Aplicações Multilineares Alternadas

E, F e.n. $\mathcal{L}_k(E,F)$, $\mathcal{L}_k^s(E,F)$

Def.: Ap. mult. alternada $f: E^k \to F$
$\forall x \in E^k$, $\forall i \in \{1, \dots, k-1\}$, $x_i = x_{i+1} \Rightarrow f(x) = 0$

$A_k(E,F) \doteq \{ f: E^k \to F$ alternada $\}$

é fechado em $\mathcal{L}_k(E,F)$ $\therefore$ Banach se F Banach

Por esta def., $A_1(E,F) = \mathcal{L}(E,F)$; pomos $A_0(E,F) \doteq F$.

(1)

/ /

· generalidades sobre grupos de permutações

· S_n age à esq. em $\{1, \dots, n\}$
" " " dir. em $E^n = \{x : \{1, \dots, n\} \to E\}$
" " " esq. em $L_n(E,F) \subset F^{E^n}$

dados $\begin{cases} f \in L_n(E,F) \\ \sigma \in S_n \end{cases}$ $\sigma f(x) = f(x \circ \sigma)$

Prop.: (H) $f \in A_n(E,F)$
(T) (i) $x \in E^n$, $1 \leq i < j \leq n$, $x_i = x_j \Rightarrow f(x) = 0$
(ii) $\forall \sigma \in S_n$, $\sigma f = \varepsilon(\sigma) f$

Def.: $\mathrm{Alt} : L_n(E,F) \circlearrowleft$
$$f \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) \sigma f$$

Prop.: (i) $\mathrm{Alt}(L_n(E,F)) \subset A_n(E,F)$
(ii) $\forall f \in A_n(E,F)$, $\mathrm{Alt}(f) = f$
(iii) $\mathrm{Alt}^2 = \mathrm{Alt}$.

Def.: [Multiplicação de Apl. Mult. Alternadas]
$f \in A_p(E,F)$; $g \in A_q(E,G)$, $\Phi : F \times G \to H$ bil.

$$\Phi \circ (f, g) \in A_{p,q}(E,H) \subset L_{p+q}(E,H)$$

$$f \wedge_\Phi g = \frac{(p+q)!}{p!\,q!} \mathrm{Alt}(\Phi \circ (f,g)) \in A_{p+q}(E,H)$$

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Obs.: Seja $Sh(p,q) \doteq \{\sigma \in S_{p+q} \,/\, \sigma(1) < \cdots < \sigma(p)$
e $\sigma(p+1) < \cdots < \sigma(p+q)\}$

Então, se $T \in A_{p,q}(E,H)$, tem-se:

$$Alt(T) = \frac{p!\,q!}{(p+q)!} \sum_{\sigma \in Sh(p,q)} \varepsilon(\sigma)\, \sigma T$$

Dem.: $S_p \hookrightarrow S_{p+q}$ (permutações dos p primeiros)
$S_q \hookrightarrow S_{p+q}$ (" " q últimos)
$S_p S_q = S_q S_p$

$$S_{p+q} = \dot{\bigcup_{\sigma \in Sh(p,q)}} \sigma S_p S_q$$

$\forall \tau \in S_p,\ \tau T = \varepsilon(\tau) T$
$\forall \tau \in S_q,\ \tau T = \varepsilon(\tau) T$

$$\therefore Alt(T) = \frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q}} \varepsilon(\sigma)\, \sigma \cdot T =$$

$$= \frac{1}{(p+q)!} \sum_{\sigma \in Sh(p,q)} \sum_{\alpha \in S_p} \sum_{\beta \in S_q} \varepsilon(\sigma)\varepsilon(\alpha)\varepsilon(\beta)\, \sigma \cdot \alpha \cdot \beta \cdot T =$$

$$= \frac{1}{(p+q)!} \sum_{\sigma \in Sh(p,q)} \sum_{\alpha \in S_p} \sum_{\beta \in S_q} \varepsilon(\sigma)\, \sigma \cdot T =$$

$$= \frac{p!\,q!}{(p+q)!} \sum_{\sigma \in Sh(p,q)} \varepsilon(\sigma)\, \sigma \cdot T \quad /\!/$$

Assim, $$f \wedge_{\Phi} g = \sum_{\sigma \in Sh(p,q)} \varepsilon(\sigma)\, \sigma \cdot \Phi(f,g)$$

