let $U_0 := \Omega$ and $U_{-1} := \overline{\Omega}^c$, so that $(U_i)_{-1\le i\le N}$ is a finite open cover of $\mathbb{R}^n$. We may apply corollary 6.11 to obtain a smooth partition of unity $(\xi_i)_{-1\le i\le N}$ of $\mathbb{R}^n$ with $\mathrm{spt}\ \xi_i \subset U_i$ for $-1 \le i \le N$. Besides, for $i \ge 1$, as $\mathrm{spt}\ \xi_i \subset U_i \Subset \mathbb{R}^n$, it follows that $\mathrm{spt}\ \xi_i$ is a compact subset of U_i.
3) Fix $f \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)$ and $1 \le p < \infty$. We contend that, for $1 \le i \le N$, there exists $C_i = C_i(n, p, \mathrm{Lip}\,\Gamma_i)$ such that

$$\|(\xi_i f)|_{\partial\Omega}\|_{\mathsf{L}^{\mathsf{p}}(\mathcal{H}^{n-1}|_{\partial\Omega})} \le C_i \|\xi_i\|_{\mathsf{W}^{1,\infty}(\mathbb{R}^n)} \|f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)}.$$

   Indeed, we have:

$$\begin{aligned}
\int_{\partial\Omega} |\xi_i f|^p \,\mathrm{d}\mathcal{H}^{n-1} &= \int_{\partial\Omega\cap U_i} |\xi_i f|^p \,\mathrm{d}\mathcal{H}^{n-1} = \\
&= \int |\xi_i f|^p \,\mathrm{d}\big(\mathcal{H}^{n-1} \llcorner \partial\Phi_i(\mathrm{epi}_{\mathsf{S}}\ \Gamma_i)\big) \overset{\text{lemma } 7.17}{=} \\
&= \int |\xi_i f|^p \,\mathrm{d}\big(\Phi_{i\#}\big(\mathcal{H}^{n-1} \llcorner \partial\,\mathrm{epi}_{\mathsf{S}}\ \Gamma_i\big)\big) \overset{\text{ex. } 1.70}{=} \\
&= \int |(\xi_i f) \circ \Phi_i|^p \,\mathrm{d}\big(\mathcal{H}^{n-1} \llcorner \partial\,\mathrm{epi}_{\mathsf{S}}\ \Gamma_i\big) \overset{\text{thm. } 6.48}{\le} \\
&\le C_i^p \|(\xi_i f) \circ \Phi_i\|^p_{\mathsf{W}^{1,\mathsf{p}}\left(\mathrm{epi}_{\mathsf{S}}\ \Gamma_i \cap \mathbb{C}(0, r_i, h_i)\right)} \overset{\text{lemma } 6.40}{=} \\
&= C_i^p \|\xi_i f\|^p_{\mathsf{W}^{1,\mathsf{p}}\left(U_i\cap\Omega\right)} = C_i^p \|\xi_i f\|^p_{\mathsf{W}^{1,\mathsf{p}}(\Omega)} \overset{\text{product rule } 6.26}{\le} \\
&\le C_i^p \|\xi_i\|^p_{\mathsf{W}^{1,\infty}(\mathbb{R}^n)} \|f\|^p_{\mathsf{W}^{1,\mathsf{p}}(\Omega)}.
\end{aligned}$$

That C_i depends only on n, p and $\mathrm{Lip}\,\Gamma_i$ follows from part 4) of the proof of theorem 6.48. Our contention is then proved.

4) Since $f|_{\partial\Omega} = \sum_{i=1}^{N} (\xi_i f)|_{\partial\Omega}$, we have

$$\|f|_{\partial\Omega}\|_{\mathsf{L}^{\mathsf{p}}(\mathcal{H}^{n-1}|_{\partial\Omega})} \leq \sum_{i=1}^{N} \|(\xi_i f)|_{\partial\Omega}\|_{\mathsf{L}^{\mathsf{p}}(\mathcal{H}^{n-1}|_{\partial\Omega})} \overset{3)}{\leq}$$
$$\leq \Big(\sum_{i=1}^{N} C_i \|\xi_i\|_{\mathsf{W}^{1,\infty}(\mathbb{R}^n)}\Big) \|f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)}.$$

The above inequality shows that the linear map $f \in \{f|_\Omega \mid f \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)\} \mapsto f|_{\partial\Omega} \in \mathsf{L}^{\mathsf{p}}(\mathcal{H}^{n-1}|_{\partial\Omega})$ is continuous with respect to the $\mathsf{W}^{1,\mathsf{p}}(\Omega)$ relative topology on $\{f|_\Omega \mid f \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)\}$. Since the latter subspace is dense in $\mathsf{W}^{1,\mathsf{p}}(\Omega)$, by corollary 6.43, we conclude that there exists a unique continuous linear map $\mathsf{W}^{1,\mathsf{p}}(\Omega) \to \mathsf{L}^{\mathsf{p}}(\mathcal{H}^{n-1}|_{\partial\Omega})$ which extends the map $f \mapsto f|_{\partial\Omega}$ on $\{f|_\Omega \mid f \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)\}$. Assertion i) is therefore proved.

5) In view of theorem 7.18, (6.13) holds for $f \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n)$. Given $f \in \mathsf{W}^{1,1}(\Omega)$, by corollary 6.43 there exists a sequence $(f_i)_{i\in\mathbb{N}}$ in $\mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n)$ such that $f_i|_\Omega \to f$ in $\mathsf{W}^{1,1}(\Omega)$; hence, by continuity of the trace operator, $f_i|_{\partial\Omega} = T \cdot f_i \to T \cdot f$ in $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$. Therefore,

$$\int_\Omega \nabla^{\mathsf{w}} f \, \mathrm{d}\mathcal{L}^n = \lim_{i\to\infty} \int_\Omega \nabla f_i \, \mathrm{d}\mathcal{L}^n =$$
$$= \lim_{i\to\infty} \int_{\partial\Omega} f_i \, \nu \, \mathrm{d}\mathcal{H}^{n-1} =$$
$$= \int_{\partial\Omega} T \cdot f \;\nu \, \mathrm{d}\mathcal{H}^{n-1},$$

thus proving assertion ii).

6) Equality (6.13) in divergence form reads

$$\int_\Omega \operatorname{div} f \, \mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} \langle T \cdot f, \nu\rangle \, \mathrm{d}\mathcal{H}^{n-1},$$

for all $f \in \mathsf{W}^{1,1}(\Omega, \mathbb{R}^n)$. Therefore, given $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$ and $\varphi \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n)$, (6.14) follows from the previous equality and from the product rule 6.26 applied to $f\varphi$ componentwise, which yields $\operatorname{div}(f\varphi) = f \operatorname{div} \varphi + \langle \nabla^{\mathsf{w}} f, \varphi\rangle$.

□

Corollary 6.52 (Trace theorem for Sobolev functions on Lipschitz domains). *With the hypothesis from the previous theorem, if $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega) \cap \mathsf{C}(\overline{\Omega})$, then $T \cdot f = f|_{\partial\Omega}$.*

Proof. It is identical to the proof of corollary 6.50. □

DEFINITION 6.53 (Extension by reflection with respect to Lipschitz graphs). Let $n \geq 2$, $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz and $\Omega := \operatorname{epi}_{\mathsf{S}} \Gamma$. We identify $\mathbb{R}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R}$.

1) The map $\Phi_\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ given by $(x', x_n) \mapsto \big(x', \Gamma(x') - \big(x_n - \Gamma(x')\big)\big) = \big(x', 2\Gamma(x') - x_n\big)$ is called *reflection with respect to* Γ.
2) Given $f : \mathbb{R}^n \to \mathbb{R}$, the function $f_\Gamma := f \circ \Phi_\Gamma$ is said to be obtained by f by reflection with respect to Γ.
3) Given $f : \overline{\Omega} \to \mathbb{R}$, the *extension of f by reflection with respect to* Γ is the function $\mathsf{E}_\Gamma\, f : \mathbb{R}^n \to \mathbb{R}$ given by

$$\mathsf{E}_\Gamma\, f(x) := \begin{cases} f(x) & \text{if } x \in \overline{\Omega} \\ f \circ \Phi_\Gamma(x) & \text{if } x \in \Omega^c. \end{cases}$$

THEOREM 6.54 (Extension by reflection for Sobolev functions on Lipschitz epigraphs). *Let $n \geq 2$, $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz, $\Omega := \operatorname{epi}_{\mathsf{S}} \Gamma$ and $1 \leq p < \infty$. Then there exists a unique* extension operator $\mathsf{E} : \mathsf{W}^{1,\mathsf{p}}(\Omega) \to \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$, *i.e. a bounded linear operator with $(\mathsf{E}\, f)|_\Omega = f$ for all $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$, such that, for all $f \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)$, $\mathsf{E}(f|_\Omega) = \mathsf{E}_\Gamma(f|_{\overline{\Omega}})$ (i.e. the extension of $f|_{\overline{\Omega}}$ by reflection with respect to Γ).*

PROOF.

1) It suffices to prove that the extension by reflection E_Γ is a bounded linear operator defined on the subspace $\mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)|_\Omega := \{f|_\Omega \mid f \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)\}$ of $\mathsf{W}^{1,\mathsf{p}}(\Omega)$, taking values in $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$. Indeed, if that is the case, since $\mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)|_\Omega$ is dense in $\mathsf{W}^{1,\mathsf{p}}(\Omega)$ by lemma 6.39.i), E_Γ may be uniquely extended to a bounded linear operator $\mathsf{E} : \mathsf{W}^{1,\mathsf{p}}(\Omega) \to \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$. Then E satisfies $(\mathsf{E}\, f)|_\Omega = f$ for all $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$ because, as the restriction $R : \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n) \to \mathsf{W}^{1,\mathsf{p}}(\Omega)$ (i.e. given by $f \mapsto f|_\Omega$) is linear continuous, the composite $R \circ \mathsf{E} : \mathsf{W}^{1,\mathsf{p}}(\Omega) \to \mathsf{W}^{1,\mathsf{p}}(\Omega)$ is linear continuous and coincides with the identity on the dense subspace $\mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)|_\Omega$, hence $R \circ \mathsf{E}$ is the identity. We then reach the thesis, i.e. E is an extension operator which uniquely extends E_Γ.

   We must therefore prove that there exists $C > 0$ such that, for each $f \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)$, $\mathsf{E}_\Gamma(f|_{\overline{\Omega}}) \in \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ and $\|\mathsf{E}_\Gamma(f|_{\overline{\Omega}})\|_{\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)} \leq C\|f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)}$; then $\mathsf{E}_\Gamma : \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)|_\Omega \to \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ is a well defined map and its linearity is clear, hence it is a bounded linear operator.
2) For all $x = (x', x_n)$, $y = (y', y_n) \in \mathbb{R}^n$, we have

$$\begin{aligned}\|\Phi_\Gamma(x) - \Phi_\Gamma(y)\| &= \big\|\big(x' - y', 2[\Gamma(x') - \Gamma(y')] - (x_n - y_n)\big)\big\| \leq \\ &\leq \|x' - y'\| + 2 \operatorname{Lip}\Gamma \|x' - y'\| + |x_n - y_n| \leq \\ &\leq 2(\operatorname{Lip}\Gamma + 1)\|x - y\|,\end{aligned}$$

hence Φ_Γ is Lipschitz with $\operatorname{Lip}\Phi_\gamma \leq 2(\operatorname{Lip}\Gamma + 1)$. It then follows from Rademacher's theorem 5.12 that Φ_Γ is $\mathcal{L}^n$-a.e. Fréchet-differentiable. Moreover, for all $x \in \mathsf{D}_{\Phi_\Gamma}$ (thus for $\mathcal{L}^n$-a.e. x in $\mathbb{R}^n$),

$$\begin{aligned} \|\mathsf{D}\Phi_\Gamma(x)\| &\leq \operatorname{Lip}\Phi_\Gamma \leq 2(\operatorname{Lip}\Gamma + 1) \\ \mathsf{J}\Phi_\Gamma(x) &\leq (\operatorname{Lip}\Phi_\Gamma)^n \leq 2^n(\operatorname{Lip}\Gamma + 1)^n. \end{aligned} \tag{6.15}$$

3) Fix $f \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)$. Since both f and Φ_Γ are Lipschitz, so is $f_\Gamma = f \circ \Phi_\Gamma$. It then follows that both $\mathsf{E}_\Gamma(f|_{\overline{\Omega}})|_{\overline{\Omega}} = f|_{\overline{\Omega}}$ and $\mathsf{E}_\Gamma(f|_{\overline{\Omega}})|_{\Omega^c} = f_\Gamma|_{\Omega^c}$ are Lipschitz. Hence $\mathsf{E}_\Gamma(f|_{\overline{\Omega}})$ is Lipschitz since, if $x \in \Omega$ and $y \in \overline{\Omega}^c$, the closed segment $[x, y]$, being connected, must intersect $\partial\Omega$ at some point z; therefore

$$\begin{aligned} &\left|\mathsf{E}_\Gamma(f|_{\overline{\Omega}})(x) - \mathsf{E}_\Gamma(f|_{\overline{\Omega}})(y)\right| \leq \\ &\quad\leq \left|\mathsf{E}_\Gamma(f|_{\overline{\Omega}})(x) - \mathsf{E}_\Gamma(f|_{\overline{\Omega}})(z)\right| + \left|\mathsf{E}_\Gamma(f|_{\overline{\Omega}})(z) - \mathsf{E}_\Gamma(f|_{\overline{\Omega}})(y)\right| \leq \\ &\quad(\operatorname{Lip} f)\|x - z\| + (\operatorname{Lip} f_\Gamma)\|z - y\| \leq \\ &\quad\leq \max\{\operatorname{Lip} f, \operatorname{Lip} f_\Gamma\}\big(\|x - z\| + \|z - y\|\big) = \\ &\quad= \max\{\operatorname{Lip} f, \operatorname{Lip} f_\Gamma\}\|x - y\|, \end{aligned}$$

from which we conclude that $\operatorname{Lip}\mathsf{E}_\Gamma(f|_{\overline{\Omega}}) \leq \max\{\operatorname{Lip} f, \operatorname{Lip} f_\Gamma\} < \infty$.

In particular, from Rademacher's theorem it follows that $\mathsf{E}_\Gamma(f|_{\overline{\Omega}})$ is $\mathcal{L}^n$-a.e. Fréchet-differentiable and its classical gradient coincides with its weak gradient $\mathcal{L}^n$-almost everywhere. Since $\partial\Omega = \operatorname{gr}\Gamma$ is $\mathcal{L}^n$-null, and since $\mathsf{E}_\Gamma(f|_{\overline{\Omega}})$ coincides with f on the open set Ω and with $f_\Gamma = f \circ \Phi_\Gamma$ on the open set $\overline{\Omega}^c$, we conclude that the weak gradient of $\mathsf{E}_\Gamma(f|_{\overline{\Omega}})$ is given $\mathcal{L}^n$-a.e. by

$$\nabla^\mathsf{w}\big[\mathsf{E}_\Gamma(f|_{\overline{\Omega}})\big](x) = \begin{cases} \nabla f(x) & \text{if } x \in \Omega, \\ \nabla f_\Gamma(x) = \mathsf{D}\Phi_\Gamma(x)^* \cdot \nabla f\big(\Phi_\Gamma(x)\big) & \text{if } x \in \overline{\Omega}^c \cap D_{\Phi_\Gamma}, \end{cases}$$

where $\mathsf{D}\Phi_\Gamma(x)^*$ denotes the adjoint of $\mathsf{D}\Phi_\Gamma(x)$ with respect to the standard inner product of $\mathbb{R}^n$. In particular, since $\forall x \in D_{\Phi_\Gamma}$, $\|\mathsf{D}\Phi_\Gamma(x)^*\| = \|\mathsf{D}\Phi_\Gamma(x)\| \leq 2(\operatorname{Lip}\Gamma + 1)$ by (6.15), it follows that

$$\left\|\nabla^\mathsf{w}\big[\mathsf{E}_\Gamma(f|_{\overline{\Omega}})\big]\right\| \leq \chi_\Omega \cdot \|\nabla f\| + \chi_{\overline{\Omega}^c} \cdot 2(\operatorname{Lip}\Gamma + 1)\|(\nabla f) \circ \Phi_\Gamma\| \tag{6.16}$$

$\mathcal{L}^n$-a.e. on $\mathbb{R}^n$.

4) We estimate the L^p norms of both $\mathsf{E}_\Gamma(f|_{\overline{\Omega}})$ and of its weak gradient:

(6.17)
$$\int_{\mathbb{R}^n} |\mathsf{E}_\Gamma(f|_{\overline{\Omega}})|^p \, \mathrm{d}\mathcal{L}^n = \int_\Omega |f|^p \, \mathrm{d}\mathcal{L}^n + \int_{\overline{\Omega}^c} |f \circ \Phi_\Gamma|^p \, \mathrm{d}\mathcal{L}^n \overset{\Phi_\Gamma = \Phi_\Gamma^{-1} \text{ and AF } 5.39.ii)}{=}$$
$$= \int_\Omega |f|^p \, \mathrm{d}\mathcal{L}^n + \int_\Omega |f|^p \, \mathsf{J}\Phi_\Gamma \, \mathrm{d}\mathcal{L}^n \overset{(6.15)}{\leq}$$
$$\leq [2^n(\operatorname{Lip}\Gamma + 1)^n + 1] \int_\Omega |f|^p \, \mathrm{d}\mathcal{L}^n.$$

Similarly, it follows from (6.16) that

(6.18)
$$\int_{\mathbb{R}^n} \left\| \nabla^{\mathsf{w}} \left[\mathsf{E}_\Gamma(f|_{\overline{\Omega}}) \right] \right\|^p \mathrm{d}\mathcal{L}^n \leq$$
$$\leq \int_\Omega \|\nabla f\|^p \, \mathrm{d}\mathcal{L}^n + 2^p(\operatorname{Lip}\Gamma + 1)^p \int_{\overline{\Omega}^c} \|(\nabla f) \circ \Phi_\Gamma\|^p \, \mathrm{d}\mathcal{L}^n \overset{\text{AF } 5.39.ii)}{=}$$
$$= \int_\Omega \|\nabla f\|^p \, \mathrm{d}\mathcal{L}^n + 2^p(\operatorname{Lip}\Gamma + 1)^p \int_\Omega \|\nabla f\|^p \, \mathsf{J}\Phi_\Gamma \, \mathrm{d}\mathcal{L}^n \overset{(6.15)}{\leq}$$
$$\leq [2^{n+p}(\operatorname{Lip}\Gamma + 1)^{n+p} + 1] \int_\Omega \|\nabla f\|^p \, \mathrm{d}\mathcal{L}^n.$$

From (6.17) and (6.18), we therefore conclude that $\mathsf{E}_\Gamma(f|_{\overline{\Omega}}) \in \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ and, with $C := [2^{n+p}(\operatorname{Lip}\Gamma + 1)^{n+p} + 1]^{1/p}$, $\|\mathsf{E}_\Gamma(f|_{\overline{\Omega}})\|_{\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)} \leq C\|f\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)}$.

□

Recall our convention from remark 1.57, i.e. we consider essential supports only.

Corollary 6.55 (Extension by reflection for Sobolev functions on Lipschitz epigraphs). *With the notation from the previous theorem, if $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$ and the closure of* spt f *in $\mathbb{R}^n$ is a compact subset of an open set $V \subset \mathbb{R}^n$, then* spt $\mathsf{E} f \Subset V \cup V_\Gamma$, *where $V_\Gamma = \Phi_\Gamma(V)$.*

Proof. Let $W \subset \mathbb{R}^n$ be a relatively compact open set such that $\overline{\operatorname{spt} f} \Subset W \Subset V$. We may take, by lemma 6.39, a sequence $(g_k)_{k\in\mathbb{N}}$ in $\mathsf{C}^\infty_\mathsf{c}(W)$ such that $g_k|_\Omega \to f$ in $\mathsf{W}^{1,\mathsf{p}}(\Omega)$. Then $\mathsf{E}\, g_k \to \mathsf{E}\, f$ in $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$. Since $\forall k \in \mathbb{N}$, $\mathsf{E}\, g_k = \mathsf{E}_\Gamma\, g_k$ has compact support in $W \cup W_\Gamma$, it follows that, for all $\varphi \in \mathsf{C}^\infty_\mathsf{c}(\overline{W \cup W_\Gamma}^c)$, $\int \mathsf{E}\, f\, \varphi \, \mathrm{d}\mathcal{L}^n = \lim_{k\to\infty} \int \mathsf{E}\, g_k\, \varphi \, \mathrm{d}\mathcal{L}^n = 0$. It then follows from the fundamental lemma of the Calculus of Variations 4.34 that $\mathsf{E}\, f = 0$ $\mathcal{L}^n$-a.e. on $\overline{W \cup W_\Gamma}^c$, so that spt $\mathsf{E}\, f \subset \overline{W \cup W_\Gamma} = \overline{W} \cup \overline{W}_\Gamma \Subset V \cup V_\Gamma$, as asserted. □

THEOREM 6.56 (Extension of Sobolev functions on Lipschitz domains). *Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ a Lipschitz domain with $\partial\Omega$ bounded and $1 \leq p < \infty$. Then there exists an extension operator $\mathsf{E}^\Omega : \mathsf{W}^{1,p}(\Omega) \to \mathsf{W}^{1,p}(\mathbb{R}^n)$. Moreover, if Ω is bounded and $V \subset \mathbb{R}^n$ is an open set such that $\Omega \Subset V$, we may choose E^Ω so that, for all $f \in \mathsf{W}^{1,p}(\Omega)$,* $\operatorname{spt}(\mathsf{E}^\Omega f) \Subset V$.

PROOF.

1) For each $x \in \partial\Omega$, there exists an open set $U_x \subset \mathbb{R}^n$ such that $x \in U_x$ and U_x is obtained by rigid motion of a cylinder centered at $0 \in \mathbb{R}^n$ as in definition 6.33, i.e. there exists a rigid motion $\Phi \in \mathrm{SE}(n)$ with $\Phi(0) = x$ and there exists $r, h > 0$ and $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz with $\Gamma(0) = 0$ such that $U_x = \Phi\big(\mathbb{C}(0, r, h)\big)$, $\Phi\big(\mathrm{gr}\ \Gamma \cap \mathbb{C}(0, r, h)\big) = U_x \cap \partial\Omega$ and $\Phi\big(\mathrm{epi}_{\mathsf{S}}\ \Gamma \cap \mathbb{C}(0, r, h)\big) = U_x \cap \Omega$. If Ω is bounded and $V \subset \mathbb{R}^n$ is an open set such that $\Omega \Subset V$, we may take smaller r and h so that $U_x \subset V$. Moreover, since Γ is continuous and $\Gamma(0) = 0$, taking smaller r if necessary we may assume that $|\Gamma(y)| < h/4$ for $y \in \mathbb{U}(0, r) \subset \mathbb{R}^{n-1}$; with that assumption, using the notation from the previous corollary, we have $\mathbb{C}(0, r/2, h/2) \cup \mathbb{C}(0, r/2, h/2)_\Gamma \subset \mathbb{C}(0, r, h)$. Let $W_x := \Phi\big(\mathbb{C}(0, r/2, h/2)\big) \Subset U_x$.
2) From the open cover $(W_x)_{x \in \partial\Omega}$ of the compact set $\partial\Omega$, we may extract a finite subcover $(W_i)_{1 \leq i \leq N}$. For each $1 \leq i \leq N$, let the corresponding objects defined in the previous item be denoted with a subscript i, so that $\Phi_i\big(\mathbb{C}(0, r_i/2, h_i/2)\big) = W_i$, $\Phi_i\big(\mathbb{C}(0, r_i, h_i)\big) = U_i$, $|\Gamma_i| < h_i/4$ on $\mathbb{U}(0, r_i) \subset \mathbb{R}^{n-1}$.

   Let $W_0 := \Omega$ and $W_{-1} := \overline{\Omega}^c$, so that $(W_i)_{-1 \leq i \leq N}$ is an open cover of $\mathbb{R}^n$. We may apply corollary 6.11 to obtain a smooth partition of unity $(\xi_i)_{-1 \leq i \leq N}$ of unity of $\mathbb{R}^n$ with $\operatorname{spt} \xi_i \subset W_i$ for $-1 \leq i \leq N$. Besides, for $1 \leq i \leq N$, as $\operatorname{spt} \xi_i \subset W_i \Subset \mathbb{R}^n$, it follows that $\operatorname{spt} \xi_i$ is a compact subset of W_i.

   We now define a sequence $(\mathsf{E}_i)_{0 \leq i \leq N}$ of bounded linear operators $\mathsf{W}^{1,p}(\Omega) \to \mathsf{W}^{1,p}(\mathbb{R}^n)$ whose sum will be the desired extension operator.
3) For $i = 0$. For each $f \in \mathsf{W}^{1,p}(\Omega)$, let $\mathsf{E}_0(f) := \xi_0 \cdot f$. We contend that $\xi_0 \in \mathsf{W}^{1,\infty}(\mathbb{R}^n)$. Indeed, $\xi_0 \in \mathsf{L}^\infty(\mathbb{R}^n)$ (because $0 \leq \xi_0 \leq 1$) and, since $\sum_{i=0}^N \xi_i \equiv 1$ on Ω, we have $\nabla\xi_0 = -\sum_{i=1}^N \nabla\xi_i$ on Ω, hence $\nabla\xi_0|_\Omega \in \mathsf{L}^\infty(\Omega, \mathbb{R}^n)$ (because $\xi_i \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n) \subset \mathsf{W}^{1,\infty}(\mathbb{R}^n)$ for $1 \leq i \leq N$). As $\operatorname{spt} \xi_0 \subset \Omega$, our contention is proved. Therefore, $\xi_0 \in \mathsf{C}^\infty(\mathbb{R}^n) \cap \mathsf{W}^{1,\infty}(\mathbb{R}^n)$, with $\operatorname{spt} \xi_0 \subset \Omega$; an application of lemma 6.35 yields $\mathsf{E}_0(f) = \xi_0 \cdot f \in \mathsf{W}^{1,p}(\mathbb{R}^n)$ and $\nabla^\mathsf{w}[\mathsf{E}_0(f)] = (\nabla\xi_0) \cdot f +$

$\xi_0 \cdot (\nabla^{\mathsf{w}} f)$. Hence

$$\begin{aligned}\|\mathsf{E}_0(f)\|_{\mathsf{L}^{\mathsf{p}}(\mathcal{L}^n)} &\le \|f\|_{\mathsf{L}^{\mathsf{p}}(\mathcal{L}^n|_\Omega)}\\ \big\|\nabla^{\mathsf{w}}[\mathsf{E}_0(f)]\big\|_{\mathsf{L}^{\mathsf{p}}(\mathcal{L}^n)} &\le \|\xi_0\|_{\mathsf{W}^{1,\infty}(\mathbb{R}^n)}\big(\|f\|_{\mathsf{L}^{\mathsf{p}}(\mathcal{L}^n|_\Omega)} + \|\nabla^{\mathsf{w}} f\|_{\mathsf{L}^{\mathsf{p}}(\mathcal{L}^n|_\Omega,\mathbb{R}^n)}\big),\end{aligned}$$

thus showing that $\mathsf{E}_0 : \mathsf{W}^{1,\mathsf{p}}(\Omega) \to \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ is a well defined bounded linear operator.

4) For $1 \le i \le N$. We define $\mathsf{E}_i : \mathsf{W}^{1,\mathsf{p}}(\Omega) \to \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ as the composite of the following sequence of continuous linear maps:

$$\mathsf{W}^{1,\mathsf{p}}(\Omega) \xrightarrow{L_{\xi_i}} \mathsf{W}^{1,\mathsf{p}}_{(\mathsf{C})}(W_i \cap \Omega) \xrightarrow{(\circ\Phi_i)} \mathsf{W}^{1,\mathsf{p}}_{(\mathsf{C})}\big(\mathbb{C}(0, r_i/2, h_i/2) \cap \operatorname{epi}_{\mathsf{S}} \Gamma_i\big) \xrightarrow{e_0}$$
$$\xrightarrow{e_0} \mathsf{W}^{1,\mathsf{p}}(\operatorname{epi}_{\mathsf{S}} \Gamma_i) \xrightarrow{\mathsf{E}} \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n) \xrightarrow{(\circ\Phi_i^{-1})} \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n),$$

where $\mathsf{W}^{1,\mathsf{p}}_{(\mathsf{C})}(W_i \cap \Omega) := \{f \in \mathsf{W}^{1,\mathsf{p}}(W_i \cap \Omega) \mid \overline{\operatorname{spt} f} \Subset W_i\}$ is a linear subspace of $\mathsf{W}^{1,\mathsf{p}}(W_i \cap \Omega)$, $\mathsf{W}^{1,\mathsf{p}}_{(\mathsf{C})}\big(\mathbb{C}(0, r_i/2, h_i/2) \cap \operatorname{epi}_{\mathsf{S}} \Gamma_i\big) := \{f \in \mathsf{W}^{1,\mathsf{p}}\big(\mathbb{C}(0, r_i/2, h_i/2) \cap \operatorname{epi}_{\mathsf{S}} \Gamma_i\big) \mid \overline{\operatorname{spt} f} \Subset \mathbb{C}(0, r_i/2, h_i/2)\}$ is a linear subspace of $\mathsf{W}^{1,\mathsf{p}}\big(\mathbb{C}(0, r_i/2, h_i/2) \cap \operatorname{epi}_{\mathsf{S}} \Gamma_i\big)$, and the linear maps are described below:

a) L_{ξ_i} is the multiplication by ξ_i. The fact that $\xi_i \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n)$ and the product rule 6.26 imply that, for each $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$, $\xi_i \cdot f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$ and $\operatorname{spt} \xi_i \cdot f \subset \operatorname{spt} \xi_i \Subset W_i$, so that $L_{\xi_i} : \mathsf{W}^{1,\mathsf{p}}(\Omega) \to \mathsf{W}^{1,\mathsf{p}}_{(\mathsf{C})}(W_i \cap \Omega)$ is a well-defined linear map. Since $\nabla^{\mathsf{w}}(\xi_i \cdot f) = (\nabla\xi_i) \cdot f + \xi_i \cdot \nabla^{\mathsf{w}} f$, we have

$$\begin{aligned}\|L_{\xi_i}(f)\|_{\mathsf{L}^{\mathsf{p}}(W_i\cap\Omega)} &\le \|\xi_i\|_u \|f\|_{\mathsf{L}^{\mathsf{p}}(\Omega)}\\ \big\|\nabla^{\mathsf{w}}[L_{\xi_i}(f)]\big\|_{\mathsf{L}^{\mathsf{p}}(W_i\cap\Omega,\mathbb{R}^n)} &\le \|\nabla\xi_i\|_u \|f\|_{\mathsf{L}^{\mathsf{p}}(\Omega)} + \|\xi_i\|_u \|\nabla^{\mathsf{w}} f\|_{\mathsf{L}^{\mathsf{p}}(\Omega,\mathbb{R}^n)},\end{aligned}$$

hence L_{ξ_i} is continuous.

b) By lemma 6.40, $(\circ\Phi_i) : \mathsf{W}^{1,\mathsf{p}}(W_i \cap \Omega) \to \mathsf{W}^{1,\mathsf{p}}\big(\mathbb{C}(0, r_i/2, h_i/2) \cap \operatorname{epi}_{\mathsf{S}} \Gamma_i\big)$ is a surjective linear isometry and maps $\mathsf{W}^{1,\mathsf{p}}_{(\mathsf{C})}(W_i \cap \Omega)$ onto $\mathsf{W}^{1,\mathsf{p}}_{(\mathsf{C})}\big(\mathbb{C}(0, r_i/2, h_i/2) \cap \operatorname{epi}_{\mathsf{S}} \Gamma_i\big)$, since $\operatorname{spt} f \circ \Phi_i = \Phi_i^{-1}(\operatorname{spt} f)$. Hence $(\circ\Phi_i) : \mathsf{W}^{1,\mathsf{p}}_{(\mathsf{C})}(W_i \cap \Omega) \to \mathsf{W}^{1,\mathsf{p}}_{(\mathsf{C})}\big(\mathbb{C}(0, r_i/2, h_i/2) \cap \operatorname{epi}_{\mathsf{S}} \Gamma_i\big)$ is a surjective linear isometry.

c) $e_0 : \mathsf{W}^{1,\mathsf{p}}_{(\mathsf{C})}\big(\mathbb{C}(0, r_i/2, h_i/2) \cap \operatorname{epi}_{\mathsf{S}} \Gamma_i\big) \to \mathsf{W}^{1,\mathsf{p}}(\operatorname{epi}_{\mathsf{S}} \Gamma_i)$ is the extension by 0. Note that, for each $f \in \mathsf{W}^{1,\mathsf{p}}_{(\mathsf{C})}\big(\mathbb{C}(0, r_i/2, h_i/2) \cap \operatorname{epi}_{\mathsf{S}} \Gamma_i\big)$,
   - $e_0(f) \in \mathsf{L}^{\mathsf{p}}(\operatorname{epi}_{\mathsf{S}} \Gamma_i)$ and

$$\|e_0(f)\|_{\mathsf{L}^{\mathsf{p}}(\operatorname{epi}_{\mathsf{S}} \Gamma_i)} = \|f\|_{\mathsf{L}^{\mathsf{p}}\left(\mathbb{C}(0, r_i/2, h_i/2) \cap \operatorname{epi}_{\mathsf{S}} \Gamma_i\right)};$$

- since $\text{epi}_{\mathsf{S}}\ \Gamma_i$ is the union of the open sets $\mathbb{C}(0, r_i/2, h_i/2) \cap \text{epi}_{\mathsf{S}}\ \Gamma_i$ and $\text{epi}_{\mathsf{S}}\ \Gamma_i \setminus \text{spt}\ f$, and since $e_0(f)$ has weak gradients on both open sets (as on the latter its restriction is null), by the locality of the weak derivative 6.13 we conclude that $e_0(f)$ has weak gradient given by $\nabla^{\mathsf{w}}[e_0(f)] = e_0(\nabla^{\mathsf{w}} f) \in \mathsf{L}^{\mathsf{p}}(\text{epi}_{\mathsf{S}}\ \Gamma_i, \mathbb{R}^n)$ and

$$\|\nabla^{\mathsf{w}}[e_0(f)]\|_{\mathsf{L}^{\mathsf{p}}(\text{epi}_{\mathsf{S}}\ \Gamma_i, \mathbb{R}^n)} = \|\nabla^{\mathsf{w}} f\|_{\mathsf{L}^{\mathsf{p}}\left(\mathbb{C}(0, r_i/2, h_i/2) \cap \text{epi}_{\mathsf{S}}\ \Gamma_i, \mathbb{R}^n\right)}.$$

We therefore conclude that e_0 is a well defined linear isometry into $\mathsf{W}^{1,\mathsf{p}}(\text{epi}_{\mathsf{S}}\ \Gamma_i)$.

d) $\mathsf{E} : \mathsf{W}^{1,\mathsf{p}}(\text{epi}_{\mathsf{S}}\ \Gamma_i) \to \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ is the extension by reflection with respect to Γ, cf. theorem 6.54, hence it is linear continuous.

e) $(\circ\Phi_i^{-1}) : \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n) \to \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ is a surjective linear isometry, cf. lemma 6.40.

5) It follows from the two previous items that $\mathsf{E}^{\Omega} := \sum_{i=0}^{N} \mathsf{E}_i$ is a well defined bounded linear operator $\mathsf{W}^{1,\mathsf{p}}(\Omega) \to \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$. We shall prove that (a) it is an extension operator, i.e. for each $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$, $(\mathsf{E}^{\Omega} f)|_{\Omega} = f$ and (b) in the case Ω bounded, for each $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$, $\text{spt}\,(\mathsf{E}^{\Omega} f) \Subset V$ (recall that V is given in the statement of the theorem).

Fix $1 \leq i \leq N$ and $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$. Since the closure in $\mathbb{R}^n$ of the support of $(\xi_i \cdot f) \circ \Phi_i \in \mathsf{W}^{1,\mathsf{p}}_{(\mathsf{C})}\big(\mathbb{C}(0, r_i/2, h_i/2) \cap \text{epi}_{\mathsf{S}}\ \Gamma_i\big)$ is a compact subset of $\mathbb{C}(0, r_i/2, h_i/2)$, and since $\mathbb{C}(0, r_i/2, h_i/2) \cup \mathbb{C}(0, r_i/2, h_i/2)_{\Gamma_i} \subset \mathbb{C}(0, r_i, h_i)$, cf. the end of part 1) of the proof, it follows from corollary 6.55 that $\text{spt}\ \mathsf{E}[(\xi_i \cdot f) \circ \Phi_i] \Subset \mathbb{C}(0, r_i, h_i)$. Thus, $\text{spt}\,(\mathsf{E}_i f) = \Phi_i\big(\text{spt}\ \mathsf{E}[(\xi_i \cdot f) \circ \Phi_i]\big) \Subset \Phi_i\big(\mathbb{C}(0, r_i, h_i)\big) = U_i$. We then conclude that:

- $\mathsf{E}_i f = 0$ $\mathcal{L}^n$-a.e. on $\Omega \setminus U_i$;
- if $x \in U_i \cap \Omega$, $\Phi_i^{-1}(x) \in \mathbb{C}(0, r_i, h_i) \cap \text{epi}_{\mathsf{S}}\ \Gamma_i$, hence $\mathsf{E}[(\xi_i \cdot f) \circ \Phi_i] \circ \Phi_i^{-1}(x) = (\xi_i \cdot f)) \circ \Phi_i \circ \Phi_i^{-1}(x) = (\xi_i \cdot f)(x)$.

