CONNECTIONS BETWEEN THE CAUSAL BOUNDARY AND ISOCAUSALITY

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(joint works with J. Herrera and M. Sánchez)
MOTIVATION:

- The c-boundary has been computed in multiple classes of spacetimes of physical interest, as *standard stationary* ones.
- It seems natural to argue that spacetimes with similar causal structure will present similar c-boundary.
- This is the case of conformal equivalent spacetimes, which have the same c-boundary.
- *Isocausality* is a generalization of conformal equivalence, but adding more flexibility.
- Is the c-boundary also preserved under isocausal equivalence? No.
- We will show a precise relation between the c-boundaries of stationary spacetimes and spacetimes isocausal to them.
1. C-BOUNDARY OF SPACETIMES
CLASSICAL CAUSAL BOUNDARY:
- Introduced by Geroch, Kronheimer and Penrose’72.
- Conformally invariant and applicable to any strongly causal spacetime.

FURTHER REDEFINITIONS:
- Budic and Sachs’74, Racz’87’88, Szabados’88’89:
  Study the problems derived from the “identifications”.
- Harris’98’00:
  Study the topology of the partial boundaries.
- Marolf y Ross’03:
  “Identifications” ↔ representations of boundary points by pairs.
- F, Herrera y Sánchez’11:
  New re-definition of causal boundary (c-boundary).
FUTURE C-BOUNDARY:

- **Past Set:** $\emptyset \neq P \subset V$ such that $I^-(P) = P$.
- **IP:** Past set which is not the union of two proper past sets.
- **PIP:** IP, $P \subset V$ such that $P = I^-(p)$ for some $p \in V$.
- **TIP:** IP, $P \subset V$ such that $P \neq I^-(\gamma)$ for $\gamma$ inext. fut. tmkl. curve.

Then, we have:

$$\hat{\partial} V \equiv TIPs, \quad V \equiv PIPs, \quad \hat{V} \equiv IPs$$
PAST C-BOUNDARY:

- Future Set: $\emptyset \neq F \subset V$ such that $I^+(F) = F$.
- IF: Future set which is not the union of two proper future sets.
- PIF: IF, $F \subset V$ such that $F = I^+(p)$ for some $p \in V$.
- TIF: IF, $F \subset V$ such that $F \neq I^+(\gamma)$ for $\gamma$ inext. past timelike curve.

Then, we have:

$$\partial V \equiv TIFs, \quad V \equiv PIFs, \quad \check{V} \equiv IFs$$
S-Relation (Szabados’88)

$P, F$ are $S$-related, $P \sim_S F$, iff:

- $P$ is a maximal IP in $\downarrow F := I^-(\{q \in V : q \ll p, \forall p \in F\})$.
- $F$ is a maximal IF in $\uparrow P := I^+(\{p \in V : q \ll p, \forall q \in P\})$.

Moreover, $P \sim_S \emptyset, \emptyset \sim_S F$ otherwise.

\[ I^-(p) \sim_S I^+(p) \ \forall \ p \in V \ (Szabados’88); \]
Definition of c-boundary: as point set

- **C-completion:** \( \overline{V} := \{(P, F) \in (\hat{V} \cup \{\emptyset\}) \times (\check{V} \cup \{\emptyset\}), P \sim_S F\} \).
- **C-boundary:** \( \partial V := \overline{V} \setminus V \), where \( V \equiv \{((l^-(p), l^+(p)) : p \in V\} \).
Definition of c-boundary: as chronological set

The extended chronological relation \( \ll \) on \( \overline{V} \) is defined as:

\[(P, F) \ll (P', F') \iff F \cap P' \neq \emptyset.\]
Causal and horismotical relation

The extended causal relation $\leq$ on $\overline{V}$ is defined as (assuming $P \neq \emptyset \neq F'$):

$$(P, F) \leq (P', F') \iff P \subset P' \text{ and } F' \subset F.$$ 

Two different pairs in $\overline{V}$ are *horismotically related* if they are causally but not chronologically related.
Definition of c-boundary: as a topological space

- **Limit Operator:** The limit operator $L$ is defined as:

$$(P, F) \in L\left(\{(P_n, F_n)\}\right) \iff \begin{cases} P \in \hat{L}(P_n) & \text{if } P \neq \emptyset \\ F \in \hat{L}(F_n) & \text{if } F \neq \emptyset \end{cases}$$

with $P \in \hat{L}(P_n)$ iff $P \subset LI(P_n)$ and $P$ maximal in $LS(P_n)$ and $F \in \hat{L}(F_n)$ iff $F \subset LI(F_n)$ and $F$ maximal in $LS(F_n)$.
Definition of c-boundary: as a topological space