Exemplos:

(1) $p = q = 1$, $f \in L(E,F)$, $g \in L(E,G)$

$f \wedge_\Phi g \in A_2(E,H)$

$f \wedge_\Phi g = \sum_{\sigma \in Sh(1,1)} \varepsilon(\sigma)\, \sigma \cdot \Phi(f,g)$

$\therefore f \wedge_\Phi g\,(x_1, x_2) = \Phi(f(x_1), g(x_2)) - \Phi(f(x_2), g(x_1))$

(2) $p = 1$, q qualquer, $f \in L(E,F)$, $g \in L_q(E,G)$

$f \wedge_\Phi g \in A_{q+1}(E,G)$

$f \wedge_\Phi g\,(x) = \sum_{\sigma \in Sh(1,q)} \varepsilon(\sigma)\, \sigma \cdot \Phi(f,g)(x) =$

$= \sum_{i=0}^{q} (-1)^i\, \Phi(f(x_i), g(x_0, \ldots, \hat{x}_i, \ldots, x_q))$

Caso Particular: $F = \mathbb{R}$, G, $H = G$, $\Phi : \mathbb{R} \times G \to \mathbb{R}$ multiplicação por escalares. Omitir-se-á Φ de $\wedge_\Phi$. Se $H = G = \mathbb{R}$, obtém-se o produto exterior de formas multilineares.

Prop.: (i) $L(E,\mathbb{R}) \times L(E,\mathbb{R}) \longrightarrow A_2(E,\mathbb{R})$

$(f,g) \longmapsto f \wedge g$

é bilinear alternada

(ii) $L(E,F) \times L(E,G) \longrightarrow A_2(E,H)$ é bilinear

$(f,g) \longmapsto f \wedge_\Phi g$

Prop.:

(H) $f \in A_p(E, \mathbb{R})$, $g \in A_q(E, \mathbb{R})$

(T) $f \wedge g = (-1)^{pq}\, g \wedge f$

Dem.: Seja $\tau \in S_{p+q}$, $\tau = (p+1, \dots, p+q, 1, \dots, p)$
$\mid \tau \in Sh(p,q)$, $\varepsilon(\tau) = (-1)^{pq}$

Então $\mid R_\tau : Sh(q,p) \longrightarrow Sh(p,q)$

$\sigma \longmapsto \sigma\tau$

$$f \wedge g = \sum_{\sigma \in Sh(p,q)} \varepsilon(\sigma)\, \sigma \cdot (f \otimes g) = \sum_{\sigma' \in Sh(q,p)} \varepsilon(\sigma'\tau)\, \sigma'\tau \cdot (f \otimes g) =$$

$$= \sum_{\sigma' \in Sh(q,p)} (-1)^{pq}\, \varepsilon(\sigma')\, \sigma' \cdot (g \otimes f) =$$

$$= (-1)^{pq}\, g \wedge f \quad \#$$

Prop.:

(1) Se $S \in L_k(E, \mathbb{R})$; $T \in L_m(E, \mathbb{R})$ e $Alt(S) = 0$, então $Alt(S \otimes T) = Alt(T \otimes S) = 0$

(2) $Alt(Alt(\omega \otimes \eta) \otimes \theta) = Alt(\omega \otimes \eta \otimes \theta) = Alt(\omega \otimes Alt(\eta \otimes \theta))$

(3) $\omega \in A_k(E, \mathbb{R})$, $\eta \in A_l(E, \mathbb{R})$, $\theta \in A_m(E, \mathbb{R})$:

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) =$$

$$= \frac{(k+l+m)!}{k!\,l!\,m!}\, Alt(\omega \otimes \eta \otimes \theta)$$

Obs.: Com a mesma dem. de (1), abaixo, vale: $S \in L_k(E, F)$, $T \in L_m(E, G)$, $\Phi : F \times G \to H$, $Alt(S) = 0 \Rightarrow Alt[\Phi(S, T)] = 0$.