We have thus proved that $(E_i f)|_{\Omega} = \xi_i \cdot f$ $\mathcal{L}^n$-a.e. on Ω. Therefore, $\mathcal{L}^n$-a.e. on Ω,

$$\begin{aligned}(\mathsf{E}^{\Omega} f)|_{\Omega} &= \sum_{i=0}^{N} (\mathsf{E}_i f)|_{\Omega} = \\ &= \sum_{i=0}^{N} \xi_i \cdot f = f,\end{aligned}$$

since $\sum_{i=0}^{N} \xi_i \equiv 1$ on Ω. That is, as elements of $\mathsf{W}^{1,\mathsf{p}}(\Omega)$, $(\mathsf{E}^{\Omega} f)|_{\Omega} = f$, as asserted. Furthermore, in the case Ω bounded, $\text{spt}\,(\mathsf{E}_0 f) =$

$\mathrm{spt}\,(\xi_0 \cdot f) \Subset \Omega \subset V$ and, for $1 \le i \le N$, $\mathrm{spt}\,(\mathsf{E}_i\, f) \Subset U_i \subset V$, whence $\mathrm{spt}\,(\mathsf{E}^{\Omega}\, f) \subset \cup_{i=0}^{N} \mathrm{spt}\,(\mathsf{E}_i\, f) \Subset V$, which concludes the proof.

□

REMARK 6.57. With the same proof and notation above, in the case in which Ω is unbounded and $V \subset \mathbb{R}^n$ is an open set which contains Ω, we may choose E^{Ω} so that, for all $f \in \mathsf{W}^{1,p}(\Omega)$, $\mathrm{spt}\,(\mathsf{E}^{\Omega}\, f) \subset V$, but not necessarily compact.

6.6. Sobolev Inequalities

In this section, for $1 \le p \le \infty$ we want to find continuous injections of $\mathsf{W}^{1,p}(\mathbb{R}^n)$ into $\mathsf{L}^{q}(\mathcal{L}^n)$ for some q. We divide the problem into cases: $1 \le p < n$, $n < p \le \infty$ and the limit case $p = n$.

6.6.1. Case $1 \le p < n$.

DEFINITION 6.58. Let $1 \le p < n$. We define the *Sobolev conjugate* exponent p^* to p by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n},$$

that is

$$p^* = \frac{np}{n-p}.$$

THEOREM 6.59 (Sobolev-Gagliardo-Nirenberg inequality). *Let $1 \le p < n$. Then there exists a constant $C = C(n,p)$ such that, for all $f \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)$,*

$$\|f\|_{p^*} \le C\|\nabla f\|_p.$$

REMARK 6.60. For $1 \le p < n$, there can exist only one $q \in [1, \infty]$ for which Sobolev's inequality holds: it is precisely the Sobolev conjugate to p. That can be deduced by a scaling argument: suppose that $q \in [1, \infty]$ and that for all $f \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)$, $\|f\|_q \le C\|\nabla f\|_p$ for some constant $C = C(n,p)$. Fix $f \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)$ and $\lambda > 0$. Then f_λ given by $x \mapsto f(\lambda x)$ belongs to $\mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)$ and $\nabla f_\lambda(x) = \lambda \nabla f(\lambda x)$, so that $\|f_\lambda\|_q = \lambda^{-n/q}\|f\|_q$ and $\|\nabla f_\lambda\|_p = \lambda^{1-n/p}\|\nabla f\|_p$. Therefore, $\|f_\lambda\|_q \le C\|\nabla(\lambda f)\|_p$ is equivalent to $\|f\|_q \le C\lambda^{1-n/p+n/q}\|\nabla f\|_p$. The latter inequality must hold for all $f \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)$ and $\lambda > 0$; if the exponent of λ is not 0, sending λ to 0 or to ∞ yields a contradiction. Hence $1 - n/p + n/q = 0$, i.e. $1/q = 1/p - 1/n$.

NOTATION. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. For $1 \le i \le n$, we denote by $\hat{x}_i \in \mathbb{R}^{n-1}$ the point obtained by deleting the i-th coordinate of x, i.e. $\hat{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$.

LEMMA 6.61. *Let $n \geq 2$ and $f_1, \dots, f_n : \mathbb{R}^{n-1} \to [0, \infty]$ Borelian functions on $\mathbb{R}^{n-1}$. Define $f : \mathbb{R}^n \to [0, \infty]$ by*

$$f(x) := \prod_{i=1}^{n} f_i(\hat{x}_i).$$

Then f is Borelian on $\mathbb{R}^n$ and

$$\|f\|_{\mathsf{L}^1(\mathcal{L}^n)} \leq \prod_{i=1}^{n} \|f_i\|_{\mathsf{L}^{n-1}(\mathcal{L}^{n-1})}.$$

In particular, $f \in \mathsf{L}^1(\mathcal{L}^n)$ if $f_i \in \mathsf{L}^{n-1}(\mathcal{L}^{n-1})$ for $1 \leq i \leq n$.

PROOF. For $1 \leq i \leq n$, let $\mathsf{pr}_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection $x \mapsto \hat{x}_i$, which is continuous, hence Borelian; then $f_i \circ \mathsf{pr}_i$ is Borelian, and so is the product $f = \prod_{i=1}^{n} f_i \circ \mathsf{pr}_i$.

We prove the asserted inequality by induction on n:

1) For $n = 2$,

$$\begin{aligned}\|f\|_1 &= \int f_1(x_2) f_2(x_1) \, \mathrm{d}\mathcal{L}^2(x_1, x_2) \overset{\text{Tonelli } 1.84}{=} \\ &= \int f_1(x_2) \, \mathrm{d}x_2 \cdot \int f_2(x_1) \, \mathrm{d}x_1 = \|f_1\|_1 \|f_2\|_1.\end{aligned}$$

2) Induction step. Suppose that the inequality holds for n. We identify $\mathbb{R}^{n+1} \equiv \mathbb{R}^n \times \mathbb{R}$ and use the notation $x = (x', x_{n+1})$ for $x \in \mathbb{R}^{n+1}$. Fix $x_{n+1} \in \mathbb{R}$. It follows from Hölder's inequality 1.73 that

$$\begin{aligned}\int f(x_1, \dots, x_n, x_{n+1}) \, \mathrm{d}\mathcal{L}^n(x_1, \dots, x_n) &= \int \prod_{i=1}^{n+1} f_i(\hat{x}_i) \, \mathrm{d}\mathcal{L}^n(x_1, \dots, x_n) \overset{\text{Hölder } 1.73}{\leq} \\ &\leq \|f_{n+1}\|_n \Big[\int \prod_{i=1}^{n} f_i(\hat{x}_i)^{n'} \, \mathrm{d}\mathcal{L}^n(x_1, \dots, x_n)\Big]^{1/n'},\end{aligned}$$

where $n' = \frac{n}{n-1}$ is the conjugate exponent of n. By the induction hypothesis with $f_i^{n'}(\cdot, x_{n+1})$ in place of f_i, we have:

$$\int \prod_{i=1}^{n} f_i(\hat{x}_i)^{n'} \, \mathrm{d}\mathcal{L}^n(x_1, \dots, x_n) \leq \prod_{i=1}^{n} [\int f_i(\hat{x}_i)^{n'(n-1)} \, \mathrm{d}\mathcal{L}^{n-1}(\widehat{x'}_i)]^{1/(n-1)}.$$

It then follows from the two previous equalities that

$$\int f(x_1, \dots, x_n, x_{n+1}) \, \mathrm{d}\mathcal{L}^n(x_1, \dots, x_n) \leq \|f_{n+1}\|_n \prod_{i=1}^{n} [\int f_i(\hat{x}_i)^n \, \mathrm{d}\mathcal{L}^{n-1}(\widehat{x'}_i)]^{1/n}.$$

The equality above holds for an arbitrarily fixed $x_{n+1} \in \mathbb{R}$. Therefore, integrating both members on x_{n+1} and applying Tonelli's theorem, we obtain:

$$\begin{aligned}
\int f \,\mathrm{d}\mathcal{L}^{n+1} &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} f(x_1, \dots, x_n, x_{n+1}) \,\mathrm{d}\mathcal{L}^n(x_1, \dots, x_n) \,\mathrm{d}\mathcal{L}^1(x_{n+1}) \le \\
&\le \int_{\mathbb{R}} \|f_{n+1}\|_n \prod_{i=1}^n [\int f_i(\hat{x}_i)^n \,\mathrm{d}\mathcal{L}^{n-1}(\widehat{x'}_i)]^{1/n} \,\mathrm{d}\mathcal{L}^1(x_{n+1}) = \\
&= \|f_{n+1}\|_n \int_{\mathbb{R}} \prod_{i=1}^n [\int f_i(\hat{x}_i)^n \,\mathrm{d}\mathcal{L}^{n-1}(\widehat{x'}_i)]^{1/n} \,\mathrm{d}\mathcal{L}^1(x_{n+1}) \overset{\text{gen. Hölder 1.74}}{\le} \\
&\le \|f_{n+1}\|_n \prod_{i=1}^n \Big[\int_{\mathbb{R}} \int_{\mathbb{R}^n} f_i(\hat{x}_i)^n \,\mathrm{d}\mathcal{L}^{n-1}(\widehat{x'}_i) \,\mathrm{d}\mathcal{L}^1(x_{n+1})\Big]^{1/n} \overset{\text{Tonelli}}{=} \\
&= \|f_{n+1}\|_n \prod_{i=1}^n \Big[\int f_i(\hat{x}_i)^n \,\mathrm{d}\mathcal{L}^n(\hat{x}_i)\Big]^{1/n} = \\
&= \prod_{i=1}^{n+1} \|f_i\|_n,
\end{aligned}$$

thus proving the induction step.

$\square$

Proof of theorem 6.59. Let $f \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n)$. For $1 \le i \le n$ and for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$:

$$f(x) = \int_{-\infty}^{x_i} \frac{\partial f}{\partial x_i}(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n) \,\mathrm{d}t,$$

hence

$$|f(x)| \le \int_{-\infty}^{\infty} \Big|\frac{\partial f}{\partial x_i}(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)\Big| \,\mathrm{d}t.$$

It then follows that

$$|f(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^n \underbrace{\Big[\int_{\mathbb{R}} \Big|\frac{\partial f}{\partial x_i}(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)\Big| \,\mathrm{d}t\Big]^{1/(n-1)}}_{=:f_i(\hat{x}_i)}.$$

We therefore conclude from lemma 6.61 that

$$\int |f|^{n/(n-1)}\,\mathrm{d}\mathcal{L}^n \leq \prod_{i=1}^n \Big[\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \Big|\frac{\partial f}{\partial x_i}(x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n)\Big|\,\mathrm{d}t\,\mathrm{d}\hat{x}_i\Big]^{1/(n-1)} \overset{\text{Tonelli}}{=}$$

$$= \prod_{i=1}^n \Big[\int \Big|\frac{\partial f}{\partial x_i}\Big|\,\mathrm{d}\mathcal{L}^n\Big]^{1/(n-1)} \leq$$

$$\leq \prod_{i=1}^n \Big[\int \|\nabla f\|\,\mathrm{d}\mathcal{L}^n\Big]^{1/(n-1)} = \Big[\int \|\nabla f\|\,\mathrm{d}\mathcal{L}^n\Big]^{n/(n-1)},$$

which proves the thesis for $p = 1$ with $C = 1$.

For $1 < p < \infty$, let $f \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)$ and $g := |f|^\gamma$, with $\gamma > 1$ to be chosen later. Note that $g \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n)$ and $\nabla g = \gamma \cdot \operatorname{sgn} f \cdot |f|^{\gamma-1} \cdot \nabla f$. We may therefore apply to g the inequality already proved, i.e. with $p = 1$, which yields

$$\Big(\int |f|^{\frac{\gamma n}{n-1}}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{n}} \leq \gamma \int |f|^{(\gamma-1)} \|\nabla f\|\,\mathrm{d}\mathcal{L}^n \overset{\text{Hölder}}{\leq}$$

$$\leq \gamma \Big(\int |f|^{\frac{(\gamma-1)p}{p-1}}\Big)^{\frac{p-1}{p}} \|\nabla f\|_p.$$

We choose γ satisfying

$$\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1},$$

i.e.

$$\gamma = \frac{(n-1)p}{n-p} > 1.$$

Then

$$p^* = \frac{np}{n-p} = \frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1}.$$

We then conclude that

$$\Big(\int |f|^{p^*}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{n}} \leq \gamma \Big(\int |f|^{p^*}\Big)^{\frac{p-1}{p}} \|\nabla f\|_p,$$

which yields the thesis with $C = \gamma = \gamma(n,p)$.

□

Corollary 6.62. *For $1 \leq p < n$, $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n) \subset \mathsf{L}^{\mathsf{p}^*}(\mathbb{R}^n)$ and the Sobolev-Gagliardo-Nirenberg inequality 6.59 holds for all $f \in \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$. In particular, the inclusion $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n) \subset \mathsf{L}^{\mathsf{p}^*}(\mathbb{R}^n)$ is continuous.*

PROOF. Let $f \in \mathsf{W}^{1,p}(\mathbb{R}^n)$. By corollary 6.21, there exists a sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ such that $f_k \to f$ in $f \in \mathsf{W}^{1,p}(\mathbb{R}^n)$. Passing to a subsequence, we may assume that $f_k \to f$ $\mathcal{L}^n$-almost everywhere. On the other hand, by theorem 6.59, for all $j,k \in \mathbb{N}$, $\|f_j - f_k\|_{p^*} \le C\|\nabla f_j - \nabla f_k\|_p \le C\|f_j - f_k\|_{\mathsf{W}^{1,p}(\mathbb{R}^n)}$. That is, $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathsf{L}^{p^*}(\mathcal{L}^n)$. Hence it is convergent in $\mathsf{L}^{p^*}(\mathcal{L}^n)$, and since it converges $\mathcal{L}^n$-a.e. fo f, we conclude that $f \in \mathsf{L}^{p^*}(\mathcal{L}^n)$ and $f_k \to f$ in $\mathsf{L}^{p^*}(\mathcal{L}^n)$. Therefore, since for all $k \in \mathbb{N}$, $\|f_k\|_{p^*} \le C\|\nabla f_k\|_p$, taking the limit as $k \to \infty$ in both members yields

$$\|f\|_{p^*} \le C\|\nabla^{\mathsf{w}} f\|_p,$$

as asserted. □

COROLLARY 6.63. *Let $1 \le p < n$ and $\Omega \subset \mathbb{R}^n$ a Lipschitz domain with $\partial\Omega$ bounded. Then $\mathsf{W}^{1,p}(\Omega) \subset \mathsf{L}^{p^*}(\mathcal{L}^n|_\Omega)$ with continuous inclusion.*

PROOF. Let $\mathsf{E} : \mathsf{W}^{1,p}(\Omega) \to \mathsf{W}^{1,p}(\mathbb{R}^n)$ be an extension operator, cf. theorem 6.56, and $C = C(n,p)$ given by the Sobolev-Gagliardo-Nirenberg inequality 6.59. Then, for all $f \in \mathsf{W}^{1,p}(\Omega)$,

$$\begin{aligned}\|f\|_{\mathsf{L}^{p^*}(\mathcal{L}^n|_\Omega)} \le \|\mathsf{E}\, f\|_{\mathsf{L}^{p^*}(\mathcal{L}^n)} \le&\\ &\le C\|\nabla^{\mathsf{w}}(\mathsf{E}\, f)\|_{\mathsf{L}^p(\mathcal{L}^n,\mathbb{R}^n)} \le C\|\mathsf{E}\, f\|_{\mathsf{W}^{1,p}(\mathbb{R}^n)} \le\\ &\le C\|\mathsf{E}\|\|f\|_{\mathsf{W}^{1,p}(\Omega)}.\end{aligned}$$

□

The Poincaré's inequality, proved below, is a kind of local version of the Sobolev-Gagliardo-Nirenberg inequality 6.59, 6.62.

LEMMA 6.64. *Let X, Y be metric spaces and $f : X \to Y$ bi-Lipschitz with $\operatorname{Lip} f^{-1} = (\operatorname{Lip} f)^{-1}$. Then, for all $s \ge 0$ and $A \subset X$, $\mathcal{H}^s\big(f(A)\big) = (\operatorname{Lip} f)^s\mathcal{H}^s(A)$, i.e. ${f^{-1}}_\#\mathcal{H}^s = (\operatorname{Lip} f)^s\mathcal{H}^s$.*

PROOF.

$$\begin{aligned}\mathcal{H}^s\big(f(A)\big) \le (\operatorname{Lip} f)^s\mathcal{H}^s(A) \le&\\ &\le (\operatorname{Lip} f)^s(\operatorname{Lip} f^{-1})^s\mathcal{H}^s\big(f(A)\big) = \mathcal{H}^s\big(f(A)\big).\end{aligned}$$

□

LEMMA 6.65. *For each $1 \le p < \infty$, there exists a constant $C = C(n,p)$ such that, for all $\mathbb{B}(x,r) \subset \mathbb{R}^n$, $f \in \mathsf{C}^1(\mathbb{R}^n)$ and $z \in \mathbb{B}(x,r)$,*

$$\int_{\mathbb{B}(x,r)} |f(y) - f(z)|^p \,\mathrm{d}y \le Cr^{n+p-1}\int_{\mathbb{B}(x,r)} \|\nabla f(y)\|^p\|y-z\|^{1-n}\,\mathrm{d}y.$$

PROOF. 1) For $y, z \in \mathbb{B}(x,r)$, we have

$$f(y)-f(z) = \int_0^1 \frac{d}{dt} f\big(z+t(y-z)\big)\,\mathrm{d}t = \int_0^1 \nabla f\big(z+t(y-z)\big)\cdot(y-z)\,\mathrm{d}t.$$

Then

$$\begin{aligned}\big|f(y)-f(z)\big|^p &\le |y-z|^p\Big(\int_0^1 \|\nabla f\big(z+t(y-z)\big)\,\mathrm{d}t\|\Big)^p \overset{\text{Hölder}}{\le} \\ &\le |y-z|^p \int_0^1 \|\nabla f\big(z+t(y-z)\big)\|^p\,\mathrm{d}t.\end{aligned}$$

For $s>0$, $\mathcal{H}^{n-1} \llcorner \partial\mathbb{B}(z,s)$ is a finite Radon measure - the trace $\mathcal{H}^{n-1}|_{\partial\mathbb{B}(z,s)}$ actually coincides with the usual Lebesgue measure of the sphere, cf. exercise 5.42. We may therefore apply Fubini-Tonelli's theorem to the product measure $\mathcal{L}^1 \otimes (\mathcal{H}^{n-1} \llcorner \partial\mathbb{B}(z,s))$ in equality $(*)$ of the following computation:

$$\begin{aligned}
&\int_{\mathbb{B}(x,r)\cap\partial\mathbb{B}(x,s)} \big|f(y)-f(z)\big|^p\,\mathrm{d}\mathcal{H}^{n-1}(y) \le \\
&\le \int_{\mathbb{B}(x,r)\cap\partial\mathbb{B}(x,s)} |y-z|^p \int_0^1 \|\nabla f\big(z+t(y-z)\big)\|^p\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{n-1}(y) \overset{(*)}{=} \\
&= s^p \int_0^1 \int_{\mathbb{B}(x,r)\cap\partial\mathbb{B}(x,s)} \|\nabla f\big(\underbrace{z+t(y-z)}_{=:g(y)}\big)\|^p\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}t = \\
&= s^p \int_0^1 \int \chi_{\mathbb{B}((1-t)z+tx,tr)\cap\partial\mathbb{B}(z,ts)} \circ g(y) \|(\nabla f)\circ g(y)\|^p\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}t = \\
&= s^p \int_0^1 \int \chi_{\mathbb{B}((1-t)z+tx,tr)\cap\partial\mathbb{B}(z,ts)} \|(\nabla f)\|^p\,\mathrm{d}(\underbrace{g_\#\mathcal{H}^{n-1}}_{=t^{1-n}\mathcal{H}^{n-1}\text{ by lemma 6.64}})\,\mathrm{d}t = \\
&= s^p \int_0^1 \frac{1}{t^{n-1}} \int_{\mathbb{B}((1-t)z+tx,tr)\cap\partial\mathbb{B}(z,ts)} \|\nabla f(w)\|^p\,\mathrm{d}\mathcal{H}^{n-1}(w)\,\mathrm{d}t \overset{\mathbb{B}((1-t)z+tx,tr)\subset\mathbb{B}(x,r)}{\le} \\
&\le s^p \int_0^1 \frac{1}{t^{n-1}} \int_{\mathbb{B}(x,r)\cap\partial\mathbb{B}(z,ts)} \|\nabla f(w)\|^p\,\mathrm{d}\mathcal{H}^{n-1}(w)\,\mathrm{d}t = \\
&= s^{n+p-1} \int_0^1 \frac{1}{(ts)^{n-1}} \int_{\mathbb{B}(x,r)\cap\partial\mathbb{B}(z,ts)} \|\nabla f(w)\|^p\,\mathrm{d}\mathcal{H}^{n-1}(w)\,\mathrm{d}t = \\
&= s^{n+p-1} \int_0^1 \int_{\mathbb{B}(x,r)\cap\partial\mathbb{B}(z,ts)} \|\nabla f(w)\|^p \|w-z\|^{1-n}\,\mathrm{d}\mathcal{H}^{n-1}(w)\,\mathrm{d}t.
\end{aligned}$$

The latter integral may now be computed by means of the coarea formula 5.50 with the Lipschitz map $\phi : \mathbb{R}^n \to \mathbb{R}$ given by

$$\phi(w) = \frac{\|w - z\|}{s}, \quad \mathsf{J}\phi = \frac{1}{s}\ \mathcal{L}^n\text{-q.s.}, \quad \phi^{-1}\{t\} = \partial\mathbb{B}(z, ts)$$

which yields, for all $s > 0$,

$$\begin{aligned}
&\int_{\mathbb{B}(x,r)\cap\partial\mathbb{B}(x,s)} \big|f(y) - f(z)\big|^p \, \mathrm{d}\mathcal{H}^{n-1}(y) \leq \\
&\qquad \leq s^{n+p-2} \int_{\mathbb{B}(x,r)\cap\mathbb{B}(z,s)} \|\nabla f(w)\|^p \|w - z\|^{1-n} \, \mathrm{d}\mathcal{L}^n(w) \leq \\
&\qquad \leq s^{n+p-2} \int_{\mathbb{B}(x,r)} \|\nabla f(w)\|^p \|w - z\|^{1-n} \, \mathrm{d}\mathcal{L}^n(w).
\end{aligned}$$

We now integrate both members of the inequality above from $s = 0$ to $s = 2r$, applying once more the coarea formula 5.52 for the integral of the first member, which yields

$$\begin{aligned}
&\int_{\underbrace{\mathbb{B}(x,r)\cap\mathbb{B}(z,2r)}_{=\mathbb{B}(x,r)}} \big|f(y) - f(z)\big|^p \, \mathrm{d}\mathcal{L}^n(y) \leq \\
&\qquad \leq \frac{(2r)^{n+p-1}}{n+p-1} \int_{\mathbb{B}(x,r)} \|\nabla f(w)\|^p \|w - z\|^{1-n} \, \mathrm{d}\mathcal{L}^n(w),
\end{aligned}$$

whence the thesis with

$$C = \frac{2^{n-p+1}}{n-p+1}.$$

□

NOTATION. For $f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n)$ we define the *average* $(f)_{x,r} = \fint_{\mathbb{B}(x,r)} f \, \mathrm{d}\mathcal{L}^n$ of f on $\mathbb{B}(x,r)$ by

$$(f)_{x,r} := \fint_{\mathbb{B}(x,r)} f \, \mathrm{d}\mathcal{L}^n := \frac{1}{\mathcal{L}^n\big(\mathbb{B}(x,r)\big)} \int f \, \mathrm{d}\mathcal{L}^n.$$

THEOREM 6.66 (Poincaré's inequality). *For $1 \leq p < n$, there exists a constant $C = C(n,p)$ such that, for all $\mathbb{B}(x,r)$ and $f \in \mathsf{W}^{1,\mathsf{p}}\big(\mathbb{U}(x,r)\big)$,*

$$\Big(\fint_{\mathbb{B}(x,r)} |f - (f)_{x,r}|^{p^*} \, \mathrm{d}\mathcal{L}^n\Big)^{1/p^*} \leq Cr\Big(\fint_{\mathbb{B}(x,r)} \|\nabla^{\mathsf{w}} f\|^p \, \mathrm{d}\mathcal{L}^n\Big)^{1/p}.$$

Equivalently,

$$\|f - (f)_{x,r}\|_{\mathsf{L}^{\mathsf{p}^*}(\mathbb{B}(x,r))} \leq C' \|\nabla^{\mathsf{w}} f\|_{\mathsf{L}^{\mathsf{p}^*}(\mathbb{B}(x,r))}$$

for some constant $C' = C'(n,p)$ (compare with the Sobolev-Gagliardo-Nirenberg inequality 6.59).

PROOF. 1) Let $f \in \mathsf{C}_\mathsf{c}^\infty(\mathbb{R}^n)$. For all $\mathbb{B}(x,r) \subset \mathbb{R}^n$, we compute

$$\begin{aligned}
&⨍_{\mathbb{B}(x,r)} |f-(f)_{x,r}|^p \,\mathrm{d}\mathcal{L}^n = \\
&\quad = ⨍_{\mathbb{B}(x,r)} \Big| ⨍_{\mathbb{B}(x,r)} [f(y)-f(z)]\,\mathrm{d}z \Big|^p \,\mathrm{d}y \overset{\text{Hölder}}{\leq} \\
&\quad \leq ⨍_{\mathbb{B}(x,r)} ⨍_{\mathbb{B}(x,r)} \big|f(y)-f(z)\big|^p \,\mathrm{d}z\,\mathrm{d}y \overset{\text{lemma 6.65}}{\leq} \\
&\quad \leq ⨍_{\mathbb{B}(x,r)} C\frac{r^{n+p-1}}{\alpha(n)r^n} \int_{\mathbb{B}(x,r)} \|\nabla f(z)\|^p \|y-z\|^{1-n} \,\mathrm{d}z\,\mathrm{d}y \overset{\text{Fubini}}{=} \\
&\quad = \frac{Cr^{p-1}}{\alpha(n)} \int_{\mathbb{B}(x,r)} \|\nabla f(z)\|^p ⨍_{\mathbb{B}(x,r)} \|y-z\|^{1-n} \,\mathrm{d}y\,\mathrm{d}z \leq \\
&\quad \leq 2Cnr^p ⨍_{\mathbb{B}(x,r)} \|\nabla f(z)\|^p \,\mathrm{d}z.
\end{aligned}$$

where, in the last inequality, we have estimated, for $z \in \mathbb{B}(x,r)$,

$$\begin{aligned}
\int_{\mathbb{B}(x,r)} \|y-z\|^{1-n}\,\mathrm{d}y \quad &\leq \int_{\mathbb{B}(z,2r)} \|y-z\|^{1-n}\,\mathrm{d}y = \\
&= n\alpha(n)\int_0^{2r} \rho^{1-n}\rho^{n-1}\,\mathrm{d}\rho = 2rn\alpha(n).
\end{aligned}$$

2) Claim: there exists a constant $C' = C'(n,p)$ such that, for all $g \in \mathsf{C}_\mathsf{c}^\infty(\mathbb{R}^n)$ and $\mathbb{B}(x,r) \subset \mathbb{R}^n$,

$$\Big(⨍_{\mathbb{B}(x,r)} |g|^{p^*}\,\mathrm{d}\mathcal{L}^n\Big)^{1/p^*} \leq C'\Big(r^p ⨍_{\mathbb{B}(x,r)} \|\nabla g\|^p\,\mathrm{d}\mathcal{L}^n + ⨍_{\mathbb{B}(x,r)} |g|^p\,\mathrm{d}\mathcal{L}^n\Big)^{1/p}.$$

Indeed:

a) For $x = 0$ and $r = 1$, we have, taking $\overline{g} := g|_{\mathbb{U}}(0,1)$ and $\mathsf{E} : \mathsf{W}^{1,p}(\mathbb{U}(0,1)) \to \mathsf{W}^{1,p}(\mathbb{R}^n)$ an extension operator, cf. theorem 6.56:

$$\begin{aligned}
\Big(\int_{\mathbb{B}(0,1)} |g|^{p^*}\,\mathrm{d}\mathcal{L}^n\Big)^{1/p^*} &\leq \Big(\int_{\mathbb{R}^n} |\mathsf{E}\,\overline{g}|^{p^*}\,\mathrm{d}\mathcal{L}^n\Big)^{1/p^*} \overset{\text{cor. 6.62}}{\leq} \\
&\leq C\Big(\int_{\mathbb{R}^n} \|\nabla^{\mathsf{w}}(\mathsf{E}\,\overline{g})\|^p\,\mathrm{d}y\Big)^{1/p} \leq \\
&\leq \underbrace{C\|\mathsf{E}\|}_{=:C'=C'(n,p)} \|\overline{g}\|_{\mathsf{W}^{1,p}(\mathbb{U}(0,1))} = \\
&= C'\Big(\int_{\mathbb{U}(0,1)} \|\nabla g\|^p + |g|^p\,\mathrm{d}\mathcal{L}^n\Big)^{1/p}
\end{aligned}$$

b) For arbitrary $x \in \mathbb{R}^n$ and $r > 0$, let $\widetilde{g} \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ be given by $\widetilde{g}(y) := g(ry + x)$, so that

$$\fint_{\mathbb{B}(0,1)} |\widetilde{g}|^{p^*} \,\mathrm{d}\mathcal{L}^n = \fint_{\mathbb{B}(x,r)} |g|^{p^*} \,\mathrm{d}\mathcal{L}^n$$

$$\fint_{\mathbb{B}(0,1)} |\widetilde{g}|^{p} \,\mathrm{d}\mathcal{L}^n = \fint_{\mathbb{B}(x,r)} |g|^{p} \,\mathrm{d}\mathcal{L}^n$$

$$\begin{aligned}\fint_{\mathbb{B}(0,1)} \|\nabla\widetilde{g}(y)\|^p \,\mathrm{d}\mathcal{L}^n(y) &= \fint_{\mathbb{B}(0,1)} \|r\nabla g(ry+x)\|^p \,\mathrm{d}\mathcal{L}^n(y) = \\ &= r^p \fint_{\mathbb{B}(x,r)} \|\nabla g\|^p \,\mathrm{d}\mathcal{L}^n,\end{aligned}$$

hence the claim follows from part a) applied to $\widetilde{g}$ in place of g.

3) Applying part 2) to $g := f - (f)_{x,r} \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}&\Big(\fint_{\mathbb{B}(x,r)} |f-(f)_{x,r}|^{p^*} \,\mathrm{d}\mathcal{L}^n\Big)^{1/p^*} \le \\ &\le C'\Big(r^p \fint_{\mathbb{B}(x,r)} \|\nabla f\|^p \,\mathrm{d}\mathcal{L}^n + \fint_{\mathbb{B}(x,r)} |f-(f)_{x,r}|^p \,\mathrm{d}\mathcal{L}^n\Big)^{1/p} \overset{\text{by 1)}}{\le} \\ &\le C'\Big(r^p \fint_{\mathbb{B}(x,r)} \|\nabla f\|^p \,\mathrm{d}\mathcal{L}^n + 2Cnr^p \fint_{\mathbb{B}(x,r)} \|\nabla f\|^p \,\mathrm{d}\mathcal{L}^n\Big)^{1/p} \\ &\le \underbrace{\big(C' + C'(2Cn)^{1/p}\big)}_{=C(n,p)} r\Big(\fint_{\mathbb{B}(x,r)} \|\nabla f\|^p \,\mathrm{d}\mathcal{L}^n\Big)^{1/p},\end{aligned}$$

thus reaching the thesis for $f \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$.

4) Let $x \in \mathbb{R}^n$, $r > 0$ and $f \in \mathsf{W}^{1,\mathsf{p}}\big(\mathbb{U}(x,r)\big)$. By corollary 6.43, there exists a sequence $(f_i)_{i\in\mathbb{N}}$ in $\mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ such that $f_i \to f$ in $\mathsf{W}^{1,\mathsf{p}}\big(\mathbb{U}(x,r)\big)$. For each $i \in \mathbb{N}$, it follows from the previous step of the proof that

$$\Big(\fint_{\mathbb{B}(x,r)} |f_i-(f_i)_{x,r}|^{p^*} \,\mathrm{d}\mathcal{L}^n\Big)^{1/p^*} \le Cr\Big(\fint_{\mathbb{B}(x,r)} \|\nabla f_i\|^p \,\mathrm{d}\mathcal{L}^n\Big)^{1/p}.$$

As $i \to \infty$, the second member of the inequality above has limit $Cr\Big(\fint_{\mathbb{B}(x,r)} \|\nabla^\mathsf{w} f\|^p \,\mathrm{d}\mathcal{L}^n\Big)^{1/p}$, since $\nabla f_i \to \nabla^\mathsf{w} f$ in $\mathsf{L}^\mathsf{p}\big(\mathbb{U}(x,r),\mathbb{R}^n\big)$. We contend that the first member has limit $\Big(\fint_{\mathbb{B}(x,r)} |f-(f)_{x,r}|^{p^*} \,\mathrm{d}\mathcal{L}^n\Big)^{1/p^*}$, whence the thesis.

Indeed, applying the previous inequality with $f_i - f_j$ in place of f_i, we conclude that the sequence $\{f_i - (f_i)_{x,r}\}_{i\in\mathbb{N}}$ is Cauchy in $\mathsf{L}^{\mathsf{p}^*}\big(\mathbb{U}(x,r)\big)$, hence convergent in that space. Its limit must be $f - (f)_{x,r}$ because

$(f_i)_{x,r} \to (f)_{x,r}$ and, passing to a subsequence if necessary, $f_i \to f$ $\mathcal{L}^n$-a.e., hence $f_i - (f_i)_{x,r} \to f - (f)_{x,r}$ $\mathcal{L}^n$-a.e., thus proving our contention. $\square$

6.6.2. Case $n < p$.

DEFINITION 6.67 (Hölder spaces). Let Ω be an open subset of $\mathbb{R}^n$ and $0 < \gamma \leq 1$. We say that $f : \Omega \to \mathbb{R}$ is *Hölder continuous with exponent* γ on Ω if there exists a constant $C \geq 0$ such that, for all $x, y \in \Omega$,

$$|f(x) - f(y)| \leq C\|x - y\|^\gamma.$$

Such functions form a linear subspace of $\mathbb{R}^\Omega$. We shall denote by $\mathsf{C}^{0,\gamma}(\overline{\Omega})$ the linear subspace of $\mathbb{R}^\Omega$ of all bounded Hölder continuous functions on Ω.

Note that, for $\gamma = 1$, the definition above is equivalent to f being Lipschitz.

DEFINITION 6.68 (Hölder seminorm). Let Ω be an open subset of $\mathbb{R}^n$, $0 < \gamma \leq 1$ and $f : \Omega \to \mathbb{R}^n$. We define the $\mathsf{C}^{0,\gamma}$ *seminorm of* f by

$$[f]_{\mathsf{C}^{0,\gamma}(\overline{\Omega})} := \sup\{\frac{|f(x) - f(y)|}{\|x - y\|^\gamma} \mid x \neq y \in \Omega\} \in [0, \infty].$$

With the notation above, note that f is Hölder continuous with exponent γ on Ω if, and only if, $[f]_{\mathsf{C}^{0,\gamma}(\overline{\Omega})} < \infty$.

PROPOSITION 6.69 (Hölder spaces are Banach). *Let Ω be an open subset of $\mathbb{R}^n$ and $0 < \gamma \leq 1$. Then $\mathsf{C}^{0,\gamma}(\overline{\Omega})$ is a Banach space endowed with the norm*

$$\|f\|_{\mathsf{C}^{0,\gamma}(\overline{\Omega})} := \|f\|_u + [f]_{\mathsf{C}^{0,\gamma}(\overline{\Omega})}$$

DEFINITION 6.70. Let Ω be an open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$. We say that $f^* : \Omega \to \mathbb{R}$ is a *version of* f if $f^* = f$ $\mathcal{L}^n$-a.e. on Ω.

THEOREM 6.71 (Morrey's inequality). *Fix $n < p < \infty$.*

i) There exists a constant $C = C(n, p)$ such that, for all $\mathbb{B}(x, r) \subset \mathbb{R}^n$ and all $f \in \mathsf{W}^{1,\mathsf{p}}\big(\mathbb{U}(x, r)\big)$,

$$|f(y) - f(z)| \leq Cr\Big(\fint_{\mathbb{B}(x,r)} \|\nabla^{\mathsf{w}} f\|^p \, \mathrm{d}\mathcal{L}^n\Big)^{1/p} \tag{6.19}$$

for $\mathcal{L}^n$-a.e. y, z in $\mathbb{U}(x, r)$.

ii) If $f \in \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$, then the limit

$$f^*(x) := \lim_{r \to 0} (f)_{x,r}$$

exists for every $x \in \mathbb{R}^n$ and f^* is a Hölder continuous version of f with exponent $\gamma = 1 - n/p$, with

$$[f^*]_{\mathsf{C}^{0,\gamma}(\mathbb{R}^n)} \leq C \|\nabla^{\mathsf{w}} f\|_{\mathsf{L}^p(\mathbb{R}^n)},$$

where $C = C(n,p)$ is the constant from part i).

Remark 6.72. See theorem 6.44 for the case $p = \infty$.

Proof.