- **Limit Operator**: The limit operator $L$ is defined as:

$$(P, F) \in L(\{(P_n, F_n)\}) \iff \begin{cases} P \in \hat{L}(P_n) & \text{if } P \neq \emptyset \\ F \in \check{L}(F_n) & \text{if } F \neq \emptyset \end{cases}$$

with $P \in \hat{L}(P_n)$ iff $P \subset LI(P_n)$ and $P$ maximal in $LS(P_n)$ and $F \in \check{L}(F_n)$ iff $F \subset LI(F_n)$ and $F$ maximal in $LS(F_n)$.

- **Chr. Topology**: $C \subset \overline{V}$ closed if $L(\sigma) \subset C$ for any sequence $\sigma \subset C$.

**Remarks**

1. $L$ may not provide all the topological limits of the sequence.
2. A topology on $\hat{V}$ (resp. $\check{V}$) is defined by considering $\hat{L}$ (resp. $\check{L}$).
### Properties of the c-boundary

1. **(a)** Any timelike curve in $V$ admits some limit in $\overline{V}$.
2. **(b)** $V$ is chronologically and topologically embedded in $\overline{V}$. Moreover, $V$ is dense in $\overline{V}$.
3. **(c)** $\partial V$ is closed in $\overline{V}$.
4. **(d)** The future and past elements of $\overline{V}$ are open in $\overline{V}$.
5. **(e)** $\overline{V}$ is a $T_1$ topological space (but not necessarily $T_2$).

### Remark
- The c-boundary can be deduced from first principles.
- The c-boundary coincides with the accessible part of the conformal boundary when it is well-behaved.
2. ISOCAUSAL COMPARISON
Isocausality

-Introduced by García-Parrado, Senovilla '03:

Causally related

$V$ is *causally related* with $V'$, denoted $V \prec V'$, if there exists a diffeomorphism

$$\phi : V \to V'$$

mapping causal vectors to causal vectors (preserving time orientation).

Isocausality

$V$ and $V'$ are *isocausal* if they are causally related in both directions. In particular, it happens if there exists $V''$ conformal to $V'$ such that

$$V'' \prec_0 V \prec_0 V' \quad (\prec_0 \text{ means } \phi \equiv Id).$$
Isocausality

**Properties**

- Based on the conformal structure of the spacetime.
- More flexible than the conformal equivalence.
- It preserves some relevant global properties associated to the conformal structure (but not all of them, $G^a$-Parrado, Sánchez'05).

**Important Fact**

There exist spacetimes

\[(V, g_{cl}) \prec_0 (V, g) \prec_0 (V, g_{op}), \quad \text{with } g_{op}, g_{cl} \text{ conformal,}\]

such that $(V, g)$, $(V, g_{op})$ have different future c-boundaries.
Construction of $V_{cl} \equiv (V, g_{cl})$

$V = (-\infty, 0) \times \mathbb{R}$

$g_{cl} = -dt^2 + dx^2$

Future c-boundary of $(V, g_{cl})$
Construction of $V_{op} \equiv (V, g_{op})$

$V = (-\infty, 0) \times \mathbb{R}$

$g_{op} = -dt^2 + \frac{1}{4}dx^2$

Future c-boundary of $(V, g_{op})$
Construction of $V \equiv (V, g)$

$V = (-\infty, 0) \times \mathbb{R}$

$g = -dt^2 + \beta(t/x)dx^2$

$\beta(u) = \begin{cases} 
1/4 & \text{if } u \leq \frac{1}{2} \\
1 & \text{if } u \geq 1 \\
incr. & \text{otherwise}
\end{cases}$

$(V, g_{cl}) \prec_0 (V, g) \prec_0 (V, g_{op})$
Future c-boundary of $V \equiv (V, g)$

- Conformal map between $(V,g)$ and the open region $V_0$ of $\mathbb{L}^2$.