Dem.:

(1) $\mathrm{Alt}(S\otimes T) = \dfrac{1}{(k+m)!}\sum_{\sigma\in S_{k+m}} \varepsilon(\sigma)\,\sigma\cdot(S\otimes T) =$

$= \dfrac{1}{(k+m)!}\sum_{\sigma\in Sh(k,m)}\sum_{\alpha\in S_k}\sum_{\beta\in S_m} \varepsilon(\sigma)\varepsilon(\alpha)\varepsilon(\beta)\,\sigma\cdot\alpha\cdot\beta(S\otimes T) =$

$= \dfrac{1}{(k+m)!}\sum_{\sigma\in Sh(k,m)}\sum_{\beta\in S_m} \varepsilon(\sigma)\varepsilon(\beta)\,\sigma\cdot\beta\cdot\left\{\sum_{\alpha\in S_k}\varepsilon(\alpha)\,\alpha\cdot(S\otimes T)\right\}$

Como $\sum_{\alpha\in S_k}\varepsilon(\alpha)\,\alpha\cdot(S\otimes T) = \underbrace{\left(\sum_{\alpha\in S_k}\varepsilon(\alpha)\,\alpha\cdot S\right)}_{=k!\,\mathrm{Alt}(S)=0}\otimes T = 0$

segue-se $\mathrm{Alt}(S\otimes T) = 0$.

Analogamente, $\mathrm{Alt}(T\otimes S) = 0$

(2) $\mathrm{Alt}(\mathrm{Alt}(\omega\otimes\eta) - \omega\otimes\eta) = \mathrm{Alt}^2(\omega\otimes\eta) - \mathrm{Alt}(\omega\otimes\eta) =$
$= \mathrm{Alt}(\omega\otimes\eta) - \mathrm{Alt}(\omega\otimes\eta) = 0$

$\therefore\ \mathrm{Alt}[(\mathrm{Alt}(\omega\otimes\eta) - \omega\otimes\eta)\otimes\theta] = 0$

$\parallel$

$\mathrm{Alt}(\mathrm{Alt}(\omega\otimes\eta)\otimes\theta) - \mathrm{Alt}(\omega\otimes\eta\otimes\theta)$

Analogamente, $\mathrm{Alt}(\omega\otimes\mathrm{Alt}(\eta\otimes\theta)) = \mathrm{Alt}(\omega\otimes\eta\otimes\theta)$.

(3) $(\omega\wedge\eta)\wedge\theta = \dfrac{(k+l+m)!}{(k+l)!\,m!}\mathrm{Alt}((\omega\wedge\eta)\otimes\theta) =$

$= \dfrac{(k+l+m)!}{(k+l)!\,m!}\,\dfrac{(k+l)!}{k!\,l!}\,\underbrace{\mathrm{Alt}(\mathrm{Alt}(\omega\otimes\eta)\otimes\theta)}_{=\mathrm{Alt}(\omega\otimes\eta\otimes\theta)}$ #

Corolário: (H) $\begin{cases} f_1, \dots, f_k \in L(E, \mathbb{R}) \\ v_1, \dots, v_k \end{cases}$

(T) $(f_1 \wedge \dots \wedge f_k)(v_1, \dots, v_k) = \det(f_i(v_j))$

Dem.: $(f_1 \wedge \dots \wedge f_k)(v) = k! \, \mathrm{Alt}(f_1 \otimes \dots \otimes f_k)(v) =$

$$= \sum_{\sigma \in S_k} \varepsilon(\sigma) f_1(v_{\sigma(1)}) \dots f_k(v_{\sigma(k)}) = \det[f_i(v_j)]$$

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Obs.: O mesmo argumento usado na dem. do item (c) da última prop. mostra que, se $S \in L_k(E, \mathbb{R})$ e $\exists\ 1 \leq i < j \leq k \ / \ S \circ (i,j) = S$, então $\mathrm{Alt}(S) = 0$. Isto tb pode ser verificado diretamente:

Dem.: $A_n \doteq \{\sigma \in S_n \ / \ \varepsilon(\sigma) = 1\}$

$S_n = A_n(i,j) \dot{\cup} A_n$ e $f_{(i,j)} : A_n \to A_n(i,j)$ bijeção

$$\mathrm{Alt}\, S = \frac{1}{k!} \sum_{\sigma \in S_n} \varepsilon(\sigma) \sigma \cdot S = \frac{1}{k!} \sum_{\sigma \in A_n} \Big[\varepsilon(\sigma) \sigma \cdot S +$$

$$+ \varepsilon((i,j)\sigma) \sigma \cdot \underbrace{((i,j) \cdot S)}_{= S} \Big] = \frac{1}{k!} \sum_{\sigma \in A_n} [\sigma \cdot S - \sigma \cdot S] = 0$$

