Pseudo-Riemannian manifolds all of whose geodesics of one causal type are closed

Stefan Suhr (Hamburg University)

July 23, 2013
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**Riemannian case:**

A large theory contained in the first book by A. Besse. E.g. Theorem (Bott, Samelson)

Let $(M, g)$ be a Riemannian manifold such that all geodesics are simply closed. Then $H^* (M, \mathbb{Z}) \cong H^* (\text{CROSS}, \mathbb{Z})$ where CROSS $\in \{ S^n, \mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n, \mathbb{C}aP^2 \}$.

Proof with Morse theory $\rightsquigarrow$ not directly applicable in pseudo-Riemannian geometry (future development).
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Known so far:

**Proposition (Guillemin)**

Let \((M, g)\) be a compact pseudo-Riemannian 2-manifold such that all lightlike geodesics are closed. Then \((M, g)\) is finitely covered by \((T^2, \bar{g})\) which is globally conformal to \((\mathbb{R}^2/\mathbb{Z}^2, dx dy)\).

▶ Easily extended to non-compact 2-manifolds.
▶ For \((S^n \times S^1, g_{\lambda})\) with \(\lambda \in \mathbb{Q}\) all lightlike geodesics are closed.
▶ Similar problem known for refocussing spacetimes.

What about examples with all spacelike/timelike geodesics closed?

Example Consider \(S^{n+1}(r) = \{x \in \mathbb{R}^{n+1} | \langle x, x \rangle = r^2\}\). Then all spacelike geodesics of the induced metric are closed. By change of sign obtain examples with all timelike geodesics closed.
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Topological classification for 2-dimensional spacetimes.

Theorem (Mounoud/–)

Let \((M, g)\) be a pseudo-Riemannian and non-Riemannian 2-manifold all of whose timelike/spacelike geodesics are closed. Then \((M, g)\) is finitely covered by \((S^1 \times \mathbb{R}, g)\) such that all timelike/spacelike \(g\)-geodesics are simply closed (timelike/spacelike Zoll).

▶ The result is optimal, due to the previous examples.

▶ Zoll surfaces are Riemannian 2-manifolds all of whose geodesics are simply closed, i.e. metrics on \(S^2\) and \(\mathbb{R}P^2\).

▶ For 2-manifolds: If all timelike/spacelike geodesics are closed then all non-timelike/non-spacelike geodesics are non-closed. Due to the theorem and the Poincaré-Bendixson theorem.

Corollary (Mounoud/–)

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Connection to geodesic foliations

Proposition

Let \((M, g)\) be a pseudo-Riemannian manifold. Then there exists a pseudo-Riemannian metric \(G\) on \(TM\) such that the tangent curves of \(g\)-geodesics are \(G\)-geodesics of the same causal type.

Sketch of proof.

Consider the connection map \(\nabla_g\) of the Levi-Civita connection of \(g\).

\[
\nabla_g \circ T : TM \to \text{ker}(\pi_{TM})^* \oplus \text{ker}K_g = T_pM \oplus T_pM
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Define \(G\) such that this isomorphism induces an isometry with \(g \oplus g\).

Then \(\pi_{TM} : TM \to M\) becomes an pseudo-Riemannian submersion. Note that the tangent curves of geodesics are parallel lifts.

Remark

If all geodesics of one causal type (say timelike) are closed then \(\{v \in TM | g(v, v) < 0\}\) is foliated by closed geodesics.

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Theorem (Wadsley, Mounoud/–)

Let $\mathcal{F}$ be a smooth foliation by circles of $M$. The following conditions are equivalent:

1. There is a smooth pseudo-Riemannian metric rendering $\mathcal{F}$ a geodesic foliation by non-degenerate geodesics of the same causal character, i.e. the leaves of $\mathcal{F}$ are either timelike or spacelike geodesics.

2. For any compact subset $K$ of $M$, the circles meeting $K$ have bounded length with respect to some (hence every) Riemannian metric.

