Recent progress on the Lorentz-Finsler correspondence

Miguel Sánchez

Universidad de Granada

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Stationary to Randers correspondence. Equivalence:

Conformal structure of stationary spacetimes

←→ Geometry of Randers spaces
Stationary to Randers correspondence. Equivalence:

\[\text{Conformal structure of stationary spacetimes} \leftrightarrow \text{Geometry of Randers spaces}\]

Applicability:

- \(\leftarrow\) Precise description of spacetime elements in terms of Finsler counterparts
- \(\rightarrow\) New geometric elements and results in Randers spaces — some of them extensible to general Finsler manifolds
Introduction

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Broad relation

\textit{Lorentzian Geometry} \[\leftrightarrow\] \textit{Finsler Geometry}

(including the Riemannian one!)
Starting point

Normalized standard stationary spacetime:
\[ V = (\mathbb{R} \times M, g_L = -1 dt^2 + \pi^* \omega \otimes dt + dt \otimes \pi^* \omega + \pi^* g) \]

\( \omega \) 1-form, \( g \) Riemannian metric on \( M \), \( \pi : \mathbb{R} \times M \rightarrow M \) projection

- \( \partial_t \) timelike (future-directed) Killing vector field
  \[ g_L \equiv -dt^2 + 2 \omega dt + g \]
- Normalized: \(-1 dt^2\). Useful for conformal elements such as lightlike vectors/geodesics (otherwise: \(-\Lambda dt^2\))
- Global “standard” (but not unique) splitting not too restrictive: it always hold locally and [Javaloyes & — ’08]:

M. Sánchez  Lorentz-Finsler correspondence
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- Global “standard” (but not unique) splitting not too restrictive: it always hold locally and [Javaloyes & — ’08]: A spacetime is (globally) conformal to a standard stationary one iff it admits a complete timelike conformal vector field and it is distinguishing (and, so, strongly causal and causally continuous).
Starting point

**Appearance of Finsler Geometry**

With these elements \( \omega, g \) construct the functions \( F^\pm : TM \to \mathbb{R} \)

\[
F^\pm(v) = \sqrt{g(v, v) + \omega(v)^2} \pm \omega(v)
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Finsler metrics of Randers type on \( M \), “Fermat metrics”
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Connection with the spacetime geometry (Cap, Jav. Mas. ’11):
A curve \( \gamma(t) = (\pm t, c(t)), t \in [a, b] \) is lightlike and future/past directed iff \( F^\pm(\dot{c}) = 1 \). In this case:

- the arrival time \( b - a \) is equal to the \( F^\pm \)-length of \( c \), \( \int_a^b F^\pm(\dot{c}) \)
- (Fermat principle) \( \gamma \) is a pregeodesic iff \( c \) is a geodesic for \( F^\pm \) (i.e., a critical point of the arrival time/length functional \( c \mapsto \int F^\pm(\dot{c}) \) parametrized with \( F^\pm \)-length).
Starting point

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1. Causal structure $\longleftrightarrow$ Finslerian distances  
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These elementary considerations suggest the possibility to relate the **conformal** geometry of standard stationary spacetimes and the geometry of the corresponding class of Finsler manifolds, i.e. **Randers** spaces. Aims:

1. Causal structure \(\longleftrightarrow\) Finslerian distances
   (Caponio, Javaloyes, — Rev. Mat. Iberoam, ’11)
2. Visibility and gravitational lensing \(\longleftrightarrow\)
   convexity of Finsler hypersurfaces
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3. Causal boundaries $\leftrightarrow$ Cauchy, Gromov and Busemann boundaries in Finslerian (and Riemannian) settings
   (Flores, Herrera, — ATMP’11, Memoirs AMS’13).
1. CAUSAL STRUCTURE
Notion of Finsler and Randers metric

\[ F : TM \to \mathbb{R} \]  
**Finsler metric**: continuous, smooth away 0  
+ positively homog. strongly convex norm at each \( p \in M \)

- Positively homogeneous: \( F(\lambda v) = \lambda F(v) \) for \( \lambda > 0 \)
- Strongly convex: the second fundamental form of the unit sphere is positive definite
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**Randers metric:** \( R = \sqrt{g + \omega^2} + \omega \) for some Riemannian \( g \) (and \( h := g + \omega^2 \)) and 1-form \( \omega \).