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Caso em que E tem dim finita

Prop.: $\forall\, \omega \in L_k(\mathbb{R}^n, F)$, $\exists!\ c_{i_1 \dots i_k} \in F$, $1 \leq i_j \leq n$,

t.q. $\omega = \sum_{1 \leq i_1, \dots, i_k \leq n} c_{i_1 \dots i_k} e^{i_1} \otimes \dots \otimes e^{i_k}$

Dem.:

(i) Existência: Sejam $v_1, \dots, v_k \in \mathbb{R}^n$

$(\forall 1 \le j \le k)\ v_j = \sum_{i_j=1}^{n} a_j^{i_j} e_{i_j}$

$$\therefore \omega(v_1, \dots, v_k) = \sum_{1 \le i_1, \dots, i_k \le n} \underbrace{a_1^{i_1} \cdots a_k^{i_k}}_{= e^{i_1} \otimes \cdots \otimes e^{i_k}(v_1, \dots, v_k)} \overbrace{\omega(e_{i_1}, \dots, e_{i_k})}^{= c_{i_1 \dots i_k}}$$

i.e. $\omega(v_1, \dots, v_k) = \sum_{1 \le i_1, \dots, i_k \le n} c_{i_1 \dots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k}(v_1, \dots, v_k)$

(ii) Unicidade

Suponha $\sum_{i_1, \dots, i_k = 1}^{n} c_{i_1 \dots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k} = \sum_{i_1, \dots, i_k = 1}^{n} c'_{i_1 \dots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k}$

Aplicando os dois membros em $(e_{i_1}, \dots, e_{i_k})$ obtém-se:

$c_{i_1 \dots i_k} = c'_{i_1 \dots i_k}$. #

Prop.: (H) $\omega \in A_k(\mathbb{R}^n, F)$

(T) $\exists!\ c_{i_1 \dots i_k} \in F$, $1 \le i_1 < \cdots < i_k \le n$ /

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} c_{i_1 \dots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}$$

Dem.: (i) Pela prop. anterior, $\omega = \sum_{i_1, \dots, i_k = 1}^{n} c'_{i_1 \dots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k}$

$\therefore \omega = Alt(\omega) = \sum_{i_1, \dots, i_k = 1}^{n} c'_{i_1, \dots, i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}$

e isto prova a existência.

(ii) Unicidade:

Se $\omega = \sum_{1 \le i_1 < \dots < i_k \le n} c_{i_1 \dots i_k} \, e^{i_1} \wedge \dots \wedge e^{i_k}$

Então, fixados $1 \le i_1 < \dots < i_k \le n$:

$$\omega(e_{i_1}, \dots, e_{i_k}) = c_{i_1 \dots i_k}.$$

Corolário: Se $k > n$, $A_k(\mathbb{R}^n, F) = \{0\}$

Corolário: Se $k = n$, $\forall \omega \in A_n(\mathbb{R}^n, F)$, $\exists! c \in F /$
$\omega = c \, e^1 \wedge \dots \wedge e^n$.

Formas Diferenciais

$U \overset{ab}{\subset} E$, F Banach.

Def: $\omega : U \to A_p(E, F)$ forma dif. de grau p

$$\Omega_p^{(n)}(U, F) = \{\omega : U \to A_p(E,F) \, / \, \omega \in C^n\}$$

Notação: ω p-forma, $\xi_1, \dots, \xi_p \in E$, $x \in U$
$\omega(x) \cdot (\xi_1, \dots, \xi_n) = \omega(x; \xi_1, \dots, \xi_n) \in F$

Operações c/ Formas Dif.

1) Prod. Ext.: $\omega \in \Omega_p^{(n)}(U, F)$, $\eta \in \Omega_q^{(n)}(U, G)$, $\Phi : F \times G \to H$
$\omega \wedge_\Phi \eta \in \Omega_{p+q}^{(n)}(U, G)$, $x \mapsto \omega(x) \wedge_\Phi \eta(x)$