3. Let $\tilde{M}$ be the double cover of $M$ obtained by taking the two different possible local orientations of the leaves. There is a smooth action of the orthogonal group $O(2)$ on $\tilde{M}$ and the non-trivial deck transformation $\sigma: \tilde{M} \to \tilde{M}$ is an element of the non-trivial component of $O(2)$. Each orbit under the $O(2)$-action consists of two components and each component is mapped diffeomorphically onto a leaf of $\mathcal{F}$ by the covering projection.
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- Epstein: If $M^3$ is compact every foliation by circles has locally bounded length of the leaves.
- $(1)$ cannot be extended to possibly lightlike geodesics. $\rightsquigarrow$ Thurston-Sullivan examples
Sketch of proof of the pseudo-Riemannian Wadsley theorem.

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1) There is a smooth pseudo-Riemannian metric rendering $F$ a geodesic foliation by non-degenerate geodesics of the same causal character, i.e., the leaves of $F$ are either timelike or spacelike.

1') There is a smooth Riemannian metric rendering $F$ a geodesic foliation. This is the condition in the known formulation of Wadsley's theorem.

1) $\Rightarrow$ 1': Let $X$ be a locally defined unit-tangent field to the foliation. Choose any Riemannian metric $h$ on the orthogonal complement $X^\perp$ and define the Riemannian metric $h = h^\perp + X^\sharp \otimes X^\sharp$. The claim follows from Koszul's formula.

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(1) There is a smooth pseudo-Riemannian metric rendering $\mathcal{F}$ a geodesic foliation by non-degenerate geodesics of the same causal character, i.e. the leaves of $\mathcal{F}$ are either timelike or spacelike.

(1’) There is a smooth Riemannian metric rendering $\mathcal{F}$ a geodesic foliation.

This is the condition in the known formulation of Wadsley’s theorem.

(1) $\Rightarrow$ (1’): Let $X$ be a locally defined unit-tangent field to the foliation. Choose any Riemannian metric $h^\perp$ on the orthogonal complement $X^\perp$ and define the Riemannian metric

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Sketch of proof of the pseudo-Riemannian Wadsley theorem. The following are equivalent:
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The claim follows from Koszul’s formula. $h$ is well defined independent of orientability of $\mathcal{F}$. 
Theorem (”Signature-rigidity-theorem”, Mounoud/–)

A pseudo-Riemannian manifold having a geodesic flow that can be periodically reparametrized is Riemannian or anti-Riemannian.

Proposition

Let $F$ be an oriented $1$-dimensional geodesic foliation on a pseudo-Riemannian manifold $(M, g)$. If the leaves of $F$ are circles with locally bounded Riemannian(!) length then they all have the same type.

Remark

▶ If examples exist of pseudo-Riemannian manifolds with all geodesics closed, then their geodesics flow is complicated.
▶ The problem lies on the lightcones.
▶ There exist examples of foliations by circles such that the length of the leafs are not locally bounded (Thurston-Sullivan examples).

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Thurston-Sullivan examples:

Consider $H/\Gamma \times S^1 \times S^1$, where $H$ is the 3-dimensional Heisenberg group and $\Gamma$ is the lattice of integer matrices in $H$. Denote with $(x, y, z, t, u)$ coordinates on $H \times \mathbb{R} \times \mathbb{R}$. Set $X = \sin(2u)(-\sin(t) \partial_x + \cos(t) \partial_y) + (x \sin(2u) \cos(t) - \cos^2(u)) \partial_z + 2 \sin^2(u) \partial_t$.

$X$ descends to the quotient $H/\Gamma \times S^1 \times S^1$ ($S^1 = \mathbb{R}/2\pi \mathbb{Z}$).

Note that $X$ is tangent to $H \times \mathbb{R}$ and therefore the projection is tangent to $H/\Gamma \times S^1 \times S^1$.

The flowlines of $X$ are all closed and of unbounded length (period $2\pi \sin^2(u)$ for $u \neq 0, \pi$ and $2\pi$ for $u = 0, \pi$).
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Consider the frame $(X, \partial_u, V, W, 2\partial_t + \partial_z)$ with $V = \cos(t) \partial_x + \sin(t)(\partial_y + x \partial_z)$ and $W = -\sin(t) \partial_x + \cos(t)(\partial_y + x \partial_z)$.

Note that the frame descends to the quotient.

Define the Lorentzian metric $g$ to be lightlike on $X$ and $\partial_u$ and unit Riemannian on the other vector fields.

Clearly the flowlines of $X$ form a $g$-geodesic foliations by lightlike geodesics.