1) Let $f \in \mathsf{C}^1(\mathbb{R}^n)$. Taking $C = C(n,1)$ given by lemma 6.65 with $p = 1$, we compute, for all $\mathbb{B}(x,r) \subset \mathbb{R}^n$ and $y, z \in \mathbb{U}(x,r)$,

$$\begin{aligned}
|f(y) - f(z)| &= \fint_{\mathbb{B}(x,r)} |f(y) - f(z)| \,\mathrm{d}w \leq \\
&\leq \fint_{\mathbb{B}(x,r)} \big(|f(y) - f(w)| + |f(z) - f(w)|\big) \,\mathrm{d}w \overset{6.65}{\leq} \\
&\leq \frac{C}{\alpha(n) r^n} r^n \int_{\mathbb{B}(x,r)} \|\nabla f(w)\| \big(\|y - w\|^{1-n} + \|z - w\|^{1-n}\big) \,\mathrm{d}w \overset{\text{Hölder}}{\leq} \\
&\leq \frac{C}{\alpha(n)} \Big(\int_{\mathbb{B}(x,r)} \big(\|y - w\|^{1-n} + \|z - w\|^{1-n}\big)^{\frac{p}{p-1}} \,\mathrm{d}w\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{B}(x,r)} \|\nabla f\|^p \,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}}.
\end{aligned}$$

Since

$$\begin{aligned}
&\Big(\int_{\mathbb{B}(x,r)} \big(\|y - w\|^{1-n} + \|z - w\|^{1-n}\big)^{\frac{p}{p-1}} \,\mathrm{d}w\Big)^{\frac{p-1}{p}} \overset{\text{Minkowski}}{\leq} \\
&\leq \Big(\int_{\mathbb{B}(x,r)} \big(\|y - w\|^{1-n}\big)^{\frac{p}{p-1}} \,\mathrm{d}w\Big)^{\frac{p-1}{p}} + \Big(\int_{\mathbb{B}(x,r)} \big(\|z - w\|^{1-n}\big)^{\frac{p}{p-1}} \,\mathrm{d}w\Big)^{\frac{p-1}{p}} \leq \\
&\leq \Big(\int_{\mathbb{B}(y,2r)} \big(\|y - w\|^{1-n}\big)^{\frac{p}{p-1}} \,\mathrm{d}w\Big)^{\frac{p-1}{p}} + \Big(\int_{\mathbb{B}(z,2r)} \big(\|z - w\|^{1-n}\big)^{\frac{p}{p-1}} \,\mathrm{d}w\Big)^{\frac{p-1}{p}} \leq \\
&\leq 2\Big(\alpha(n) \int_0^{2r} \underbrace{\rho^{n-1} \rho^{\frac{(1-n)p}{p-1}}}_{=\rho^{\frac{1-n}{p-1}}} \,\mathrm{d}\rho\Big)^{\frac{p-1}{p}} \overset{p \geq n}{=} \\
&= 2\alpha(n)^{\frac{p-1}{p}} \Big(\frac{(2r)^{\frac{-n+p}{p-1}}}{\frac{-n+p}{p-1}}\Big)^{\frac{p-1}{p}} = \\
&= C(n,p) r^{\frac{-n+p}{p}},
\end{aligned}$$

we obtain

$$|f(y)-f(z)| \le C(n,p)r^{1-n/p}\Big(\int_{\mathbb{B}(x,r)} \|\nabla f\|^p \,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}} = \\ = C(n,p)r\Big(\fint_{\mathbb{B}(x,r)} \|\nabla f\|^p \,\mathrm{d}\mathcal{L}^n\Big)^{1/p}$$

thus proving part i) for $f \in \mathsf{C}^1(\mathbb{R}^n)$.

2) Let $x \in \mathbb{R}^n$, $r > 0$ and $f \in \mathsf{W}^{1,\mathsf{p}}\big(\mathbb{U}(x,r)\big)$. By corollary 6.43, there exists a sequence $(f_i)_{i\in\mathbb{N}}$ in $\mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ such that $f_i \to f$ in $\mathsf{W}^{1,\mathsf{p}}\big(\mathbb{U}(x,r)\big)$. Passing to a subsequence, if necessary, we may assume that $f_i \to f$ on the complement of a $\mathcal{L}^n$-null set $N \subset \mathbb{U}(x,r)$. With $C = C(n,p)$ obtained in part 1), we have, for all $i \in \mathbb{N}$ and all $y,z \in \mathbb{U}(x,r)$,

$$|f_i(y)-f_i(z)| \le Cr\Big(\fint_{\mathbb{B}(x,r)} \|\nabla f_i\|^p \,\mathrm{d}\mathcal{L}^n\Big)^{1/p}.$$

Taking $i \to \infty$, it follows that (6.19) holds for all $y,z \in \mathbb{U}(x,r)\setminus N$, which concludes the proof of part i).

3) Let $f \in \mathsf{C}^1(\mathbb{R}^n)$ and $x \neq y$ in $\mathbb{R}^n$. We take $r = \|x-y\|$ and $C = C(n,p)$ obtained in step 1 of the proof, which yields the estimate

$$|f(x)-f(y)| \le C\|x-y\|^{1-n/p}\Big(\int_{\mathbb{B}(x,r)} \|\nabla f\|^p \,\mathrm{d}\mathcal{L}^n\Big)^{1/p} \le \\ \le C\|\nabla f\|_{\mathsf{L}^\mathsf{p}(\mathcal{L}^n,\mathbb{R}^n)}\|x-y\|^{1-n/p}.$$

4) Let $f \in \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$. By corollary 6.21, there exists a sequence $(f_i)_{i\in\mathbb{N}}$ in $\mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ such that $f_i \to f$ in $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$. Passing to a subsequence, if necessary, we may assume that $f_i \to f$ on the complement of a $\mathcal{L}^n$-null set $N \subset \mathbb{R}^n$. By the previous step, for each $i \in \mathbb{N}$ and $x \neq y$ in $\mathbb{R}^n$, we have

$$|f_i(x)-f_i(y)| \le C\|\nabla f_i\|_{\mathsf{L}^\mathsf{p}(\mathcal{L}^n,\mathbb{R}^n)}\|x-y\|^{1-n/p}.$$

Therefore, taking $i \to \infty$ in the previous equality, we conclude that, for $x,y \in \mathbb{R}^n \setminus N$,

$$|f(x)-f(y)| \le C\|\nabla^\mathsf{w} f\|_{\mathsf{L}^\mathsf{p}(\mathcal{L}^n,\mathbb{R}^n)}\|x-y\|^{1-n/p}. \tag{6.20}$$

In particular, f is uniformly continuous on $\mathbb{R}^n \setminus N$, which is dense in $\mathbb{R}^n$ because N is $\mathcal{L}^n$-null (thus it has empty interior). Hence $f|_{\mathbb{R}^n\setminus N}$ may be extended to a continuous function $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$, which therefore coincides with f $\mathcal{L}^n$-a.e. on $\mathbb{R}^n$, i.e. $\widetilde{f}$ is a version of

f. By continuity, $\widetilde{f}$ satisfies (6.20) for all $x, y \in \mathbb{R}^n$, i.e. $\widetilde{f}$ is Holder continuous with exponent $\gamma = 1 - 1/n$ and

$$[\widetilde{f}]_{\mathsf{C}^{0,\gamma}(\mathbb{R}^n)} \leq C \|\nabla^{\mathsf{w}} f\|_{\mathsf{L}^{\mathsf{p}}(\mathbb{R}^n)}.$$

Finally, for all $x \in \mathbb{R}^n$ and all $r > 0$, $(f)_{x,r} = (\widetilde{f})_{x,r}$, because $f = \widetilde{f}$ $\mathcal{L}^n$ almost everywhere. Since $\widetilde{f}$ is continuous, it follows that $\exists \lim_{r\to 0}(f)_{x,r} = \lim_{r \to 0}(\widetilde{f})_{x,r} = \widetilde{f}(x)$, i.e. $f^* = \widetilde{f}$, which concludes the proof of part ii).

□

REMARK 6.73. With the notation from the previous theorem, for $n < p$ the map $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n) \to \mathsf{C}(\mathbb{R}^n)$ given by $f \mapsto f^*$ is injective: if $f^* = g^*$, then $f = g$ $\mathcal{L}^n$-a.e., hence they represent the same equivalence class in $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$. Thus, identifying each element f of $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ with its continuous version f^*, we obtain an inclusion $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n) \subset \mathsf{C}(\mathbb{R}^n)$. We shall see in the next corollary that we actually have a continuous inclusion $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n) \subset \mathsf{C}^{0,1-\mathsf{n}/\mathsf{p}}(\mathbb{R}^n)$.

COROLLARY 6.74. *If $n < p < \infty$, then $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n) \subset \mathsf{C}^{0,\gamma}(\mathbb{R}^n)$, where $\gamma = 1 - n/p$, with continuous inclusion.*

PROOF.

1) Let $f \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ and fix $x \in \mathbb{R}^n$. Taking $C = C(n,1)$ given by lemma 6.65 with $p = 1$, we compute,

$$\begin{aligned}
|f(x)| &\leq \fint_{\mathbb{B}(x,1)} |f(x) - f(y)| \, \mathrm{d}y + \fint_{\mathbb{B}(x,1)} |f(y)| \, \mathrm{d}y \overset{6.65}{\leq} \\
&\leq \frac{C}{\alpha(n)} \int_{\mathbb{B}(x,1)} \|\nabla f(y)\| \|y - x\|^{1-n} \, \mathrm{d}y + C(n,p) \|f\|_{\mathsf{L}^{\mathsf{p}}\left(\mathbb{B}(x,1)\right)} \overset{\text{Hölder}}{\leq} \\
&\leq C(n,p) \|\nabla f\|_{\mathsf{L}^{\mathsf{p}}(\mathbb{R}^n)} \underbrace{\left(\int_{\mathbb{B}(x,1)} \|y - x\|^{\frac{(1-n)p}{p-1}} \, \mathrm{d}y \right)^{\frac{p-1}{p}}}_{=C(n,p)<\infty, \text{ since } \frac{(1-n)p}{p-1} > -n} + C(n,p) \|f\|_{\mathsf{L}^{\mathsf{p}}(\mathbb{R}^n)} \leq \\
&\leq C(n,p) \|f\|_{\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)}.
\end{aligned}$$

Taking the sup over $x \in \mathbb{R}^n$ on the first member of the above inequality, it follows that f is bounded and

$$\|f\|_u \leq C(n,p) \|f\|_{\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)}.$$

2) Let $f \in \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$. By corollary 6.21, there exists a sequence $(f_i)_{i \in \mathbb{N}}$ in $\mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ such that $f_i \to f$ in $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$. Passing to a subsequence, if necessary, we may assume that $f_i \to f$ on the complement of a

$\mathcal{L}^n$-null set $N \subset \mathbb{R}^n$. By the previous step, for each $i \in \mathbb{N}$ and x in $\mathbb{R}^n$, we have

$$|f_i(x)| \le C(n,p)\|f_i\|_{\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)}.$$

Therefore, taking $i \to \infty$ in the previous equality, we conclude that, for $x \in \mathbb{R}^n \setminus N$,

$$|f(x)| \le C(n,p)\|f\|_{\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)}.$$

In view of part ii) of theorem 6.71, we therefore conclude that f^* is bounded and Hölder continuous with exponent $\gamma = 1 - n/p$, i.e. $f^* \in \mathsf{C}^{0,\gamma}(\mathbb{R}^n)$, with

$$\|f^*\|_{\mathsf{C}^{0,\gamma}(\mathbb{R}^n)} \le C(n,p)\|f\|_{\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)}.$$

$\square$

Corollary 6.75. *If $n < p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain with $\partial\Omega$ bounded, then $\mathsf{W}^{1,\mathsf{p}}(\Omega) \subset \mathsf{C}^{0,\gamma}(\overline{\Omega})$, where $\gamma = 1 - n/p$, with continuous inclusion.*

Proof. Let $\mathsf{E} : \mathsf{W}^{1,\mathsf{p}}(\Omega) \to \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ be an extension operator, cf. theorem 6.56. The inclusion $\mathsf{W}^{1,\mathsf{p}}(\Omega) \subset \mathsf{C}^{0,\gamma}(\overline{\Omega})$ is the composite of the following sequence of continuous linear maps:

$$\mathsf{W}^{1,\mathsf{p}}(\Omega) \xrightarrow{\mathsf{E}} \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n) \to \mathsf{C}^{0,\gamma}(\mathbb{R}^n) \to \mathsf{C}^{0,\gamma}(\overline{\Omega}),$$

where the last arrow is the restriction $f \mapsto f|_\Omega$ and the middle arrow is the inclusion from the previous corollary. $\square$

6.7. Compactness

Lemma 6.76. *Let $1 \le p < n$ and $1 \le q < p^*$, where p^* is the Sobolev conjugate of p. Let $(f_i)_{i\in\mathbb{N}}$ be a bounded sequence in $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$. Suppose that there is a relatively compact open set $V \subset \mathbb{R}^n$ such that, for all $i \in \mathbb{N}$, $\operatorname{spt} f_i \Subset V$. Then there exists a subsequence $(f_{i_k})_{k\in\mathbb{N}}$ of $(f_i)_i$ which is convergent in $\mathsf{L}^\mathsf{q}(\mathcal{L}^n)$.*

Note that, for $1 \le q \le p^*$ and for each $i \in \mathbb{N}$, $f_i \in \mathsf{L}^\mathsf{q}(\mathcal{L}^n)$, in view of Sobolev-Gagliardo-Nirenberg inequality 6.62 and of the fact that $\operatorname{spt} f_i \Subset V$. However, we cannot ensure the existence of a subsequence as in the statement of the lemma for $q = p^*$.

We give two proofs for this lemma.

Proof 1. Let $(\phi_\epsilon)_{\epsilon>0}$ be the standard mollifier on $\mathbb{R}^n$. For each $\epsilon > 0$ and $i \in \mathbb{N}$, we define

$$f_i^\epsilon := \phi_\epsilon * f_i \in \mathsf{C}_\mathsf{c}^\infty(\mathbb{R}^n).$$

Substituting V with $V + \mathbb{U}(0,1)$, we may assume that, for all $i \in \mathbb{N}$ and for all $0 < \epsilon < 1$,

$$\text{spt } f_i^\epsilon \Subset V.$$

1) Claim 1: $f_i^\epsilon \to f_i$ as $\epsilon \to 0$ on $\mathsf{L}^\mathsf{q}(\mathcal{L}^n)$, uniformly on $i \in \mathbb{N}$.
   Indeed:
   a) Fix $0 < \epsilon < 1$. For each $i \in \mathbb{N}$, by corollary 6.43 we may take $g_i \in \mathsf{C}_\mathsf{c}^\infty(\mathbb{R}^n)$ such that $\|f_i - g_i\|_{\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)} < \epsilon$. Moreover, since spt $f_i \Subset V$, we may assume spt $g_i \Subset V$.
   For each $i \in \mathbb{N}$ and $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
g_i * \phi_\epsilon(x) - g_i(x) &= \int_{\mathbb{B}(0,\epsilon)} [g_i(x-y) - g_i(x)]\phi_\epsilon(y)\,\mathrm{d}y \overset{z=\epsilon^{-1}y}{=} \\
&= \int_{\mathbb{B}(0,1)} [g_i(x-\epsilon z) - g_i(x)]\phi(z)\,\mathrm{d}z = \\
&= \int_{\mathbb{B}(0,1)} \phi(z) \int_0^1 \frac{d}{dt}\,[g_i(x - t\epsilon z)]\,\mathrm{d}t\,\mathrm{d}z = \\
&= -\int_{\mathbb{B}(0,1)} \phi(z) \int_0^1 \nabla g_i(x - t\epsilon z)\cdot \epsilon z\,\mathrm{d}t\,\mathrm{d}z.
\end{aligned}$$

Thus

$$\begin{aligned}
\|g_i^\epsilon - g_i\|_1 &= \int_{\mathbb{R}^n} |g_i^\epsilon(x) - g_i(x)|\,\mathrm{d}x \le \\
&\le \epsilon \int_{\mathbb{R}^n}\int_{\mathbb{B}(0,1)} \phi(z) \int_0^1 \|\nabla g_i(x - t\epsilon z)\|\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}x \overset{\text{Tonelli}}{=} \\
&= \epsilon \int_{\mathbb{B}(0,1)} \phi(z) \int_0^1 \int_{\mathbb{R}^n} \|\nabla g_i(x - t\epsilon z)\|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}x = \\
&= \epsilon \int_{\mathbb{R}^n} \|\nabla g_i\|\,\mathrm{d}\mathcal{L}^n \overset{\text{spt } g_i \subset V}{=} \epsilon \int_V \|\nabla g_i\|\,\mathrm{d}\mathcal{L}^n \overset{\text{Hölder}}{\le} \\
&\le \mathcal{L}^n(V)^{\frac{p-1}{p}} \epsilon \|\nabla g_i\|_p \le \\
&\le \mathcal{L}^n(V)^{\frac{p-1}{p}} \epsilon\big(\|\nabla f_i\|_p + \epsilon\big).
\end{aligned}$$

b) Hence, for each $i \in \mathbb{N}$, we have

$$\begin{aligned}\|f_i^\epsilon - f_i\|_1 &\le \|f_i^\epsilon - g_i^\epsilon\|_1 + \|g_i^\epsilon - g_i\|_1 + \|g_i - f_i\|_1 \overset{\text{Hölder, Young 1.108.g) and a)}}{\le} \\ &\le 2\mathcal{L}^n(V)^{\frac{p-1}{p}}\|f_i - g_i\|_p + \mathcal{L}^n(V)^{\frac{p-1}{p}}\epsilon\big(\|\nabla f_i\|_p + \epsilon\big) \le \\ &\le \epsilon\mathcal{L}^n(V)^{\frac{p-1}{p}}\big(2 + \epsilon + \|\nabla f_i\|_p\big) \le \\ &\le \epsilon \underbrace{\mathcal{L}^n(V)^{\frac{p-1}{p}}\big(3 + \sup\{\|\nabla f_i\|_p \mid i \in \mathbb{N}\}\big)}_{=:C<\infty}.\end{aligned}$$

c) It then follows from the interpolation inequality 1.77, with $\lambda \in (0,1]$ given by $\frac{1}{q} = \lambda + \frac{1-\lambda}{p^*}$, that, for each $i \in \mathbb{N}$,

$$\begin{aligned}\|f_i^\epsilon - f_i\|_q &\le \|f_i^\epsilon - f_i\|_1^\lambda \|f_i^\epsilon - f_i\|_{p^*}^{1-\lambda} \overset{6.62}{\le} \\ &\le (C\epsilon)^\lambda \underbrace{\|\nabla(f_i^\epsilon - f_i)\|_p^{1-\lambda}}_{\le\left(2\sup\{\|\nabla f_i\|_p \mid i\in\mathbb{N}\}\right)^{1-\lambda}} \le \\ &\le C'\epsilon^\lambda,\end{aligned}$$

where $C' = C^\lambda\big(2\sup\{\|\nabla f_i\|_p \mid i \in \mathbb{N}\}\big)^{1-\lambda} < \infty$, which concludes the proof of claim 1.

2) Claim 2: for each $0 < \epsilon < 1$ fixed, $(f_i^\epsilon)_{i\in\mathbb{N}}$ is uniformly bounded and equicontinuous.

Indeed, for all $i \in \mathbb{N}$, it follows from Young's inequality 1.108.g) that:

$$\|f_i^\epsilon\|_\infty = \|\phi_\epsilon * f_i\|_\infty \le \|\phi_\epsilon\|_\infty \|f_i\|_1 \le \frac{C}{\epsilon^n} < \infty,$$

$$\|\nabla f_i^\epsilon\|_\infty = \|\nabla\phi_\epsilon * f_i\|_\infty \le \|\nabla\phi_\epsilon\|_\infty \|f_i\|_1 \le \frac{C'}{\epsilon^{n+1}} < \infty,$$

whence the claim.

3) Claim 3: for each $\delta > 0$, there exists a subsequence $(f_{i_k})_{k\in\mathbb{N}}$ of $(f_i)_i$ such that

$$\limsup_{j,k\to\infty} \|f_{i_j} - f_{i_k}\|_{\mathsf{L}^q(\mathcal{L}^n)} \le \delta.$$

To prove claim 3, we firstly apply claim 1 to find $0 < \epsilon < 1$ such that, for all $i \in \mathbb{N}$, $\|f_i^\epsilon - f_i\|_{\mathsf{L}^q(\mathcal{L}^n)} \le \frac{\delta}{2}$.

Since $\operatorname{spt} f_i^\epsilon \Subset V \Subset \mathbb{R}^n$ for all $i \in \mathbb{N}$, in view of claim 2 we may apply Arzelà-Ascoli's theorem to find a subsequence $(f_{i_k})_{k\in\mathbb{N}}$ such that $f_{i_k}^\epsilon$ is uniformly convergent on $\mathbb{R}^n$. Since $\mathcal{L}^n(V) < \infty$, this subsequence is also convergent in $\mathsf{L}^q(\mathcal{L}^n)$, so that

$$\limsup_{j,k\to\infty} \|f_{i_j}^\epsilon - f_{i_k}^\epsilon\|_{\mathsf{L}^q(\mathcal{L}^n)} = 0.$$

Finally, since $\|f_{i_j} - f_{i_k}\|_q \leq \|f_{i_j} - f^\epsilon_{i,j}\|_q + \|f^\epsilon_{i_j} + f^\epsilon_{i_k}\|_q + \|f^\epsilon_{i_k} - f_{i_k}\|_q$, the claim follows.

4) We now apply claim 3 for $\delta = 1/m$, $m \in \mathbb{N}$, yielding for each $m \in \mathbb{N}$ a subsequence $f^m = (f_i^m)_{i\in\mathbb{N}}$ of $(f_i^{m-1})_{i\in\mathbb{N}}$, with $f^0 = (f_i)_{i\in\mathbb{N}}$, such that, for all $m \in \mathbb{N}$,

$$\limsup_{j,k\to\infty} \|f_j^m - f_k^m\|_{\mathsf{L}^{\mathsf{q}}(\mathcal{L}^n)} \leq \frac{1}{m}.$$

The diagonal $(f_m^m)_{m\in\mathbb{N}}$ is therefore a subsequence of $(f_i)_i$ which is Cauchy in $\mathsf{L}^{\mathsf{q}}(\mathcal{L}^n)$, hence convergent in that space.

□

Proof 2. We apply the Kolmogorov-Riesz-Fréchet compactness criterion 1.80.

Note that, in view of Sobolev-Gagliardo-Nirenberg inequality 6.62, $(f_i)_{i\in\mathbb{N}}$ is bounded on $\mathsf{L}^{\mathsf{p}^*}(\mathcal{L}^n)$. Thus, since $\mathcal{L}^n(V) < \infty$, it follows that, for $1 \leq q < p^*$,

$$\sup\{\|f_i\|_{\mathsf{L}^{\mathsf{q}}(\mathcal{L}^n)} \mid i \in \mathbb{N}\} < \infty. \tag{6.21}$$

On the other hand, it follows from exercise 6.25 for $p > 1$ and from exercise 7.38 for $p = 1$ that, for all $h \in \mathbb{R}^n$ and all $i \in \mathbb{N}$,

$$\|\tau_h f_i - f_i\|_{\mathsf{L}^{\mathsf{p}}(\mathcal{L}^n)} \leq \|h\| \underbrace{\sup\{\|\nabla^{\mathsf{w}} f_i\|_{\mathsf{L}^{\mathsf{p}}(\mathcal{L}^n)} \mid i \in \mathbb{N}\}}_{=:C<\infty}.$$

Thus, in view of Hölder's inequality, for all $h \in \mathbb{R}^n$ and all $i \in \mathbb{N}$,

$$\|\tau_h f_i - f_i\|_{\mathsf{L}^1(\mathcal{L}^n)} \leq \|h\| C \mathcal{L}^n(V)^{\frac{p-1}{p}}.$$

Therefore, applying the interpolation inequality 1.77, it follows that, for $1 \leq q < p^*$, $h \in \mathbb{R}^n$ and $i \in \mathbb{N}$,

$$\|\tau_h f_i - f_i\|_{\mathsf{L}^{\mathsf{q}}(\mathcal{L}^n)} \leq \|h\|^\lambda \underbrace{\left(C\mathcal{L}^n(V)^{\frac{p-1}{p}}\right)^\lambda \sup\{\|\tau_h f_i - f_i\|_{\mathsf{L}^{\mathsf{p}^*}(\mathcal{L}^n)} \mid i \in \mathbb{N}\}^{1-\lambda}}_{:=C'},$$

where $\lambda \in (0, 1]$ is given by

$$\frac{1}{q} = \lambda + \frac{1-\lambda}{p^*}.$$

Since

$$\sup\{\|\tau_h f_i - f_i\|_{\mathsf{L}^{\mathsf{p}^*}(\mathcal{L}^n)} \mid i \in \mathbb{N}\} \leq 2\sup\{\|f_i\|_{\mathsf{L}^{\mathsf{p}^*}(\mathcal{L}^n)} \mid i \in \mathbb{N}\} < \infty,$$

we have $C' < \infty$ and, since $\lambda > 0$, we conclude that

$$\lim_{h\to 0} \|\tau_h f_i - f_i\|_{\mathsf{L}^{\mathsf{q}}(\mathcal{L}^n)} = 0 \tag{6.22}$$

uniformly in $i \in \mathbb{N}$.

With (6.21) and (6.22) in force, we may apply the Kolmogorov-Riesz-Fréchet compactness criterion 1.80 to $\mathcal{F} := \{f_i \mid i \in \mathbb{N}\}$. Therefore, $\mathcal{F}|_V$ has compact closure in $\mathsf{L}^\mathsf{q}(\mathcal{L}^n|_V)$; since each $f \in \mathcal{F}$ has support in V, we conclude that $\mathcal{F}$ has compact closure in $\mathsf{L}^\mathsf{q}(\mathcal{L}^n)$, whence the thesis.

$\square$

THEOREM 6.77 (Rellich-Kondrachov). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $1 \leq p < n$ and $1 \leq q < p^*$, where p^* is the Sobolev conjugate of p. Then*

$$\mathsf{W}^{1,\mathsf{p}}(\Omega) \Subset \mathsf{L}^\mathsf{q}(\mathcal{L}^n|_\Omega),$$

i.e. $\mathsf{W}^{1,\mathsf{p}}(\Omega) \subset \mathsf{L}^\mathsf{q}(\mathcal{L}^n|_\Omega)$ *with compact inclusion.*

PROOF. We have continuous inclusions $\mathsf{W}^{1,\mathsf{p}}(\Omega) \subset \mathsf{L}^{\mathsf{p}^*}(\mathcal{L}^n|_\Omega) \subset \mathsf{L}^\mathsf{q}(\mathcal{L}^n|_\Omega)$, the first in view of corollary 6.63 and the second in view of the fact that $\mathcal{L}^n(\Omega) < \infty$ and of Hölder's inequality.

Therefore, it suffices to show that each bounded sequence $(f_i)_{i\in\mathbb{N}}$ in $\mathsf{W}^{1,\mathsf{p}}(\Omega)$ has a subsequence which is convergent in $\mathsf{L}^\mathsf{q}(\mathcal{L}^n|_\Omega)$.

Let V be an open relatively compact subset of $\mathbb{R}^n$ such that $\Omega \Subset V \Subset \mathbb{R}^n$ and $\mathsf{E} : \mathsf{W}^{1,\mathsf{p}}(\Omega) \to \mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ an extension operator, cf. theorem 6.56, such that spt $\mathsf{E}\, f \Subset V$ for all $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$. Then $(\mathsf{E}\, f_i)_{i\in\mathbb{N}}$ is a bounded sequence in $\mathsf{W}^{1,\mathsf{p}}(\mathbb{R}^n)$ with spt $f_i \Subset V$ for all $i \in \mathbb{N}$. We may therefore apply lemma 6.76 to obtain a subsequence $(f_{i_j})_{j\in\mathbb{N}}$ such that $(\mathsf{E}\, f_{i_j})_{j\in\mathbb{N}}$ is convergent in $\mathsf{L}^\mathsf{q}(\mathcal{L}^n)$. Since $f_i = \mathsf{E}\, f_i|_\Omega$ for all $i \in \mathbb{N}$, we conclude that $(f_{i_j})_{j\in\mathbb{N}}$ is convergent in $\mathsf{L}^\mathsf{q}(\mathcal{L}^n|_\Omega)$. $\square$

COROLLARY 6.78. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $1 \leq p < n$ and $(f_i)_{i\in\mathbb{N}}$ a bounded sequence in $\mathsf{W}^{1,\mathsf{p}}(\Omega)$. Then there exists a subsequence $(f_{i_j})_{j\in\mathbb{N}}$ of $(f_i)_{i\in\mathbb{N}}$ which is convergent in each $\mathsf{L}^\mathsf{q}(\Omega)$, for $1 \leq q < p^*$.*

PROOF. Let $(q_m)_{m\in\mathbb{N}}$ be a sequence in $[1, p^*)$ which increases to p^*. For each $m \in \mathbb{N}$, we may apply theorem 6.77 to find a subsequence $f^m = (f_i^m)_{i\in\mathbb{N}}$ of f^{m-1}, with $f^0 = (f_i)_{i\in\mathbb{N}}$, such that f^m is convergent in $\mathsf{L}^{\mathsf{q_m}}(\Omega)$ (hence on $\mathsf{L}^\mathsf{q}(\Omega)$ for $1 \leq q \leq q_m$, since $\mathcal{L}^n(\Omega) < \infty$). The diagonal $(f_m^m)_{m\in\mathbb{N}}$ is therefore a subsequence of $(f_i)_{i\in\mathbb{N}}$ which is convergent in each $\mathsf{L}^\mathsf{q}(\Omega)$, for $1 \leq q < p^*$. $\square$

COROLLARY 6.79. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $1 < p < n$ and $(f_i)_{i\in\mathbb{N}}$ a bounded sequence in $\mathsf{W}^{1,\mathsf{p}}(\Omega)$. Then there exists $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$ and a subsequence $(f_{i_j})_{j\in\mathbb{N}}$ of $(f_i)_{i\in\mathbb{N}}$ such that $f_{i_j} \to f$ in each $\mathsf{L}^\mathsf{q}(\Omega)$, for $1 \leq q < p^*$.*

PROOF. Let $(f_{i_j})_{j\in\mathbb{N}}$ be a subsequence of $(f_i)_{i\in\mathbb{N}}$ which is convergent in each $\mathsf{L}^\mathsf{q}(\Omega)$, for $1 \le q < p^*$, cf. corollary 6.78. We contend that its limit f belongs to $\mathsf{W}^{1,\mathsf{p}}(\Omega)$. Indeed, since $1 \le p < p^*$, $f \in \mathsf{L}^\mathsf{p}(\Omega)$ and $f_{i_j} \to f$ in $\mathsf{L}^\mathsf{p}(\Omega)$. As $(\nabla f_{i_j})_{j\in\mathbb{N}}$ is bounded in $\mathsf{L}^\mathsf{p}(\mathcal{L}^n|_\Omega, \mathbb{R}^n)$, it follows from proposition 6.3.ii) that $f \in \mathsf{W}^{1,\mathsf{p}}(\Omega)$, as asserted. $\square$

COROLLARY 6.80. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and $(f_i)_{i\in\mathbb{N}}$ a bounded sequence in $\mathsf{W}^{1,1}(\Omega)$. Then there exists $f \in \mathsf{L}^{1^*}(\mathcal{L}^n|_\Omega)$ and a subsequence $(f_{i_j})_{j\in\mathbb{N}}$ of $(f_i)_{i\in\mathbb{N}}$ such that $f_{i_j} \to f$ in each $\mathsf{L}^\mathsf{q}(\Omega)$, for $1 \le q < 1^*$.*

PROOF. Let $(f_{i_j})_{j\in\mathbb{N}}$ be a subsequence of $(f_i)_{i\in\mathbb{N}}$ which is convergent in each $\mathsf{L}^\mathsf{q}(\Omega)$, for $1 \le q < 1^*$, cf. corollary 6.78. We contend that its limit f belongs to $\mathsf{L}^{1^*}(\mathcal{L}^n|_\Omega)$. Indeed, since $(f_{i_j})_{j\in\mathbb{N}}$ is bounded in $\mathsf{L}^{1^*}(\mathcal{L}^n|_\Omega)$ (because it is bounded in $\mathsf{W}^{1,\mathsf{p}}(\Omega)$ and corollary 6.63 may be applied), it follows from Banach-Alaoglu's theorem that there exists a subsequence of $(f_{i_j})_{j\in\mathbb{N}}$ which is weak-star convergent to some $g \in \mathsf{L}^{1^*}(\mathcal{L}^n|\Omega)$; since it also converges to $f \in \mathsf{L}^1(\mathcal{L}^n|_\Omega)$, we conclude that $f = g \in \mathsf{L}^{1^*}(\mathcal{L}^n|_\Omega)$, as asserted. $\square$

CHAPTER 7

Functions of Bounded Variation and Sets of Finite Perimeter

Let $\Omega \subset \mathbb{R}^n$ open. We define functions of bounded variation on Ω as $\mathsf{L}^1_{\mathsf{loc}}$ functions on Ω whose distributional gradient is an $\mathbb{R}^n$-valued Radon measure on Ω. For that purpose, we make the following generalization of the notion of weak derivatives introduced in 5.3:

Definition 7.1 (weak derivatives and gradients, bis). Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$. We say that:

i) For $1 \leq i \leq n$, u has *weak i-th partial derivative* $\mu_i \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}) \equiv \mathsf{C}_\mathsf{c}(\Omega, \mathbb{R})^*$ if $\forall \varphi \in \mathsf{C}^\infty_\mathsf{c}(\Omega)$,
$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}\mathcal{L}^n = -\int_\Omega \varphi \,\mathrm{d}\mu_i.$$
ii) u has *weak gradient* $\mu \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^n) \equiv \mathsf{C}_\mathsf{c}(\Omega, \mathbb{R}^n)^*$ if $\forall \varphi \in \mathsf{C}^\infty_\mathsf{c}(\Omega, \mathbb{R}^n)$,
$$\int_\Omega u \,\mathrm{div}\ \varphi \,\mathrm{d}\mathcal{L}^n = -\int_\Omega \varphi \cdot \,\mathrm{d}\mu. \tag{7.1}$$

We use the same notations for weak derivatives introduced in definition 5.3.

Remark 7.2. With the notation from the definition above, it follows from the definition of $\mathbb{R}^n$-valued Radon measures 4.1 and remark 4.4 that:

1) For $1 \leq i \leq n$, $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ admits weak i-th partial derivative if, for each compact $K \subset \Omega$, there exists $C_K < \infty$ such that
$$\sup\{\int_\Omega u \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}\mathcal{L}^n \mid \varphi \in \mathsf{C}^\infty_\mathsf{c}(\Omega), \mathrm{spt}\ \varphi \subset K, \|\varphi\|_u \leq 1\} \leq C_K.$$
2) $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ admits weak gradient if, for each compact $K \subset \Omega$, there exists $C_K < \infty$ such that
$$\sup\{\int_\Omega u \,\mathrm{div}\ \varphi \,\mathrm{d}\mathcal{L}^n \mid \varphi \in \mathsf{C}^\infty_\mathsf{c}(\Omega, \mathbb{R}^n), \mathrm{spt}\ \varphi \subset K, \|\varphi\|_u \leq 1\} \leq C_K.$$
3) Weak partial derivatives or weak gradients, if exist, are unique.

4) $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ has weak gradient $\mu = (\mu_1, \dots, \mu_n) \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ iff it has weak partial derivatives of first order $\mu_i \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R})$ for $1 \le i \le n$.
5) If $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ has weak i-th partial derivative $v_i \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ in the sense of definition 5.3, then it has weak i-th partial derivative $\mathcal{L}^n \llcorner v_i \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R})$ in the sense of definition 7.1. Thus, considering the injection $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega) \subset \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R})$ given by $v \mapsto \mathcal{L}^n \llcorner v$, we see that definition 5.3 may be considered a particular case of definition 7.1.
6) It is clear that the set of functions $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ which admit weak gradient is a linear subspace of $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ and that weak derivatives and weak gradient are linear in this subspace. We denote it by $\mathsf{BV}_{\mathrm{loc}}(\Omega)$, cf. definition 7.5 below.

Exercises 5.4 and 5.5 admit the following counterparts for the extended notion of weak derivatives.

Exercise 7.3 (weak gradients, bis). Weak gradients may be also characterized by means of Gauss-Green identity in gradient form. That is, let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$; then u admits weak gradient $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ iff $\forall \varphi \in \mathsf{C}^\infty_{\mathsf{c}}(\Omega)$,

$$\int_\Omega u \nabla \varphi \, \mathrm{d}\mathcal{L}^n = -\int_\Omega \varphi \, \mathrm{d}\mu. \tag{7.2}$$

Exercise 7.4. Let Ω be an open subset of $\mathbb{R}^n$, $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ and $1 \le i \le n$. If there exists $\mu_i = \frac{\partial^{\mathsf{w}} u}{\partial x_i} \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R})$, then $\forall \varphi \in \mathsf{C}^1_{\mathsf{c}}(\Omega)$,

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}\mathcal{L}^n = -\int_\Omega \varphi \, \mathrm{d}\mu_i$$

Definition 7.5. Let Ω be an open subset of $\mathbb{R}^n$.

i) We denote by $\mathsf{BV}_{\mathrm{loc}}(\Omega)$ the set of functions $u \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ which admit weak partial gradient $\nabla^{\mathsf{w}} u \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Such functions are called of *locally bounded variation* on Ω.
ii) We say that u is a function of *bounded variation* on Ω if $u \in \mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and u admits weak gradient $\nabla^{\mathsf{w}} u \in \mathcal{M}(\Omega, \mathbb{R}^n)$, i.e. its weak gradient is a finite $\mathbb{R}^n$-valued Radon measure on Ω. We denote by $\mathsf{BV}(\Omega)$ the set of functions of bounded variation on Ω.
iii) We say that $E \subset \Omega$ is a *set of locally finite perimeter* in Ω if $\chi_E \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$. We say that E is a *Caccioppoli set* or a *set of finite perimeter* in Ω if $\chi_E \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$ and $\nabla^{\mathsf{w}} \chi_E \in \mathcal{M}(\Omega, \mathbb{R}^n)$.

Example 7.6. Let $\Omega \subset \mathbb{R}^n$ open and $f \in \mathsf{W}^{1,1}_{\mathsf{loc}}(\Omega)$. It follows from remark 7.2.5) that $f \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$ and its measure-weak gradient is given by $\mathcal{L}^n \llcorner \nabla^{\mathsf{w}} f \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.

The inclusion $\mathsf{W}^{1,1}_{\mathsf{loc}}(\Omega) \subset \mathsf{BV}_{\mathrm{loc}}(\Omega)$ is strict; for instance, if $u = \chi_{(0,\infty)}$ on $\Omega = \mathbb{R}$, $\nabla^{\mathsf{w}} u$ coincides with the Dirac measure $\delta_0 \in \mathcal{M}(\mathbb{R}, \mathbb{R})$, so that $\nabla^{\mathsf{w}} u \perp \mathcal{L}^n$, hence $u \in \mathsf{BV}(\mathbb{R}) \setminus \mathsf{W}^{1,1}_{\mathsf{loc}}(\mathbb{R})$.