2) Derivação exterior

Def.: $\omega \in \Omega_p^{(n)}(U, F)$, i.e. $\omega : U \xrightarrow{C^n} A_p(E,F)$

$$D\omega : U \longrightarrow L(E, A_p(E,F)) \equiv A_{1,p}(E,F)$$

$$L(E, L_p(E,F)) \equiv L_{p+1}(E,F)$$

$$d\omega(x) \doteq \frac{\overbrace{(p+1)!}^{=p+1}}{p!\,1!} \operatorname{Alt}[D\omega(x)] = \frac{(p+1)!}{p!\,1!} \cdot \frac{1}{(p+1)!}\, p!\,1! \sum_{\sigma \in Sh(1,p)} \varepsilon(\sigma)\, \sigma \cdot D\omega(x) =$$

$$= \sum_{\sigma \in Sh(1,p)} \varepsilon(\sigma)\, \sigma \cdot D\omega(x) \in \Omega_p^{(n-1)}(U, F)$$

$$\therefore \boxed{d\omega(x; \xi_0, \dots, \xi_p) = \sum_{i=0}^{p} (-1)^i\, D\omega(x; \xi_i, \xi_0, \dots, \hat{\xi}_i, \dots, \xi_p)}$$

Propriedades:

1) Prop.: (H) $U \overset{ab}{\subset} E$, F, G, H Banach

$\omega \in \Omega_p^{(n)}(U, F)$, $\eta \in \Omega_q^{(n)}(U, G)$, $\Phi : F \times G \xrightarrow{\text{bil. cont.}} H$

(T) $d(\omega \wedge_\Phi \eta) = d\omega \wedge_\Phi \eta + (-1)^p \omega \wedge_\Phi d\eta$

Dem.: $d(\omega \wedge_\Phi \eta)_{(x)}(\xi_0, \dots, \xi_{p+q}) = (p+q+1) \operatorname{Alt}[D(\omega \wedge_\Phi \eta)(x)](\xi) =$

$$= \frac{(p+q+1)(p+q)!}{p!\,q!} \operatorname{Alt}[D\Phi(\omega, \eta)(x)](\xi) \quad (*)$$

$$D\Phi(\omega, \eta)(x)(\xi) = \Phi(D\omega(x)\cdot\xi_0, \eta(x))(\xi_1, \dots, \xi_{p+q}) + $$
$$+ \Phi(\omega(x), D\eta(x)\cdot\xi_0)(\xi_1, \dots, \xi_{p+q}) =$$

$= \Phi\left(D\omega(x)(\xi_0,\dots,\xi_p), \eta(x)(\xi_{p+1},\dots,\xi_{p+q})\right) +$
$+ \Phi\left(\omega(x)(\xi_1,\dots,\xi_p), D\eta(x)(\xi_0,\dots,\xi_{p+q})\right) =$

$= \Phi(D\omega(x), \eta(x))\cdot\xi + \Phi(\omega(x), D\eta(x))\cdot\xi\circ\sigma$

onde $\sigma = \begin{pmatrix} 0, 1, \dots, p, p+1, \dots, q \\ 1, \dots, p, 0, p+1, \dots, q \end{pmatrix}$ (**)

$\therefore D\Phi(\omega,\eta) = \Phi(D\omega,\eta) + \sigma\cdot\Phi(\omega, D\eta)$
onde σ como em (**).

Assim, de (*):

$d(\omega\wedge_\Phi\eta) = \dfrac{(p+q+1)!}{p!\,q!}\,\mathrm{Alt}[\Phi(D\omega,\eta) + \sigma\Phi(\omega,D\eta)] =$

$= \dfrac{(p+q+1)!}{p!\,q!}\left\{\mathrm{Alt}[\Phi(D\omega,\eta)] + \overbrace{(-1)^p}^{=\varepsilon(\sigma)}\mathrm{Alt}[\Phi(\omega,D\eta)]\right\}$ (Δ)

Como $d\omega = (p+1)\,\mathrm{Alt}[D\omega]$, tem-se $\mathrm{Alt}[d\omega - (p+1)D\omega] = 0$, donde $\mathrm{Alt}\{\Phi([d\omega - (p+1)D\omega], \eta)\} = 0 \Leftrightarrow$
$\Leftrightarrow \mathrm{Alt}[\Phi(d\omega,\eta)] = (p+1)\,\mathrm{Alt}[\Phi(D\omega,\eta)]$

Analogamente, $\mathrm{Alt}[\Phi(\omega,d\eta)] = (q+1)\,\mathrm{Alt}[\Phi(\omega,D\eta)]$

$\therefore (\Delta) = \dfrac{(p+q+1)!}{p!\,q!\,(p+1)}\,\mathrm{Alt}[\Phi(d\omega,\eta)] + (-1)^p\dfrac{(p+q+1)!}{p!\,q!\,(q+1)}\,\mathrm{Alt}[\Phi(\omega,d\eta)]$