In particular, Fermat metrics \( F^\pm \) are Randers (and viceversa)
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**Reversed Finsler metric:** \( F^{rev}(v) := F(-v) \)

In particular,
- for Fermat metrics \( (F^+)^{rev}(v) = F^-(v) \)
- if \( \omega \neq 0 \), Randers metrics are non-reversible \( (R \neq R^{rev}) \)
Notion of generalized distance

Taking infimum of lengths of curves connecting two points, each Finsler metric induces a \textit{generalized distance} \(d\). This means:

\[d_{\text{sym}}(x, y) = \frac{d(x, y) + d(y, x)}{2}\]

Remark
Even in the Finslerian case, \(d_{\text{sym}}\) does not come from a length space.
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Taking infimum of lengths of curves connecting two points, each Finsler metric induces a *generalized distance* $d$. This means:

1. all the axioms of a distance hold but symmetry
2. for sequences $\{x_n\}$: $d(x, x_n) \to 0 \iff d(x_n, x) \to 0$
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Centered at any point \( x_0 \), there are forward balls \( (d(x_0, x) < r) \) and backward balls \( (d(x, x_0) < r) \) that may differ but generate the same topology (in the Finslerian case, the manifold topology)
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- Symmetrized distance: $d_s(x, y) = (d(x, y) + d(y, x))/2$

**Remark** Even in the Finslerian case, $d_s$ does not come from a length space.
Let \((M, F)\) be a (connected) Finsler manifold with generalized distance \(d\). They are equivalent:

(a) \(d\) is forward (resp. backward) complete (Cauchy sequences).

(b) The Finsler manifold \((M, F)\) is forward (resp. backward) geodesically complete.

(c) At some (and then all) point \(p \in M\), \(\exp_p\) (resp. \(\tilde{\exp}_p\)) is defined on all of \(T_pM\).

(d) Heine-Borel property: every closed and forward (resp. backward) bounded subset of \((M, d)\) is compact.

Moreover, in this case \((M, F)\) is convex, i.e., every pair of points \(p, q \in M\) can be joined by a minimizing geodesic from \(p\) to \(q\).
*Remark.* Relation with $d_s$:

1. $d$ is either forward or backward complete $\implies$
2. $d_s$ satisfies Heine-Borel $\iff$ all $\bar{B}_s(x, r)$ compact $\implies$
3. $d_s$ complete
Basic idea:

For \( p \in M \), \( d^+ \equiv d \) distance of \( F^+ \equiv F \):

\[
l^+(0, p) \text{ determined by the graph of } d^+(p, \cdot):
\{t_0\} \times B^+(p, t_0) = l^+(0, p) \cap (\{t_0\} \times M)
\]
Ladder of causality for stationary s-p

Theorem

- The slices \( \{t_0\} \times M \) are Cauchy hypersurfaces of \((\mathbb{R} \times M, g_L)\) iff \(d^+\) is forward and backward complete.
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\((\mathbb{R} \times M, g)\) is globally hyperbolic (causal + \(J^+(z) \cap J^-(z')\) compact) iff \(\overline{B}_s^+(p, r)\) are compact \(\forall p \in M, r > 0\) (Heine-Borel property)
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\( (\mathbb{R} \times M, g_L) \) is causally simple (causal + \( J^\pm(z) \) closed) iff \( (M, F) \) is convex

(Full characterization of Causality, as standard stationary s-t are always causally continuous.)
Consequences for Finsler manifolds

Remark. Compactness of $\bar{B}_s^+(p, r)$ for a Randers metric
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$M_f = \{(f(x), x) : x \in M\}$
$\implies$ The spacetime admits a splitting $\mathbb{R} \times M_f$ with Cauchy slices
and Fermat metric $F_f \equiv F - df$
Consequences for Finsler manifolds

Remark. Compactness of $\tilde{B}_s^+(p, r)$ for a Randers metric

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$\implies$ for some $f$, the new metric $F - df$ is a forward and backward complete Randers metric (with the same pregeodesics as $F$) $\iff$

current property is extensible to any Finsler metric (Matveev’12).
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*In general, the compactness of $\bar{B}_s^+(p, r)$ (Heine-Borel) is the optimal assumption for classical Finsler theorems (Myers, sphere...)*
Cauchy developments