THEOREM 7.7 (locality of the weak derivative). *Let $\Omega \subset \mathbb{R}^n$ open, $f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ and $\mathcal{F} \subset 2^\Omega$ an open cover of Ω. Then f admits weak partial derivatives of first order on Ω* iff *$\forall U \in \mathcal{F}$, $f|_U$ admits weak partial derivatives of first order on U. Moreover, weak derivatives commute with restrictions (for a Radon measure, "restriction" here means "trace").*

LEMMA 7.8. *Let $U \subset \mathbb{R}^n$ open.*

i) If $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $f \in \mathsf{L}^1(|\mu|)$, then $\mu \llcorner f \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^n)$ and $|\mu \llcorner f| = |\mu| \llcorner |f|$.

ii) If $\xi \in \mathsf{C}^\infty_{\mathsf{c}}(U)$ and $f \in \mathsf{BV}_{\mathrm{loc}}(U)$, then $\xi \cdot f$ (defined as 0 on $\mathbb{R}^n \setminus U$) belongs to $\mathsf{BV}(\mathbb{R}^n)$ and

$$\nabla^{\mathsf{w}}(\xi \cdot f) = \mathcal{L}^n \llcorner (f \nabla \xi) + \nabla^{\mathsf{w}} f \llcorner \xi.$$

PROOF.

i) Let $(\nu, |\mu|)$ be the polar decomposition of μ. Then, for all $\varphi \in \mathsf{C}_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n)$, $|\mu \llcorner f \cdot \varphi| = |\int_\Omega \langle \varphi f, \nu \rangle \,\mathrm{d}|\mu|| \leq \|\varphi\|_u \|f\|_{\mathsf{L}^1(|\mu|)}$, thus showing that $\mu \llcorner f \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^n)$. Besides, the same computation shows that, for all $\varphi \in \mathsf{C}_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n)$, $\mu \llcorner f \cdot \varphi = \int_{\mathbb{R}^n} \langle \varphi, \frac{f\nu}{|f|} \rangle \,\mathrm{d}(\mu \llcorner |f|)$, hence the polar decomposition of $\mu \llcorner f$ is $(\frac{f\nu}{|f|}, \mu \llcorner |f|)$.

ii) It follows from lemma 6.14 that $f \nabla \xi \in \mathsf{L}^1(\mathcal{L}^n, \mathbb{R}^n)$. Let $\mu := \mathcal{L}^n \llcorner (f \nabla \xi) + \nabla^{\mathsf{w}} f \llcorner \xi$. Then $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^n)$ by the previous item; it therefore suffices to show that $\xi \cdot f$ admits weak gradient equal to μ. Indeed, $\forall \varphi \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\mathbb{R}^n} (\xi \cdot f) \cdot \nabla \varphi \,\mathrm{d}\mathcal{L}^n &= \int_\Omega f \cdot \Big(\nabla(\xi \cdot \varphi) - \nabla \xi \cdot \varphi \Big) \,\mathrm{d}\mathcal{L}^n = \\ &= -\int_\Omega \xi \varphi \,\mathrm{d}\, \nabla^{\mathsf{w}} f - \int_\Omega f \cdot \nabla \xi \cdot \varphi \,\mathrm{d}\mathcal{L}^n = \\ &= -\int_{\mathbb{R}^n} \varphi \,\mathrm{d}\mu,\end{aligned}$$

as asserted.

$\square$

PROOF OF THEOREM 7.7. The implication "$\Rightarrow$" and the fact that weak derivatives commute with restrictions are clear. We must prove the converse implication, i.e. if $\forall U \in \mathcal{F}$, $f|_U$ admits weak gradient $\nabla^{\mathsf{w}}(f|_U) \in \mathcal{M}_{\mathrm{loc}}(U, \mathbb{R}^n)$, then f admits weak gradient on Ω.

1) We may assume that $\mathcal{F}$ is locally finite and $\forall U \in \mathcal{F}$, $U \Subset \Omega$. Indeed, in the general case, take a locally finite open refinement $\mathcal{G}$ of $\mathcal{F}$ such that $\forall U \in \mathcal{G}$, $U \Subset \Omega$. For each $V \in \mathcal{G}$, there exists $U \in \mathcal{F}$ such that $V \subset U$; since $f|_U$ admits weak gradient $\nabla^{\mathsf{w}}(f|_U) \in \mathcal{M}_{\mathrm{loc}}(U, \mathbb{R}^n)$, it follows that $f|_V = (f|_U)|_V$ admits weak gradient, so that we may replace $\mathcal{F}$ by $\mathcal{G}$.
2) Take a smooth partition of unity $(\xi_V)_{V \in \mathcal{F}}$ of Ω, given by theorem 6.8, such that $\forall V \in \mathcal{F}$, $\xi_V \in \mathsf{C}^\infty_{\mathsf{c}}(V)$. We contend that $\mu := \sum_{V \in \mathcal{F}} \nabla^{\mathsf{w}}(f|_V) \llcorner \xi_V \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Indeed, for each compact $K \subset \Omega$, there are finitely many $V_1, \ldots, V_N \in \mathcal{F}$ which intersect K, so that $\mu|_{\mathsf{C}^{\mathsf{K}}_{\mathsf{c}}(\Omega, \mathbb{R}^n)} = \sum_{j=1}^N \nabla^{\mathsf{w}}(f|_{V_i}) \llcorner \xi_{V_i}$ is linear continuous by lemma 7.8.i), thus proving our contention.
3) Let $\varphi \in \mathsf{C}^\infty_{\mathsf{c}}(\Omega, \mathbb{R}^n)$. Since spt φ is compact, there are finitely many $V_1, \ldots, V_N \in \mathcal{F}$ which intersect K. We then have

$$
\begin{aligned}
\int_\Omega \varphi \cdot \mathrm{d}\mu &= \sum_{j=1}^N \int_\Omega \xi_j \varphi \cdot \mathrm{d}\, \nabla^{\mathsf{w}}(f|_{V_j}) = \\
&= \sum_{j=1}^N \int_{V_j} (\xi_j \varphi) \cdot \mathrm{d}\, \nabla^{\mathsf{w}}(f|_{V_j}) = \\
&= -\sum_{j=1}^N \int_{V_j} f|_{V_j} \cdot \mathrm{div}\,(\xi_j \varphi)\, \mathrm{d}\mathcal{L}^n = \\
&= -\sum_{j=1}^N \int_\Omega f \cdot \mathrm{div}\,(\xi_j \varphi)\, \mathrm{d}\mathcal{L}^n = \\
&= -\int_\Omega f \cdot \mathrm{div}\,(\sum_{j=1}^N \xi_j \varphi)\, \mathrm{d}\mathcal{L}^n = \\
&= -\int_\Omega f \cdot \mathrm{div}\, \varphi\, \mathrm{d}\mathcal{L}^n,
\end{aligned}
$$

thus proving that $\nabla^{\mathsf{w}} f = \mu$ on Ω.

□

COROLLARY 7.9. *Let $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}$ Lebesgue measurable. Then $f \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$ iff for all open $V \Subset \Omega$, $f|_V \in \mathsf{BV}(V)$.*

PROOF. The implication "⇒" is clear, in view of the fact that weak derivatives commute with restrictions. Conversely, assume that for all open $V \Subset \Omega$, $f|_V \in \mathsf{BV}(V)$. In particular, for all open $V \Subset \Omega$, $f|_V \in \mathsf{L}^1(\mathcal{L}^n|_V)$, hence $f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$. It then follows from theorem 7.7 that $\exists\, \nabla^{\mathsf{w}} f \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, thus $f \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$. □

PROPOSITION 7.10. *Let Ω be an open subset of $\mathbb{R}^n$. Then* $\mathsf{BV}(\Omega)$ *is a Banach space with the norm*

$$\|f\|_{\mathsf{BV}(\Omega)} := \|f\|_{\mathsf{L}^1(\Omega)} + |\nabla^{\mathsf{w}} f|(\Omega). \tag{7.3}$$

PROOF. It suffices to show that the graph of $\nabla^{\mathsf{w}} : \mathsf{BV}(\Omega) \to \mathcal{M}(\Omega, \mathbb{R}^n) \equiv \mathsf{C}_0(\Omega, \mathbb{R}^n)^*$ is closed in the Banach space $H := \mathsf{L}^1(\mathcal{L}^n|_\Omega) \times \mathcal{M}(\Omega, \mathbb{R}^n)$. Indeed, let $(u_k, v_k)_{k\in\mathbb{N}}$ be a sequence in gr ∇^{w} such that $(u_k, v_k) \to (u, v) \in H$. We must show that u is weakly differentiable and $\nabla^{\mathsf{w}} u = v$. Indeed, $\forall \varphi \in \mathsf{C}^\infty_{\mathsf{c}}(\Omega, \mathbb{R}^n)$, $\forall k \in \mathbb{N}$,

$$\int_\Omega u_k \operatorname{div} \varphi \, \mathrm{d}\mathcal{L}^n = -\int_\Omega \varphi \cdot \mathrm{d}v_k.$$

Since $u_k \to u$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and $v_k \to v$ in in $\mathcal{M}(\Omega, \mathbb{R}^n)$ (in particular, $v_k \overset{*\mathsf{f}}{\rightharpoonup} v$), the above equality holds with u in place of u_k and v in place of v_k, thus proving our contention. $\square$

REMARK 7.11. If Ω is an open subset of $\mathbb{R}^n$, $\mathsf{BV}_{\mathrm{loc}}(\Omega)$ admits a Fréchet space topology induced by the family of seminorms $\{\|\cdot\|_{\mathsf{BV}(V)} \mid V \Subset \Omega \text{ open}\}$.

7.1. Gauss-Green Measures and Generalized Divergence Theorem

DEFINITION 7.12 (Gauss-Green measure, exterior normal and perimeter measure of a set of locally finite perimeter). Let Ω be an open subset of $\mathbb{R}^n$ and $E \subset \Omega$ be a set of locally finite perimeter in Ω, i.e. such that $\chi_E \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$ (in particular, if E is a Caccioppoli set in Ω, cf. definition 7.5). The $\mathbb{R}^n$-valued Radon measure $\mu_E := -\nabla^{\mathsf{w}} \chi_E \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ (attention to the minus sign) is called the *Gauss-Green measure* of E.

Let $(\nu_E, |\mu_E|)$ be the polar decomposition of μ_E. We call the positive Radon measure $\mathsf{P}(E, \cdot) := |\mu_E|$ on Ω the *perimeter measure of* E and ν_E the *exterior normal* to E.

REMARK 7.13. With the notation from the previous definition, let E be a set of locally finite perimeter in Ω and $\partial^\Omega E = \Omega \cap \partial E$ be the topological boundary of E in Ω.

1) It is clear that spt $\mu_E \subset \partial^\Omega E$. Since ν_E is determined up to $|\mu_E|$-null sets, we may and do assume henceforth that $\nu_E = 0$ on $\Omega \setminus \partial^\Omega E$ and we identify ν_E with a Borelian map $\partial^\Omega E \to \mathbb{R}^n$.
2) It follows from the definition of the polar decomposition and from exercise 7.4 that, for all $\varphi \in \mathsf{C}^1_{\mathsf{c}}(\Omega, \mathbb{R}^n)$,

$$\int_E \operatorname{div} \varphi \, \mathrm{d}\mathcal{L}^n = \int_{\partial^\Omega E} \varphi \cdot \nu_E \, \mathrm{d}|\mu_E|. \tag{7.4}$$

We call the above equality the *generalized Gauss-Green theorem*.

EXERCISE 7.14 (Complements of sets of locally finite perimeter). Let Ω be an open subset of $\mathbb{R}^n$ and $E \subset \Omega$ be a set of locally finite perimeter in Ω. Then $\Omega \setminus E$ has locally finite perimeter in Ω and

$$\mu_{\Omega\setminus E} = -\mu_E.$$

EXERCISE 7.15 (Sets of finite perimeter under scaling and translation). Let E be a set of locally finite perimeter in $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and $\lambda > 0$. Then $x + \lambda E$ is a set of locally finite perimeter in $\mathbb{R}^n$ and

$$\mu_{x+\lambda E} = \Phi_\# \mu_E,$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is given by $y \mapsto x + \lambda y$. In particular, if E has finite perimeter, so does $x + \lambda E$ and $P(x + \lambda E, \mathbb{R}^n) = \lambda^{n-1} P(E, \mathbb{R}^n)$.

PROPOSITION 7.16 (Lipschitz epigraphs have locally finite perimeter). *Let $n \geq 2$, $f : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz and $\Omega := \operatorname{epi_S} f$. Then Ω is a set of locally finite perimeter in $\mathbb{R}^n$, $|\mu_\Omega| = \mathcal{H}^{n-1} \llcorner \partial\Omega$ and ν_Ω coincides with the unit outer normal to $\partial\Omega$ in the sense of definition 6.46, i.e.*

$$\nu(x) = \frac{\big(\nabla f(x'), -1\big)}{\sqrt{1 + \|\nabla f(x')\|^2}}$$

on each point point $x = \big(x', f(x')\big)$ in $\partial\Omega = \operatorname{gr} f$ whose abscissa x' is a differentiability point of f.

PROOF. It follows from theorem 6.45 and remark 6.47 that χ_E admits weak gradient $\nabla^{\mathsf{w}} \chi_E = (-\nu, \mathcal{H}^{n-1} \llcorner \partial E)$. □

We next generalize the previous proposition to Lipschitz domains.

LEMMA 7.17. *Let $n \geq 2$, $f : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz, U' an open subset of $\mathbb{R}^n$, $E' := U' \cap \operatorname{epi_S} f$ and $\nu' : \partial \operatorname{epi_S} f \to \mathbb{R}^n$ the unit outer normal to $\partial \operatorname{epi_S} f$ in the sense of definition 6.46 (which, in view of the previous proposition, coincides with the exterior normal to $\operatorname{epi_S} f$ in the sense of definition 7.12); see figure 1. Let $\Phi \in \mathrm{SE}(n)$ be a rigid motion, $U := \Phi(U')$, $E := \Phi(E')$ and $\nu := \Phi_* \nu'$, i.e. $\nu : \partial\Phi(\operatorname{epi_S} f) \to \mathbb{R}^n$ is given by $x \mapsto \mathsf{D}\Phi\big(\Phi^{-1}(x)\big) \cdot \nu'\big(\Phi^{-1}(x)\big)$. Then:*

i) $\Phi_\#\big(\mathcal{H}^{n-1} \llcorner \partial \operatorname{epi_S} f\big) = \mathcal{H}^{n-1} \llcorner \partial\Phi(\operatorname{epi_S} f)$.
ii) E is a set of locally finite perimeter in U, $|\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^U E$ and its exterior normal is given by $\nu_E = \nu|_{\partial^U E}$.

In particular, $\mathcal{H}^{n-1} \llcorner \partial\Phi(\operatorname{epi_S} f)$ is a Radon measure. Note that $\partial \operatorname{epi_S} f = \operatorname{gr} f$, $\partial^{U'} E' = U' \cap \partial \operatorname{epi_S} f$ and $\partial^U E = \Phi(\partial^{U'} E') = U \cap \partial\Phi(\operatorname{epi_S} f)$.

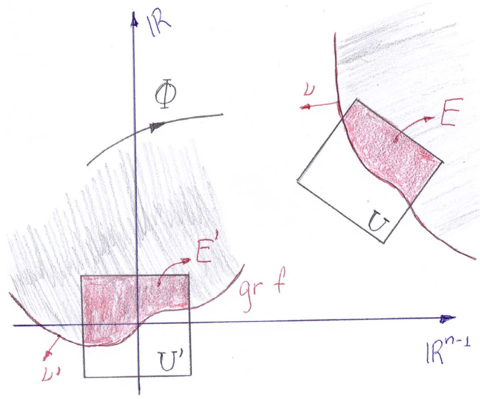

FIGURE 1. Gauss-Green measure of a Lipschitz Domain

PROOF.

i) Since Φ is an isometry onto $\mathbb{R}^n$, $\Phi_{\#}\mathcal{H}^{n-1} = \mathcal{H}^{n-1}$. Therefore, for all $A \subset \mathbb{R}^n$,

$$\begin{aligned}\Phi_{\#}\big(\mathcal{H}^{n-1} \llcorner \partial\, \mathrm{epi}_{\mathsf{S}}\, f\big)(A) &= \mathcal{H}^{n-1} \llcorner \partial\, \mathrm{epi}_{\mathsf{S}}\, f\big(\Phi^{-1}(A)\big) = \\ &= \mathcal{H}^{n-1}\big(\partial\, \mathrm{epi}_{\mathsf{S}}\, f \cap \Phi^{-1}(A)\big) = \\ &= \mathcal{H}^{n-1}\big[\Phi^{-1}\big(\partial\Phi(\mathrm{epi}_{\mathsf{S}}\, f) \cap A\big)\big] = \\ &= \Phi_{\#}\mathcal{H}^{n-1}\big(\partial\Phi(\mathrm{epi}_{\mathsf{S}}\, f) \cap A\big) = \\ &= \mathcal{H}^{n-1} \llcorner \partial\Phi(\mathrm{epi}_{\mathsf{S}}\, f)(A).\end{aligned}$$

ii)
1) Since $\chi_{E'} = \chi_{\mathrm{epi}_{\mathsf{S}}\, f}|_{U'}$, it follows from proposition 7.16 and from theorem 7.7 that E' is a set of locally finite perimeter in U' and its Gauss-Green measure $\mu_{E'}$ coincides with the trace $\mu_{\mathrm{epi}_{\mathsf{S}}\, f}|_{U'}$. Moreover, by proposition 4.36, the polar decomposition of $\mu_{E'}$ is $(\nu'|_{\partial^{U'}E'}, \mathcal{H}^{n-1})$.

2) Let $R \in \mathrm{SO}(n)$ be the linear part of Φ, so that $\mathsf{D}\Phi(x) =$cte.$= R$. We have, for all $\varphi \in \mathsf{C}^\infty_\mathsf{c}(U,\mathbb{R}^n)$:

$$\begin{aligned}\int_E \operatorname{div}\varphi \,\mathrm{d}\mathcal{L}^n &\overset{\text{AF 5.39,J}\Phi\equiv 1}{=} \int_{E'} (\operatorname{div}\varphi)\circ\Phi\,\mathrm{d}\mathcal{L}^n \overset{(*)}{=}\\
&= \int_{E'} \operatorname{div}\big(\underbrace{R^{-1}\circ\varphi\circ\Phi}_{\in\mathsf{C}^\infty_\mathsf{c}(U',\mathbb{R}^n)}\big)\,\mathrm{d}\mathcal{L}^n \overset{7.4\text{ and }1)}{=}\\
&= \int_{\partial^{U'}E'} \langle R^{-1}\circ\varphi\circ\Phi, \nu'\rangle\,\mathrm{d}\mathcal{H}^{n-1} =\\
&= \int \langle R^{-1}\circ\varphi, \nu'\circ\Phi^{-1}\rangle\circ\Phi \;\;\mathrm{d}\big(\mathcal{H}^{n-1} \llcorner \partial^{U'}E'\big) =\\
&= \int \underbrace{\langle R^{-1}\circ\varphi, \nu'\circ\Phi^{-1}\rangle}_{\overset{R\in\mathrm{SO}(n)}{=}\langle\varphi, R\circ\nu'\circ\Phi^{-1}\rangle}\,\mathrm{d}\,\underbrace{\Phi_\#\big(\mathcal{H}^{n-1}\llcorner\partial^{U'}E'\big)}_{\overset{\text{by } i)}{=}\mathcal{H}^{n-1}\llcorner\partial^U E} =\\
&= \int \langle\varphi, \nu|_{\partial^U E}\rangle\,\mathrm{d}\big(\mathcal{H}^{n-1}\llcorner\partial^U E\big),\end{aligned}$$

where equality $(*)$ is justified by, for all $x \in U'$,

$$\begin{aligned}\operatorname{div}(R^{-1}\circ\varphi\circ\Phi)(x) &= \operatorname{tr}\,\mathsf{D}(R^{-1}\circ\varphi\circ\Phi)(x) \overset{\text{chain rule}}{=}\\
&= \operatorname{tr}\big[R^{-1}\circ\mathsf{D}\varphi(\Phi\cdot x)\circ R\big] =\\
&= \operatorname{tr}\,\mathsf{D}\varphi(\Phi\cdot x) = (\operatorname{div}\varphi)\circ\Phi(x).\end{aligned}$$

□

THEOREM 7.18 (Gaus-Green theorem for Lipschitz domains). *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Then Ω is a set of locally finite perimeter in $\mathbb{R}^n$ and $|\mu_\Omega| = \mathcal{H}^{n-1}\llcorner\partial\Omega$.*

PROOF.

1) For each $x \in \partial\Omega$, there exists an open set $U_x \subset \mathbb{R}^n$ such that $x \in U_x$ and U_x is obtained by rigid motion of a cylinder centered at $0 \in \mathbb{R}^n$ as in definition 6.33, i.e. there exists a rigid motion $\Phi \in \mathrm{SE}(n)$ with $\Phi(0) = x$ and there exists $r, h > 0$ and $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz with $\Gamma(0) = 0$ such that $U_x = \Phi\big(\mathbb{C}(0,r,h)\big)$, $\Phi\big(\mathrm{gr}\ \Gamma \cap \mathbb{C}(0,r,h)\big) = U_x \cap \partial\Omega$ and $\Phi\big(\mathrm{epi}_\mathsf{S}\ \Gamma \cap \mathbb{C}(0,r,h)\big) = U_x \cap \Omega$.
2) From the open cover $(U_x)_{x\in\partial\Omega}$ of $\partial\Omega$, we may extract a countable subcover $(U_i)_{i\in\mathbb{N}}$ by means of Lindelöf's theorem. For each $i \in \mathbb{N}$, let the corresponding objects defined in the previous item be denoted with a subscript i, so that $\Phi_i\big(\mathbb{C}(0,r_i,h_i)\big) = U_i$.

Let $U_0 := \Omega$ and $U_{-1} := \overline{\Omega}^c$, so that $(U_i)_{i\geq -1}$ is a countable open cover of $\mathbb{R}^n$. We may apply corollary 6.11 to obtain a smooth partition of unity $(\xi_i)_{i\geq -1}$ of $\mathbb{R}^n$ with $\operatorname{spt}\,\xi_i \subset U_i$ for $i \geq -1$. Besides, for $i \geq 1$, as $\operatorname{spt}\,\xi_i \subset U_i \Subset \mathbb{R}^n$, it follows that $\operatorname{spt}\,\xi_i$ is a compact subset of U_i.

3) Claim 1: for each $i \geq -1$, $\mu_i : \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$ given by $\varphi \mapsto \int_\Omega \operatorname{div}(\xi_i\varphi)\,\mathrm{d}\mathcal{L}^n$ is a finite $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^n$. Indeed, it is clear that $\mu_{-1} = \mu_0 = 0$, and for $i \geq 1$ and $\varphi \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int_\Omega \operatorname{div}(\xi_i\varphi)\,\mathrm{d}\mathcal{L}^n \overset{\operatorname{spt}\,\xi_i \Subset U_i}{=} \int_{U_i\cap\Omega} \operatorname{div}(\xi_i\varphi)\,\mathrm{d}\mathcal{L}^n \overset{\text{lemma } 7.17}{=}$$
$$= \int \langle \varphi\xi_i, \nu_i\rangle\,\mathrm{d}\big(\mathcal{H}^{n-1} \llcorner \partial^{U_i}\Omega\big),$$

where $\nu_i = \Phi_{i*}\nu_i'$, cf. lemma 7.17. It then follows that, for all $\varphi \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n)$,

$$|\mu_i \cdot \varphi| \leq \big(\mathcal{H}^{n-1} \llcorner \partial^{U_i}\Omega\big)(\operatorname{spt}\,\xi_i)\|\varphi\|_u.$$

Since $\mathcal{H}^{n-1} \llcorner \partial^{U_i}\Omega$ is a Radon measure on U_i and $\operatorname{spt}\,\xi_i$ is a compact subset of U_i, we conclude that $\|\mu_i\|_{\mathsf{C}_0(\mathbb{R}^n,\mathbb{R}^n)^*} \leq \big(\mathcal{H}^{n-1} \llcorner \partial^{U_i}\Omega\big)(\operatorname{spt}\,\xi_i) < \infty$, thus proving the claim.

4) Claim 2: For each compact subset K of $\mathbb{R}^n$, $\mu_K : \mathsf{C}^\infty_{\mathsf{c}}(K, \mathbb{R}^n) := \{\varphi \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n) \mid \operatorname{spt}\,\varphi \subset K\} \to \mathbb{R}$ given by $\varphi \mapsto \int_\Omega \operatorname{div}\,\varphi\,\mathrm{d}\mathcal{L}^n$ is continuous with respect to the topology of uniform convergence.

Indeed, since K is compact and $(\operatorname{spt}\,\xi_i)_{i\geq -1}$ is a locally finite family in $\mathbb{R}^n$, K intersects the members of this family for at most finitely many indices. That is, there exists $N \in \mathbb{N}$ such that $K \cap \operatorname{spt}\,\xi_i = \emptyset$ for $i > N$. Thus, for all $\varphi \in \mathsf{C}^\infty_{\mathsf{c}}(K, \mathbb{R}^n)$, $\varphi = \sum_{i=-1}^N \xi_i\varphi$, hence $\operatorname{div}\,\varphi = \sum_{i=-1}^N \operatorname{div}(\xi_i\varphi)$, which implies $\mu_K = \sum_{i=-1}^N \mu_i|_{\mathsf{C}^\infty_{\mathsf{c}}(K,\mathbb{R}^n)}$, where the μ_i's were defined in the previous item. Therefore, claim 2 follows from claim 1.

5) It follows from claim 2 that $\varphi \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n) \mapsto \int_\Omega \operatorname{div}\,\varphi\,\mathrm{d}\mathcal{L}^n$ is linear continuous in the LF topology of $\mathsf{C}_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n)$, i.e. it is an $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^n$. We have thus proved that $\chi_\Omega \in \mathsf{BV}_{\mathrm{loc}}(\mathbb{R}^n)$, i.e. Ω is a set of locally finite perimeter in $\mathbb{R}^n$.

Let $(\nu_\Omega, |\mu_\Omega|)$ be the polar decomposition of μ_Ω.

6) Claim 3: with the notation from claim 1, for $i \geq 1$, the trace of μ_Ω on U_i has polar decomposition $(\nu_i, \mathcal{H}^{n-1} \llcorner \partial^{U_i}\Omega)$. In particular, from the uniqueness of the polar decomposition it follows that $|\mu_\Omega|\big|_{U_i} = \mathcal{H}^{n-1} \llcorner \partial^{U_i}\Omega$ and $\nu_\Omega = \nu_i$ $|\mu_\Omega|$-a.e. on $\partial^{U_i}\Omega$.

Indeed, for each $\varphi \in \mathsf{C}_\mathsf{c}^\infty(U_i, \mathbb{R}^n)$,

$$\begin{aligned}\mu_\Omega \cdot \varphi = \int_\Omega \operatorname{div}\, \varphi \, \mathrm{d}\mathcal{L}^n &\overset{\text{spt } \varphi \Subset U_i}{=} \\ = \int_{U_i} \operatorname{div}\, \varphi \, \mathrm{d}\mathcal{L}^n &\overset{\text{lemma 7.17}}{=} \\ = \int \langle \varphi, \nu_i \rangle \, \mathrm{d}\big(\mathcal{H}^{n-1} \llcorner \partial^{U_i}\Omega\big),\end{aligned}$$

whence the claim, since $\mathcal{H}^{n-1} \llcorner \partial^{U_i}\Omega$ is a Radon measure on U_i and $\|\nu_i\| = 1$ almost everywhere on U_i with respect to $\mathcal{H}^{n-1} \llcorner \partial^{U_i}\Omega$.

7) For $i \geq 1$, $\partial\Omega \cap U_i = \partial^{U_i}\Omega$. It then follows from claim 3 that the Borel regular measure $\mathcal{H}^{n-1} \llcorner \partial\Omega$ and the positive Radon measure $|\mu_\Omega|$ have the same traces on U_i, namely, $\mathcal{H}^{n-1} \llcorner \partial^{U_i}\Omega$. Since both measures have support on $\partial\Omega$, and since $(U_i)_{i\geq 1}$ is a countable open cover of $\partial\Omega$, we conclude that $|\mu_\Omega| = \mathcal{H}^{n-1} \llcorner \partial\Omega$ (since, for each $A \in \mathscr{B}_{\mathbb{R}^n}$, we may write $A \cap \partial\Omega$ as a countable disjoint union $\dot\bigcup_{i\geq 1} A_i$ with A_i a Borel subset of U_i for each $i \geq 1$).

□

Corollary 7.19. *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Then $\mathcal{H}^{n-1} \llcorner \partial\Omega$ is a Radon measure.*

Remark 7.20 (outer normal to a Lipschitz domain). With the notation from the proof of the previous theorem, for each $i \geq 1$, the exterior normal to Ω coincides $\mathcal{H}^{n-1} \llcorner \partial\Omega$-a.e. with ν_i on $\partial\Omega \cap U_i = \partial^{U_i}\Omega$. In particular, it follows from remark 6.47 that, if $\partial\Omega$ is a C^1 hypersurface on a neighborhood of $p \in \partial\Omega$, we may choose ν_Ω on this neighborhood as the usual outer unit normal from Differential Geometry.

7.2. Regularization of Radon measures and BV functions

Proposition 7.21. *Let $(\phi_t)_{t>0}$ be the standard mollifier on $\mathbb{R}^m$. Then, for each $\epsilon > 0$, the convolution with ϕ_ϵ defines a continuous linear map $\phi_\epsilon * : \mathsf{C}_\mathsf{c}(\mathbb{R}^m, \mathbb{R}^n) \to \mathsf{C}_\mathsf{c}(\mathbb{R}^m, \mathbb{R}^n)$.*

Proof. Given $K \subset \mathbb{R}^m$ compact, ϕ_ϵ maps $\mathsf{C}_\mathsf{c}^\mathsf{K}(\mathbb{R}^m, \mathbb{R}^n)$ to $\mathsf{C}_\mathsf{c}^{\mathsf{K}_\epsilon}(\mathbb{R}^m, \mathbb{R}^n)$, where $K_\epsilon := K + \mathbb{B}(0, \epsilon)$ is the ϵ-neighborhood of K. Since, for all $\varphi \in \mathsf{C}_\mathsf{c}^\mathsf{K}(\mathbb{R}^m, \mathbb{R}^n)$, $\|\phi_\epsilon * \varphi\|_u \leq \|\varphi\|_u$, the linear map $\phi_\epsilon * : \mathsf{C}_\mathsf{c}^\mathsf{K}(\mathbb{R}^m, \mathbb{R}^n) \to \mathsf{C}_\mathsf{c}^{\mathsf{K}_\epsilon}(\mathbb{R}^m, \mathbb{R}^n)$ is bounded with respect to the norm of uniform convergence. □

With the same proof, given an open subset $\Omega \subset \mathbb{R}^m$, the convolution with ϕ_ϵ defines a continuous linear map $\phi_\epsilon * : \mathsf{C}_\mathsf{c}(\Omega_\epsilon, \mathbb{R}^n) \to \mathsf{C}_\mathsf{c}(\Omega, \mathbb{R}^n)$, where Ω_ϵ is given by definition 6.17. It then follows from proposition 4.39 that $(\phi_\epsilon *)^{\mathsf{t}} : \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n) \to \mathcal{M}_{\mathrm{loc}}(\Omega_\epsilon, \mathbb{R}^n)$ is a well defined linear map. We shall omit the "t" in the notation of this transpose, i.e. we denote it with the same notation "$\phi_\epsilon *$".

Definition 7.22 (regularization of $\mathbb{R}^n$-valued Radon measures). *Let Ω be an open subset of $\mathbb{R}^m$, $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $(\phi_t)_{t>0}$ the standard mollifier on $\mathbb{R}^m$. We define the t-approximation or t-regularization of μ by $\mu_t := \phi_t * \mu \in \mathcal{M}_{\mathrm{loc}}(\Omega_t, \mathbb{R}^n)$.*

Remark 7.23. The definition above extends definition 6.17 for $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^m|_\Omega, \mathbb{R}^n)$. That is, considering the embedding $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^m|_\Omega, \mathbb{R}^n) \subset \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ given by $f \mapsto \mathcal{L}^m|_\Omega \llcorner f$, we have

$$(\mathcal{L}^m|_\Omega \llcorner f)_\epsilon = \mathcal{L}^m|_\Omega \llcorner (f_\epsilon) \in \mathcal{M}_{\mathrm{loc}}(\Omega_\epsilon, \mathbb{R}).$$

Indeed, for all $f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^m|_\Omega, \mathbb{R}^n)$ and all $\varphi \in \mathsf{C}_{\mathsf{c}}(\Omega_\epsilon, \mathbb{R}^n)$,

$$\begin{aligned}(\mathcal{L}^m|_\Omega \llcorner f)_\epsilon \cdot \varphi &= \int_\Omega (\phi_\epsilon * \varphi) \cdot f \,\mathrm{d}\mathcal{L}^m \overset{1.108.i),\check{\phi}_\epsilon = \phi_\epsilon}{=} \\ &= \int_\Omega \varphi \cdot (\phi_\epsilon * f) \,\mathrm{d}\mathcal{L}^m = \\ &= \mathcal{L}^m|_\Omega \llcorner (f_\epsilon) \cdot \varphi.\end{aligned}$$

Proposition 7.24. *With the notation from the previous definition, let $\Omega \subset \mathbb{R}^m$ open and $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Define $\mu^\epsilon : \Omega_\epsilon \to \mathbb{R}^n$ by*

$$\mu^\epsilon(x) := \int_\Omega \phi_\epsilon(x - y) \,\mathrm{d}\mu(y).$$

Then $\mu^\epsilon \in \mathsf{C}^\infty(\Omega_\epsilon, \mathbb{R}^n)$ and

$$\mu_\epsilon = \mathcal{L}^m|_{\Omega_\epsilon} \llcorner \mu^\epsilon.$$

In particular, $\mu_\epsilon \ll \mathcal{L}^m|_{\Omega_\epsilon}$ and $|\mu_\epsilon| \le |\mu|_\epsilon$.

Proof. 1) Let $(\nu, |\mu|)$ be the polar decomposition of μ. For each closed ball $B \subset \Omega_\epsilon$ and for each multi-index $\alpha \in \mathbb{Z}^m_+$, we have, for all $x \in B$ and all $y \in \Omega$,

$$\left|\partial^\alpha_x \big(\phi_\epsilon(\cdot - y)\nu(y)\big)\right| = \left|\partial^\alpha \phi_\epsilon(x-y)\nu(y)\right| \le \|\partial^\alpha \phi_\epsilon\|_u \chi_{B + \mathbb{B}(0,\epsilon)}(y).$$

Since $\chi_{B+\mathbb{B}(0,\epsilon)} \in \mathsf{L}^1(|\mu|)$, we may apply the dominated convergence theorem to conclude that, for all $x \in B^{\mathrm{o}}$,

$$\exists \partial^\alpha \mu^\epsilon(x) = \int \partial^\alpha \phi_\epsilon(x-y) \,\mathrm{d}\mu(y).$$

2) For all $\varphi \in \mathsf{C_c}(\Omega_\epsilon, \mathbb{R}^n)$, we have, for all $x \in \Omega_\epsilon$ and all $y \in \Omega$,

$$\phi_\epsilon(x-y)\|\varphi(x)\| \leq \|\phi_\epsilon\|_u \|\varphi\|_u \chi_{\mathrm{spt}\ \varphi}(x)\chi_{\mathrm{spt}\ \varphi+\mathbb{B}(0,\epsilon)}(y),$$

hence $(x,y) \in \Omega_\epsilon \times \Omega \mapsto \phi_\epsilon(x-y)\varphi(x) \in \mathbb{R}^n$ is summable with respect to $\mathcal{L}^m|_{\Omega_\epsilon} \otimes |\mu|$. That justifies the application of Fubini's theorem in the following computation:

$$\begin{aligned}
&\int_{\Omega_\epsilon} \varphi(x) \cdot \mu^\epsilon(x)\, \mathrm{d}\mathcal{L}^m(x) = \\
&= \int_{\Omega_\epsilon} \varphi(x) \Big(\int_\Omega \phi_\epsilon(x-y) \cdot \nu(y)\, \mathrm{d}|\mu|(y) \Big)\, \mathrm{d}\mathcal{L}^m(x) \overset{\text{Fubini}}{=} \\
&= \int_\Omega \Big(\int_{\Omega_\epsilon} \phi_\epsilon(x-y)\varphi(x)\, \mathrm{d}\mathcal{L}^m(x) \Big) \cdot \nu(y)\, \mathrm{d}|\mu|(y) \overset{\check{\phi}_\epsilon = \phi_\epsilon}{=} \\
&= \int_\Omega \phi_\epsilon * \varphi(y) \cdot \mathrm{d}\mu(y) = \\
&= \mu_\epsilon \cdot \varphi,
\end{aligned}$$

thus showing that $\mu_\epsilon = \mathcal{L}^m|_{\Omega_\epsilon} \llcorner \mu^\epsilon$, as asserted. In particular, since $\|\mu^\epsilon\| \leq |\mu|^\epsilon$ (by the triangle inequality), it follows that $|\mu_\epsilon| = \mathcal{L}^m|_{\Omega_\epsilon} \llcorner \|\mu^\epsilon\| \leq \mathcal{L}^m|_{\Omega_\epsilon} \llcorner |\mu|^\epsilon = |\mu|_\epsilon$.