$= \boxed{d\omega\wedge_\Phi\eta + (-1)^p\,\omega\wedge_\Phi d\eta}$ //

II.) **Prop.:** $\forall \omega \in \Omega_p^{(n)}(\mathcal{U}, F)$, $n \geq 2$,
$d^2\omega = 0$

Dem.:

$$d\omega = (p+1)\, \mathrm{Alt}[D\omega] \in \Omega_{p+1}^{(n-1)}(\mathcal{U}, F)$$

$$d^2\omega = (p+2)(p+1)\, \mathrm{Alt}[D\,\mathrm{Alt}(D\omega)] =$$

$$= (p+2)(p+1)\, \underbrace{\mathrm{Alt}^2}_{=\mathrm{Alt}}[D^2\omega] \in \Omega_{p+2}^{(n-2)}(\mathcal{U}, F)$$

Como $D^2\omega(x; \xi_1, \xi_2, \xi') \overset{\text{Schwartz}}{\underset{\text{teo.}}{=}} D^2\omega(x; \xi_2, \xi_1, \xi')$,

tem-se $\mathrm{Alt}\, D^2\omega = 0 \therefore d^2\omega = 0$. #

Pull Back

Def.: $f : E' \longrightarrow E$

$$f^* : L_k(E, F) \longrightarrow L_k(E', F)$$
$$T \longmapsto ((v_1, \dots, v_k) \longmapsto T\cdot(fv_1, \dots, fv_k))$$

$\forall \sigma \in S_k$, $\forall T \in L_k(E,F)$: $f^*(\sigma \cdot T) = \sigma \cdot (f^* T) \therefore \boxed{f^*\mathrm{Alt} = \mathrm{Alt}\, f^*}$

$\therefore f^*(A_k(E,F)) \subset A_k(E', F)$

Def.: $\mathcal{U}' \overset{ab}{\subset} E'$, $\mathcal{U} \overset{ab}{\subset} E$, $f : \mathcal{U}' \xrightarrow{C^{(n+1)}} \mathcal{U}$

$$f^* : \Omega_p^{(n)}(\mathcal{U}, F) \longrightarrow \Omega_p^{(n)}(\mathcal{U}', F)$$
$$\omega \longmapsto (y \in \mathcal{U}' \longmapsto Df(y)^* \omega(f(y)))$$

i.e. $(f^*\omega)(y; v_1, \dots, v_p) = \omega(f(y); Df(y)v_1, \dots, Df(y)\cdot v_p)$

e $f^*\omega = \omega \circ f$ se $p = 0$.

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Propriedades do Pull Back

1.) $(f\circ g)^* w = g^*(f^* w)$ e $id^* w = w$

2.) $f^*(w \wedge_\Phi \eta) = f^* w \wedge_\Phi f^* \eta$

3.) $d f^* w = f^* dw$ $(f \in C^2, w \in C^1)$

Dem.: Seja $w \in \Omega_p^{(n)}(M,F)$, $n \geq 1$.

$d f^* w = (p+1) \, Alt[D f^* w]$

$(f^* w)(y) \cdot (\xi_1, \dots, \xi_p) = w(f(y); Df(y)\cdot\xi_1, \dots, Df(y)\cdot \xi_p)$

$\therefore D(f^* w)(y) \cdot (\xi_0, \dots, \xi_p) = Dw(f(y); Df(y)\cdot\xi_0, \dots, Df(y)\cdot\xi_p) +$

$+ \sum_{i=1}^{p} \underbrace{w(f(y); Df(y)\cdot\xi_1, \dots, D^2 f(y)\cdot\xi_0\cdot\xi_i, \dots, Df(y)\cdot\xi_p)}_{\text{simétrico em } \xi_0, \xi_i \text{ pelo teo. de Schwartz}}$

$\therefore Alt[D(f^* w)(y)] = Alt[Df(y)^* Dw(f(y))] =$

$= Df(y)^* \underbrace{Alt[Dw(f(y))]}_{= dw(f(y))}$

$\therefore d(f^* w)(y) = (p+1) \, Alt[D(f^* w)(y)] = Df(y)^* dw(y) = (f^* dw)(y)$

i.e. $d(f^* w) = f^* dw$. #

Caso em que E tem dimensão finita

Sejam $E = \mathbb{R}^n$, $U \overset{ab}{\subset} \mathbb{R}^n$, $\omega \in \Omega_p^{(k)}(U, F)$.