\( A \subset V \) achronal set
\[ D^+(A) = \{ z \in V : \gamma \cap A \neq \emptyset \ \text{for all} \ \gamma \ \text{past-inextensible causal curve starting at} \ p \} \]
\[ H^+(A) = \{ z \in \overline{D}^+(A) : I^+(z) \cap D^+(A) = \emptyset \} \]
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**Proposition**

For \( A \subset M \equiv \{0\} \times M \) (slice of \( \mathbb{R} \times S \)):

\( D^+(A) = \{(t, y) : 0 \leq t < d^+(x, y) \ \forall x \notin A \} \)

\( H^+(A) = \{(t, y) : t = \inf_{x \notin A} d^+(x, y) (= d^+(M \setminus A, y)) \} \)

\( H^+(A) \) is constructed from the level sets of \( d^+(M \setminus A, \cdot) \)
Remark. The results on horizons also yield results on the distance function to a set in a Randers manifold. [This extends the viewpoint for the static case (and symmetric distances) by Chrusciel, Fu, Galloway and Howard '02.]
Applications to Finsler

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An example is the following translation of a result by Beem and Krolak ’98:

**Theorem**

Let $C \subset M$ a closed subset, $p \in M \setminus C$ is a differentiable point of the distance from $C$ iff $p$ is crossed by exactly one minimizing segment.
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Let $C \subset M$ a closed subset, $p \in M \setminus C$ is a differentiable point of the distance from $C$ iff $p$ is crossed by exactly one minimizing segment $\rightsquigarrow$ generalizable to any Finsler manifold (Sabau-Tanaka’12)
SECOND PART

2. VISIBILITY AND LENSOING
Visibility of particles

Problem studied by many authors: Giannoni, Fortunato, Masiello, Perlick, Piccione...
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**Fermat principle** (Kovner’90, Perlick’90): if a first arriving (or critical for the arrival time) causal curve connecting a point and a line (stationary trajectory) exists then it is a lightlike geodesic.

BUT:
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- Will it remain in a (reasonably big) realistic region \( \mathbb{R} \times D \)?
  (where the conformally stationary model remains valid)
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Problems related to **convexity**.
(\(M, g_R\)) Riemannian, \(D \subset M\) open domain with smooth \(\partial D\)

Notions of convexity for \(\partial D\):

1. **Infinitesimal convexity**: second fundamental form positive semi-definite with respect to the inner normal
   \(\iff\) \(\text{Hess} \phi\) negative semidefinite on \(T(\partial D)\) for any smooth \(\phi : \overline{D} \to [0, \infty)\) with \(\partial D = \phi^{-1}(0)\) and \(\phi\) regular on \(\partial D\).

2. **Local convexity**: locally each \(\exp(T_p(\partial D))\) does not touch \(D\).
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   - Local $\implies$ infinitesimal trivially (and pointwise)
   - Non-trivial converse (Do Carmo, Warner '70)
Previous: Riemannian convexity

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- Local \(\implies\) infinitesimal trivially (and pointwise)
- Non-trivial converse (Do Carmo, Warner ’70)
- Bishop ’74 proved the converse but
  - his proof required smoothness \(C^4\) (also for \(g\))
  - it cannot be extended to the Finslerian setting (Borisenko, Olin ’10)
General Finslerian results

Approach by Bartolo, Caponio, Germinario, — ’10:

- previous notions extensible to Finslerian manifolds
- intermediate notion: $\partial D$ is geometrically convex when: no geodesic in $\overline{D}$ connecting some $p, q \in D$ touches $\partial D$. 
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**Theorem**

For any Finsler manifold, and domain $D$ with $\mathcal{C}^{1,1}_{loc}$ boundary $\partial D$:

1. The infinitesimal, geometric and local notions of convexity are equivalent.
General Finslerian results

**Theorem (Bartolo, Caponio, Germinario, — ’10)**

*For any Finsler manifold, and domain $D$ with $C^{1,1}_{loc}$ boundary $\partial D$:

1. *The infinitesimal, geometric and local notions of convexity are equivalent.*

2. *If all $\overline{B}_s^D(p, r)$ (in particular, if $\overline{B}_s(p, r) \cap \overline{D}$) are compact: $\partial D$ is convex iff $D$ is convex.*
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In the last case, if \( D \) is not contractible then any \( p, q \in D \) can be connected by infinitely many geodesics contained in \( D \) with diverging lengths.
Basic ideas: existence

**Existence** of connecting causal geodesics in a prescribed $\mathbb{R} \times D$

- **Lightlike geodesics.** Optimal conditions:
  1. (geometric) light-convexity of $\mathbb{R} \times \partial D$ in $\mathbb{R} \times M$
     $\iff$ Convexity of $\partial D$ in $(M, F)$. 

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  2. + completeness of the space of connecting curves:
     $\iff$ Compactness of $\bar{B}_s^D(p, r)$ ($\iff \bar{B}_s(p, r) \cap \bar{D}$)
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- **Timelike geodesics.** Reduction to the lightlike case: 
  $\gamma$ timelike geodesic in $\mathbb{R} \times M$ with $g_{L}(\gamma', \gamma') = -c^{2}$
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     \[ \iff \text{Convexity of } \partial D \text{ in } (M, F). \]
  2. + completeness of the space of connecting curves:
     \[ \iff \text{Compactness of } \bar{B}^D_s(p, r) (\subseteq \bar{B}_s(p, r) \cap \bar{D}) \]

- **Timelike geodesics.** Reduction to the lightlike case:
  $\gamma$ timelike geodesic in $\mathbb{R} \times M$ with $g_L(\gamma', \gamma') = -c^2$
  \[ \iff \tilde{\gamma}(s) = (cs, \gamma(s)) \text{ is a lightlike geodesic in } \mathbb{R} \times (\mathbb{R} \times M) \]
  with the product metric ($\equiv du^2 + g_L$)
Basic ideas: multiplicity

Multiplicity (lensing)

- Local, around a given lightlike pregeodesic $\gamma(t) = (t, c(t))$:
  existence of conjugate points for $\gamma$
  $\rightsquigarrow$ $c$ admits conjugate point as a Finsler geodesic
  (tidal lensing)
Basic ideas: multiplicity

Multiplicity (lensing)

- Local, around a given lightlike pregeodesic $\gamma(t) = (t, c(t))$: existence of conjugate points for $\gamma$
  $\leadsto c$ admits conjugate point as a Finsler geodesic
  (tidal lensing)

- Global: non trivial topology of $\mathbb{R} \times D$
  $\leadsto D$ non-contractible
  (topological lensing)
Results: lightlike geodesics

$\mathbb{R} \times M$ standard stationary, $D \subset M$ a $C^2$ domain

**Theorem**

Assume that all $\overline{B}^D_s(p, r)$ are compact (which happens, in particular, when $\overline{B}_s(p, r) \cap \overline{D}$ are compact). They are equivalent:
Results: lightlike geodesics

\( \mathbb{R} \times M \) standard stationary, \( D \subset M \) a \( C^2 \) domain

**Theorem**

Assume that all \( \bar{B}_s^D(p, r) \) are compact (which happens, in particular, when \( \bar{B}_s(p, r) \cap \bar{D} \) are compact). They are equivalent:

1. \( (\mathbb{R} \times D, g_L) \) is causally simple (\( \iff (D, F) \) is convex)
Results: lightlike geodesics

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4. Any point $w \in \mathbb{R} \times D$ and any line $l_q, q \in D$ connected in $\mathbb{R} \times D$ by a future–pointing lightlike geodesic minimizing the (future) arrival time $T$ ($\iff$ idem for past)
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In this case, if \(D\) is not contractible, infinitely many connecting lightlike geodesics with diverging arrival times exist.
Results: timelike geodesics

$\mathbb{R} \times M$ standard stationary, non-necessarily normalized $(-\Lambda dt^2)$, product metric for $\mathbb{R}_u \equiv (\mathbb{R}, du^2)$ and $(\mathbb{R} \times M, g_L)$, $\Pi_u$ projection on the first factor, in addition to $\Pi_M$. Fermat $F_{\Lambda}$ on $\mathbb{R} \times M$:

$$F_{\Lambda} = \sqrt{\Pi_M^* h + \frac{\Pi_u^* d u^2}{\Lambda \circ \Pi_M}} + \Pi_M^* \omega = \sqrt{h_{\Lambda}} + \omega_1.$$  