□

Theorem 7.25 (Weak-star convergence of regularized Radon measures). *Let Ω be an open subset of $\mathbb{R}^m$ and $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Then, as $\epsilon \downarrow 0$,*

$$\mu_\epsilon \overset{*}{\rightharpoonup} \mu \text{ and } |\mu_\epsilon| \overset{*}{\rightharpoonup} |\mu|,$$

in the sense that, for all $\varphi \in \mathsf{C_c}(\Omega, \mathbb{R}^n)$, $\mu_\epsilon \cdot \varphi \to \mu \cdot \varphi$ and similarly for the total variations. Moreover, for all $\epsilon > 0$ and $E \in \mathscr{B}_{\Omega_\epsilon}$,

$$|\mu_\epsilon|(E) \leq |\mu|(E_\epsilon),$$

where $E_\epsilon := E + \mathbb{U}(0,\epsilon)$ is the ϵ-neighborhood of E.

Proof.

1) Let $\varphi \in \mathsf{C_c}(\Omega, \mathbb{R}^n)$ and take $\epsilon_0 > 0$ such that $\mathrm{spt}\ \varphi \subset \Omega_{\epsilon_0}$. Put $K := \mathrm{spt}\ \varphi + \mathbb{B}(0,\epsilon_0) \Subset \Omega$ and $\varphi_\epsilon := \phi_\epsilon * \varphi$, where $(\phi_\epsilon)_{\epsilon>0}$ is the standard mollifier on $\mathbb{R}^m$. Then, for all $0 < \epsilon < \epsilon_0$, $\mathrm{spt}\ \varphi_\epsilon \subset K$ and $\varphi_\epsilon \to \varphi$ uniformly, by 1.111.ii). It then follows that

$$\mu_\epsilon \cdot \varphi = \mu \cdot \varphi_\epsilon \to \mu \cdot \varphi$$

as $\epsilon \downarrow 0$, thus showing that $\mu_\epsilon \overset{*}{\rightharpoonup} \mu$ as $\epsilon \downarrow 0$.

2) For all $\epsilon > 0$, it follows from proposition 7.24 and remark 4.32 that $|\mu_\epsilon| = \mathcal{L}^m|_{\Omega_\epsilon} \llcorner \|\mu^\epsilon\|$. On the other hand, it follows from the triangle inequality that, for all $x \in \Omega_\epsilon$, $\|\mu^\epsilon(x)\| \leq \int_\Omega \phi_\epsilon(x-y)\,\mathrm{d}|\mu|(y)$. Therefore, for all $\epsilon > 0$ and all $E \in \mathscr{B}_{\Omega_\epsilon}$,

$$
\begin{aligned}
|\mu_\epsilon|(E) &= \int_E \|\mu^\epsilon(x)\|\,\mathrm{d}\mathcal{L}^m(x) \leq \\
&\leq \int_{\Omega_\epsilon} \chi_E(x)\Big(\int_\Omega \phi_\epsilon(x-y)\,\mathrm{d}|\mu|(y)\Big)\,\mathrm{d}\mathcal{L}^m(x) \overset{\text{Tonelli}}{=} \\
&= \int_\Omega \underbrace{\int_{\Omega_\epsilon} \phi_\epsilon(x-y)\chi_E(x)\,\mathrm{d}\mathcal{L}^m(x)}_{\leq \chi_{E_\epsilon}(y)}\,\mathrm{d}|\mu|(y) \leq \\
&\leq |\mu|(E_\epsilon).
\end{aligned}
$$

3) Let V be a relatively compact open subset of Ω. Take $\epsilon_0 > 0$ such that $V \Subset \Omega_{\epsilon_0}$ and $(\epsilon_k)_{k\in\mathbb{N}}$ a sequence in $(0,\epsilon_0)$ with $\epsilon_k \downarrow 0$.

In view of part 1), $(\mu_{\epsilon_k}|_V)$ is a sequence in $\mathcal{M}_{\mathrm{loc}}(V,\mathbb{R}^n)$ weak-star convergent to $\mu|_V$. Thus, for all $U \subset V$ open, it follows from proposition 4.57 that $|\mu|\big|_V(U) = \big|\mu|_V\big|(U) \leq \liminf\big|\mu_{\epsilon_k}|_V\big|(U) = \liminf|\mu_{\epsilon_k}|\big|_V(U)$.

On the other hand, given $K \subset V$ compact, in view of part 2) we have $|\mu_{\epsilon_k}|(K) \leq |\mu|(K_{\epsilon_k}) \to |\mu|(K)$ as $k \to \infty$, since the sequence of relatively compact open sets $(K_{\epsilon_k})_{k\in\mathbb{N}}$ decreases to K. Hence, $\limsup|\mu_{\epsilon_k}|\big|_V(K) = \limsup|\mu_{\epsilon_k}|(K) \leq |\mu|(K) = |\mu|\big|_V(K)$. Therefore, applying theorem 4.54, we conclude that the sequence of traces $(|\mu_{\epsilon_k}|\big|_V)_{k\in\mathbb{N}}$ is weak-star convergent to $|\mu|\big|_V$. Since the decreasing sequence $(\epsilon_k)_{k\in\mathbb{N}}$ in $(0,\epsilon_0)$ was arbitrarily taken, we conclude that, for all $\varphi \in \mathsf{C_c}(V,\mathbb{R})$, $\int_\Omega \varphi\,\mathrm{d}|\mu_\epsilon| \to \int_\Omega \varphi\,\mathrm{d}|\mu|$ as $\epsilon \to 0$. Since the relatively compact open subset $V \subset \Omega$ was arbitrarily taken, we conclude that $|\mu_\epsilon| \overset{*}{\rightharpoonup} |\mu|$ and the thesis follows.

□

Proposition 7.26 (regularization of BV functions). *Let Ω be an open subset of $\mathbb{R}^n$, $f \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$, $(\phi_\epsilon)_{\epsilon>0}$ the standard mollifier on $\mathbb{R}^n$, $f_\epsilon := \phi_\epsilon * f \in \mathsf{C}^\infty(\Omega_\epsilon)$ and $(\nabla^{\mathsf{w}} f)_\epsilon := \phi_\epsilon * \nabla^{\mathsf{w}} f \in \mathcal{M}_{\mathrm{loc}}(\Omega_\epsilon,\mathbb{R}^n)$. Then:*

i) $(\nabla^{\mathsf{w}} f)_\epsilon = \mathcal{L}^n|_{\Omega_\epsilon} \llcorner \nabla(f_\epsilon)$.

ii) $f_\epsilon \to f$ in the sense of $\mathsf{L}^1_{\mathsf{loc}}(\Omega)$.

iii) For each open $V \Subset \Omega$,

$$\big(\mathcal{L}^n|_{\Omega_\epsilon} \llcorner \nabla(f_\epsilon)\big)|_V \overset{*\mathsf{f}}{\rightharpoonup} (\nabla^{\mathsf{w}} f)|_V \text{ and } \big(\mathcal{L}^n|_{\Omega_\epsilon} \llcorner \|\nabla(f_\epsilon)\|\big)|_V \overset{*\mathsf{f}}{\rightharpoonup} |\nabla^{\mathsf{w}} f|\big|_V$$

as $\epsilon \downarrow 0$.

PROOF. For each $\varphi \in \mathsf{C}^\infty_\mathsf{c}(\Omega_\epsilon, \mathbb{R}^n)$,

$$\begin{aligned}\mathcal{L}^n|_{\Omega_\epsilon} \llcorner \nabla(f_\epsilon) \cdot \varphi &= \int \varphi \cdot \nabla(f_\epsilon) \,\mathrm{d}\mathcal{L}^n = \\ &= -\int \mathrm{div}\ \varphi f_\epsilon \,\mathrm{d}\mathcal{L}^n \overset{\text{prop. 1.108.}i)}{=} \\ &= -\int (\mathrm{div}\ \varphi)_\epsilon f \,\mathrm{d}\mathcal{L}^n \overset{\text{prop. 1.108.}j)}{=} \\ &= -\int \mathrm{div}\,(\varphi_\epsilon) f \,\mathrm{d}\mathcal{L}^n = \\ &= \int \varphi_\epsilon \cdot \,\mathrm{d}\,\nabla^\mathsf{w} f \overset{\text{def. 7.22}}{=} \\ &= \int \varphi \cdot \,\mathrm{d}(\nabla^\mathsf{w} f)_\epsilon,\end{aligned}$$

thus showing assertion i).

Assertion ii) was already proved in 6.20.

To prove assertion iii), let $V \Subset \Omega$ open. Take $\epsilon_0 > 0$ such that $V \Subset \Omega_{\epsilon_0}$. It follows from theorem 7.25 that $(\nabla^\mathsf{w} f)_\epsilon|_V \overset{*}{\rightharpoonup} (\nabla^\mathsf{w} f)|_V$ and $|(\nabla^\mathsf{w} f)_\epsilon|\big|_V \overset{*}{\rightharpoonup} |\nabla^\mathsf{w} f|\big|_V$. Since, by theorem 7.25, $\sup\{|(\nabla^\mathsf{w} f)_\epsilon|(V) \mid 0 < \epsilon < \epsilon_0\} \leq |\nabla^\mathsf{w} f|(V_{\epsilon_0}) < \infty$, the thesis follows from part i) and from proposition 4.49.

□

7.3. First properties of BV functions

PROPOSITION 7.27. *Let Ω be an open subset of $\mathbb{R}^n$ and $(f_k)_{k\in\mathbb{N}}$ a sequence in $\mathsf{BV}_{\mathrm{loc}}(\Omega)$.*

i) If $f \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$ and $f_k \to f$ in $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$, then $\nabla^\mathsf{w} f_k \overset{*}{\rightharpoonup} \nabla^\mathsf{w} f$.
ii) If $f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$, $f_k \to f$ in $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ and there exists $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ such that $\nabla^\mathsf{w} f_k \overset{*}{\rightharpoonup} \mu$, then $f \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$ and $\nabla^\mathsf{w} f = \mu$.

PROOF.

i) For all $\varphi \in \mathsf{C}_\mathsf{c}^\infty(\Omega, \mathbb{R}^n)$,

$$\begin{aligned}\int \varphi \cdot \mathrm{d}\,\nabla^\mathsf{w} f_k = -\int \operatorname{div}\, \varphi f_k \,\mathrm{d}\mathcal{L}^n &\overset{k\to\infty}{\to} \\ \to -\int \operatorname{div}\, \varphi f \,\mathrm{d}\mathcal{L}^n &= \\ = \int \varphi \cdot \mathrm{d}\,\nabla^\mathsf{w} f&,\end{aligned}$$

thus showing that $\nabla^\mathsf{w} f_k \overset{*}{\rightharpoonup} \nabla^\mathsf{w} f$.

ii) For all $\varphi \in \mathsf{C}_\mathsf{c}^\infty(\Omega, \mathbb{R}^n)$,

$$\begin{aligned}\int \varphi \cdot \mathrm{d}\mu &= \lim_{k\to\infty} \int \varphi \cdot \mathrm{d}\,\nabla^\mathsf{w} f_k = \\ &= -\lim_{k\to\infty} \int \operatorname{div}\, \varphi f_k \,\mathrm{d}\mathcal{L}^n = \\ &= -\int \operatorname{div}\, \varphi f \,\mathrm{d}\mathcal{L}^n,\end{aligned}$$

hence f admits weak gradient $\nabla^\mathsf{w} f = \mu \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, i.e. $f \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$.

□

COROLLARY 7.28. *Let Ω be an open subset of $\mathbb{R}^n$, $f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ and $(f_i)_{i\in\mathbb{N}}$ a sequence in $\mathsf{BV}_{\mathrm{loc}}(\Omega)$ such that $f_i \to f$ in $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$.*

i) If, for each compact $K \subset \Omega$, $\sup\{|\nabla^\mathsf{w} f_i|(K) \mid i \in \mathbb{N}\} < \infty$, then $f \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$ and $\nabla^\mathsf{w} f_i \overset{*}{\rightharpoonup} \nabla^\mathsf{w} f$.

ii) If $\sup\{|\nabla^\mathsf{w} f_i|(\Omega) \mid i \in \mathbb{N}\} < \infty$, then $f \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$ with $\nabla^\mathsf{w} f \in \mathcal{M}(\Omega, \mathbb{R}^n)$ and $\nabla^\mathsf{w} f_i \overset{*\mathsf{f}}{\rightharpoonup} \nabla^\mathsf{w} f$.

PROOF.

i) By corollary 4.63, there exists a subsequence $(f_{i_j})_{j\in\mathbb{N}}$ of $(f_i)_{i\in\mathbb{N}}$ and $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ such that $\nabla^\mathsf{w} f_{i_j} \overset{*}{\rightharpoonup} \mu$. It then follows from proposition 7.27.ii) that $f \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$ and $\nabla^\mathsf{w} f = \mu$. Then, from 7.27.i) we conclude that $\nabla^\mathsf{w} f_i \overset{*}{\rightharpoonup} \nabla^\mathsf{w} f$, as asserted.

ii) By the previous item, $f \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$ and $\nabla^\mathsf{w} f_i \overset{*}{\rightharpoonup} \nabla^\mathsf{w} f$. By proposition 4.49, it then follows that $\nabla^\mathsf{w} f \in \mathcal{M}(\Omega, \mathbb{R}^n)$ and $\nabla^\mathsf{w} f_i \overset{*\mathsf{f}}{\rightharpoonup} \nabla^\mathsf{w} f$.

□

PROPOSITION 7.29 (Product rule for BV, part I). *Let Ω be an open subset of $\mathbb{R}^n$, $f \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$ and $g : \Omega \to \mathbb{R}$ locally Lipschitz. Then $fg \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$ and $\nabla^\mathsf{w}(fg) = \nabla^\mathsf{w} f \llcorner g + \mathcal{L}^n \llcorner f\, \nabla^\mathsf{w} g$.*

PROOF.

1 Case 1: $g \in \mathsf{C}^\infty(\Omega)$. Then, for all $\varphi \in \mathsf{C}^\infty_{\mathsf{c}}(\Omega)$,

$$\int \nabla\varphi f g \,\mathrm{d}\mathcal{L}^n = \int \underbrace{\nabla(\varphi g)}_{\in \mathsf{C}^\infty_{\mathsf{c}}(\Omega)} f \,\mathrm{d}\mathcal{L}^n - \int \varphi \nabla g\, f \,\mathrm{d}\mathcal{L}^n =$$

$$= - \int \varphi g \,\mathrm{d}\, \nabla^{\mathsf{w}} f - \int \varphi f \nabla g \,\mathrm{d}\mathcal{L}^n,$$

whence the thesis.

2 General case. It is clear that $fg \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ and that $\mu := \nabla^{\mathsf{w}} f \llcorner g + \mathcal{L}^n \llcorner f\, \nabla^{\mathsf{w}} g \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. We must show that the weak gradient of fg exists and coincides with μ. By the locality of the weak derivative, cf. theorem 7.7, it suffices to prove the latter assertion for the restriction of fg to a given $V \Subset \Omega$ open. Let $\epsilon_0 > 0$ such that $V \Subset \Omega_{\epsilon_0}$ and $(\phi_\epsilon)_{\epsilon>0}$ the standard mollifier on $\mathbb{R}^n$. Fix a sequence $(\epsilon_i)_{i\in\mathbb{N}}$ in $(0, \epsilon_0)$ decreasing to 0. Denoting by a subscript "ϵ" the convolutions with ϕ_ϵ, as usual, we have:

(a) $g_i := g_{\epsilon_i} \in \mathsf{C}^\infty(\Omega_{\epsilon_0})$ and, in view of theorem 6.20.iv), $g_i \to g$ uniformly on V. Hence $fg_i \to fg$ in $\mathsf{L}^1_{\mathsf{loc}}(V)$.

(b) For each $i \in \mathbb{N}$, we may apply case 1 with V in place of Ω to fg_i to conclude that $fg_i \in \mathsf{BV}_{\mathrm{loc}}(V)$ and $\nabla^{\mathsf{w}}(fg_i) = \nabla^{\mathsf{w}} f \llcorner g_i + \mathcal{L}^n \llcorner f \nabla g_i \in \mathcal{M}_{\mathrm{loc}}(V, \mathbb{R}^n)$.

(c) For each $\varphi \in \mathsf{C}^\infty_{\mathsf{c}}(V, \mathbb{R}^n)$, $g_i\varphi \to g\varphi$ pointwise on V and $\|g_i\varphi\| \leq \sup\{\|g_i|_{\mathrm{spt}\,\varphi}\|_u \mid i \in \mathbb{N}\} \cdot \|\varphi\| \in \mathsf{L}^1(|\nabla^{\mathsf{w}} f|)$, hence we may apply the dominated convergence theorem to conclude that

$$\int \varphi \cdot \mathrm{d}(\nabla^{\mathsf{w}} f \llcorner g_i) \to \int \varphi \cdot \mathrm{d}(\nabla^{\mathsf{w}} f \llcorner g),$$

i.e. $\nabla^{\mathsf{w}} f \llcorner g_i \overset{*}{\rightharpoonup} \nabla^{\mathsf{w}} f \llcorner g$ on $\mathcal{M}_{\mathrm{loc}}(V, \mathbb{R}^n)$.

(d) Since $g \in \mathsf{W}^{1,\infty}_{\mathsf{loc}}(\Omega)$, it follows from theorem 6.20.vi) that, for all $i \in \mathbb{N}$, $\nabla g_i = (\nabla^{\mathsf{w}} g)_{\epsilon_i}$. Thus $\|\nabla g_i\| \leq \|\nabla^{\mathsf{w}} g\|_{\mathsf{L}^\infty(V_{\epsilon_0})}$ and, by 6.20.iii), $\nabla g_i \to \nabla^{\mathsf{w}} g$ $\mathcal{L}^n$-a.e. on V. Hence, for each $\varphi \in \mathsf{C}^\infty_{\mathsf{c}}(V, \mathbb{R}^n)$, $\varphi f \nabla g_i \to \varphi f\, \nabla^{\mathsf{w}} g$ $\mathcal{L}^n$-a.e. on V, with $\|\varphi f \nabla g_i\| \leq \|\nabla^{\mathsf{w}} g\|_{\mathsf{L}^\infty(V_{\epsilon_0})} |f| \|\varphi\| \in \mathsf{L}^1(\mathcal{L}^n|_V)$; therefore, by the dominated convergence theorem, $\int \varphi \cdot f \nabla g_i \,\mathrm{d}\mathcal{L}^n \to \int \varphi \cdot f\, \nabla^{\mathsf{w}} g \,\mathrm{d}\mathcal{L}^n$, whence $\mathcal{L}^n \llcorner f \nabla g_i \overset{*}{\rightharpoonup} \mathcal{L}^n \llcorner f\, \nabla^{\mathsf{w}} g$ on $\mathcal{M}_{\mathrm{loc}}(V, \mathbb{R}^n)$.

(e) From the two previous steps we conclude that $\nabla^{\mathsf{w}}(fg_i) \overset{*}{\rightharpoonup} \mu$ on $\mathcal{M}_{\mathrm{loc}}(V, \mathbb{R}^n)$. By proposition 7.27.ii) with V in place of Ω, it follows that $fg \in \mathsf{BV}_{\mathrm{loc}}(V)$ and $\nabla^{\mathsf{w}}(fg) = \mu$, as we wanted to show.

□

Given $\Omega \subset \mathbb{R}^n$ open, it will be useful in the subsequent developments to consider the *variation* of a function in $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$, in the sense of the definition below, even if it does not belong to $\mathsf{BV}_{\mathrm{loc}}(\Omega)$.

DEFINITION 7.30 (variation of a function in $\mathsf{L}^1_{\mathsf{loc}}$). Let $\Omega \subset \mathbb{R}^n$ open and $f \in \mathsf{L}^1_{\mathsf{loc}}(\Omega)$. We define, for each open $V \subset \Omega$,

$$\mathsf{Var}(f, V) := \sup\{\int f \operatorname{div} \varphi \, \mathrm{d}\mathcal{L}^n \mid \varphi \in \mathsf{C}^\infty_{\mathsf{c}}(V, \mathbb{R}^n), \|\varphi\|_u \leq 1\}.$$

EXERCISE 7.31 (variation of a function in $\mathsf{L}^1_{\mathsf{loc}}$). Let $\Omega \subset \mathbb{R}^n$ open and $f \in \mathsf{L}^1_{\mathsf{loc}}(\Omega)$. Define, for each $B \subset \Omega$,

$$\mathsf{Var}_f(B) := \inf\{\mathsf{Var}(f, U) \mid U \text{ open}, B \subset U\}.$$

Then Var_f is a Borel regular measure on U which extends the variation $\mathsf{Var}(f, \cdot)$. Moreover, $f \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$ if, and only if, Var_f is a positive Radon measure on Ω, in which case it coincides with $|\nabla^{\mathsf{w}} f|$.

We call Var_f the *variation measure* of f.

PROPOSITION 7.32 (lower semicontinuity of the variation). *Let $\Omega \subset \mathbb{R}^n$ open, $(f_i)_{i\in\mathbb{N}}$ a sequence in $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ and $f \in \mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$ such that $f_i \to f$ in $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n|_\Omega)$. Then, for all $V \subset \Omega$ open,*

$$\mathsf{Var}(f, V) \leq \liminf \mathsf{Var}(f_i, V).$$

In particular, if $f_i \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$ for all $i \in \mathbb{N}$ and the second member of the equality above is finite for each open $V \Subset \Omega$, then $f \in \mathsf{BV}_{\mathrm{loc}}(\Omega)$.

PROOF. For each $\varphi \in \mathsf{C}^\infty_{\mathsf{c}}(V, \mathbb{R}^n)$ with $\|\varphi\|_u \leq 1$,

$$\int f \operatorname{div} \varphi \, \mathrm{d}\mathcal{L}^n = \lim \int f_i \operatorname{div} \varphi \, \mathrm{d}\mathcal{L}^n \leq \liminf \mathsf{Var}(f_i, V),$$

and taking the sup on the first member yields the thesis. □

We now prove a theorem on approximation of BV functions by smooth functions.

THEOREM 7.33 (Almgren). *Let Ω be an open subset of $\mathbb{R}^n$ and $f \in \mathsf{BV}(\Omega)$. There exists a sequence $(f_i)_{i\in\mathbb{N}} \in \mathsf{BV}(\Omega) \cap \mathsf{C}^\infty(\Omega)$ such that $f_i \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and $|\nabla^{\mathsf{w}} f_i|(\Omega) \to |\nabla^{\mathsf{w}} f|(\Omega)$.*

PROOF. 1) Fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ sufficiently large so that, putting

$$U := \underbrace{\Omega_{\frac{1}{N}}}_{=\{x\in\Omega \mid d(x,\Omega^c) > \frac{1}{N}\}} \cap \mathbb{U}(0, N),$$

we have $|\nabla^{\mathsf{w}} f|(\Omega - U) < \epsilon$. We can choose such N because $|\nabla^{\mathsf{w}} f|$ is a finite Radon measure and the second member above increases to Ω as $N \to \infty$. We now define $(U_i)_{i\in\mathbb{N}}$ by

$$\begin{aligned} &U_0 := U \\ &U_i := \Omega_{\frac{1}{N+i}} \cap \mathbb{U}(0, N+i), \qquad i \geq 1. \end{aligned}$$

Then $(U_i)_i$ is an increasing sequence of open relatively compact subsets of Ω which increases to Ω.

Set $U_{-1} := \emptyset$ and define $(V_i)_{i\in\mathbb{N}}$ by

$$V_i := U_{i+1} \setminus \overline{U_{i-1}}, \qquad i \geq 0.$$

Note that, for each $i < j \in \mathbb{N}$, $V_i \cap V_j = \emptyset$ if $j - 1 \geq i + 1$, i.e. if $j \geq i + 2$; hence, each V_i meets at most 3 other V_j's (including itself). Thus, $(V_i)_{i\in\mathbb{N}}$ is a locally finite open cover of Ω with $V_i \Subset \Omega$ for each $i \geq 0$. By theorem 6.8, there exists a smooth partition of unity $(\xi_i)_{i\in\mathbb{N}}$ of Ω such that, for all $i \geq 0$, $\xi_i \in \mathsf{C}^\infty_{\mathsf{c}}(V_i)$.

2) Let $(\phi_t)_{t>0}$ be the standard mollifier on $\mathbb{R}^n$. Note that $f\xi_i \in \mathsf{L}^1(\mathcal{L}^n)$, $f\nabla\xi_i \in \mathsf{L}^1(\mathcal{L}^n, \mathbb{R}^n)$ and both functions have compact support contained in spt $\xi_i \Subset V_i$. Then, by proposition 1.108.d) and by theorem 1.111.i) we may choose, for each $i \geq 0$, $\epsilon_i > 0$ sufficiently small so that

$$\begin{aligned} &\operatorname{spt} \phi_{\epsilon_i} * (\xi_i f) \Subset V_i \\ &\int |\phi_{\epsilon_i} * (f\xi_i) - f\xi_i| \, \mathrm{d}\mathcal{L}^n < \epsilon/2^{i+1} \\ &\int \|\phi_{\epsilon_i} * (f\nabla\xi_i) - f\nabla\xi_i\| \, \mathrm{d}\mathcal{L}^n < \epsilon/2^{i+1}. \end{aligned} \tag{7.5}$$

3) Define

$$f_\epsilon := \sum_{i=0}^{\infty} \phi_{\epsilon_i} * (f\xi_i).$$

Since spt $\phi_{\epsilon_i} * (f\xi_i) \Subset V_i$ for each $i \geq 0$ and since $(V_i)_{i\geq 0}$ is a locally finite family of subsets of Ω, the sum above is locally finite, hence $f_\epsilon \in \mathsf{C}^\infty(\Omega)$.

Since $f = \sum_{i=0}^{\infty} f\xi_i$, it follows from (7.5) and from the monotone convergence theorem that

$$\|f_\epsilon - f\|_{\mathsf{L}^1(\mathcal{L}^n|_\Omega)} \leq \sum_{i=0}^{\infty} \|\phi_{\epsilon_i} * (f\xi_i) - f\xi_i\|_{\mathsf{L}^1(\mathcal{L}^n|_\Omega)} < \epsilon,$$

i.e. $f_\epsilon \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$ as $\epsilon \to 0$.

It remains to show that $f_\epsilon \in \mathsf{BV}(\Omega)$ and $|\nabla f_\epsilon|(\Omega) \to |\nabla^{\mathsf{w}} f|(\Omega)$ as $\epsilon \to 0$.

4) It follows from the previous step and proposition 7.32 that

$$|\nabla^{\mathsf{w}} f|(\Omega) = \mathsf{Var}(f,\Omega) \le \liminf_{\epsilon\to 0} \mathsf{Var}(f_\epsilon,\Omega).$$

We will then achieve the thesis once we show that $\limsup_{\epsilon\to 0}\mathsf{Var}(f_\epsilon,\Omega) \le |\nabla^{\mathsf{w}} f|(\Omega)$.

5) For all $\varphi \in \mathsf{C}^\infty_{\mathsf{c}}(\Omega,\mathbb{R}^n)$ with $\|\varphi\|\le 1$, we have, noting that $f_\epsilon \operatorname{div}\,\varphi = \sum_{i=0}^\infty \phi_{\epsilon_i} *(f\xi_i)\operatorname{div}\,\varphi$ is a finite sum (because $\operatorname{spt}\,\varphi$ is compact subset of Ω and $(\operatorname{spt}\,\xi_i)_{i\in\mathbb{N}}$ is a locally finite family of subsets of Ω):

$$\int_\Omega f_\epsilon \operatorname{div}\,\varphi\,\mathrm{d}\mathcal{L}^n = \sum_{i=0}^\infty \int_\Omega \phi_{\epsilon_i} *(f\xi_i)\operatorname{div}\,\varphi\,\mathrm{d}\mathcal{L}^n \overset{\text{prop. 1.108.}i,j)}{=}$$

$$= \sum_{i=0}^\infty \int_\Omega f\xi_i \operatorname{div}\,(\phi_{\epsilon_i} * \varphi)\,\mathrm{d}\mathcal{L}^n =$$

$$= \sum_{i=0}^\infty \int_\Omega f \operatorname{div}\,(\xi_i[\phi_{\epsilon_i} * \varphi])\,\mathrm{d}\mathcal{L}^n - \sum_{i=0}^\infty \int_\Omega \underbrace{\langle f\nabla\xi_i, \phi_{\epsilon_i} * \varphi\rangle}_{=\langle \phi_{\epsilon_i} *(f\nabla\xi_i),\varphi\rangle}\,\mathrm{d}\mathcal{L}^n \overset{\sum_{i=0}^\infty \nabla\xi_i\equiv 0}{=}$$

$$= \underbrace{\sum_{i=0}^\infty \int_\Omega f \operatorname{div}\,(\xi_i[\phi_{\epsilon_i} * \varphi])\,\mathrm{d}\mathcal{L}^n}_{=:I_1} - \underbrace{\sum_{i=0}^\infty \int_\Omega \langle \varphi, \phi_{\epsilon_i} *(f\nabla\xi_i) - f\nabla\xi_i\rangle\,\mathrm{d}\mathcal{L}^n}_{=:I_2}.$$

It follows from (7.5) that $|I_2| < \epsilon$. On the other hand, since $\|\xi_i[\phi_{\epsilon_i} * \varphi]\| \le 1$ for all $i \ge 0$, we have

$$|I_1| = \left|\int_\Omega f \operatorname{div}\,(\xi_0[\phi_{\epsilon_0} * \varphi])\,\mathrm{d}\mathcal{L}^n + \sum_{i=1}^\infty \int_\Omega f \operatorname{div}\,(\xi_i[\phi_{\epsilon_i} * \varphi])\,\mathrm{d}\mathcal{L}^n\right| \le$$

$$\le |\nabla^{\mathsf{w}} f|(\Omega) + \sum_{i=1}^\infty |\nabla^{\mathsf{w}} f|(V_i).$$

Note that, since V_i does not intersect V_j if $j \ge i+2$, we have

$$V_1 \,\dot\cup\, V_3 \,\dot\cup\, V_5 \cdots \subset \Omega\setminus U$$
$$V_2 \,\dot\cup\, V_4 \,\dot\cup\, V_6 \cdots \subset \Omega\setminus U,$$

whence $\sum_{i=1}^\infty |\nabla^{\mathsf{w}} f|(V_i) \le 2|\nabla^{\mathsf{w}} f|(\Omega\setminus U) < 2\epsilon$ by our choice of U in part 1). It then follows that $|I_1| \le |\nabla^{\mathsf{w}} f|(\Omega) + 2\epsilon$, whence

$$\mathsf{Var}(f_\epsilon,\Omega) \le |\nabla^{\mathsf{w}} f|(\Omega) + 3\epsilon.$$

Therefore, $\limsup_{\epsilon\to 0}\mathsf{Var}(f_\epsilon\Omega) \le |\nabla^{\mathsf{w}} f|(\Omega)$, as asserted.

$\square$

REMARK 7.34. With the same hypothesis from theorem 7.33, if $f \in \mathsf{BV}(\Omega)\cap\mathsf{L}^\infty(\mathcal{L}^n|_\Omega)$, there exists a sequence $(f_i)_{i\in\mathbb{N}} \in \mathsf{BV}(\Omega)\cap\mathsf{C}^\infty(\Omega)$

such that $f_i \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$, $|\nabla^{\mathsf{w}} f_i|(\Omega) \to |\nabla^{\mathsf{w}} f|(\Omega)$ and, for all $i \in \mathbb{N}$, $\|f_i\|_{\mathsf{L}^\infty(\mathcal{L}^n|_\Omega)} \leq 3\|f\|_{\mathsf{L}^\infty(\mathcal{L}^n|_\Omega)}$. That follows from the same proof of theorem 7.33, noting that, for each $\epsilon > 0$ and for each $x \in \Omega$, the sum in step 3 of the proof defining $f_\epsilon(x)$ has at most 3 nonzero terms (since x belongs to at most 3 of the V_i's), each of which bounded by $\|f\|_{\mathsf{L}^\infty(\mathcal{L}^n|_\Omega)}$.

Corollary 7.35 (approximation by smooth functions). *Let $\Omega = \mathbb{R}^n$ or Ω be a Lipschitz domain in $\mathbb{R}^n$, and $f \in \mathsf{BV}(\Omega)$. There exists a sequence $(f_i)_{i\in\mathbb{N}} \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n)$ such that $f_i|_\Omega \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and $|\nabla^{\mathsf{w}} f_i|(\Omega) \to |\nabla^{\mathsf{w}} f|(\Omega)$.*

Proof. For each $i \in \mathbb{N}$, by theorem 7.33 there exists $g_i \in \mathsf{BV}(\Omega) \cap \mathsf{C}^\infty(\Omega)$ such that $\|g_i - f\|_{\mathsf{L}^1(\mathcal{L}^n|_\Omega)} < \frac{1}{i}$ and $\big||\nabla^{\mathsf{w}} g_i|(\Omega) - |\nabla^{\mathsf{w}} f|(\Omega)\big| < 1/i$. Since $\mathsf{BV}(\Omega) \cap \mathsf{C}^\infty(\Omega) \subset \mathsf{W}^{1,1}(\Omega)$, we may apply corollary 6.43 (or 6.21 for $\Omega = \mathbb{R}^n$) to find, for each $i \in \mathbb{N}$, $f_i \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n)$ such that $\|g_i - f_i\|_{\mathsf{L}^1(\mathcal{L}^n|_\Omega)} < \frac{1}{i}$ and $\left|\int_\Omega \big(\|\nabla f_i\| - \|\nabla g_i\|\big)\,\mathrm{d}\mathcal{L}^n\right| < \frac{1}{i}$. It then follows that

$$f_i|_\Omega \to f \in \mathsf{L}^1(\mathcal{L}^n|_\Omega) \quad \text{and} \quad |\nabla^{\mathsf{w}} f_i|(\Omega) = \int_\Omega \|\nabla f_i\|\,\mathrm{d}\mathcal{L}^n \to |\nabla^{\mathsf{w}} f|(\Omega).$$

□

Proposition 7.36 (Product rule for BV, part II). *Let Ω be an open subset of $\mathbb{R}^n$. If $f, g \in \mathsf{BV}(\Omega) \cap \mathsf{L}^\infty(\mathcal{L}^n|_\Omega)$, then $fg \in \mathsf{BV}(\Omega)$.*

Proof. By theorem 7.33 and remark 7.34, there exist sequences $(f_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$ in $\mathsf{BV}(\Omega) \cap \mathsf{C}^\infty(\Omega) \cap \mathsf{L}^\infty(\mathcal{L}^n|_\Omega)$ such that

- $f_i \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and $g_i \to g$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$;
- $|\nabla^{\mathsf{w}} f_i|(\Omega) \to |\nabla^{\mathsf{w}} f|(\Omega)$ and $|\nabla^{\mathsf{w}} g_i|(\Omega) \to |\nabla^{\mathsf{w}} g|(\Omega)$;
- for all $i \in \mathbb{N}$, $\|f_i\|_{\mathsf{L}^\infty(\mathcal{L}^n|_\Omega)} \leq C$ and $\|g_i\|_{\mathsf{L}^\infty(\mathcal{L}^n|_\Omega)} \leq C$, where $C = 3(\|f\|_{\mathsf{L}^\infty(\mathcal{L}^n|_\Omega)} + \|g\|_{\mathsf{L}^\infty(\mathcal{L}^n|_\Omega)}) < \infty$.

It is then clear that, for all $i \in \mathbb{N}$, $f_i g_i \in \mathsf{L}^1(\mathcal{L}^n|_\Omega)$, $fg \in \mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and $f_i g_i \to fg$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$. Moreover,

$$\begin{aligned}\liminf|\nabla^{\mathsf{w}}(f_i g_i)|(\Omega) &\leq \liminf \int_\Omega \big(|f_i|\|\nabla g_i\| + |g_i|\|\nabla f_i\|\big)\,\mathrm{d}\mathcal{L}^n \leq \\ &\leq C\big(|\nabla^{\mathsf{w}} f|(\Omega) + |\nabla^{\mathsf{w}} g|(\Omega)\big) < \infty.\end{aligned}$$

It then follows from proposition 7.32 that $\mathsf{Var}(fg, \Omega) \leq C\big(|\nabla^{\mathsf{w}} f|(\Omega) + |\nabla^{\mathsf{w}} g|(\Omega)\big) < \infty$, so that $fg \in \mathsf{BV}(\Omega)$, as asserted.

□

Remark 7.37. With the notation from the previous proposition, it is not true, in general, that $\nabla^{\mathsf{w}}(fg) = \nabla^{\mathsf{w}} f \,\llcorner\, g + \nabla^{\mathsf{w}} g \,\llcorner\, f$. For instance, take $\Omega = \mathbb{R}^n$, E a closed subset of $\mathbb{R}^n$ such that $\chi_E \in \mathsf{BV}(\mathbb{R}^n)$ and $f = g = \chi_E$, so that $fg = \chi_E^2 = \chi_E$. Then $\nabla^{\mathsf{w}} \chi_E = \nabla^{\mathsf{w}} \chi_E \llcorner E$ does not coincide with $2\nabla^{\mathsf{w}} \chi_E \llcorner E$ if $0 < \mathcal{L}^n(E) < \infty$.

EXERCISE 7.38. If $p = 1$ in exercise 6.25, we have $(i) \Rightarrow (ii) \Leftrightarrow (iii)$. Moreover,

- (ii) or (iii) are equivalent to $f \in \mathsf{BV}(\Omega)$ and we may take $C = \mathsf{Var}(f, \Omega) = |\nabla^{\mathsf{w}} f|(\Omega)$ in both cases;
- If $\Omega = \mathbb{R}^n$, for all $h \in \mathbb{R}^n$

$$\|\tau_h f - f\|_{\mathsf{L}^1(\mathcal{L}^n)} \leq \|h\| \cdot |\nabla^{\mathsf{w}} f|(\mathbb{R}^n).$$

7.4. Traces and Extensions

THEOREM 7.39 (Trace theorem for BV functions on Lipschitz epigraphs). *Let $n \geq 2$, $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz and $\Omega := \operatorname{epi}_{\mathsf{S}} \Gamma$. Then:*

i) There exists a unique bounded linear operator $T : \mathsf{BV}(\Omega) \to \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$ such that, for all $f \in \mathsf{BV}(\Omega)$ and all $\varphi \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int_\Omega f \operatorname{div} \varphi \, \mathrm{d}\mathcal{L}^n = -\int_\Omega \varphi \cdot \mathrm{d}\, \nabla^{\mathsf{w}} f + \int_{\partial\Omega} Tf \, \varphi \cdot \nu \, \mathrm{d}\mathcal{H}^{n-1}, \tag{7.6}$$

where ν the unit outer normal to $\partial\Omega$.

ii) For all $f \in \mathsf{BV}(\Omega)$ and for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$,

$$\lim_{r\to 0} \fint_{\mathbb{B}(x,r)\cap\Omega} \big|f(y) - Tf(x)\big| \, \mathrm{d}\mathcal{L}^n(y) = 0, \tag{7.7}$$

so that, for such x,

$$Tf(x) = \lim_{r\to 0} \fint_{\mathbb{B}(x,r)\cap\Omega} f \, \mathrm{d}\mathcal{L}^n.$$

PROOF.