$(e_1, \dots, e_n)$ b.c. de $\mathbb{R}^n$, $(e^1, \dots, e^n)$ base dual

Denotaremos por $x_i : U \longrightarrow \mathbb{R}$ a restrição de e^i a U, $1 \leq i \leq n$.

Assim, $dx_i = Dx_i : U \longrightarrow L(\mathbb{R}^n, \mathbb{R}) = A_1(\mathbb{R}^n, \mathbb{R})$
$$p \longmapsto e^i$$

Prop.: Com a notação acima, existem únicas $C_{i_1 \dots i_p} : U \longrightarrow F$ de classe C^k, para $1 \leq i_1 < \dots < i_p \leq n$, tais que
$$\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} C_{i_1 \dots i_p} \, dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

i.e. $(\forall x \in U)$ $\omega(x) = \sum_{1 \leq i_1 < \dots < i_p \leq n} C_{i_1 \dots i_p}(x) \, e^{i_1} \wedge \dots \wedge e^{i_p}$

Dem.: Basta tomar $C_{i_1 \dots i_p}(x) = \omega(x; e_{i_1}, \dots, e_{i_p})$.

Prop.: Com a notação acima, tem-se:

(i) Se $f \in \Omega_0^{(k \geq 1)}(U, F)$, $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx^i$

(ii) Se $\omega \in \Omega_p^{(k \geq 1)}(U, F)$, $\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} C_{i_1 \dots i_p} \, dx_{i_1} \wedge \dots \wedge dx_{i_p}$,
$d\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} dC_{i_1 \dots i_p} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}$.

(iii) Se $f : U' \overset{ab}{\subset} E \longrightarrow U$ de classe C^{k+1} e ω como em (ii):
$$f^* \omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} f^* C_{i_1 \dots i_p} (f^* dx_{i_1}) \wedge \dots \wedge (f^* dx_{i_p}) =$$
$$= \sum_{1 \leq i_1 < \dots < i_p \leq n} (C_{i_1 \dots i_p} \circ f) \, d(x_{i_1} \circ f) \wedge \dots \wedge d(x_{i_p} \circ f)$$

Formas Diferenciais em Superfícies

Def.: Seja $M^m \subset \mathbb{R}^n$ sup. de classe $C^{k \geq 2}$. Uma p-forma ω em M a valores no espaço de Banach F é uma aplicação $x \in M \longmapsto \omega(x) \in A_p(T_xM, F)$.

Dados $\varphi : \underset{\subset \mathbb{R}^m}{\overset{ab}{U_0}} \longrightarrow U \in \mathcal{A}(M)$ e $x \in U$, $x_0 = \varphi^{-1}(x)$, consideremos em T_xM a base $\left(\frac{\partial}{\partial x_1}\Big|_x = \varphi'(x_0) \cdot e_1, \ldots, \frac{\partial}{\partial x_m}\Big|_x = \varphi'(x_0) \cdot e_m \right)$ induzida por φ, e em T_x^*M a base dual $(dx_1(x), \ldots, dx_m(x))$. Então, pelo que vimos na seção anterior, existem únicas $\omega_{i_1 \ldots i_p} : U \longrightarrow F$, p/ $1 \leq i_1 < \ldots < i_p \leq m$, tais que

$$\omega|_U = \sum_{1 \leq i_1 < \ldots < i_p \leq n} \omega_{i_1 \ldots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p} \qquad (*)$$

i.e. $(\forall x \in U)$ $\omega(x) = \sum_{1 \leq i_1 < \ldots < i_p \leq n} \omega_{i_1 \ldots i_p}(x)\, dx_{i_1}(x) \wedge \ldots \wedge dx_{i_p}(x)$

Def.: Com a notação acima, diz-se que ω é de classe C^r em U, $0 \leq r \leq k-1$, se (p/ $1 \leq i_1 < \cdots < i_p \leq n$) $\omega_{i_1 \ldots i_p} : U \longrightarrow F$ for de classe C^r. O fato de esta condição ser independente da carta tomada será verificado na lista de exercícios #7. Notação: $\Omega_p^{(r)}(U, F)$.