(extra dimension plus non-conformal invariance)
Results: timelike geodesics

**Theorem**

Assume that all $\overline{B}^D(p, r)$ are compact (which happens, in particular, when $\overline{B}_s(p, r) \cap \overline{D}$ are compact). Then:

- $(\mathbb{R}^u \times \partial D; F_\Lambda)$ is convex $\iff$
  
  for any length $l > 0$, each point $w \in \mathbb{R} \times D$ and line $l_q$ are joined by a future (and a past) pointing timelike geodesic in $\mathbb{R} \times D$, with length $l$ minimizing the arrival time (among causal curves of length $l$).

*In this case, if $D$ is not contractible, a sequence of such connecting geodesics with diverging arrival times exists.*
Asymptotically flat stationary spacetime:

- $(M, g)$ complete, outside a compact subset $C$, is diffeomorphic to $\mathbb{R}^n \setminus B(0, R_0)$, elements $g, \omega, \Lambda$ turning Euclidean with large radial coordinate $r$.
- Model isolated systems - gravity outside a star.
- Makes sense to speak on (stationary) large balls $B(0, R) \times \mathbb{R} \subset \mathbb{R} \times M$ and spheres.
Further results: asympt. flat spacetimes

**Asymptotically flat stationary spacetime:**

- $(M, g)$ complete, outside a compact subset $C$, is diffeomorphic to $\mathbb{R}^n \setminus B(0, R_0)$, elements $g, \omega, \Lambda$ turning Euclidean with large radial coordinate $r$.
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- Makes sense to speak on (stationary) large balls $B(0, R) \times \mathbb{R} \subset \mathbb{R} \times M$ and spheres.

**Application:** Large spheres in asympt. flat spacetimes are always light-convex (and all the previous results are applicable) but, typically, (including reasonable matter, when gravity attracts) they are not time-convex.
3. CAUSAL BOUNDARY
Introduction

Causal boundary $\partial V$ of a spacetime $V$:

- **Involved structure**: conformal structure (Causality). Intrinsinc alternative to common Penrose conformal boundary applicable to any strongly causal spacetime.
- **Purpose**: attach a boundary endpoint $P \in \partial V$ to any inextensible future or past directed timelike curve $\gamma$. 
Basic idea: the boundary point would be represented by $P = I^-(\gamma)$ or $F = I^+(\gamma)$ or, more precisely, a pair $(P, F)$. 
Introduction

Long story from Geroch, Kronheimer & Penrose ’72 until its recent redefinition Flores, Herrera & — ’11 (with contributions by many authors: Budic & Sachs ’74, Szabados ’88, ’89, Harris ’97-’07, Marolf & Ross ’03...):

- **As a point set**, the completion $\overline{V}$ is composed by (S related) pairs $(P, F)$ (“(IP,IF)”, for example $V \ni p \equiv (I^+(p), I^-(p)) \in \overline{V}$)
- **Chronological relation**: $(P, F) \ll (P', F') \iff F \cap P' \neq \emptyset$
Introduction

- Subtle topology ... non always Hausdorff
Introduction

When computed for standard stationary spacetimes, relations with

- Cauchy boundary
- Gromov boundary
- Busemann-type boundary

for Randers manifolds:
Introduction

When computed for standard stationary spacetimes, relations with

- Cauchy boundary
- Gromov boundary
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for Randers manifolds:

\[ \sim \text{previous study for any Riemannian and Finslerian manifold with interest by itself} \]
Cauchy boundary for Riemannian manifolds

Remark: it may be non-locally compact
Gromov boundary for Riemannian manifolds

Classical Gromov’s compactification for complete Riemannian manifolds (Gromov ’81)

- $\mathcal{L}_1(M, g)$ 1-Lipschitz functions (pointwise topology)
- $x \in M$ can be seen in $\mathcal{L}_1(M, g)$ as $d_x : y \mapsto d(x, y)$ and also $d_x + C$ for any $C \in \mathbb{R}$
- $f \sim f' \iff f - f' =$constant (quotient topology)
- each $x \in M$ is represented in $\mathcal{L}_1(M, g)/\sim$ as the class of $-d_x$