As usual, we identify $\mathbb{R}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R}$ and, by means of this identification, we write, for each $y \in \mathbb{R}^n$, $y = (y', y_n)$.

1) Given $f \in \mathsf{BV}(\Omega)$, suppose that there exist $Tf, T'f \in \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$ such that (7.6) holds for all $\varphi \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n)$. Then, for all such φ,

$$\int_{\partial\Omega} (Tf - T'f) \, \varphi \cdot \nu \, \mathrm{d}\mathcal{H}^{n-1} = 0,$$

hence the $\mathbb{R}^n$-valued Radon measure $(\mathcal{H}^{n-1} \llcorner \partial\Omega) \llcorner (Tf - T'f)\nu$ is null. Then so is its total variation $(\mathcal{H}^{n-1} \llcorner \partial\Omega) \llcorner |Tf - T'f|$, which means that $Tf = T'f$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$.

In particular, if the bounded operator T satisfying (7.6) exists, it must be unique.

2) We define T on $\mathsf{C}_\mathsf{c}^\infty(\mathbb{R}^n)|_\Omega := \{f|_\Omega \mid f \in \mathsf{C}_\mathsf{c}^\infty(\mathbb{R}^n)\}$ by $Tf := f|_{\partial\Omega} \in \mathsf{C}_\mathsf{c}(\partial\Omega) \subset \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$.

   Since Ω is the epigraph of a Lipschitz function, for all $\varphi \in \mathsf{C}_\mathsf{c}^1(\mathbb{R}^n, \mathbb{R}^n)$ we may apply the Gauss-Green theorem 6.45 to $f\varphi \in \mathsf{C}_\mathsf{c}^1(\mathbb{R}^n, \mathbb{R}^n)$, which yields
$$\int_\Omega \operatorname{div}(f\varphi)\, \mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} f\varphi \cdot \nu \, \mathrm{d}\mathcal{H}^{n-1},$$
   where ν is the outer unit normal to $\partial\Omega = \operatorname{gr}\Gamma$. Taking into account that $\operatorname{div}(f\varphi) = \nabla f \cdot \nabla\varphi + f \operatorname{div}\varphi$, we obtain (7.6) for $f \in \mathsf{C}_\mathsf{c}^\infty(\mathbb{R}^n)$.

3) Fix $\epsilon > 0$ and $f \in \mathsf{C}_\mathsf{c}^\infty(\mathbb{R}^n)$. Let $f_\epsilon : \partial\Omega \to \mathbb{R}$ be defined by, for all $y = \big(y', \Gamma(y')\big) \in \operatorname{gr}\Gamma = \partial\Omega$,
$$f_\epsilon(y) := f\big(y', \Gamma(y') + \epsilon\big).$$
   Note that $f_\epsilon \in \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$, because $f_\epsilon \in \mathsf{C}_\mathsf{c}(\partial\Omega)$ and $\mathcal{H}^{n-1}|_{\partial\Omega}$ is a Radon measure on $\partial\Omega$.

   We also define:
$$\Omega_\epsilon := \{y = (y', y_n) \in \mathbb{R}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R} \mid \Gamma(y') < y_n < \Gamma(y') + \epsilon\},$$
$$\Omega^\epsilon := \Omega \setminus \overline{\Omega_\epsilon} = \operatorname{epi}_\mathsf{S}(\Gamma + \epsilon).$$

   For all $y = \big(y', \Gamma(y')\big) \in \partial\Omega$, $f_\epsilon(y) - Tf(y) = \int_0^\epsilon \frac{\partial f}{\partial x_n}\big(y', \Gamma(y') + t\big)\, \mathrm{d}t$, so that
$$\begin{aligned}|f_\epsilon(y) - Tf(y)| &\le \int_0^\epsilon \Big|\frac{\partial f}{\partial x_n}\big(y', \Gamma(y') + t\big)\Big|\, \mathrm{d}t \le \\ &\le \int_0^\epsilon \big\|\nabla f\big(y', \Gamma(y') + t\big)\big\|\, \mathrm{d}t.\end{aligned}$$
   Therefore, computing by means of the area formula,

(7.8)
$$\begin{aligned}&\int_{\partial\Omega} |f_\epsilon(y) - Tf(y)|\, \mathrm{d}\mathcal{H}^{n-1}(y) \\ &\le \int_{\partial\Omega} \Big(\int_0^\epsilon \big\|\nabla f(y + te_n)\big\|\, \mathrm{d}t\Big)\, \mathrm{d}\mathcal{H}^{n-1}(y) \overset{\text{AF 5.40.2)}}{=} \\ &= \int_{\mathbb{R}^{n-1}} \Big(\int_0^\epsilon \big\|\nabla f\big(y', \Gamma(y') + t\big)\big\|\, \mathrm{d}t\Big) \underbrace{\sqrt{1 + \|\nabla\Gamma(y')\|^2}}_{\le \sqrt{1 + (\operatorname{Lip}\Gamma)^2} =: C}\, \mathrm{d}\mathcal{L}^{n-1}(y') \overset{\text{Tonelli}}{\le} \\ &\le C \int_{\Omega_\epsilon} \|\nabla f\|\, \mathrm{d}\mathcal{L}^n.\end{aligned}$$

4) For an arbitrary $f \in \mathsf{BV}(\Omega)$, we may apply corollary 7.35 to obtain a sequence $(f_i)_{i\in\mathbb{N}}$ in $\mathsf{C}_\mathsf{c}^\infty(\mathbb{R}^n)$ such that $f_i|_\Omega \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and

$|\nabla^{\mathsf{w}} f_i|(\Omega) \to |\nabla^{\mathsf{w}} f|(\Omega)$. In particular, it follows from propositions 7.27.i), 4.49 and 4.58.ii) that

$$\nabla^{\mathsf{w}} f_i|_\Omega \overset{*\mathrm{f}}{\to} \nabla^{\mathsf{w}} f \text{ and } \left|\nabla^{\mathsf{w}} f_i|_\Omega\right| \overset{*\mathrm{f}}{\to} |\nabla^{\mathsf{w}} f|.$$

Fix $\epsilon > 0$. For each $i \in \mathbb{N}$, let $f_i^\epsilon : \partial\Omega \to \mathbb{R}$ be defined by, for all $y = \big(y', \Gamma(y')\big) \in \partial\Omega$,

$$\begin{aligned} f_i^\epsilon(y) &:= \frac{1}{\epsilon}\int_0^\epsilon f_i\big(y', \Gamma(y') + t\big)\,\mathrm{d}t = \\ &= \frac{1}{\epsilon}\int_0^\epsilon (f_i)_t(y)\,\mathrm{d}t. \end{aligned}$$

Note that, for all $i \in \mathbb{N}$, $f_i^\epsilon \in \mathsf{C_c}(\partial\Omega)$ and, by (7.8) (applied to $f_i \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n)$),

$$\begin{aligned} \int_{\partial\Omega} |Tf_i - f_i^\epsilon|\,\mathrm{d}\mathcal{H}^{n-1} &\leq \frac{1}{\epsilon}\int_{\partial\Omega}\int_0^\epsilon |Tf_i(y) - (f_i)_t(y)|\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}t \overset{\text{Tonelli}}{=} \\ &= \frac{1}{\epsilon}\int_0^\epsilon\int_{\partial\Omega} |Tf_i(y) - (f_i)_t(y)|\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}t \overset{7.8}{\leq} \\ &\leq C|\nabla^{\mathsf{w}} f_i|(\Omega_\epsilon). \end{aligned}$$

Hence, for all $i, j \in \mathbb{N}$,

(7.9)
$$\begin{aligned} &\int_{\partial\Omega} |Tf_i - Tf_j|\,\mathrm{d}\mathcal{H}^{n-1} \leq \\ &\quad \int_{\partial\Omega} |Tf_i - f_i^\epsilon|\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial\Omega} |f_i^\epsilon - f_j^\epsilon|\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial\Omega} |Tf_j - f_j^\epsilon|\,\mathrm{d}\mathcal{H}^{n-1} \leq \\ &\quad \leq C\big(|\nabla^{\mathsf{w}} f_i|(\Omega_\epsilon) + |\nabla^{\mathsf{w}} f_j|(\Omega_\epsilon)\big) + \int_{\partial\Omega} |f_i^\epsilon - f_j^\epsilon|\,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

We now estimate

(7.10)
$$\begin{aligned} &\int_{\partial\Omega} |f_i^\epsilon - f_j^\epsilon|\,\mathrm{d}\mathcal{H}^{n-1} \overset{\text{Tonelli}}{\leq} \\ &\quad \leq \frac{1}{\epsilon}\int_0^\epsilon\int_{\partial\Omega} \big|(f_i)_t - (f_j)_t\big|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t \overset{\text{AF } 5.40.2)}{=} \\ &= \frac{1}{\epsilon}\int_0^\epsilon\int_{\mathbb{R}^{n-1}} \big|f_i\big(y', \Gamma(y') + t\big) - f_j\big(y', \Gamma(y') + t\big)\big| \underbrace{\sqrt{1 + \|\nabla\Gamma(y')\|^2}}_{\leq\sqrt{1 + (\operatorname{Lip}\Gamma)^2} = C}\,\mathrm{d}y'\,\mathrm{d}t \overset{\text{Tonelli}}{\leq} \\ &\leq \frac{C}{\epsilon}\int_{\Omega_\epsilon} |f_i - f_j|\,\mathrm{d}\mathcal{L}^n \overset{i,j\to\infty}{\to} 0, \end{aligned}$$

since $f_i \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$. On the other hand,

$$|\nabla^{\mathsf{w}} f_i|(\Omega_\epsilon) \leq |\nabla^{\mathsf{w}} f_i|(\overline{\Omega_\epsilon} \cap \Omega) = |\nabla^{\mathsf{w}} f_i|(\Omega) - |\nabla^{\mathsf{w}} f_i|(\Omega^\epsilon).$$

Since $|\nabla^{\mathsf{w}} f_i|(\Omega) \to |\nabla^{\mathsf{w}} f|(\Omega)$ and, by proposition 4.57, $|\nabla^{\mathsf{w}} f|(\Omega^\epsilon) \leq \liminf|\nabla^{\mathsf{w}} f_i|(\Omega^\epsilon)$, we conclude that

$$\begin{aligned} \limsup|\nabla^{\mathsf{w}} f_i|(\Omega_\epsilon) &= |\nabla^{\mathsf{w}} f|(\Omega) - \liminf|\nabla^{\mathsf{w}} f_i|(\Omega^\epsilon) \leq \\ &\leq |\nabla^{\mathsf{w}} f|(\Omega) - |\nabla^{\mathsf{w}} f|(\Omega^\epsilon) = \\ &= |\nabla^{\mathsf{w}} f|(\overline{\Omega_\epsilon} \cap \Omega). \end{aligned} \tag{7.11}$$

It then follows from (7.9), (7.10) and (7.11) that

$$\limsup_{i,j\to\infty} \int_{\partial\Omega} |Tf_i - Tf_j| \, \mathrm{d}\mathcal{H}^{n-1} \leq 2C|\nabla^{\mathsf{w}} f|(\overline{\Omega_\epsilon} \cap \Omega).$$

Therefore, since $\epsilon > 0$ was arbitrarily taken, $|\nabla^{\mathsf{w}} f|$ is a finite Radon measure and $\overline{\Omega_\epsilon} \cap \Omega$ decreases to $\emptyset$ as $\epsilon \to 0$, it follows that $(Tf_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$, thus it is convergent in that space. We define

$$Tf := \lim Tf_i \in \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega}).$$

As it was seen in step 2 of the proof, since $f_i \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ for each $i \in \mathbb{N}$, equality (7.6) holds for f_i in place of f. Thus, taking $i \to \infty$ and taking into account that $f_i \to f$ in $\mathsf{L}^1(\Omega)$ and $Tf_i \to Tf$ in $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$, we conclude that (7.6) also holds for f. In particular, our definition of Tf is independent of the choice of the sequence $(f_i)_{i\in\mathbb{N}}$ in $\mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ such that $f_i|_\Omega \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and $|\nabla^{\mathsf{w}} f_i|(\Omega) \to |\nabla^{\mathsf{w}} f|(\Omega)$. Indeed, if $(f'_i)_{i\in\mathbb{N}}$ is another such sequence and $T'f := \lim Tf'_i$ in $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$, then both $T'f$ and Tf satisfy (7.6), which implies, in view of step 1 of the proof, that $T'f = Tf$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$.

The map $T : \mathsf{BV}(\Omega) \to \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$ is therefore well-defined, it is clearly linear and (7.6) is verified for all $f \in \mathsf{BV}(\Omega)$ and all $\varphi \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n, \mathbb{R}^n)$.

5) Let $(f_i)_{i\in\mathbb{N}}$ be a sequence in $\mathsf{BV}(\Omega)$ and $f \in \mathsf{BV}(\Omega)$. We contend that, if $f_i \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and $|\nabla^{\mathsf{w}} f_i|(\Omega) \to |\nabla^{\mathsf{w}} f|(\Omega)$, then $Tf_i \to Tf$ in $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$. In particular, that proves the continuity of $T : \mathsf{BV}(\Omega) \to \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$, thus reaching the conclusion of the proof of part i).

Indeed, for each $i \in \mathbb{N}$ we may take a sequence $(g^i_j)_{j\in\mathbb{N}}$ in $\mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n)$ such that $\lim_{j\to\infty} g^i_j|_\Omega \to f_i$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and $\lim_{j\to\infty}|\nabla^{\mathsf{w}} g^i_j|(\Omega) \to |\nabla^{\mathsf{w}} f_i|(\Omega)$. By the previous step, it then follows that $\lim_{j\to\infty} Tg^i_j =$

Tf_i in $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$. Hence, we may take $j = j(i)$ sufficiently large in order that $g_i := g^i_{j(i)}$ satisfy

$$\begin{aligned}
&\|f_i - g_i\|_{\mathsf{L}^1(\mathcal{L}^n|_\Omega)} < \frac{1}{i},\\
&\big||\nabla^{\mathsf{w}} f_i|(\Omega) - |\nabla^{\mathsf{w}} g_i|(\Omega)\big| < \frac{1}{i} \quad \text{and}\\
&\|Tf_i - Tg_i\|_{\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})} < \frac{1}{i}.
\end{aligned}$$

Then $(g_i)_{i\in\mathbb{N}}$ is a sequence in $\mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n)$ such that $g_i \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and $|\nabla^{\mathsf{w}} g_i|(\Omega) \to |\nabla^{\mathsf{w}} f|(\Omega)$; by the previous step of the proof, it follows that $Tg_i \to Tf$ in $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$. Then $\lim Tf_i = \lim Tg_i = Tf$ in $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$, which proves our contention.

6) We now prove part ii). Fix $f \in \mathsf{BV}(\Omega)$ up to the end of the proof.

For all $x \in \partial\Omega$ and all $r > 0$, we estimate

(7.12)
$$\begin{aligned}
&\frac{1}{\mathcal{L}^n\big(\mathbb{B}(x,r)\cap\Omega\big)} \int_{\mathbb{B}(x,r)\cap\Omega} \big|f(y) - Tf(x)\big| \,\mathrm{d}\mathcal{L}^n(y) \leq\\
&\quad \leq \underbrace{\frac{1}{\mathcal{L}^n\big(\mathbb{B}(x,r)\cap\Omega\big)} \int_{\mathbb{B}(x,r)\cap\Omega} \big|f(y) - Tf\big(y',\Gamma(y')\big)\big| \,\mathrm{d}\mathcal{L}^n(y)}_{①} +\\
&\quad\quad + \underbrace{\frac{1}{\mathcal{L}^n\big(\mathbb{B}(x,r)\cap\Omega\big)} \int_{\mathbb{B}(x,r)\cap\Omega} \big|Tf\big(y',\Gamma(y')\big) - Tf(x)\big| \,\mathrm{d}\mathcal{L}^n(y)}_{②}
\end{aligned}$$

We estimate ① and ② along the following steps.

7) For $x' \in \mathbb{R}^{n-1}$ and $r \in (0,\infty]$, we generalize the estimate (7.8) with the infinite closed cylinder $\overline{\mathbb{C}}(x',r,\infty) = \mathbb{B}(x',r) \times \mathbb{R}$ in place of $\mathbb{R}^{n-1} \times \mathbb{R}$.

For $\epsilon > 0$, we define f_ϵ, Ω_ϵ and Ω^ϵ as in step 3) of the proof. Note that, assuming f Borelian (which we may assume without loss of generality — i.e. in each equivalence class of $\mathsf{BV}(\Omega)$ we may take a Borelian representative, in view of corollary 1.118), $f_\epsilon : \partial\Omega \to \mathbb{R}$ is clearly Borelian.

We extend the notation from step 3) to denote intersections with the closed cylinder $\overline{\mathbb{C}}(x',r,\infty)$:

$$\begin{aligned}\Omega(x',r) &:= \Omega \cap \overline{\mathbb{C}}(x',r,\infty),\\ \Omega(x',r)_\epsilon &:= \Omega_\epsilon \cap \overline{\mathbb{C}}(x',r,\infty),\\ \Omega(x',r)^\epsilon &:= \Omega^\epsilon \cap \overline{\mathbb{C}}(x',r,\infty).\end{aligned}$$

Note that $\Omega(x',\infty) = \Omega$.

Claim 1: for $\mathcal{L}^1$-a.e. $\epsilon > 0$,

$$\int_{\partial\Omega\cap\overline{\mathbb{C}}(x',r,\infty)} |Tf - f_\epsilon| \,\mathrm{d}\mathcal{H}^{n-1} \le C|\nabla^{\mathsf{w}} f|\big(\Omega(x',r)_\epsilon\big), \tag{7.13}$$

where $C = \sqrt{1 + \|\operatorname{Lip}\Gamma\|^2}$.

If $f \in \mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n)$, the claim follows from the same argument used in estimate (7.8), with $\partial\Omega \cap \overline{\mathbb{C}}(x',r,\infty)$ in place of $\partial\Omega$, $\mathbb{B}(x',r)$ in place of $\mathbb{R}^{n-1}$ and $\Omega(x',r)_\epsilon$ in place of Ω_ϵ.

To prove claim 1 for $f \in \mathsf{BV}(\Omega)$, we shall apply the coarea formula with the Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ given by, for all $y = (y', y_n) \in \mathbb{R}^n$,

$$g(y) = y_n - \Gamma(y'),$$

whose level sets are translations of $\operatorname{gr}\Gamma$ in the e_n direction. Note that, for all $y = (y', y_n) \in \mathbb{R}^n$ such that $y' \in D_\Gamma$, $\nabla g(y) = \big(-\nabla\Gamma(y'), 1\big)$, hence $\mathsf{J}g(y) = \sqrt{1 + \|\nabla\Gamma(y)\|^2} \le \sqrt{1 + (\operatorname{Lip}\Gamma)^2} = C$.

Take $(f_i)_{i\in\mathbb{N}}$ in $\mathsf{C}^\infty_{\mathsf{c}}(\mathbb{R}^n)$ such that $f_i|_\Omega \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and $|\nabla^{\mathsf{w}} f_i|(\Omega) \to |\nabla^{\mathsf{w}} f|(\Omega)$. We have, for all $i \in \mathbb{N}$,

$$\begin{aligned}\int_0^\infty \int_{g^{-1}\{t\}\cap\Omega(x',r)} |f_i - f| \,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t &\overset{\text{coarea f. 5.50}}{=}\\ &= \int_{\Omega(x',r)} |f_i - f| \,\mathsf{J}g \,\mathrm{d}\mathcal{L}^n \le\\ &\le C\int_{\Omega(x',r)} |f_i - f| \,\mathrm{d}\mathcal{L}^n \overset{i\to\infty}{\to} 0,\end{aligned}$$

since $f_i \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$. Thus, passing to a subsequence, if necessary, we conclude that, for $\mathcal{L}^1$-a.e. $t > 0$,

$$\int_{\partial\Omega\cap\overline{\mathbb{C}}(x',r,\infty)} \big|(f_i)_t - f_t\big| \,\mathrm{d}\mathcal{H}^{n-1} \overset{*}{=} \int_{g^{-1}\{t\}\cap\Omega(x',r)} |f_i - f| \,\mathrm{d}\mathcal{H}^{n-1} \overset{i\to\infty}{\to} 0, \tag{7.14}$$

where in equality $(*)$ we have used the fact that the isometry $y \mapsto y + te_n$ of $\partial\Omega = g^{-1}\{0\}$ onto $g^{-1}\{t\}$ preserves $\mathcal{H}^{n-1}$ measure.

On the other hand, we may estimate, for all $\epsilon > 0$ and all $i \in \mathbb{N}$,

(7.15)
$$\int_{\partial\Omega\cap\overline{\mathbb{C}}(x',r,\infty)} |Tf - f_\epsilon| \,\mathrm{d}\mathcal{H}^{n-1} \leq \underbrace{\int_{\partial\Omega\cap\overline{\mathbb{C}}(x',r,\infty)} \big|(f_i)_\epsilon - f_\epsilon\big| \,\mathrm{d}\mathcal{H}^{n-1}}_{\overset{i\to\infty}{\to} 0 \text{ for a.e. } \epsilon>0, \text{ by (7.14)}} +$$
$$+ \underbrace{\int_{\partial\Omega\cap\overline{\mathbb{C}}(x',r,\infty)} \big|Tf_i - (f_i)_\epsilon\big| \,\mathrm{d}\mathcal{H}^{n-1}}_{\leq C|\nabla^{\mathsf{w}} f_i|(\Omega(x',r)_\epsilon) \text{ by the case } f\in\mathsf{C}_c^\infty(\mathbb{R}^n)} + \underbrace{\int_{\partial\Omega\cap\overline{\mathbb{C}}(x',r,\infty)} |Tf - Tf_i| \,\mathrm{d}\mathcal{H}^{n-1}}_{\overset{i\to\infty}{\to} 0 \text{ by step 5}}.$$

Besides, adapting the estimate from (7.11), we have, for all $\epsilon > 0$,

$$\begin{aligned}\limsup|\nabla^{\mathsf{w}} f_i|(\Omega(x,y)_\epsilon) &\leq \limsup|\nabla^{\mathsf{w}} f_i|(\overline{\Omega(x,y)_\epsilon} \cap \Omega) \leq \\ &\leq |\nabla^{\mathsf{w}} f|(\overline{\Omega(x,y)_\epsilon} \cap \Omega).\end{aligned}$$

But, since $|\nabla^{\mathsf{w}} f|$ is a Radon measure on Ω, it follows from proposition 4.53 that, for $\mathcal{L}^n$-a.e. $\epsilon > 0$, $|\nabla^{\mathsf{w}} f|(g^{-1}\{\epsilon\}) = 0$; for such ϵ,

$$|\nabla^{\mathsf{w}} f|(\overline{\Omega(x,y)_\epsilon} \cap \Omega) = |\nabla^{\mathsf{w}} f|(\Omega(x,y)_\epsilon).$$

The claim therefore follows taking $\limsup_{i\to\infty}$ on both members of (7.15).

8) Note that, for all $x = \big(x', \Gamma(x')\big) \in \partial\Omega$ and all $r > 0$, if $y = (y', y_n) \in \mathbb{B}(x,r) \cap \Omega$, then

$$\begin{aligned}0 < g(y) = y_n - \Gamma(y') &= (y_n - x_n) + \big(\Gamma(x') - \Gamma(y')\big) \leq \\ &\leq |y_n - x_n| + (\operatorname{Lip}\Gamma)\|y' - x'\| \leq r(1 + \operatorname{Lip}\Gamma).\end{aligned}$$

That is, $\mathbb{B}(x,r) \cap \Omega \subset g^{-1}\big(]0, r(1 + \operatorname{Lip}\Gamma)]\big) \cap \overline{\mathbb{C}}(x', r, \infty)$. Hence, using the estimate from claim 1 in the previous step and the coarea formula, we compute:

$$\begin{aligned}
&\int_{\mathbb{B}(x,r)\cap\Omega}\left|f(y)-Tf\big(y',\Gamma(y')\big)\right|\mathrm{d}\mathcal{L}^n(y) \overset{\mathsf{J}g\geq 1}{\leq}\\
&\leq \int_{g^{-1}\left(]0,r(1+\operatorname{Lip}\Gamma)]\right)\cap\overline{\mathbb{C}}(x',r,\infty)}\left|f(y)-Tf\big(y',\Gamma(y')\big)\right|\mathsf{J}g(y)\,\mathrm{d}\mathcal{L}^n(y) \overset{5.50}{=}\\
&= \int_0^{r(1+\operatorname{Lip}\Gamma)}\int_{g^{-1}\{t\}\cap\overline{\mathbb{C}}(x',r,\infty)}\left|f(y)-Tf\big(y',\Gamma(y')\big)\right|\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}t \overset{\mathcal{H}^{n-1}\text{ invariant by isometries}}{=}\\
&= \int_0^{r(1+\operatorname{Lip}\Gamma)}\int_{\partial\Omega\cap\overline{\mathbb{C}}(x',r,\infty)}\left|f_t(y)-Tf\big(y',\Gamma(y')\big)\right|\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}t \overset{\text{claim 1}}{\leq}\\
&\leq Cr(1+\operatorname{Lip}\Gamma)|\nabla^{\mathsf{w}} f|\big(\Omega(x',r)_{r(1+\operatorname{Lip}\Gamma)}\big),
\end{aligned}$$

where $C=\sqrt{1+(\operatorname{Lip}\Gamma)^2}$.

On the other hand, if $y=(y',y_n)\in\Omega(x',r)_{r(1+\operatorname{Lip}\Gamma)}$, then $\|y'-x'\|\leq r$ and $\Gamma(y')<y_n<\Gamma(y')+r(1+\operatorname{Lip}\Gamma)$, hence

$$\begin{aligned}
y_n-x_n&=y_n-\Gamma(x')\leq\\
&\leq r(1+\operatorname{Lip}\Gamma)+\Gamma(y')-\Gamma(x')\leq r(1+2\operatorname{Lip}\Gamma),\\
-(y_n-x_n)&=-\big(y_n-\Gamma(x')\big)\leq-\big(\Gamma(y')-\Gamma(x')\big)\leq r\operatorname{Lip}\Gamma,
\end{aligned}$$

whence $|y_n-x_n|\leq r(1+2\operatorname{Lip}\Gamma)$. Thus $\|y-x\|\leq r(2+2\operatorname{Lip}\Gamma)$, i.e. $\Omega(x',r)_{r(1+\operatorname{Lip}\Gamma)}\subset\mathbb{B}(x,r(2+2\operatorname{Lip}\Gamma))\cap\Omega$. We therefore conclude that

$$\begin{aligned}
&\int_{\mathbb{B}(x,r)\cap\Omega}\left|f(y)-Tf\big(y',\Gamma(y')\big)\right|\mathrm{d}\mathcal{L}^n(y)\leq\\
&\qquad\leq Cr(1+\operatorname{Lip}\Gamma)|\nabla^{\mathsf{w}} f|\big(\mathbb{B}(x,r(2+2\operatorname{Lip}\Gamma))\cap\Omega\big).
\end{aligned}\tag{7.16}$$

Similarly, for all $x=\big(x',\Gamma(x')\big)\in\partial\Omega$ and all $r>0$,

$$\begin{aligned}
&\int_{\mathbb{B}(x,r)\cap\Omega}\left|Tf\big(y',\Gamma(y')\big)-Tf(x)\right|\mathrm{d}\mathcal{L}^n(y) \overset{\mathsf{J}g\geq 1}{\leq}\\
&\leq \int_{g^{-1}\left(]0,r(1+\operatorname{Lip}\Gamma)]\right)\cap\overline{\mathbb{C}}(x',r,\infty)}\left|Tf\big(y',\Gamma(y')\big)-Tf(x)\right|\mathsf{J}g(y)\,\mathrm{d}\mathcal{L}^n(y) \overset{5.50}{=}\\
&= \int_0^{r(1+\operatorname{Lip}\Gamma)}\int_{g^{-1}\{t\}\cap\overline{\mathbb{C}}(x',r,\infty)}\left|Tf\big(y',\Gamma(y')\big)-Tf(x)\right|\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}t \overset{\mathcal{H}^{n-1}\text{ invariant by isometries}}{=}\\
&= \int_0^{r(1+\operatorname{Lip}\Gamma)}\int_{\partial\Omega\cap\overline{\mathbb{C}}(x',r,\infty)}\left|Tf(y)-Tf(x)\right|\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}t =\\
&= r(1+\operatorname{Lip}\Gamma)\int_{\partial\Omega\cap\overline{\mathbb{C}}(x',r,\infty)}\left|Tf(y)-Tf(x)\right|\mathrm{d}\mathcal{H}^{n-1}(y)
\end{aligned}$$

Besides, if $y = (y', y_n) \in \partial\Omega \cap \overline{\mathbb{C}}(x', r, \infty)$, then $\|y' - x'\| \le r$ and $|y_n - x_n| = \big|\Gamma(y') - \Gamma(x')\big| \le r \operatorname{Lip}\Gamma$. Thus, $y \in \mathbb{B}(x, r(1 + \operatorname{Lip}\Gamma))$, which implies $\partial\Omega \cap \overline{\mathbb{C}}(x', r, \infty) \subset \mathbb{B}(x, r(1 + \operatorname{Lip}\Gamma)) \cap \partial\Omega$. It then follows that

$$\begin{aligned}(7.17)\qquad &\int_{\mathbb{B}(x,r)\cap\Omega} \big|Tf\big(y', \Gamma(y')\big) - Tf(x)\big| \,\mathrm{d}\mathcal{L}^n(y) \le \\ &\le r(1 + \operatorname{Lip}\Gamma) \int_{\mathbb{B}(x, r(1+\operatorname{Lip}\Gamma))\cap\partial\Omega} \big|Tf(y) - Tf(x)\big| \,\mathrm{d}\mathcal{H}^{n-1}(y)\end{aligned}$$

9) Claim 2: for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$,

$$\lim_{r\to 0} \frac{|\nabla^{\mathsf{w}} f|\big(\mathbb{B}(x,r)\cap\Omega\big)}{r^{n-1}} = 0. \tag{7.18}$$

Indeed, it suffices to prove that $\mathcal{H}^{n-1}(A_\eta) = 0$ for each $\eta > 0$, where

$$A_\eta := \Big\{x \in \partial\Omega \mid \limsup_{r\to 0} \frac{|\nabla^{\mathsf{w}} f|\big(\mathbb{B}(x,r)\cap\Omega\big)}{r^{n-1}} > \eta\Big\}.$$

Fix $\eta > 0$ and $\delta > 0$; we shall estimate $\mathcal{H}^{n-1}_{10\delta}(A_\eta)$. For each $x \in A_\eta$ and each $0 < \epsilon < \delta$, there exists $0 < r < \epsilon$ such that

$$\frac{|\nabla^{\mathsf{w}} f|\big(\mathbb{B}(x,r)\cap\Omega\big)}{r^{n-1}} > \eta. \tag{7.19}$$

It then follows that $\mathcal{F}_\epsilon := \{\mathbb{B}(x,r) \mid x \in A_\eta, 0 < r < \epsilon \text{ and } (7.19) \text{ holds}\}$ is a cover of A_η by nondegenerate closed balls with diameters less than $2\epsilon < 2\delta$. We may therefore apply the 5-times covering lemma 2.10 to obtain a countable disjoint subfamily $\mathcal{G}_\epsilon \subset \mathcal{F}_\epsilon$ such that $A \subset \cup\mathcal{F}_\epsilon \subset \cup_{B\in\mathcal{G}_\epsilon} 5B$. Hence, denoting by U_ϵ the open subset of Ω given by $\{x \in \Omega \mid d(x, \Omega^c) < \epsilon\}$, we compute

$$\begin{aligned}\mathcal{H}^{n-1}_{10\delta}(A_\eta) &\le \sum_{B=\mathbb{B}(x,r)\in\mathcal{G}_\epsilon} \alpha(n-1)5^{n-1}r^{n-1} \le \\ &\le \sum_{B=\mathbb{B}(x,r)\in\mathcal{G}_\epsilon} \frac{\alpha(n-1)5^{n-1}}{\eta} |\nabla^{\mathsf{w}} f|\big(\mathbb{B}(x,r)\cap\Omega\big) \overset{r<\epsilon}{\le} \\ &\le \frac{\alpha(n-1)5^{n-1}}{\eta} |\nabla^{\mathsf{w}} f|(U_\epsilon).\end{aligned}$$

Since ϵ with $0 < \epsilon < \delta$ was arbitrarily taken, $|\nabla^{\mathsf{w}} f|$ is a finite Radon measure and U_ϵ decreases to $\emptyset$ as $\epsilon \to 0$, we conclude that $\mathcal{H}^{n-1}_{10\delta}(A_\eta) = 0$, for all $\delta > 0$. It then follows that $\mathcal{H}^{n-1}(A_\eta) = 0$, which concludes the proof of the claim.

10) Claim 3: Let K be the cone with vertex at the origin given by $\{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y_n > (\operatorname{Lip}\Gamma)\|y'\|\}$ and, for each $r > 0$, $K_r := \mathbb{B}(0, r) \cap K$. Then, for all $x \in \partial\Omega$,

$$\frac{1}{\mathcal{L}^n\big(\mathbb{B}(x,r) \cap \Omega\big)} \leq \frac{\alpha(n)}{\mathcal{L}^n(K_1)} \frac{1}{\mathcal{L}^n\big(\mathbb{B}(x,r)\big)}. \tag{7.20}$$

Indeed, in view of the translation invariance of Lebesgue measure, replacing Γ by $\Gamma(\cdot + x') - x_n$, we may assume $x = 0$. If $y = (y', y_n) \in \operatorname{hyp}|\Gamma|$, i.e. if $y_n \leq |\Gamma(y')|$, then

$$y_n \leq |\Gamma(y')| = |\Gamma(y') - \Gamma(x')| \leq (\operatorname{Lip}\Gamma)\|y' - x'\| = (\operatorname{Lip}\Gamma)\|y'\|,$$

hence $y \in K^c$. That is, $\operatorname{hyp}|\Gamma| \subset K^c$. Thus, for each $r > 0$,

$$\begin{aligned} K_r = \mathbb{B}(x,r) \cap K \subset \mathbb{B}(x,r) \cap \operatorname{epi}_{\mathsf{S}} |\Gamma| \subset \\ \subset \mathbb{B}(x,r) \cap \operatorname{epi}_{\mathsf{S}} \Gamma = \mathbb{B}(x,r) \cap \Omega. \end{aligned}$$

It then follows that

$$r^n \mathcal{L}^n(K_1) = \mathcal{L}^n(K_r) \leq \mathcal{L}^n\big(\mathbb{B}(x,r) \cap \Omega\big)$$

$$\begin{aligned} \mathcal{L}^n\big(\mathbb{B}(x,r) \cap \Omega\big) \geq \mathcal{L}^n(K_r) = \\ = \mathcal{L}^n(K_1) r^n = \frac{\mathcal{L}^n(K_1)}{\alpha(n)} \mathcal{L}^n\big(\mathbb{B}(x,r)\big), \end{aligned}$$

whence the claim.

11) Fix $x \in \partial\Omega$ such that

$$\begin{aligned} &\lim_{r\to 0} \frac{|\nabla^{\mathsf{w}} f|\big(\mathbb{B}(x,r) \cap \Omega\big)}{r^{n-1}} = 0 \text{ and} \\ &\lim_{r\to 0} \fint_{\mathbb{B}(x,r)\cap\partial\Omega} \big|Tf - Tf(x)\big| \,\mathrm{d}\mathcal{H}^{n-1} = 0. \end{aligned} \tag{7.21}$$

For such x, we estimate below ① and ② from step 6). Note that (7.21) holds $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$: the first equality holds $\mathcal{H}^{n-1}$-a.e. in view of claim 2 in step 9) of the proof, and the second holds $\mathcal{H}^{n-1}$-a.e. by the Lebesgue differentiation theorem 3.30 (which may be applied because $Tf \in \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$ and $\mathcal{H}^{n-1}|_{\partial\Omega}$ is a Radon measure on $\partial\Omega$).