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*This construction works for type-changing foliations as well. But nothing is known for geodesic foliations on tangent bundles.*
Idea of the topological classification:

Non-compact case:

- The fundamental class of the closed geodesics lies in the center of $\pi_1(M)$.
- $\pi_1(M)$ is a free group.
- No pseudo-Riemannian $2$-manifold contains contractible non-spacelike/non-timelike loops.
- $\pi_1(M) \cong \mathbb{Z}$.
- $M$ is covered by $S^1 \times \mathbb{R}$.
- The Zoll property follows since the geodesic flow on the unit tangent bundle is induced by an $S^1$-action.

Compact case:

- The closed unit-speed geodesics all intersect a fixed compact subset of the tangent bundle, i.e. the unit tangents to a timelike/spacelike foliation with the same rotation number (image under the Hurewicz homomorphism) as the geodesics.
- The unit tangents are unbounded and the geodesic flow is continuous.
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Semi-Riemannian manifolds all of whose geodesics are closed
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**Definition (Guillemin)**

A compact 3-dimensional pseudo-Riemannian manifold \((M, g)\) is *Zollfrei*, if the geodesic flow on the lightlike vectors induces a fibration by circles.
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- \textit{Zollfrei} is inspired by notion of Zoll surfaces.
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Theorem (Tollefson)
The only diffeomorphism types of compact manifolds covered by $S^2 \times S^1$ are $S^2 \times S^1$ itself, $\mathbb{R}P^2 \times S^1$, $\mathbb{R}P^3 \sharp \mathbb{R}P^3$ and the unique non-orientable 2-sphere bundle over $S^1$. 

Stefan Suhr (Hamburg University)
Semi-Riemannian manifolds all of whose geodesics are closed
Remark (Guillemin)

The metrics $g_{can} - \lambda d\theta^2$ descend to all quotients. They are Zollfrei iff $\lambda \in \mathbb{Q}$ and are called the standard examples.

Conjecture (Guillemin)
Every Zollfrei manifold has the diffeomorphism type of one of the standard examples.

Theorem
Every non-trivial orientable circle bundle over a closed and orientable surface admits a Zollfrei metric.

Corollary
Guillemin's conjecture is wrong. By the Gysin sequence all diffeomorphism types in the theorem are different and none is one of the standard examples.
Remark (Guillemin)

*The metrics* \( g_{\text{can}} - \lambda d\theta^2 \) *descend to all quotients. They are Zollfrei iff* \( \lambda \in \mathbb{Q} \) *and are called the standard examples.*

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Guillemin gives a weaker version of his conjecture assuming causality of the universal cover.

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\(h_\phi = \pi_\# - \cot^2(\phi) \alpha \otimes \alpha\)

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For \((B, g) = (\mathbb{C}P^1, g_{FS}) (K = 4)\) we have \(M \cong S^3\) and

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If the ”magnetic” term \(\text{dvol}^g\) is exact the arrival time functional is retained.
(iii) The critical points of \( cp_\phi \) have closed “magnetic geodesics” as boundary. A curve \( \gamma \) is a magnetic geodesic if

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Proposition $(M, h\phi)$ is Zollfrei iff $cp\phi \in 2\pi\mathbb{Q}$ on its critical points.

For $B = \mathbb{C}P^1$ we have $cp\phi = 4\tan(\phi) - 1/\sqrt{1+4\tan^2\phi}$ on the critical points.
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Open problem:

- Give a topological classification of manifolds with all lightlike/timelike/spacelike geodesics closed in dimension 3 or higher. Note that anti-deSitter 3-space admits compact quotients, so the structure will be richer.
- What can be said about the modular space of timelike/spacelike Zoll surfaces?
- Are there non-obvious Zollfrei manifolds in higher dimension?
- Is every Zollfrei 3-manifold geometrizable? The constructed example cover 4 out of 8 possible geometries.
- Maybe better (due to P. Mounoud): Is every Zollfrei manifold a Seifert fibration? If counterexamples exist they are not stationary by Flores/Javaloyes/Piccione.
- Does every (nontrivial) Seifert fibration admit a Zollfrei metric? It is probably easy to construct Lorentzian metrics with all lightlike geodesics closed.
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