Diz-se que ω é de classe C^r se, $\forall x \in M$, $\exists\ \varphi : U_0 \to U \in A_p(M)$ tal que ω seja de classe C^r na viz. coordenada U de p. Notação: $\Omega_p^{(r)}(U, F)$.

Obs.: Com a notação acima, sejam ω uma p-forma em M^m e $\varphi : U_0 \to U \in A(M)$, $U_0 \overset{ab}{\subset} \mathbb{R}^m$. Definimos $\omega_\varphi \doteq \varphi^* \omega : U_0 \to A_p(\mathbb{R}^m, F)$, chamada representante de ω na carta φ. Se $\omega|_U$ é dada por (*) da pág. anterior, i.e. $\omega = \sum_{1 \leq i_1 < \dots < i_p \leq m} \omega_{i_1 \dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}$,

então $\omega_\varphi = \sum_{1 \leq i_1 < \dots < i_p \leq m} (\omega_{i_1 \dots i_p} \circ \varphi)\, dx_{i_1} \wedge \dots \wedge dx_{i_p}$,

donde se conclui que $\omega \in \Omega_p^{(r)}(U, F) \Leftrightarrow \omega_\varphi \in \Omega_p^{(r)}(U_0, F)$

Operações c/ formas diferenciais em superfícies:

- Produto exterior
- Pull back

$\Big\}$ definidos da mesma forma como foram definidos p/ formas em abertos de esp. de Banach

- Derivada exterior:

Prop.: Seja M^m superfície e $\omega \in \Omega_p^{(r)}(M, F)$, F esp. de Banach, $r \geq 1$. Então $\exists!\ \eta \in \Omega_{p+1}^{(r-1)}(M, F)$ t.q. para toda $\varphi : U_0 \to U \in A(M)$, $\varphi^* \eta = d \varphi^* \omega$.

Def.: $d\omega \doteq \eta$ (conforme a prop.)

Dem.:

(1) Dada $\varphi : U_0 \to U \in \mathcal{A}(M)$, $\exists!\ \eta_\varphi \in \Omega^{(r-1)}_{p+1}(U,F)$ tal que $\varphi^* \eta_\varphi = d\varphi^*\omega$. Com efeito, η_φ deve ser dada por $\eta_\varphi(x) = \sum_{1 \le i_1 < \dots < i_{p+1} \le m} d(\varphi^*\omega)(\varphi^{-1}(x); e_{i_1}, \dots, e_{i_{p+1}})\, dx_{i_1}(x) \wedge \dots \wedge dx_{i_{p+1}}(x)$

(2) Se $\varphi : U_0 \to U \in \mathcal{A}(M)$ e $\psi : V_0 \to U \in \mathcal{A}(M)$, mostremos que $\eta_\varphi = \eta_\psi$. Isto concluirá a demonstração (basta, para cada viz. coordenada U, definir $\eta|_U = \eta_\varphi$). Com efeito, tem-se:

$$\varphi^*\omega = (\psi \circ \psi^{-1} \circ \varphi)^*\omega = (\psi^{-1}\circ\varphi)^*\psi^*\omega$$

$$\therefore\ d\varphi^*\omega \overset{(*)}{=} (\psi^{-1}\circ\varphi)^* d\psi^*\omega$$

Assim, $\varphi^*\eta_\psi = (\psi\circ\psi^{-1}\circ\varphi)^*\eta_\psi = (\psi^{-1}\circ\varphi)^*\overbrace{\psi^*\eta_\psi}^{= d\psi^*\omega} =$

$\overset{\text{por } (*)}{=} d\varphi^*\omega$

$\therefore\ \eta_\psi = \eta_\varphi$ pela unicidade em (1). #

Assim, definimos $d : \Omega^{(r)}_p(M,F) \to \Omega^{(r-1)}_{p+1}(M,F)$ de modo que, para toda $\varphi : U_0 \to U \in \mathcal{A}(M)$, o seguinte diagrama comuta:

$$\begin{tikzcd}
\Omega^{(r)}_p(M,F) \arrow[r, "d"] \arrow[d, "\varphi^*"'] \arrow[dr, phantom, "\circlearrowright"] & \Omega^{(r-1)}_{p+1}(M,F) \arrow[d, "\varphi^*"] \\
\Omega^{(r)}_p(U_0,F) \arrow[r, "d"] & \Omega^{(r-1)}_{p+1}(U_0,F)
\end{tikzcd}$$