Questão 1

$$V = \int_1^e \pi f(x)^2 \, dx = \pi \int_1^e \ln^2 x \, dx$$

$$\int \ln^2 x \, dx = x(\ln x)^2 - 2(x \ln x - x) + K$$

Portanto, pelo TFC,

$$V = \pi \left[x \ln^2 x - 2x \ln x + 2x \right]_1^e =$$

$$= \pi\left[(e - 2e + 2e) - 2\right] = \boxed{\pi(e-2)}$$

Questão 2: $\dfrac{x^{12}}{x^3-1} = x^9 + x^6 + x^3 + 1 + \dfrac{1}{x^3-1}$

$$\frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \; ; \quad A = 1/3, \; B = -1/3, \; C = -2/3$$

$$\int \frac{Bx+C}{x^2+x+1} \, dx = -\frac{1}{3} \int \frac{x+2}{\underbrace{x^2+x+1}_{=\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}} \, dx =$$

$$= -\frac{1}{3} \left[\int \frac{x+1/2}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \, dx + \int \frac{3/2}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \, dx \right] =$$

$$= -\frac{1}{3} \left\{ \frac{1}{2} \ln\left[\left(x+\tfrac{1}{2}\right)^2 + \left(\sqrt{3}/2\right)^2\right] + \frac{3}{2} \cdot \frac{4}{3} \operatorname{arctg}\left(\frac{x+1/2}{\sqrt{3}/2}\right) \cdot \frac{\sqrt{3}}{2} + K \right.$$

$$\therefore \int \frac{x^{12}}{x^3-1} \, dx = \frac{x^{10}}{10} + \frac{x^7}{7} + \frac{x^4}{4} + x + \frac{1}{3} \ln|x-1| -$$

$$- \frac{1}{6} \ln(x^2+x+1) - \frac{1}{\sqrt{3}} \operatorname{arctg}\left(\frac{x+1/2}{\sqrt{3}/2}\right) + K$$

Questão 3

$$\int \frac{\ln x}{x^2}\,dx \overset{\text{integrando por partes com } u=\ln x,\ dv = x^{-2}dx}{=}$$

$$= -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + K$$

Portanto, pelo TFC, $\forall y > 1$:

$$\int_1^y \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^y = 1 - \left(\frac{\ln y}{y} + \frac{1}{y}\right)$$

Como $\lim_{y\to\infty} \frac{1}{y} = 0$ e $\lim_{y\to\infty} \frac{\ln y}{y} \overset{\text{l'Hôpital}}{=} \lim_{y\to\infty} \frac{1}{y} = 0$,

segue-se $\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{y\to\infty} \int_1^y \frac{\ln x}{x^2}\,dx = \boxed{1}$

Questão 4

Pelo 2º TFC, F é derivável e $F'(x) = \sqrt{1+x^3}$. Portanto, integrando por partes, $\int x F(x)\,dx = F(x)\cdot\frac{x^2}{2} - \int \frac{x^2}{2} F'(x)\,dx = F(x)\cdot \frac{x^2}{2} - \int \frac{x^2}{2}\sqrt{\underbrace{1+x^3}_{=w}}\,dx =$

$= F(x)\cdot\frac{x^2}{2} - \frac{(1+x^3)^{3/2}}{9}$. Portanto, pelo 1º TFC,

$$\int_0^2 x F(x)\,dx = \underbrace{F(2)}_{A}\cdot\frac{2^2}{2} - \frac{(1+2^3)^{3/2}}{9} + \frac{1}{9} =$$

$$= 2A - 3 + \frac{1}{9} = \boxed{2A - \frac{26}{9}}$$

$x^2 + x + \frac{1}{4} + \frac{3}{4}$

$(x+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2$