For each $r > 0$, with $C = \sqrt{1 + (\operatorname{Lip}\Gamma)^2}$,

$$
\begin{aligned}
① &= \frac{1}{\mathcal{L}^n\big(\mathbb{B}(x,r) \cap \Omega\big)} \int_{\mathbb{B}(x,r)\cap\Omega} \big|f(y) - Tf\big(y', \Gamma(y')\big)\big| \,\mathrm{d}\mathcal{L}^n(y) \overset{(7.16)}{\leq} \\
&\leq \frac{Cr(1+\operatorname{Lip}\Gamma)}{\mathcal{L}^n\big(\mathbb{B}(x,r)\cap\Omega\big)} |\nabla^{\mathsf{w}} f|\big(\mathbb{B}(x, r(2+2\operatorname{Lip}\Gamma)) \cap \Omega\big) \overset{\text{claim 3}}{\leq} \\
&\leq \frac{Cr(1+\operatorname{Lip}\Gamma)}{\mathcal{L}^n(K_1) r^n} |\nabla^{\mathsf{w}} f|\big(\mathbb{B}(x, r(2+2\operatorname{Lip}\Gamma)) \cap \Omega\big) = \\
&= \frac{C(1+\operatorname{Lip}\Gamma)(2+2\operatorname{Lip}\Gamma)^{n-1}}{\mathcal{L}^n(K_1)} \frac{|\nabla^{\mathsf{w}} f|\big(\mathbb{B}(x, r(2+2\operatorname{Lip}\Gamma)) \cap \Omega\big)}{r^{n-1}(2+2\operatorname{Lip}\Gamma)^{n-1}}.
\end{aligned}
$$

Thus, in view of (7.21), we conclude that

$$\lim_{r\to 0} ① = 0. \tag{7.22}$$

Similarly, for each $r > 0$,

$$
\begin{aligned}
② &= \frac{1}{\mathcal{L}^n\big(\mathbb{B}(x,r)\cap\Omega\big)} \int_{\mathbb{B}(x,r)\cap\Omega} \big|Tf\big(y', \Gamma(y')\big) - Tf(x)\big| \,\mathrm{d}\mathcal{L}^n(y) \overset{(7.17)}{\leq} \\
&\leq \frac{r(1+\operatorname{Lip}\Gamma)}{\mathcal{L}^n\big(\mathbb{B}(x,r)\cap\Omega\big)} \int_{\mathbb{B}(x, r(1+\operatorname{Lip}\Gamma))\cap\partial\Omega} \big|Tf(y) - Tf(x)\big| \,\mathrm{d}\mathcal{H}^{n-1}(y) \overset{\text{claim 3}}{\leq} \\
&\leq \frac{(1+\operatorname{Lip}\Gamma)}{\mathcal{L}^n(K_1) r^{n-1}} \int_{\mathbb{B}(x, r(1+\operatorname{Lip}\Gamma))\cap\partial\Omega} \big|Tf(y) - Tf(x)\big| \,\mathrm{d}\mathcal{H}^{n-1}(y)
\end{aligned}
$$

On the other hand, since $\mathsf{pr}_{\mathbb{R}^{n-1}}\big(\mathbb{B}(x, r(1+\operatorname{Lip}\Gamma)) \cap \partial\Omega\big) \subset \mathbb{B}(x', r(1+\operatorname{Lip}\Gamma))$, it follows from the area formula that

$$
\begin{aligned}
&\mathcal{H}^{n-1}\big(\mathbb{B}(x, r(1+\operatorname{Lip}\Gamma)) \cap \partial\Omega\big) \overset{5.39}{=} \\
&\quad= \int_{\mathsf{pr}_{\mathbb{R}^{n-1}}\left(\mathbb{B}(x,r(1+\operatorname{Lip}\Gamma))\cap\partial\Omega\right)} \sqrt{1 + \nabla\Gamma(y')} \,\mathrm{d}\mathcal{L}^{n-1}(y') \leq \\
&\quad\leq C\mathcal{L}^{n-1}\big(\mathbb{B}(x', r(1+\operatorname{Lip}\Gamma))\big) = C\alpha(n-1) r^{n-1} (1+\operatorname{Lip}\Gamma)^{n-1},
\end{aligned}
$$

whence

$$\frac{1}{r^{n-1}} \leq \frac{C\alpha(n-1)(1+\operatorname{Lip}\Gamma)^{n-1}}{\mathcal{H}^{n-1}\big(\mathbb{B}(x, r(1+\operatorname{Lip}\Gamma)) \cap \partial\Omega\big)}.$$

We therefore conclude that

$$② \leq \frac{C'}{\mathcal{H}^{n-1}\big(\mathbb{B}(x, r(1+\operatorname{Lip}\Gamma)) \cap \partial\Omega\big)} \int_{\mathbb{B}(x, r(1+\operatorname{Lip}\Gamma))\cap\partial\Omega} \big|Tf(y) - Tf(x)\big| \,\mathrm{d}\mathcal{H}^{n-1}(y).$$

where

$$C' = \frac{\sqrt{1 + (\operatorname{Lip}\Gamma)^2}\,\alpha(n-1)(1+\operatorname{Lip}\Gamma)^n}{\mathcal{L}^n(K_1)}. \tag{7.23}$$

That implies, in view of (7.21),

$$\lim_{r\to 0} ② = 0. \tag{7.24}$$

Finally, from (7.12), (7.22) and (7.24), it follows that (7.7) holds for $x \in \partial\Omega$ satisfying (7.21), i.e. it holds $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, which concludes the proof.

□

COROLLARY 7.40. *With the same hypothesis of theorem 7.39, if $f \in \mathsf{BV}(\Omega) \cap \mathsf{C}(\overline{\Omega})$, then $Tf = f|_{\partial\Omega}$.*

REMARK 7.41. With the notation from theorem 7.39:

1) We have actually proved in step 5) of the proof that the continuity of $T : \mathsf{BV}(\Omega) \to \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$ holds in a stronger sense, i.e. if a sequence $(f_i)_{i\in\mathbb{N}}$ in $\mathsf{BV}(\Omega)$ and $f \in \mathsf{BV}(\Omega)$ are such that $f_i \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and $|\nabla^{\mathsf{w}} f_i|(\Omega) \to |\nabla^{\mathsf{w}} f|(\Omega)$, then $Tf_i \to Tf$ in $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$.
2) The trace operator from theorem 6.48 for $\mathsf{W}^{1,1}(\Omega)$ is the restriction of the trace operator from theorem 7.39.

LEMMA 7.42. *Let $U \subset \mathbb{R}^n$ open, $\Phi \in \mathrm{SE}(n)$ a rigid motion and $U' = \Phi(U)$. If $f \in \mathsf{BV}_{\mathrm{loc}}(U')$, then $f\circ\Phi \in \mathsf{BV}_{\mathrm{loc}}(U)$. Moreover, if $\mathsf{D}\Phi = R \in \mathrm{SO}(n)$ and if $(\nu, |\nabla^{\mathsf{w}} f|)$ is the polar decomposition of $\nabla^{\mathsf{w}} f$, then the polar decomposition of $\nabla^{\mathsf{w}}(f\circ\Phi)$ is*

$$\big(\Phi_*^{-1}\nu, {\Phi^{-1}}_\#|\nabla^{\mathsf{w}} f|\big),$$

where $\Phi_*^{-1}\nu = R^{-1}\circ\nu\circ\Phi$. In particular, $f\circ\Phi \in \mathsf{BV}(U)$ if $f \in \mathsf{BV}(U')$.

PROOF. We have, for all $\varphi \in \mathsf{C}_{\mathsf{c}}^\infty(U, \mathbb{R}^n)$:

$$\begin{aligned}
\int_U (f\circ\Phi)\operatorname{div}\varphi\,\mathrm{d}\mathcal{L}^n &\overset{\text{AF } 5.39, \mathsf{J}\Phi^{-1}\equiv 1}{=} \int_{U'} f(\operatorname{div}\varphi)\circ\Phi^{-1}\,\mathrm{d}\mathcal{L}^n \overset{(*)}{=}\\
&= \int_{U'} f\operatorname{div}\big(\underbrace{R\circ\varphi\circ\Phi^{-1}}_{\in\mathsf{C}_{\mathsf{c}}^\infty(U',\mathbb{R}^n)}\big)\,\mathrm{d}\mathcal{L}^n \overset{7.4}{=}\\
&= -\int_{U'} \langle R\circ\varphi\circ\Phi^{-1}, \nu\rangle\,\mathrm{d}|\nabla^{\mathsf{w}} f| =\\
&= -\int \langle R\circ\varphi, \nu\circ\Phi\rangle\circ\Phi^{-1}\ \mathrm{d}|\nabla^{\mathsf{w}} f| =\\
&= -\int \underbrace{\langle R\circ\varphi, \nu\circ\Phi\rangle}_{\overset{R\in\mathrm{SO}(n)}{=}\langle\varphi, R^{-1}\circ\nu\circ\Phi\rangle}\ \mathrm{d}\big({\Phi^{-1}}_\#|\nabla^{\mathsf{w}} f|\big) =\\
&= \int \langle\varphi, \Phi_*^{-1}\nu\rangle\,\mathrm{d}\big({\Phi^{-1}}_\#|\nabla^{\mathsf{w}} f|\big),
\end{aligned}$$

where equality $(*)$ is justified by, for all $x \in U'$,

$$\begin{aligned}\operatorname{div}(R \circ \varphi \circ \Phi^{-1})(x) &= \operatorname{tr}\, \mathsf{D}(R \circ \varphi \circ \Phi^{-1})(x) \overset{\text{chain rule}}{=} \\ &= \operatorname{tr}\left[R \circ \mathsf{D}\varphi(\Phi^{-1} \cdot x) \circ R^{-1}\right] = \\ &= \operatorname{tr}\, \mathsf{D}\varphi(\Phi^{-1} \cdot x) = (\operatorname{div}\, \varphi) \circ \Phi^{-1}(x).\end{aligned}$$

□

THEOREM 7.43 (Trace theorem for BV functions on Lipschitz domains). *Let $n \geq 2$ and $\Omega \subset \mathbb{R}$ a Lipschitz domain with $\partial\Omega$ bounded. Then:*

i) There exists a unique bounded linear operator $T : \mathsf{BV}(\Omega) \to \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$ such that, for all $f \in \mathsf{BV}(\Omega)$ and all $\varphi \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int_\Omega f \operatorname{div}\, \varphi \,\mathrm{d}\mathcal{L}^n = -\int_\Omega \varphi \cdot \mathrm{d}\, \nabla^\mathsf{w} f + \int_{\partial\Omega} Tf\, \varphi \cdot \nu \,\mathrm{d}\mathcal{H}^{n-1}, \tag{7.25}$$

where ν the unit outer normal to $\partial\Omega$.

ii) For all $f \in \mathsf{BV}(\Omega)$ and for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$,

$$\lim_{r\to 0} \fint_{\mathbb{B}(x,r)\cap\Omega} \left|f(y) - Tf(x)\right| \mathrm{d}\mathcal{L}^n(y) = 0, \tag{7.26}$$

so that, for such x,

$$Tf(x) = \lim_{r\to 0} \fint_{\mathbb{B}(x,r)\cap\Omega} f \,\mathrm{d}\mathcal{L}^n.$$

PROOF. We proceed as in the proof of theorem 6.51. Fix $f \in \mathsf{BV}(\Omega)$.

1) For each $x \in \partial\Omega$, there exists an open set $U_x \subset \mathbb{R}^n$ such that $x \in U_x$ and U_x is obtained by rigid motion of a cylinder centered at $0 \in \mathbb{R}^n$ as in definition 6.33, i.e. there exists a rigid motion $\Phi \in \mathrm{SE}(n)$ with $\Phi(0) = x$ and there exists $r, h > 0$ and $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz with $\Gamma(0) = 0$ such that $U_x = \Phi\big(\mathbb{C}(0, r, h)\big)$, $\Phi\big(\mathrm{gr}\ \Gamma \cap \mathbb{C}(0, r, h)\big) = U_x \cap \partial\Omega$ and $\Phi\big(\mathrm{epi}_\mathsf{S}\ \Gamma \cap \mathbb{C}(0, r, h)\big) = U_x \cap \Omega$.
2) From the open cover $(U_x)_{x\in\partial\Omega}$ of the compact set $\partial\Omega \subset \mathbb{R}^n$, we may extract a finite subcover $(U_i)_{1\leq i\leq N}$. For each $1 \leq i \leq N$, let the corresponding objects defined in the previous item be denoted with a subscript i, so that $\Phi_i\big(\mathbb{C}(0, r_i, h_i)\big) = U_i$.

   Let $U_0 := \Omega$ and $U_{-1} := \overline{\Omega}^c$, so that $(U_i)_{-1\leq i\leq N}$ is a finite open cover of $\mathbb{R}^n$. We may apply corollary 6.11 to obtain a smooth partition of unity $(\xi_i)_{-1\leq i\leq N}$ of $\mathbb{R}^n$ with $\operatorname{spt}\, \xi_i \subset U_i$ for $-1 \leq i \leq N$. Besides, for $i \geq 1$, as $\operatorname{spt}\, \xi_i \subset U_i \Subset \mathbb{R}^n$, it follows that $\operatorname{spt}\, \xi_i$ is a compact subset of U_i.

Note that, in view of the product rule 7.29, for $0 \le i \le N$, $f_i := \xi_i f \in \mathsf{BV}(\Omega)$. Moreover, $f = \sum_{i=0}^N f_i$ and $\overline{\text{spt } f_i} \subset \text{spt } \xi_i \subset U_i$.

3) For $1 \le i \le N$, it follows from lemma 7.42 that $f_i \circ \Phi_i \in \mathsf{BV}\big(\text{epi}_{\mathsf{S}}\ \Gamma_i \cap \mathbb{C}(0, r_i, h_i)\big)$ and $\overline{\text{spt } f_i \circ \Phi_i} \subset \Phi_i^{-1}(\text{spt } \xi_i) \Subset \mathbb{C}(0, r_i, h_i)$. Extending the latter function by 0, we may consider $f_i \circ \Phi_i \in \mathsf{BV}(\text{epi}_{\mathsf{S}}\ \Gamma_i)$. Denoting by T the trace operator given by theorem 7.39 applied to $\text{epi}_{\mathsf{S}}\ \Gamma_i$, we may take $T \cdot (f_i \circ \Phi_i) \in \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\, \text{epi}_{\mathsf{S}}\ \Gamma_i})$; moreover, by (7.7), $\text{spt } T \cdot (f_i \circ \Phi_i) \subset \overline{\text{spt}\,(f_i \circ \Phi_i)} \Subset \mathbb{C}(0, r_i, h_i)$. Since the composition with Φ_i induces a linear isometry of $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega \cap U_i})$ onto $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\, \text{epi}_{\mathsf{S}}\ \Gamma_i \cap \mathbb{C}(0,r_i,h_i)})$, it makes sense to define

$$T_i \cdot f := T \cdot (f_i \circ \Phi_i) \circ \Phi_i^{-1} \in \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega \cap U_i}) \subset \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega}),$$

where the latter inclusion is given by the extension by 0. Note that $\text{spt } T_i f \subset \text{spt } \xi_i$.

The map $T_i : \mathsf{BV}(\Omega) \to \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$ is clearly linear continuous, since it is the composition of the sequence of linear continuous maps described in its definition above. Actually, the continuity of T_i holds in a stronger sense: if a sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathsf{BV}(\Omega)$ and $f \in \mathsf{BV}(\Omega)$ are such that $f_k \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and $|\nabla^{\mathsf{w}} f_k|(\Omega) \to |\nabla^{\mathsf{w}} f|(\Omega)$, then $T_i f_k \to T_i f$ in $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$. Indeed,

- It is clear that $\xi_i f_k \overset{k\to\infty}{\to} \xi_i f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$. Moreover, in view of propositions 7.27.i) and 4.58.ii), we have $|\nabla^{\mathsf{w}} f_k| \overset{*\,\mathsf{nc}}{\rightharpoonup} |\nabla^{\mathsf{w}} f|$ (see exercise 4.56 for the definition of the narrow convergence $\overset{*\,\mathsf{nc}}{\rightharpoonup}$), so that $\int_\Omega \xi_i \,\mathrm{d}|\nabla^{\mathsf{w}} f_k| \to \int_\Omega \xi_i \,\mathrm{d}|\nabla^{\mathsf{w}} f|$ (because $\xi_i|_\Omega \in \mathsf{C}_{\mathsf{b}}(\Omega)$). It then follows that

$$\begin{aligned}|\nabla^{\mathsf{w}}(\xi_i f_k)|(\Omega) &\overset{\text{product rule 7.29}}{=} \int_\Omega \xi_i \,\mathrm{d}|\nabla^{\mathsf{w}} f_k| + \int_\Omega f_k \|\nabla \xi_i\| \,\mathrm{d}\mathcal{L}^n \overset{k\to\infty}{\to} \\ &\overset{k\to\infty}{\to} \int_\Omega \xi_i \,\mathrm{d}|\nabla^{\mathsf{w}} f| + \int_\Omega f \|\nabla \xi_i\| \,\mathrm{d}\mathcal{L}^n = \\ &= |\nabla^{\mathsf{w}}(\xi_i f)|(\Omega).\end{aligned}$$

- It follows from the previous item that $(\xi_i f_k) \circ \Phi_i \overset{k\to\infty}{\to} (\xi_i f) \circ \Phi_i$ in $\mathsf{L}^1(\mathcal{L}^n|_{\text{epi}_{\mathsf{S}}\ \Gamma_i})$ and, since $|\nabla^{\mathsf{w}}[(\xi_i f_k) \circ \Phi_i]| = {\Phi_i^{-1}}_{\#}|\nabla^{\mathsf{w}}(\xi_i f_k)|$ by lemma 7.42,

$$|\nabla^{\mathsf{w}}[(\xi_i f_k) \circ \Phi_i]|(\text{epi}_{\mathsf{S}}\ \Gamma_i) \overset{k\to\infty}{\to} |\nabla^{\mathsf{w}}[(\xi_i f) \circ \Phi_i]|(\text{epi}_{\mathsf{S}}\ \Gamma_i).$$

We then conclude from remark 7.41.1) that

$$T \cdot [(\xi_i f_k) \circ \Phi_i] \overset{k\to\infty}{\to} T \cdot [(\xi_i f) \circ \Phi_i]$$

in $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\, \text{epi}_{\mathsf{S}}\ \Gamma_i})$, whence $T_i f_k \to T_i f$ in $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$, as asserted.

4) For $1 \le i \le N$, with the definition of $T_i f$ in step 3), we have (recall that $f_i = \xi_i f$):
   - For each $x \in \partial\Omega \cap \mathrm{spt}\ \xi_i$ and each $r > 0$ such that $\mathbb{B}(x,r) \subset U_i$,

$$\fint_{\mathbb{B}(x,r)\cap\Omega} \big|f_i(y) - T_i f(x)\big|\, \mathrm{d}\mathcal{L}^n(y) = \\ = \fint_{\mathbb{B}(\Phi_i^{-1}(x),r)\cap \mathrm{epi}_{\mathsf{S}}\ \Gamma_i} \big|f_i \circ \Phi_i(y) - T \cdot (f_i \circ \Phi_i)\big(\Phi_i^{-1}(x)\big)\big|\, \mathrm{d}\mathcal{L}^n(y).$$

   Since Φ_i is a linear isometry of $\partial\, \mathrm{epi}_{\mathsf{S}}\ \Gamma_i \cap \mathbb{C}(0, r_i, h_i)$ onto $\partial\Omega \cap U_i$ (hence it preserves $\mathcal{H}^{n-1}$ measure), by (7.7) it follows that

$$\lim_{r\to 0} \fint_{\mathbb{B}(x,r)\cap\Omega} \big|f_i(y) - T_i f(x)\big|\, \mathrm{d}\mathcal{L}^n(y) = 0 \tag{7.27}$$

   for $\mathcal{H}^{n-1}$ a.e. $x \in \partial\Omega \cap \mathrm{spt}\ \xi_i$. Since the above equality holds trivially if $x \in \partial\Omega \setminus \mathrm{spt}\ \xi_i$ (because $\mathrm{spt}\ T_i f \subset \mathrm{spt}\ \xi_i$, as it was noted in step 3), and because $\mathbb{B}(x,r) \cap \mathrm{spt}\ \xi_i = \emptyset$ for sufficiently small $r > 0$), we conclude that the latter equality holds for $\mathcal{H}^{n-1}$ a.e. $x \in \partial\Omega$.
   - For all $\varphi \in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n)$, denoting by ν' the unit outer normal to $\mathrm{epi}_{\mathsf{S}}\ \Gamma_i$ and by $(\nu_i, |\nabla^{\mathsf{w}} f_i|)$ the polar decomposition of $\nabla^{\mathsf{w}} f_i$,

we have:

(7.28)

$$
\begin{aligned}
\int_\Omega f_i \operatorname{div} \varphi \, \mathrm{d}\mathcal{L}^n &= \int_{\mathrm{epi}_{\mathsf{S}}\ \Gamma_i} f_i \circ \Phi_i \, (\operatorname{div} \varphi) \circ \Phi_i \, \mathrm{d}\mathcal{L}^n \overset{\text{put } R_i := \mathsf{D}\Phi_i \in \mathrm{SO}(n)}{=} \\
&= \int_{\mathrm{epi}_{\mathsf{S}}\ \Gamma_i} f_i \circ \Phi_i \, \operatorname{div} \big(\underbrace{R_i^{-1} \circ \varphi \circ \Phi_i}_{\in \mathsf{C}^1_{\mathsf{c}}(\mathbb{R}^n, \mathbb{R}^n)}\big) \, \mathrm{d}\mathcal{L}^n \overset{(7.7)}{=} \\
&= - \int_{\mathrm{epi}_{\mathsf{S}}\ \Gamma_i} (R_i^{-1} \circ \varphi \circ \Phi_i) \cdot \, \mathrm{d}\, \nabla^{\mathsf{w}} (f_i \circ \Phi_i) + \\
&\quad + \int_{\partial\, \mathrm{epi}_{\mathsf{S}}\ \Gamma_i} T(f_i \circ \Phi_i) \, (R_i^{-1} \circ \varphi \circ \Phi_i) \cdot \nu' \, \mathrm{d}\mathcal{H}^{n-1} \overset{7.42}{=} \\
&= - \int_{\mathrm{epi}_{\mathsf{S}}\ \Gamma_i} \langle R_i^{-1} \circ \varphi \circ \Phi_i, R_i^{-1} \circ \nu_i \circ \Phi_i \rangle \, \mathrm{d}\big(\Phi_i^{-1}{}_{\#} |\nabla^{\mathsf{w}} f_i|\big) + \\
&\quad + \int T(f_i \circ \Phi_i) \, \langle \varphi, R_i \circ \nu' \circ \Phi_i^{-1} \rangle \circ \Phi_i \, \mathrm{d}\big(\mathcal{H}^{n-1} \llcorner \partial\, \mathrm{epi}_{\mathsf{S}}\ \Gamma_i\big) \overset{7.17}{=} \\
&= - \int_\Omega \langle \varphi, \nu_i \rangle \, \mathrm{d} |\nabla^{\mathsf{w}} f_i| + \\
&\quad + \int \big[T(f_i \circ \Phi_i) \circ \Phi_i^{-1}\big] \, \langle \varphi, \nu \rangle \, \mathrm{d}\big(\mathcal{H}^{n-1} \llcorner \partial \Phi_i(\mathrm{epi}_{\mathsf{S}}\ f)\big) = \\
&= - \int_\Omega \varphi \cdot \, \mathrm{d}\, \nabla^{\mathsf{w}} f_i + \int_{\partial\Omega} T_i f \, \langle \varphi, \nu \rangle \, \mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}
$$

5) We define $T := \sum_{i=1}^N T_i : \mathsf{BV}(\Omega) \to \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$. It follows from step 3) that T is linear continuous. Besides, we have:
- For each $x \in \partial\Omega$ and each $y \in \Omega$, $\big|f(y) - Tf(x)\big| = \big|\sum_{i=0}^N f_i(y) + \sum_{i=1}^N T_i f(x)\big| \le |f_0(y)| + \sum_{i=1}^N \big|f_i(y) - T_i f(x)\big|$. Thus, for each $x \in \partial\Omega$ and each $r > 0$,

$$
\begin{aligned}
\fint_{\mathbb{B}(x,r)\cap\Omega} \big|f(y) - Tf(x)\big| \, \mathrm{d}\mathcal{L}^n(y) \le &\fint_{\mathbb{B}(x,r)\cap\Omega} \big|f_0(y)\big| \, \mathrm{d}\mathcal{L}^n(y) + \\
&+ \sum_{i=1}^N \fint_{\mathbb{B}(x,r)\cap\Omega} \big|f_i(y) - T_i f(x)\big| \, \mathrm{d}\mathcal{L}^n(y).
\end{aligned}
$$

Since $\overline{\mathrm{spt}\ f_0} \subset \mathrm{spt}\ \xi_0 \subset \Omega$, for each $x \in \partial\Omega$ and $r > 0$ sufficiently small f_0 is null on $\mathbb{B}(x,r) \cap \Omega$, hence

$$\lim_{r\to 0} \fint_{\mathbb{B}(x,r)\cap\Omega} \big|f_0(y)\big| \,\mathrm{d}\mathcal{L}^n(y) = 0.$$

It then follows from (7.27) that (7.26) holds for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$.

- Fix $\varphi \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^n, \mathbb{R}^n)$.
  Since $\xi_0\varphi \in \mathsf{C}^\infty_\mathsf{c}(\Omega, \mathbb{R}^n)$, we have

$$\begin{aligned} -\int_\Omega \xi_0\varphi \cdot \,\mathrm{d}\,\nabla^{\mathsf{w}} f &= \int_\Omega f \,\mathrm{div}\,(\xi_0\varphi)\,\mathrm{d}\mathcal{L}^n = \\ &= \int_\Omega \xi_0 f \,\mathrm{div}\ \varphi \,\mathrm{d}\mathcal{L}^n + \int_\Omega f\nabla\xi_0 \cdot \varphi \,\mathrm{d}\mathcal{L}^n. \end{aligned}$$

Thus

(7.29)

$$\begin{aligned} \int_\Omega f_0 \,\mathrm{div}\ \varphi \,\mathrm{d}\mathcal{L}^n = -\int_\Omega \varphi \cdot \,\mathrm{d}\big(\nabla^{\mathsf{w}} f \llcorner \xi_0 + \mathcal{L}^n \llcorner f\nabla\xi_0\big) &\overset{\text{product rule 7.29}}{=} \\ = -\int_\Omega \varphi \cdot \,\mathrm{d}\,\nabla^{\mathsf{w}} f_0. \end{aligned}$$

Therefore, from (7.28) and (7.29),

$$\begin{aligned} \int_\Omega f \,\mathrm{div}\ \varphi \,\mathrm{d}\mathcal{L}^n &= \int_\Omega f_0 \,\mathrm{div}\ \varphi \,\mathrm{d}\mathcal{L}^n + \sum_{i=1}^N f_i \,\mathrm{div}\ \varphi \,\mathrm{d}\mathcal{L}^n = \\ &= -\int_\Omega \varphi \cdot \,\mathrm{d}\,\nabla^{\mathsf{w}} f_0 - \sum_{i=1}^N \int_\Omega \varphi \cdot \,\mathrm{d}\,\nabla^{\mathsf{w}} f_i + \\ &\quad + \sum_{i=1}^N \int_{\partial\Omega} T_i f\,\varphi \cdot \nu \,\mathrm{d}\mathcal{H}^{n-1} = \\ &= -\int_\Omega \varphi \cdot \,\mathrm{d}\,\nabla^{\mathsf{w}} f + \int_{\partial\Omega} Tf\,\varphi \cdot \nu \,\mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

thus (7.25) is verified.

6) We have thus proved the existence of a continuous linear map $T : \mathsf{BV}(\Omega) \to \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$ satisfying (7.26) and (7.25). It remains to prove the uniqueness stated in part i), for which we reapply the argument used in the proof of the same statement for epigraphs: given $f \in \mathsf{BV}(\Omega)$, suppose that there exist $Tf, T'f \in \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$

such that (7.25) holds for all $\varphi \in \mathsf{C}^1_\mathsf{c}(\mathbb{R}^n, \mathbb{R}^n)$. Then, for all such φ,

$$\int_{\partial\Omega} (Tf - T'f)\, \varphi \cdot \nu \,\mathrm{d}\mathcal{H}^{n-1} = 0,$$

hence the $\mathbb{R}^n$-valued Radon measure $(\mathcal{H}^{n-1} \llcorner \partial\Omega) \llcorner (Tf - T'f)\nu$ is null. Then so is its total variation $(\mathcal{H}^{n-1} \llcorner \partial\Omega) \llcorner |Tf - T'f|$, which means that $Tf = T'f$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$.

□

COROLLARY 7.44. *With the same hypothesis of theorem 7.43, if* $f \in \mathsf{BV}(\Omega) \cap \mathsf{C}(\overline{\Omega})$, *then* $Tf = f|_{\partial\Omega}$.

REMARK 7.45. With the notation from theorem 7.43:

1) We have actually proved that the continuity of $T : \mathsf{BV}(\Omega) \to \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$ holds in a stronger sense, i.e. if a sequence $(f_i)_{i\in\mathbb{N}}$ in $\mathsf{BV}(\Omega)$ and $f \in \mathsf{BV}(\Omega)$ are such that $f_i \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$ and $|\nabla^\mathsf{w} f_i|(\Omega) \to |\nabla^\mathsf{w} f|(\Omega)$, then $Tf_i \to Tf$ in $\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$. Indeed, that was proved in step 3) of the proof for each $T_i : \mathsf{BV}(\Omega) \to \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$, for $1 \le i \le N$, hence it also holds for $T = \sum_{i=1}^N T_i$.
2) The trace operator from theorem 6.51 for $\mathsf{W}^{1,1}(\Omega)$ is the restriction of the trace operator from theorem 7.43.

THEOREM 7.46 (Extension of BV functions on Lipschitz epigraphs or Lipschitz domains). *Let* $n \ge 2$ *and* Ω *an open subset of* $\mathbb{R}^n$ *which is a Lipschitz epigraph or a Lipschitz domain with* $\partial\Omega$ *bounded. Given* $f \in \mathsf{BV}(\Omega)$ *and* $g \in \mathsf{BV}(\mathbb{R}^n \setminus \overline{\Omega})$, *let* F *be* $\mathcal{L}^n$*-measurable function defined by*

$$F(x) := \begin{cases} f(x) & x \in \Omega \\ g(x) & x \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

Then $F \in \mathsf{BV}(\mathbb{R}^n)$ *and*

$$\nabla^\mathsf{w} F = i_\#\nabla^\mathsf{w} f + i_\#\nabla^\mathsf{w} g - \mathcal{H}^{n-1} \llcorner \partial\Omega \llcorner (Tf - Tg)\nu, \tag{7.30}$$

where $i_\#\nabla^\mathsf{w} f$ *and* $i_\#\nabla^\mathsf{w} g$ *are the pushforwards of* $\nabla^\mathsf{w} f \in \mathcal{M}(\Omega, \mathbb{R}^n)$ *and* $\nabla^\mathsf{w} g \in \mathcal{M}(\overline{\Omega}^c, \mathbb{R}^n)$ *by the respective inclusions (the pushforward is taken in the sense of remark 4.46),* ν *is the unit outer normal of* Ω *and* T *denotes both trace operators* $\mathsf{BV}(\Omega), \mathsf{BV}(\overline{\Omega}^c) \to \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$. *In particular,*

$$|\nabla^\mathsf{w} F| = i_\#|\nabla^\mathsf{w} f| + i_\#|\nabla^\mathsf{w} g| + \mathcal{H}^{n-1} \llcorner \partial\Omega \llcorner |Tf - Tg|.$$

Note that, since $\mathcal{L}^n(\partial\Omega) = 0$ (because, as we have already seen, $\mathcal{H}$-dim $\partial\Omega = n - 1$), F is indeed an almost everywhere defined $\mathcal{L}^n$-measurable function. Besides, if Ω is a Lipschitz epigraph or a Lipschitz domain with bounded frontier, so is $\overline{\Omega}^c$, so that the trace operator

$\mathsf{BV}(\overline{\Omega}^c) \to \mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})$ exists. By exercise 7.14 and by the fact that $\chi_{\overline{\Omega}^c} = \chi_{\Omega^c}$ $\mathcal{L}^n$-a.e., the Gauss-Green measure of $\overline{\Omega}^c$ is $-\mu_\Omega$.

PROOF.

1) For all $\varphi \in \mathsf{C}_\mathsf{c}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\|\varphi\|_u \leq 1$,

(7.31)
$$\begin{aligned}\int_{\mathbb{R}^n} F \operatorname{div} \varphi \,\mathrm{d}\mathcal{L}^n = \int_\Omega f \operatorname{div} \varphi \,\mathrm{d}\mathcal{L}^n + \int_{\overline{\Omega}^c} g \operatorname{div} \varphi \,\mathrm{d}\mathcal{L}^n &\overset{(7.6) \text{ or } (7.25)}{=} \\ = -\int_\Omega \varphi \cdot \mathrm{d}\,\nabla^\mathsf{w} f - \int_{\overline{\Omega}^c} \varphi \cdot \mathrm{d}\,\nabla^\mathsf{w} g + & \\ + \int_{\partial\Omega} (Tf - Tg)\varphi \cdot \nu \,\mathrm{d}\mathcal{H}^{n-1}. &\end{aligned}$$

Therefore,

$$\mathsf{Var}(F, \mathbb{R}^n) \leq |\nabla^\mathsf{w} f|(\Omega) + |\nabla^\mathsf{w} g|)(\overline{\Omega}^c) + \int_{\partial\Omega} |Tf - Tg| \,\mathrm{d}\mathcal{H}^{n-1} < \infty,$$

which implies $F \in \mathsf{BV}(\mathbb{R}^n)$, as asserted.

2) It remains to prove the formulas for $\nabla^\mathsf{w} F$ and $|\nabla^\mathsf{w} F|$. Since $\mathbb{R}^n = \Omega \,\dot\cup\, \partial\Omega \,\dot\cup\, \overline{\Omega}^c$, we have

$$\nabla^\mathsf{w} F = \nabla^\mathsf{w} F \llcorner \Omega + \nabla^\mathsf{w} F \llcorner \overline{\Omega}^c + \nabla^\mathsf{w} F \llcorner \partial\Omega.$$

We must compute the three measures appearing in the second member above. Since Ω and $\overline{\Omega}^c$ are open sets, by the locality of the weak gradient it is clear that $\nabla^\mathsf{w} F|_\Omega = \nabla^\mathsf{w} f$ and $\nabla^\mathsf{w} F|_{\overline{\Omega}^c} = \nabla^\mathsf{w} g$, hence

$$\nabla^\mathsf{w} F \llcorner \Omega = i_\# \nabla^\mathsf{w} f \text{ and } \nabla^\mathsf{w} F \llcorner \overline{\Omega}^c = i_\# \nabla^\mathsf{w} g.$$

We contend that

$$\nabla^\mathsf{w} F \llcorner \partial\Omega = -\mathcal{H}^{n-1} \llcorner \partial\Omega \llcorner (Tf - Tg)\nu.$$

Indeed, fix $\epsilon > 0$ and let $\Omega_\epsilon := \partial\Omega + \mathbb{U}(0, \epsilon)$ be the open ϵ-neighborhood of $\partial\Omega$. By exercise 6.12 (differentiable Urysohn lemma) We may take $\zeta_\epsilon \in \mathsf{C}^\infty(\mathbb{R}^n)$ such that $0 \leq \zeta_\epsilon \leq 1$, $\zeta_\epsilon \equiv 1$ on $\partial\Omega$ and $\zeta_\epsilon \equiv 0$ on Ω_ϵ^c; in particular, $\operatorname{spt} \zeta_\epsilon \subset \overline{\Omega_\epsilon} \subset \partial\Omega + \mathbb{B}(0, \epsilon)$.

For each $\varphi \in \mathsf{C}_\mathsf{c}^\infty(\mathbb{R}^n, \mathbb{R}^n)$, applying (7.31) with $\varphi\zeta_\epsilon$ in place of φ yields

$$\begin{aligned}-\int_{\mathbb{R}^n} \varphi\zeta_\epsilon \cdot \mathrm{d}\,\nabla^\mathsf{w} F = -\int_\Omega \varphi\zeta_\epsilon \cdot \mathrm{d}\,\nabla^\mathsf{w} f - \int_{\overline{\Omega}^c} \varphi\zeta_\epsilon \cdot \mathrm{d}\,\nabla^\mathsf{w} g + & \\ + \int_{\partial\Omega} (Tf - Tg)\varphi \cdot \nu \,\mathrm{d}\mathcal{H}^{n-1}. &\end{aligned}$$

As $\epsilon \to 0$, $\varphi\zeta_\epsilon$ converges pointwise to $\varphi\chi_{\partial\Omega}$, and $\|\varphi\zeta\epsilon\| \leq \|\varphi\| \in \mathsf{L}^1(|\nabla^\mathsf{w} F|) \cap \mathsf{L}^1(|\nabla^\mathsf{w} f|) \cap \mathsf{L}^1(|\nabla^\mathsf{w} g|)$. We may therefore apply the dominated convergence theorem along a sequence convergent to 0, which yields

$$-\int_{\partial\Omega} \varphi \cdot \mathrm{d}\,\nabla^{\mathsf{w}} F = \int_{\partial\Omega} (Tf - Tg)\varphi \cdot \nu \,\mathrm{d}\mathcal{H}^{n-1},$$

thus proving our contention.

Finally, since $\nabla^{\mathsf{w}} F \llcorner \Omega$, $\nabla^{\mathsf{w}} F \llcorner \overline{\Omega}^c$ and $\nabla^{\mathsf{w}} F \llcorner \partial\Omega$ are pairwise mutually singular, it follows from proposition 4.15 that

$$|\nabla^{\mathsf{w}} F| = |\nabla^{\mathsf{w}} F \llcorner \Omega| + |\nabla^{\mathsf{w}} F \llcorner \overline{\Omega}^c| + |\nabla^{\mathsf{w}} F \llcorner \partial\Omega|$$

which yields the stated formula for $|\nabla^{\mathsf{w}} F|$.