$M_G =$ closure of $M$ in $\mathcal{L}_1(M, g)/\sim$
Gromov boundary for Riemannian manifolds

What about if \((M, g)\) is not complete? Repeat construction:
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- **Cauchy** \(M_C\): completion, no compactification (\(M_C\) may be non-locally compact)
- **Gromov** \(M_G\): compactification even in the incomplete case.
What about if \((M, g)\) is not complete? Repeat construction:

- **Cauchy** \(M_C\): completion, no compactification (\(M_C\) may be non-locally compact)
- **Gromov** \(M_G\): compactification even in the incomplete case.

1. \(M_C \hookrightarrow M_G\) in a natural way and continuous but:
   - the inclusion is an embedding \(\hookrightarrow\) \(M_C\) is locally compact
2. \(M_G = M \cup \partial_{CG}M \cup \partial_{G}M\)
   - \(\partial_{CG}M\): limits of bounded sequences (\(\partial_{C}M \subset \partial_{CG}M\))
   - \(\partial_{G}M\): limits of unbounded sequences
Busemann boundary for Riemannian manifolds

\[ M \equiv (M, g) \text{ connected Riemannian manifold} \]

- Typically, Busemann functions are defined when \( c \) is a ray (half unit geodesic with no cut locus)

\[ b_c(x_0) = \lim_{t \to \infty} (t - d(x_0, c(t))) \text{ for all } x_0 \in M \]
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- Eberlein & O’Neill ’73 developed a compactification of any Hadamard manifold in terms of Busemann functions (cone topology), which a posteriori coincides with Gromov’s one
Busemann boundary for Riemannian manifolds

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- We will admit a “Busemann” function for any curve \( c : [0, \Omega) \to M \) with \( |\dot{c}| \leq 1 \)
  \[ b_c(x_0) = \lim_{t \to \Omega} (t - d(x_0, c(t))) \]

- \( b_c \) is \( \infty \) at some \( x_0 \in M \) iff \( b_c \equiv \infty \).

\[ B(M) \] set of finite Busemann functions
Busemann boundary for Riemannian manifolds

- As a subset $B(M) \subset \mathcal{L}_1(M, g)$
  
  Busemann completion $M_B = B(M)/\sim$ (quotient by additive constant –included as a subset in $M_G$)
Busemann boundary for Riemannian manifolds

- As a subset \( B(M) \subset \mathcal{L}_1(M, g) \)
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- But \( B(M) \), and then \( M_B \), will be regarded as topological spaces with the chronological topology (\( \neq \) the induced from \( \mathcal{L}_1(M, g) \))
Busemann boundary for Riemannian manifolds

- As a subset $B(M) \subset L_1(M, g)$
  Busemann completion $M_B = B(M)/\sim$ (quotient by additive constant –included as a subset in $M_G$)

- But $B(M)$, and then $M_B$, will be regarded as topological spaces with the chronological topology (≠ the induced from $L_1(M, g)$) defined by means of a limit operator $L$
  given $\{f_n\} \subset B(M)$, the subset $L(\{f_n\}) \subset B(M)$ is defined by:
  $f \in L(\{f_n\})$ iff
  \[
  \begin{cases}
  (a) & f \leq \liminf_n f_n \quad \text{and} \\
  (b) & \forall g \in B(M) \text{ with } f \leq g \leq \limsup_n f_n, \text{ it is } g = f.
  \end{cases}
  \]
Properties of the Busemann completion:

1. $M_B$ is sequentially compact
2. $M_B$ is $T_1$, and points in $\partial_B M$ may be non-$T_2$ related
3. $M_C \hookrightarrow M_B \hookrightarrow M_G$ (naturally) but $M_B$ topology is coarser.
4. $M_B = M_G$ (as pointsets and, then, topologically)
   \[\iff\] $\partial_B M$ is Hausdorff

**Discrepancy:** $M_B$ compactifies directions (finite or asympt)

$M_G$ may contain non-endpoints of curves in $M$

5. $M_B = M \cup \partial_C M$ (finite directions) $\cup \partial_B M$ (asymptotic)
   - $M_C$: Busemann functions for curves $c$ with $\Omega < \infty$
   - $M_B$: Busemann for $c$ with $\Omega = \infty$
Cauchy boundary for Finslerian manifolds

