(1)

(H) $\begin{cases} U \overset{ab}{\subset} \mathbb{R}^n & , \quad f: U \to \mathbb{R} \quad \text{harmónica} \\ B_r(x_0) \subset U \end{cases}$

(T) $$f(x_0) = \frac{1}{\operatorname{vol} S_r(x_0)} \int_{S_r(x_0)} f \, dS$$

Dem.:

Seja $$\phi : (0, d(x_0, U^c)) \longrightarrow \mathbb{R}$$
$$\rho \longmapsto \frac{1}{\operatorname{vol} S_\rho(x_0)} \int_{S_\rho(x_0)} f \, dS$$

$$\phi(\rho) \overset{TMV}{=} \frac{1}{\cancel{\rho^{n-1}} \operatorname{vol} S_1(0)} \int_{S_1(0)} f(x_0 + \rho z) \cancel{\rho^{n-1}} \, dS(z)$$

$$\therefore \phi'(\rho) = \frac{1}{\operatorname{vol} S_1(0)} \int_{S_1(0)} f'(x_0 + \rho z) \cdot z \, dS(z) =$$

$$\overset{TMV}{=} \frac{1}{\operatorname{vol} S_\rho(x_0)} \int_{S_\rho(x_0)} \overbrace{f'(y) \cdot \left(\frac{y - x_0}{\rho}\right)}^{= \langle \nabla f(y), \nu(y) \rangle} dS(y) =$$

$$\overset{T.Div.}{=} \frac{1}{\operatorname{vol} S_\rho(x_0)} \cdot \int_{B_\rho(x_0)} \overbrace{\operatorname{div} \nabla f(x)}^{= \Delta f(x) = 0} \, dx = 0$$

$$\therefore \phi(\rho) = \text{cte.} = \lim_{\rho \to 0} \phi(\rho) = f(x_0). \quad \#$$