□

COROLLARY 7.47 (Extension of BV functions on Lipschitz epigraphs or Lipschitz domains). *Let $n \geq 2$ and Ω an open subset of $\mathbb{R}^n$ which is a Lipschitz epigraph or a Lipschitz domain with $\partial\Omega$ bounded. The extension by 0 defines a bounded linear operator $\mathsf{BV}(\Omega) \to \mathsf{BV}(\mathbb{R}^n)$.*

PROOF. For each $f \in \mathsf{BV}(\Omega)$, its extension by 0 $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ coincides $\mathcal{L}^n$-a.e. with F defined in the previous theorem by means of f and $g \equiv 0$, hence $\bar{f} \in \mathsf{BV}(\mathbb{R}^n)$. Moreover, it follows from (7.30) and from the continuity of the trace operator that $\|\bar{f}\|_{\mathsf{BV}(\mathbb{R}^n)} = \|\bar{f}\|_{\mathsf{L}^1(\mathcal{L}^n)} + |\nabla^{\mathsf{w}} \bar{f}|(\mathbb{R}^n) = \|f\|_{\mathsf{L}^1(\mathcal{L}^n|_\Omega)} + |\nabla^{\mathsf{w}} f|(\Omega) + \|Tf\|_{\mathsf{L}^1(\mathcal{H}^{n-1}|_{\partial\Omega})} \leq C\|f\|_{\mathsf{BV}(\Omega)}$. □

COROLLARY 7.48. *Let $n \geq 2$ and Ω an open subset of $\mathbb{R}^n$ which is a Lipschitz epigraph or a Lipschitz domain with $\partial\Omega$ bounded. Given $f \in \mathsf{W}^{1,1}(\Omega)$ and $g \in \mathsf{W}^{1,1}(\mathbb{R}^n \setminus \overline{\Omega})$ such that $Tf = Tg$, then F defined in theorem 7.46 belongs to $\mathsf{W}^{1,1}(\mathbb{R}^n)$.*

PROOF. We have $F \in \mathsf{BV}(\mathbb{R}^n)$ and, by (7.30), $\nabla^{\mathsf{w}} F = i_{\#}\nabla^{\mathsf{w}} f + i_{\#}\nabla^{\mathsf{w}} g = \mathcal{L}^n \llcorner \nabla^{\mathsf{w}} f + \mathcal{L}^n \llcorner \nabla^{\mathsf{w}} g \in \mathsf{L}^1(\mathcal{L}^n, \mathbb{R}^n)$. □

7.5. Compactness

THEOREM 7.49 (Compactness theorem for BV). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $(f_i)_{i\in\mathbb{N}}$ a sequence in $\mathsf{BV}(\Omega)$ such that*

$$\sup\{\|f_i\|_{\mathsf{BV}(\Omega)} \mid i \in \mathbb{N}\} < \infty.$$

Then there exists $f \in \mathsf{BV}(\Omega)$ and a subsequence $(f_{i_j})_{j\in\mathbb{N}}$ of $(f_i)_i$ such that $f_{i_j} \to f$ in $\mathsf{L}^1(\mathcal{L}^n|\Omega)$.

We present two proofs for this theorem.

PROOF 1. For each $i \in \mathbb{N}$, we may apply theorem 7.33 to obtain $g_i \in \mathsf{C}^\infty(\Omega) \cap \mathsf{BV}(\Omega)$ such that $\|f_i - g_i\|_{\mathsf{L}^1(\Omega)} \leq 1/i$ and $\int_\Omega \|\nabla g_i\| \, \mathrm{d}\mathcal{L}^n \leq |\nabla^{\mathsf{w}} f_i|(\Omega) + 1/i$. In particular,

$$\sup\{\int_\Omega \|\nabla g_i\| \, \mathrm{d}\mathcal{L}^n \mid i \in \mathbb{N}\} < \infty.$$

It then follows that $(g_i)_{i\in\mathbb{N}}$ is a bounded sequence in $\mathsf{W}^{1,1}(\Omega) \subset \mathsf{BV}(\Omega)$. We may therefore apply Rellich-Kondrachov's theorem 6.77 to obtain a subsequence $(g_{i_j})_{j\in\mathbb{N}}$ of $(g_i)_i$ and $f \in \mathsf{L}^1(\Omega)$ [1] such that $g_{i_j} \to f$ in $\mathsf{L}^1(\Omega)$. Thus, $f_{i_j} \to f$ in $\mathsf{L}^1(\Omega)$. Moreover, it follows from proposition 7.32 that

$$\mathsf{Var}(f,\Omega) \leq \liminf \underbrace{\mathsf{Var}(f_{i_j},\Omega)}_{=|\nabla^{\mathsf{w}} f_{i_j}|(\Omega)} \leq \sup\{\|f_i\|_{\mathsf{BV}(\Omega)} \mid i \in \mathbb{N}\} < \infty,$$

whence $f \in \mathsf{BV}(\Omega)$. □

PROOF 2. By means of the extension by 0, cf. corollary 7.47, we may assume that $(f_i)_{i\in\mathbb{N}}$ a sequence in $\mathsf{BV}(\mathbb{R}^n)$ and $\operatorname{spt} f_i \subset \overline{\Omega} \Subset \mathbb{R}^n$.

It is clear that $(f_i)_{i\in\mathbb{N}}$ is bounded in $\mathsf{L}^1(\mathcal{L}^n)$, since it is bounded in $\mathsf{BV}(\Omega)$. Moreover, it follows from exercise 7.38 that

$$\|\tau_h f - f\|_{\mathsf{L}^1(\mathcal{L}^n)} \leq \|h\| \cdot \underbrace{\sup\{|\nabla^{\mathsf{w}} f_i|(\mathbb{R}^n) \mid i \in \mathbb{N}\}}_{<\infty},$$

so that $\lim_{h\to 0}\|\tau_h f_i - f_i\|_{\mathsf{L}^1(\mathcal{L}^n)} = 0$ uniformly on $i \in \mathbb{N}$. The thesis then follows from the Kolmogorov-Riesz-Fréchet compactness criterion 1.80. □

7.6. Sets of Finite Perimeter and Existence of Minimal Surfaces

In this section we develop some basic properties of sets of finite perimeter and we apply the direct method of the Calculus of Variations to prove the existence of minimizers in some geometric variational problems. Recall the definitions and notations for sets of finite perimeter in 7.5 and 7.12.

7.6.1. Support of the Gauss-Green measure. Let Ω be an open set in $\mathbb{R}^n$, $E \subset \Omega$ a set of locally finite perimeter in Ω and $\mu_E \in \mathcal{M}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ its Gauss-Green measure. As we have already noted in 7.13.1), it is clear that $\operatorname{spt} \mu_E \subset \partial^\Omega E$. Actually, we have the following precise description of $\operatorname{spt} \mu_E$. We use $|\cdot|$ to denote the Lebesgue measure in $\mathbb{R}^n$.

[1] actually $f \in \mathsf{L}^{1^*}(\Omega)$, by corollary 6.80

PROPOSITION 7.50. *If $E \subset \Omega$ is a set of locally finite perimeter in the open subset Ω of $\mathbb{R}^n$, then*

$$\operatorname{spt} \mu_E = \{x \in \Omega \mid \forall r > 0, 0 < |E \cap \mathbb{U}(x,r)| < \alpha(n) r^n\} \subset \partial^\Omega E.$$

Moreover, there exists a Borel set $F \subset \Omega$ in the same $\mathsf{L}^1_{\mathsf{loc}}$ class of E such that $\mu_F = \partial^\Omega F$.

PROOF.

1) Let $x \in \Omega$. If there exists $r > 0$ such that $|E \cap \mathbb{U}(x,r)| = 0$ (respectively, such that $|E \cap \mathbb{U}(x,r)| = \alpha(n) r^n$), then $\chi_E = 0$ (respectively, $\chi_E = 1$) $\mathcal{L}^n$-a.e. on the open set $\Omega \cap \mathbb{U}(x,r)$, which implies $\nabla^{\mathsf{w}} \chi_E = 0$ on $\Omega \cap \mathbb{U}(x,r)$ by the locality if the weak derivative 7.7, hence $\Omega \cap \mathbb{U}(x,r) \subset \Omega \setminus \operatorname{spt} \mu_E$.

   Conversely, if $x \in \Omega \setminus \operatorname{spt} \mu_E$, there exists $r > 0$ such that $\mathbb{U}(x,r) \subset \Omega$ and $\nabla^{\mathsf{w}} \chi_E = 0$ on $\mathbb{U}(x,r)$. It then follows from proposition 5.7 that χ_E coincides $\mathcal{L}^n$-a.e. with a constant function on $\mathbb{U}(x,r)$, hence $\chi_E = 0$ a.e. on $\mathbb{U}(x,r)$ or $\chi_E = 1$ a.e. on $\mathbb{U}(x,r)$, which implies $|E \cap \mathbb{U}(x,r)| = 0$ or $|E \cap \mathbb{U}(x,r)| = \alpha(n) r^n$, respectively.

   We have thus proved that $x \in \Omega \setminus \operatorname{spt} \mu_E$ if, and only if, there exists $r > 0$ such that $|E \cap \mathbb{U}(x,r)| = 0$ or $|E \cap \mathbb{U}(x,r)| = \alpha(n) r^n$.
2) Up to modifying E on a $\mathcal{L}^n$-null set, we may assume that $E \in \mathscr{B}_\Omega$. Define:
$$\begin{aligned} A_0 &:= \{x \in \Omega \mid \exists r > 0, |E \cap \mathbb{U}(x,r)| = 0\}, \\ A_1 &:= \{x \in \Omega \mid \exists r > 0, |E \cap \mathbb{U}(x,r)| = \alpha(n) r^n\} = \\ &= \{x \in \Omega \mid \exists r > 0, |(\Omega \setminus E) \cap \mathbb{U}(x,r)| = 0\}. \end{aligned}$$
   Then A_0 and A_1 are disjoint open subsets of Ω with $|E \cap A_0| = 0$ and $|A_1 \setminus E| = 0$. Define $F := (E \cup A_1) \setminus A_0 \in \mathscr{B}_\Omega$. Then:
   - $E \setminus F \subset E \cap A_0$ and $F \setminus E \subset A_1 \setminus E$, so that $|E \,\triangle\, F| = 0$.
   - It follows from the previous item that $\mu_F = \mu_E$, hence $\partial^\Omega F \supset \operatorname{spt} \mu_F = \operatorname{spt} \mu_E = \Omega \setminus (A_0 \cup A_1)$ by part 1) of the proof.
   - Since $A_1 \subset F^{\circ}$ and $\overline{F}^{\Omega} \subset \Omega \setminus A_0$, we conclude that $\partial^\Omega F \subset \Omega \setminus (A_0 \cup A_1)$, whence the thesis.

□

7.6.2. Operations with Sets of Finite Perimeter, part I.

PROPOSITION 7.51. *Let Ω be an open subset of $\mathbb{R}^n$. If E, F are sets of (locally) finite perimeter in Ω, then so are $E \cup F$ and $E \cap F$. Moreover,*

$$|\mu_{E \cup F}| + |\mu_{E \cap F}| \leq |\mu_E| + |\mu_F|. \tag{7.32}$$

Proof. It follows from proposition 7.36 and of the locality of the weak derivative that both $\chi_{E\cap F} = \chi_E \chi_F$ and $\chi_{E\cup F} = \chi_E + \chi_F - \chi_E\chi_F$ belong to $\mathsf{BV}_{\mathrm{loc}}(\Omega)$.

In order to prove (7.32), it suffices to show that the inequality holds when both members are computed in each open $A \Subset \Omega$. Fix such an open $A \Subset \Omega$, let $(\phi_\epsilon)_{\epsilon>0}$ be the standard mollifier in $\mathbb{R}^n$ and take $\epsilon_0 > 0$ such that $A \Subset \Omega_{\epsilon_0}$.

Define, for $0 < \epsilon < \epsilon_0$, $f_\epsilon := \phi_\epsilon * \chi_E \in \mathsf{C}^\infty(\Omega_{\epsilon_0})$ and $g_\epsilon := \phi_\epsilon * \chi_F \in \mathsf{C}^\infty(\Omega_{\epsilon_0})$, so that $0 \le f_\epsilon, g_\epsilon \le 1$, $f_\epsilon g_\epsilon \to \chi_{E\cap F}$ in $\mathsf{L}^1_{\mathsf{loc}}(\Omega_{\epsilon_0})$ and $h_\epsilon := f_\epsilon + g_\epsilon - f_\epsilon g_\epsilon \to \chi_{E\cup F}$ in $\mathsf{L}^1_{\mathsf{loc}}(\Omega_{\epsilon_0})$. Then:

1) It follows from proposition 7.26 that, for all open $V \Subset \Omega_{\epsilon_0}$,
$$|\nabla^{\mathsf{w}} f_\epsilon|\big|_V \overset{*}{\rightharpoonup} |\mu_E|\big|_V \quad \text{and} \quad |\nabla^{\mathsf{w}} g_\epsilon|\big|_V \overset{*}{\rightharpoonup} |\mu_F|\big|_V.$$
In particular, taking an open set V such that $A \Subset V \Subset \Omega_{\epsilon_0}$, we conclude that
$$\limsup |\nabla^{\mathsf{w}} f_\epsilon|(A) \le \limsup |\nabla^{\mathsf{w}} f_\epsilon|(\overline{A}) \overset{4.54.ii)}{\le} |\mu_E|(\overline{A}),$$
and, similarly, $\limsup |\nabla^{\mathsf{w}} g_\epsilon|(A) \le |\mu_F|(\overline{A})$.

2) For $0 < \epsilon < \epsilon_0$,
$$|\nabla^{\mathsf{w}}(f_\epsilon g_\epsilon)|(A) \le \int_A \big(f_\epsilon \|g_\epsilon\| + g_\epsilon \|f_\epsilon\|\big)\, \mathrm{d}\mathcal{L}^n,$$
$$|\nabla^{\mathsf{w}} h_\epsilon|(A) \le \int_A \big((1-g_\epsilon)\|f_\epsilon\| + (1-f_\epsilon)\|g_\epsilon\|\big)\, \mathrm{d}\mathcal{L}^n,$$
hence
$$|\nabla^{\mathsf{w}}(f_\epsilon g_\epsilon)|(A) + |\nabla^{\mathsf{w}} h_\epsilon|(A) \le |\nabla^{\mathsf{w}} f_\epsilon|(A) + |\nabla^{\mathsf{w}} g_\epsilon|(A).$$

3) Taking the $\liminf$ of both members in the previous equality along the sequence $\epsilon = 1/k$, it follows from step 1) and from the lower semicontinuity of the variation 7.32 that
$$|\mu_{E\cup F}|(A) + |\mu_{E\cap F}|(A) \le |\mu_E|(\overline{A}) + |\mu_F|(\overline{A}).$$

The previous inequality holds for each open $A \Subset \Omega$. In particular, given such an open $A \Subset \Omega$, it may be applied to $A_k := \{x \in A \mid d(x, A^c) > \frac{1}{k}\}$, for each $k \in \mathbb{N}$, which yields
$$|\mu_{E\cup F}|(A_k) + |\mu_{E\cap F}|(A_k) \le |\mu_E|(\overline{A_k}) + |\mu_F|(\overline{A_k}) \le |\mu_E|(A) + |\mu_F|(A).$$
Since the sequence $(A_k)_{k\in\mathbb{N}}$ increases to A, taking $\lim_{k\to\infty}$ in the first member of the previous inequality allows us to conclude that
$$|\mu_{E\cup F}|(A) + |\mu_{E\cap F}|(A) \le |\mu_E|(A) + |\mu_F|(A),$$
which proves (7.32), whence the thesis. ☐

7.6.3. Compactness from perimeter bounds.

DEFINITION 7.52. Let $(E_i)_{i\in\mathbb{N}}$ be a sequence of Lebesgue measurable sets in $\mathbb{R}^n$ and E a Lebesgue measurable set in $\mathbb{R}^n$. We say that

$$E_i \rightharpoonup E$$

if $\|\chi_{E_i} - \chi_E\|_{\mathsf{L}^1(\mathcal{L}^n)} = |E_i \triangle E| \to 0$.

We say that $E_i \overset{\text{loc}}{\rightharpoonup} E$ if $\chi_{E_i} \to \chi_E$ in $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n)$.

THEOREM 7.53 (Compactness from perimeter bounds). *Let $R > 0$ and $(E_i)_{i\in\mathbb{N}}$ be a sequence of sets of finite perimeter in $\mathbb{R}^n$ such that*

$$\sup_{i\in\mathbb{N}} \mathsf{P}(E_i) < \infty,$$
$$E_i \subset \mathbb{U}(0,R) \quad \forall i \in \mathbb{N}.$$

Then there exists a set $E \subset \mathbb{U}(0,R)$ of finite perimeter in $\mathbb{R}^n$ and a subsequence $(E_{i_j})_{j\in\mathbb{N}}$ of $(E_i)_{i\in\mathbb{N}}$ such that

$$E_{i_j} \rightharpoonup E \quad \textit{and} \quad \mu_{E_{i_j}} \overset{*}{\rightharpoonup} \mu_E.$$

PROOF. Let $\Omega = \mathbb{U}(0,R)$, which is a bounded Lipschitz domain. Note that, given $f \in \mathsf{L}^1(\mathcal{L}^n|_\Omega) \subset \mathsf{L}^1(\mathcal{L}^n)$, it follows from corollary 7.47 that $f \in \mathsf{BV}(\Omega)$ if, and only if, its extension by 0 belongs to $\mathsf{BV}(\mathbb{R}^n)$.

The hypothesis implies that $(\chi_{E_i})_{i\in\mathbb{N}}$ is a bounded sequence in $\mathsf{BV}(\Omega)$ since, for all $i \in \mathbb{N}$, $\|\chi_{E_i}\|_{\mathsf{L}^1(\mathcal{L}^n|_\Omega)} \le \alpha(n)R^n$ and $|\nabla^{\mathsf{w}}(\chi_{E_i}|_\Omega)|(\Omega) = |\nabla^{\mathsf{w}}\chi_{E_i}|\big|_\Omega(\Omega) \le \mathsf{P}(E_i) \le \sup_{i\in\mathbb{N}} \mathsf{P}(E_i) < \infty$. It then follows from theorem 7.49 that there exists a subsequence $(E_{i_j})_{j\in\mathbb{N}}$ of $(E_i)_{i\in\mathbb{N}}$ and $f \in \mathsf{BV}(\Omega)$ such that $\chi_{E_{i_j}} \to f$ in $\mathsf{L}^1(\mathcal{L}^n|_\Omega)$. Since there exists a subsequence of $\chi_{E_{i_j}}$ which converges $\mathcal{L}^n$-a.e. to f on Ω, we conclude that there exists $E \in \mathscr{B}_\Omega$ such that $f = \chi_E$ $\mathcal{L}^n$-a.e. on Ω, hence $\chi_E \in \mathsf{BV}(\Omega)$ and $E_{i_j} \rightharpoonup E$. By the remark on the first paragraph of the proof, we have $\chi_E \in \mathsf{BV}(\mathbb{R}^n)$, i.e. E is a set of finite perimeter in $\mathbb{R}^n$. Finally, it follows from proposition 7.27.i) that $\mu_{E_{i_j}} \overset{*}{\rightharpoonup} \mu_E$, which completes the proof. □

LEMMA 7.54. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $E \subset \mathbb{R}^n$ be a set of locally finite perimeter. Then $E \cap \Omega$ is a set of finite perimeter in $\mathbb{R}^n$ and*

$$\mathsf{P}(E \cap \Omega) \le \mathsf{P}(E,\Omega) + \mathsf{P}(\Omega).$$

PROOF. We know from proposition 7.51 (with $\mathbb{R}^n$ in place of Ω and Ω in place of F) that $E \cap \Omega$ is a set of locally finite perimeter in $\mathbb{R}^n$. It then suffices to prove the asserted inequality, which implies

$\mathsf{P}(E\cap\Omega)<\infty$, since $\mathsf{P}(E,\Omega)<\infty$ (because E is a set of locally finite perimeter in $\mathbb{R}^n$ and $\Omega \Subset \mathbb{R}^n$) and $\mathsf{P}(\Omega)<\infty$ (because $\partial\Omega$ is bounded).

Consider F in theorem 7.46 given by $f = \chi_E|_\Omega$ and $g = 0$. As elements of $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n)$, we have $F = \chi_{E\cap\Omega}$; it then follows from theorem 7.46 that $\nabla^{\mathsf{w}} F = \nabla^{\mathsf{w}} \chi_{E\cap\Omega}$ is given by (7.30). In particular, since $|Tf| \leq 1$ by (7.26), it follows that

$$\begin{aligned}\mathsf{P}(E\cap\Omega) = |\nabla^{\mathsf{w}} F|(\mathbb{R}^n) &\leq |\nabla^{\mathsf{w}} f|(\Omega) + \int_{\partial\Omega} |Tf| \,\mathrm{d}\mathcal{H}^{n-1} = \\ &= \underbrace{|\nabla^{\mathsf{w}} \chi_E|\big|_\Omega(\Omega)}_{=|\mu_E|(\Omega)=\mathsf{P}(E,\Omega)} + \int_{\partial\Omega} |Tf| \,\mathrm{d}\mathcal{H}^{n-1} \leq \\ &\leq \mathsf{P}(E,\Omega) + \mathcal{H}^{n-1}(\partial\Omega) = \mathsf{P}(E,\Omega) + \mathsf{P}(\Omega),\end{aligned}$$

as asserted. □

Corollary 7.55 (Compactness from perimeter bounds). *Let $(E_i)_{i\in\mathbb{N}}$ be a sequence of sets of locally finite perimeter in $\mathbb{R}^n$ such that, for all $R>0$,*

$$\sup_{i\in\mathbb{N}} \mathsf{P}\big(E_i, \mathbb{U}(0,R)\big) < \infty.$$

Then there exists a set E of locally finite perimeter in $\mathbb{R}^n$ and a subsequence $(E_{i_j})_{j\in\mathbb{N}}$ of $(E_i)_{i\in\mathbb{N}}$ such that

$$E_{i_j} \overset{loc}{\rightharpoonup} E \quad \textit{and} \quad \mu_{E_{i_j}} \overset{*}{\rightharpoonup} \mu_E.$$

Proof. For each $N\in\mathbb{N}$, it follows from lemma 7.54 that, for all $i\in\mathbb{N}$, $E_i\cap\mathbb{U}(0,N)$ is a set of finite perimeter in $\mathbb{R}^n$ and

$$\sup_{i\in\mathbb{N}} \mathsf{P}\big(E_i\cap\mathbb{U}(0,N)\big) \leq \sup_{i\in\mathbb{N}} \mathsf{P}\big(E_i, \mathbb{U}(0,N)\big) + \mathsf{P}\big(\mathbb{U}(0,N)\big) < \infty.$$

We may therefore apply theorem 7.53 to obtain a subsequence $(E^1_j)_{j\in\mathbb{N}}$ of $(E_i)_{i\in\mathbb{N}}$ and for each $k\geq 2$ a subsequence $(E^k_j)_{j\in\mathbb{N}}$ of $(E^{k-1}_j)_{j\in\mathbb{N}}$ such that for all $k\in\mathbb{N}$, $\chi_{E^k_j\cap\mathbb{U}(0,k)}$ converges in $\mathsf{L}^1(\mathcal{L}^n)$ to a set of finite perimeter $E_k \subset \mathbb{U}(0,k)$ of $\mathbb{R}^n$. The diagonal $(E^k_k)_{k\in\mathbb{N}}$ is therefore a subsequence of $(E_i)_{i\in\mathbb{N}}$ such that $\chi_{E^k_k\cap\mathbb{U}(0,N)}$ is $\mathsf{L}^1(\mathcal{L}^n)$ convergent for each $N\in\mathbb{N}$. That is, $\chi_{E^k_k}$ is a convergent sequence in $\mathsf{L}^1_{\mathsf{loc}}(\mathcal{L}^n)$ and its limit is the characteristic function of a Borel measurable set $F\subset\mathbb{R}^n$ such that $\big|\big(F\cap\mathbb{U}(0,k)\big)\triangle E_k\big| = 0$ for each $k\in\mathbb{N}$, i.e. $F\cap\mathbb{U}(0,k)$ is a set of finite perimeter in $\mathbb{R}^n$ for each $k\in\mathbb{N}$, hence $\chi_F|_{\mathbb{U}(0,k)} \in \mathsf{BV}\big(\mathbb{U}(0,k)\big)$ for each $k\in\mathbb{N}$. We have thus proved that $\chi_F\in\mathsf{BV}_{\mathrm{loc}}(\mathbb{R}^n)$, i.e. F is a set of locally finite perimeter in $\mathbb{R}^n$, and $E^k_k \overset{\mathrm{loc}}{\rightharpoonup} F$. It then follows from proposition 7.27.i) that $\mu_{E^k_k} \overset{*}{\rightharpoonup} \mu_F$, which completes the proof. □

7.6.4. Existence of Minimizers. In this subsection we apply the direct method of the Calculus of Variations to prove the existence of minimizers of two classes of geometric variational problems. Such application rests on the compactness theorems 7.53, 7.55 and on the lower semicontinuity of the perimeter 7.32.

Firstly we consider the Plateau problem in a compact subset K of $\mathbb{R}^n$ with boundary data given by a set M of locally finite perimeter in $\mathbb{R}^n$. The problem consists in finding a set $E_0 \subset \mathbb{R}^n$ of locally finite perimeter which has least perimeter in K among the sets $E \subset \mathbb{R}^n$ with locally finite perimeter whose boundaries are "fixed" by M, in the sense that $E \setminus K = M \setminus K$ — see figure 2.

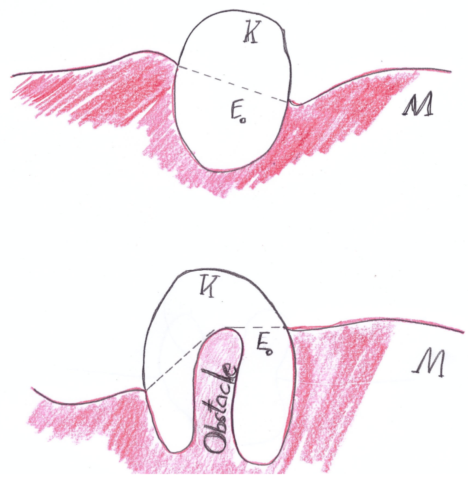

FIGURE 2. Plateau problem in K with boundary data M

PROPOSITION 7.56 (Minimizers for the Plateau problem in K with boundary data M). *Let $K \subset \mathbb{R}^n$ be a compact set and M be a set of locally finite perimeter in $\mathbb{R}^n$. Then there exists $E_0 \subset \mathbb{R}^n$ of locally finite perimeter which minimizes the functional*

$$E \mapsto \mathsf{P}(E, K)$$

in the class $\mathcal{E} := \{E \subset \mathbb{R}^n \mid \chi_E \in \mathsf{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $E \setminus K = M \setminus K\}$.

PROOF. Note that $\mathcal{E} \neq \emptyset$, since $M \in \mathcal{E}$. Let $m := \inf\{\mathsf{P}(E, K) \mid E \in \mathcal{E}\}$ (hence $0 \leq m < \infty$), and $(E_i)_{i\in\mathbb{N}}$ a sequence in $\mathcal{E}$ such that $\mathsf{P}(E_i, K) \to m$. Take $R > 0$ such that $\Omega := \mathbb{U}(0, R) \supset K$.

For all $i \in \mathbb{N}$, we have

$$\begin{aligned} \mathsf{P}(E_i, \Omega) &= \mathsf{P}(E_i, \Omega \setminus K) + \mathsf{P}(E_i, K) = \\ &= \mathsf{P}(M, \Omega \setminus K) + \mathsf{P}(E_i, K) \leq C(\Omega). \end{aligned}$$

It then follows from corollary 7.55 that there exists a set of locally finite perimeter $E_0 \subset \mathbb{R}^n$ and a subsequence $(E_{i_j})_{j\in\mathbb{N}}$ of $(E_i)_i$ such that $E_{i_j} \xrightarrow{\text{loc}} E_0$. Modifying E_0 on a $\mathcal{L}^n$-null set, if necessary, me may assume that $E_0 \setminus K = M \setminus K$, so that $E_0 \in \mathcal{E}$. Besides, by the lower semicontinuity of the variation 7.32 it follows that $\mathsf{P}(E_0, \Omega) \leq \liminf \mathsf{P}(E_{i_j}, \Omega)$, that is

$$
\begin{aligned}
&\mathsf{P}(M, \Omega \setminus K) + \mathsf{P}(E_0, K) \leq \\
&\quad \leq \liminf \big(\mathsf{P}(M, \Omega \setminus K) + \mathsf{P}(E_{i_j}, K)\big) = \\
&\quad = \mathsf{P}(M, \Omega \setminus K) + \liminf \mathsf{P}(E_{i_j}, K),
\end{aligned}
$$

whence $\mathsf{P}(E_0, K) \leq \liminf \mathsf{P}(E_{i_j}, K) = \lim \mathsf{P}(E_i, K) = m$. Since $E_0 \in \mathcal{E}$, we also have the opposite inequality $m \leq \mathsf{P}(E_0, K)$, hence $m = \mathsf{P}(E_0, K)$. $\square$

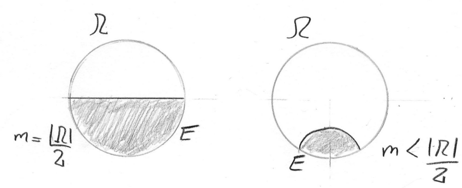

FIGURE 3. Relative isoperimetric problem in Ω

Given an open set $\Omega \subset \mathbb{R}^n$, the *relative isoperimetric problem in* Ω is the problem of finding sets with least perimeter in Ω with a fixed prescribed volume — see figure 3. Precisely, given $m \in (0, |\Omega|)$ (note that $|\Omega|$ is not assumed to be finite), we want to decide whether the following infimum is realized by a set of finite perimeter in Ω:

$$\alpha(m, \Omega) := \inf\{\mathsf{P}(E, \Omega) \mid E \subset \Omega, \chi_E \in \mathsf{BV}(\Omega), |E| = m\}.$$

We say that a set $E \subset \Omega$ of finite perimeter in Ω is a *relative isoperimetric set in* Ω if if is normalized according to proposition 7.50 so that $\operatorname{spt} \mu_E = \partial^\Omega E$ and it is a minimizer of the above problem, i.e. if $\mathsf{P}(E, \Omega) = \alpha(|E|, \Omega)$. If Ω is a bounded Lipschitz domain, the existence of such minimizers may be proved once more by a direct application of the direct method of the Calculus of Variations:

PROPOSITION 7.57 (Existence of relative isoperimetric sets on bounded Lipschitz domains). *Let Ω be a bounded Lipschitz domain and $m \in (0, |\Omega|]$. Then there exists a set $E \subset \Omega$ such that $\chi_E \in \mathsf{BV}(\Omega)$, $|E| = m$ and $\mathsf{P}(E, \Omega) = \alpha(m, \Omega)$.*

PROOF. Let $\mathcal{E} := \{\mathsf{P}(E,\Omega) \mid E \subset \Omega, \chi_E \in \mathsf{BV}(\Omega), |E| = m\}$.

1) We contend that $\mathcal{E}$ is not empty. Indeed, for each $t \in \mathbb{R}^n$, define $\Omega_t := \Omega \cap \{x \in \mathbb{R}^n \mid x_1 < t\}$, so that $\chi_{\Omega_t} \in \mathsf{BV}(\Omega)$. By a direct application of the dominated convergence theorem, $t \in \mathbb{R} \mapsto |\Omega_t| \in \mathbb{R}$ is a continuous function which is null in t_0 such that $\Omega_{t_0} = \emptyset$ and $|\Omega|$ in t_1 such that $\Omega_{t_1} = \Omega$ (such t_0 and t_1 exist because Ω is bounded). Therefore, by the intermediate value theorem, there exists $t \in [t_0, t_1]$ such that $|\Omega_t| = m$, hence $\Omega_t \in \mathcal{E}$.
2) It follows from the previous item that $0 \leq \alpha(m,\Omega) < \infty$. Let $(E_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{E}$ such that $\mathsf{P}(E_i,\Omega) \to \alpha(m,\Omega)$. Lemma 7.54 ensures that, for all $i \in \mathbb{N}$, E_i is a set of finite perimeter in $\mathbb{R}^n$ and
$$\mathsf{P}(E_i) \leq \mathsf{P}(E_i,\Omega) + \mathsf{P}(\Omega),$$
so that $\sup\{\mathsf{P}(E_i) \mid i \in \mathbb{N}\} < \infty$. Since Ω is bounded, we may therefore apply the compactness criterion 7.53 to obtain $E \subset \Omega$ such that $\chi_E \in \mathsf{BV}(\mathbb{R}^n)$ and such that, passing to a subsequence if necessary, $E_i \rightharpoonup E$. Then $|E_i| \to |E|$, so that $|E| = m$, i.e. $E \in \mathcal{E}$. Besides, by the lower semicontinuity of the variation 7.32 it follows that $\mathsf{P}(E,\Omega) \leq \liminf \mathsf{P}(E_i,\Omega) = m$; since $E \in \mathcal{E}$, we also have the opposite inequality, hence $\mathsf{P}(E,\Omega) = m$ and we are done.

□

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List of Symbols

$\mathsf{C}_\mathsf{c}(X)$	space of continuous functions with compac support 25
$\tau_y f$	$x \mapsto f(x-y)$ 26, 35
$\mu \times \nu$	product measure of μ and ν 27
$\mu \ll \nu$	μ is absolutely continuous with respect to ν 31, 74, 106
$\mu \perp \nu$	μ and ν are mutually singular 31, 74, 106
ν^+	positive part of ν 32
ν^-	negative part of ν 32
$\lvert\mu\rvert$	total variation of μ 32, 91
$\frac{d\nu}{d\mu}$	Radon-Nikodym derivative of ν with respect to μ 33
$\check{f}$	$x \mapsto f(-x)$ 35
$\partial^\alpha f$	For a multi-index $\alpha \in \mathbb{Z}^n$, $\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f$ 35
$\lvert\alpha\rvert$	For a multi-index $\alpha \in \mathbb{Z}^n$, $\alpha_1 + \cdots + \alpha_n$. 35
$\mathcal{H}^m$	Hausdorff m-dimensional measure 45
$\mathcal{H}^m_\delta$	size δ approximation of Hausdorff m-dimensional measure 45
$\mathcal{H}\text{-dim } A$	Hausdorff dimension of A 47
S_a	Steiner symmetrization with respect to $\langle a\rangle^\perp$ 51
$\Theta^{*n}(\mu, A, x)$	n-dimensional upper density of A at x with respect to μ 57
$\Theta^n_*(\mu, A, x)$	n-dimensional lower density of A at x with respect to μ 57
$\Theta^n(\mu, A, x)$	n-dimensional density of A at x with respect to μ 57
$\Theta^{*n}(\mu, x)$	n-dimensional upper density at x with respect to μ 57
$\Theta^n_*(\mu, x)$	n-dimensional lower density at x with respect to μ 57
$\Theta^n(\mu, x)$	n-dimensional density at x with respect to μ 57
$\Theta^{*\nu}(\mu, x)$	upper density of μ with respect to ν at x 62
$\Theta^\nu_*(\mu, x)$	lower density of μ with respect to ν at x 62
$\Theta^\nu(\mu, x)$	density of μ with respect to ν at x 62
SVP	symmetric Vitali property 65
$\mathsf{L}^1_{\mathsf{loc}}(\mu)$	$f : X \to \mathbb{C}$ μ-measurable and locally μ-summable 71
$f \prec U$	$f \in \mathsf{C}_\mathsf{c}(U)$ and $0 \le f \le 1$ 83
$\mathsf{C}^+_\mathsf{c}(X)$	$\{f \in \mathsf{C}_\mathsf{c}(X) \mid f \ge 0\}$ 84
$\mathscr{B}^c_X$	the set of Borel subsets of X which are relatively compact 94
$\mathcal{M}(X)^n$ or $\mathcal{M}(X, \mathbb{R}^n)$	finite $\mathbb{R}^n$-valued Radon measures on X 95
$\mathcal{M}_{\mathrm{loc}}(X)^n$ or $\mathcal{M}_{\mathrm{loc}}(X, \mathbb{R}^n)$	$\mathbb{R}^n$-valued Radon measures on X 95
$\mu \llcorner g$	restriction of μ to g 104
$\overset{*}{\rightharpoonup}$	weak star convergence 111
$\overset{*\mathsf{f}}{\rightharpoonup}$	weak star convergence for finite measures 111
$\overset{*\mathsf{nc}}{\rightharpoonup}$	narrow convergence 115
$\frac{\partial^{\mathsf{w}} u}{\partial x_i}$	i-th weak partial derivative of u 123

$\nabla^{\mathsf{w}} u$	weak gradient of u 123
$\mathsf{W}^{1,\mathsf{p}}$	space of $(1,p)$-Sobolev functions 126
$\mathsf{W}^{1,\mathsf{p}}_{\mathsf{loc}}$	space of local $(1,p)$-Sobolev functions 126
$\mathsf{D}f$	Fréchet derivative of f 127
$\mathrm{O}(n,m)$	set of orthogonal injections $\mathbb{R}^n \to \mathbb{R}^m$ 131
$\mathrm{Sym}(n)$	set of symmetric linear maps $\mathbb{R}^n \to \mathbb{R}^n$ 132
$[\![L]\!]$	Jacobian of L 133
$\Lambda(m,n)$	set of strictly increasing functions $\{1,\dots,m\} \to \{1,\dots,n\}$ 134
$\mathsf{J}f$	Jacobian of f 136
D_f	$\{x \mid \exists \mathsf{D}f(x)\}$ 136
J_f^+	$\{x \in D_f \mid \exists\, \mathsf{J}f(x) > 0\}$ 136
J_f^0	$\{x \in D_f \mid \exists\, \mathsf{J}f(x) = 0\}$ 136
$\\|f\\|_{\mathsf{W}^{1,\mathsf{p}}(\Omega)}$	$(\int_\Omega \lvert f\rvert^p + \\|\mathrm{grad}\ f\\|^p)^{1/p}\,\mathrm{d}\mathcal{L}^n$ 169
$\\|f\\|_{\mathsf{W}^{1,\infty}(\Omega)}$	$\big\\|\lvert f\rvert + \\|\nabla f\\|\big\\|_{\mathsf{L}^\infty(\Omega)}$ 169
$(f)_{x,r} = ⨍_{\mathbb{B}(x,r)} f\,\mathrm{d}\mathcal{L}^n$	average of f on $\mathbb{B}(x,r) \subset \mathbb{R}^n$, i.e. $\mathcal{L}^n\big(\mathbb{B}(x,r)\big)^{-1} \int_{\mathbb{B}(x,r)} f\,\mathrm{d}\mathcal{L}^n$ 216
$\mathsf{C}^{0,\gamma}(\overline{\Omega})$	Hölder continuous functions with exponent γ on Ω 219
$\mathsf{BV}_{\mathrm{loc}}(\Omega)$	locally bounded variation functions on Ω 230
$\mathsf{BV}(\Omega)$	bounded variation functions on Ω 230
$\mathsf{P}(E,\cdot)$	perimeter measure of a set E of locally finite perimeter 233
$\mathsf{Var}(f,V)$	variation of f on V 245