Cauchy completions for a Finsler manifold:
$(M, d), d$ associated to $F$

1. Two types of Cauchy sequences (ordering)
   Cauchy boundaries $\partial^+_C M, \partial^-_C M$ plus the symmetrized one
   $\partial^s_C M = \partial^-_C M \cap \partial^+_C M$
Cauchy boundary for Finslerian manifolds

Cauchy completions for a Finsler manifold: 
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   Cauchy boundaries \(\partial^+_C M, \partial^-_C M\) plus the symmetrized one
   \(\partial^s_C M = \partial^-_C M \cap \partial^+_C M\)

2. The extension \(d_Q\) of \(d\) to, say, \(M^+_C = M \cup \partial^+_C M\) is not a
   generalized distance but a quasidistance
   \(d_Q(x_n, x) \to 0 \iff d_Q(x, x_n) \to 0\)
   - Topology on \(M^+_C\) generated by the forward balls different to 
     generated by backward balls
   - \(\partial^+_C M\) may be only a \(T_0\) space
Gromov completions for a Finsler manifold:

1. Non-symmetric notions of Lipschitzian
   \[ \mathcal{L}^+_1(M, d): f(y) - f(x) \leq d(x, y) \]
   \[ \mathcal{L}^-_1(M, d): f(x) - f(y) \leq d(x, y) \]
Finslerian Gromov completions

Gromov completions for a Finsler manifold:

1. Non-symmetric notions of Lipschitzian
   \( \mathcal{L}_1^+(M, d) : f(y) - f(x) \leq d(x, y) \)
   \( (\mathcal{L}_1^-(M, d) : f(x) - f(y) \leq d(x, y)) \)

2. Two Gromov's compactifications \( M_G^\pm \),
   say: \( M_G^+ = \) closure of \( M \) in \( \mathcal{L}_1^+(M, d)/\sim \)
Finslerian Gromov completions

Gromov completions for a Finsler manifold:

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\[ \mathcal{L}_1^+(M, d): f(y) - f(x) \leq d(x, y) \]
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2. Two Gromov's compactifications \( M_G^\pm \),
say: \( M_G^+ = \text{closure of } M \text{ in } \mathcal{L}_1^+(M, d)/\sim \)

In a natural way, \( i: M_C^+ \hookrightarrow M_G^+ \) but:

- \( i \) continuous iff the backward balls generate a finer topology on \( M_C^+ \) than the forward balls
- \( i \) embedding when:
  - \( M_C^+ \) is locally compact [also Riemannian condition] AND
  - \( d_Q \) is a generalized distance
Finslerian Busemann completions

Busemann completions for a Finsler manifold:

1. There are also two completions $M^+_B$ constructed by using Busemann functions (which depends on the order of the arguments in $d$)
Finslerian Busemann completions

Busemann completions for a Finsler manifold:

1. There are also two completions $M_B^{\pm}$ constructed by using Busemann functions (which depends on the order of the arguments in $d$)

2. As in the Riemannian case, each $M_B^{\pm}$ is naturally included in $M_G^{\pm}$ (with a coarser topology) and
There are also two completions $M_B^\pm$ constructed by using Busemann functions (which depends on the order of the arguments in $d$).

As in the Riemannian case, each $M_B^\pm$ is naturally included in $M_G^\pm$ (with a coarser topology) and both boundaries coincide iff $\partial_B^\pm M$ is Hausdorff.

The spacetime viewpoint suggests further relations between $M_B^+$ and $M_B^-$. 
Appearance of the Busemann functions
Appearance of the Busemann functions

**Standard stationary spacetime:**

(g Riemannian on $M$ with distance $d$, $\pi : \mathbb{R} \times M \to M$ projection)

\[ V = (\mathbb{R} \times M, g_L = -dt^2 + 2dt \otimes \pi^* \omega + \pi^* g) \]

**Fermat metric**

(with associated generalized distance $d^+$)

\[ F^+(v) = \sqrt{g(v,v) + \omega(v)^2} \quad \forall v \in TM \]

