Lema: Seja $f: \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}$ tal que $(\forall x \in \mathbb{R})\ f(x,0) = 0$. Então $\exists\, g: \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}$ t.q. $(\forall (x,y) \in \mathbb{R}^2)\ f(x,y) = g(x,y)\cdot y$.

Dem.: $\forall (x,y) \in \mathbb{R}^2$:

$$f(x,y) = f(x,y) - f(x,0) \overset{TFC}{=}$$

$$= \int_0^1 \frac{d}{dt} f(x,ty)\,dt =$$

$$= \int_0^1 D_2 f(x,ty)\cdot y\,dt = y\int_0^1 D_2 f(x,ty)\,dt$$

Tome, então, $g: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$(x,y) \longmapsto \int_0^1 D_2 f(x,ty)\,dt$$

$g \in C^\infty$, pois $f \in C^\infty$ (basta aplicar a regra de Leibnitz p/ derivar sob o sinal da integral, conforme visto na lista #3).

#

Obs.: Analogamente, se $f: \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}$ é tal que $(\forall y \in \mathbb{R})\ f(0,y) = 0$, $\exists\, g: \mathbb{R}^2 \longrightarrow \mathbb{R}$ C^∞ tal que $(\forall (x,y) \in \mathbb{R}^2)\ f(x,y) = g(x,y)\cdot x$

Prop.: Seja $f : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}$ tal que

$(\forall x \in \mathbb{R})\ f(x,0) = 0$ e $(\forall y \in \mathbb{R})\ f(0,y) = 0$.

Então $\exists g : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}$ t.q.

$(\forall (x,y) \in \mathbb{R}^2)\ f(x,y) = g(x,y) . x . y$

Dem.: Pelo lema, $\exists g_1 : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}$ t.q.

$(\forall (x,y) \in \mathbb{R}^2)\ f(x,y) = g_1(x,y) \cdot y$

Como $(\forall y \in \mathbb{R})\ 0 = f(0,y) = g_1(0,y) . y$

conclui-se que $(\forall y \in \mathbb{R} \setminus \{0\})\ g_1(0,y) = 0$ e,

por continuidade, $(\forall y \in \mathbb{R})\ g_1(0,y) = 0$. Aplicando

o lema p/ g_1, conclui-se que $\exists g : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}$

t.q. $(\forall (x,y) \in \mathbb{R}^2)\ g_1(x,y) = g(x,y)\, x$

Donde $(\forall (x,y) \in \mathbb{R}^2)\ f(x,y) = g(x,y) . x . y$ #

— x —

Obs.: Por um argumento totalmente análogo, se

E Banach e $f : \mathbb{R}^n \xrightarrow{C^\infty} E$ se anular ao longo

dos eixos coordenados, $\exists g : \mathbb{R}^n \xrightarrow{C^\infty} E$ t.q.

$(\forall x = (x_1, \dots, x_n) \in \mathbb{R}^n)\ f(x) = g(x) . x_1 \cdots x_n$. #