**Aim: computation of IP’s (and dual IF’s)**

$P = I^-[\gamma]$, future-directed timelike curve

\[ \gamma(t) = (t, c(t)), t \in [\alpha, \Omega), |\dot{c}| < 1 \]
Characterization of \( \ll \):

\[
(t_0, x_0) \ll (t_1, x_1) \iff d^+(x_0, x_1) < t_1 - t_0
\]
Appearance of the Busemann functions

1. Characterization of $\ll$
   \[(t_0, x_0) \ll (t_1, x_1) \iff d^+(x_0, x_1) < t_1 - t_0\]

2. Application to $P = I^-[\gamma], \gamma(t) = (t, c(t))$:
   
   \[
P = \{(t_0, x_0) \in V : (t_0, x_0) \ll \gamma(t) \text{ for some } t \in [\alpha, \Omega)\}
   \]
   
   \[
   = \{(t_0, x_0) \in V : t_0 < t - d^+(x_0, c(t)) \text{ for some } t \in [\alpha, \Omega)\}
   \]
   
   \[
   = \{(t_0, x_0) \in V : t_0 < \lim_{t \to \Omega} (t - d^+(x_0, c(t)))\}
   \]
   
   \[
   = \{(t_0, x_0) \in V : t_0 < b^+_c(x_0)\}
   \]

   \[\sim \text{ Busemann forward function:}\]
   
   \[b^+_c(x_0) = \lim_{t \to \Omega} (t - d^+(x_0, c(t)))\]
Fundamental correspondence

\[ B^+(M) \text{: set of all Busemann functions } b_c^+ \text{ for } (M, F^+) \]

\[ \{ \text{IP's on } V \} \equiv B^+(M) \cup \{ b_c \equiv \infty \} \]

1. Past of points (PIP’s): converging \( \Omega < \infty \)
2. Past of inextensible curves (TIP’s): non-converging \( \Omega = \infty \)

**Remark:** for the causal boundary no quotient in the set of Busemann functions must be carried out
Computation: c-boundary for static spacetimes

Parts of $\partial V$

- $i^+, i^-$ apexes ($b_c \equiv \infty$) of a "double cone"

\[(P, F) : P, F \text{ from } \partial_c M\]
Parts of $\partial V$

- $i^+, i^-$ apexes ($b_c \equiv \infty$) of a “double cone”
- Horismotic (as lightlike with no cut points) lines on $\partial_B M$ starting at $i^\pm$
Computation: \( c \)-boundary for static spacetimes

Parts of \( \partial V \)

- \( i^+, i^- \) apexes (\( b_c \equiv \infty \)) of a “double cone”
- Horismotic (as lightlike with no cut points) lines on \( \partial_B M \) starting at \( i^\pm \)
- Timelike lines on \( \partial_c M \) (unique non-trivial \( S \)-pairs) connecting \( i^+, i^- \)

Recall: with the chr-topology
Computation: c-boundary for stationary spacetimes

Parts of $\partial V$

- **“Static part”**: $i^+, i^-$ “apexes” of two distinct cones
- Horismotic lines on $\partial^\pm M$
- Timelike lines on $\partial^C M$
  (composed of $S$-pairs)
Parts of $\partial V$

- **“Static part”:** $i^+, i^-$ “apexes” of two distinct cones
  Horismotic lines on $\partial^\pm_B M$
  Timelike lines on $\partial^s_C M$
  (composed of $S$-pairs)

- **Locally horismotic lines** on $\partial^\pm_C M \setminus \partial^s_C M$
  The lines arrive both $i^+$ and $i^-$ iff pairings with lines on $\partial^\pm_C M \setminus \partial^s_C M$
Further developments

1. Gibbons, Herdeiro, Warnick, Werner ’09: progress on the correspondence of curvatures, including a conjecture refined by Javaloyes, —.

2. (Caponio, Javaloyes, —, in progress). Case when the spacetime is not strictly stationary, but $K = \partial_t$ is allowed to have a changing sign. As a consequence:
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   - Geometric properties of Kropina (and Kropina-Randers) metrics: geodesics, convexity.
   - Extension of classical Finsler metrics and their applicability ("wind Finsler" metrics)
   - Some causal properties of Killing horizons
   - Extension of Fermat principle
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