Thus, $(V, g)$ and $V_0$ have the same future c-boundary.
Future c-boundary of $V \equiv (V, g)$
Different future c-boundaries

Future c-boundary of \((V, g_{cl})\) (and \((V, g_{op})\))

Future c-boundary of \((V, g)\)

\(\mathcal{J}^+\)

\(\mathcal{T}\)

\(\mathcal{J}^+\)

\(\mathcal{T}_{Str}\)

Strain
3. STANDARD STATIONARY FRAMEWORK
Standard Stationary Spacetime

\((V, g)\) is a standard stationary spacetime if \(V = \mathbb{R} \times M\) and

\[
g = -dt^2 + \omega \otimes dt + dt \otimes \omega + g_0
\]

where \(\omega\) is a one-form and \((M, g_0)\) a Riemannian manifold.

Associated Finsler Metrics

- There exist two Finsler* metrics (of Randers type) associated to every standard stationary spacetime:

\[
F^\pm(v) = \sqrt{g_0(v, v) + \omega(v)^2} \pm \omega(v).
\]

* A Finsler metric gives smoothly a positively homogeneous norm at each \(p \in V\) (i.e., \(F(\lambda v) = \lambda F(v)\) if \(\lambda \geq 0\)).
The Finsler metric $F = F^+$ defines a map $d : M \times M \to \mathbb{R}$ given by

$$d(x, y) = \inf_{\sigma \in C(x, y)} \int_0^1 F(\dot{\sigma}(t))dt,$$

where $C(x, y)$ is the set of piecewise smooth curves from $x$ to $y$.

This map is a (non-necessarily symmetric) generalized distance, i.e. it satisfies the following properties:

1. $d(x, y) \geq 0$
2. $d(x, y) = d(y, x) = 0 \iff x = y$. (quasi-distance)
3. $d(x, z) \leq d(x, y) + d(y, z)$.
4. $\lim_{n}d(x_n, x) = 0 \iff \lim_{n}d(x, x_n) = 0$.

The pair $(M, d)$ is a generalized metric space.
3.1 CAUCHY COMPLETION
Cauchy Completion

(Forward) Cauchy sequence

- \{x_n\} \subset M is a (forward) Cauchy sequence if \( \forall \epsilon > 0 \) there exists \( n_0 \) such that \( d(x_n, x_m) < \epsilon \) for all \( m \geq n \geq n_0 \).
- \( \text{Cau}^+(M, d) \equiv \) space of (forward) Cauchy sequences.
- Two Cauchy sequences are related, \( \{x_n\} \sim \{x'_n\} \), if, and only if,
  \[
  \lim_n(\lim_m d(x_n, x'_m)) = \lim_n(\lim_m d(x'_n, x_m)) = 0.
  \]

(Forward) Cauchy completion

The (forward) Cauchy completion and boundary are defined as follows:

\[
M^+_C := \text{Cau}^+(M, d)/\sim, \quad \partial^+_C M := M^+_C \setminus M.
\]
Cauchy Completion

**Topology**
- The map $d_Q : M_C^+ \times M_C^+ \to [0, \infty]$ defined by
  
  $$d_Q([\{x_n\}], [\{y_n\}]) = \lim_n (\lim_m d(x_n, y_m))$$

  is a well defined **quasi-distance** which extends $d$.
- $M_C^+$ is endowed with the topology induced by the **backward** $d_Q$-balls.

**Properties**
- Any (forward) Cauchy sequence in $M_C^+$ has limit (completeness).
- $M_C^+$ may not be locally compact.
Cauchy Completion

Backward Cauchy sequence

- \{x_n\} \subset M is a backward Cauchy sequence if \( \forall \, \epsilon > 0 \) there exists \( n_0 \) such that \( d(x_m, x_n) < \epsilon \) for all \( m \geq n \geq n_0 \).
- \( \text{Cau}^{-}(M, d) \equiv \) space of (backward) Cauchy sequences.
- Two Cauchy sequences are related, \( \{x_n\} \sim \{x'_n\} \), if, and only if,
  \[
  \lim_n(\lim_m d(x_n, x'_m)) = \lim_n(\lim_m d(x'_n, x_m)) = 0.
  \]

Backward Cauchy completion

The (backward) Cauchy completion and boundary are defined as follows:

\[
M^-_\mathcal{C} := \text{Cau}^{-}(M, d)/ \sim, \quad \partial^-\mathcal{C} M := M^-_\mathcal{C} \setminus M.
\]
In general, $M^+_C$ and $M^-_C$ do not coincide.

However, the following relation holds:

$$\partial^+_CM \cap \partial^-_CM = \partial^s_CM,$$

where $\partial^s_CM$ is the Cauchy boundary for the symmetrized distance

$$d^s(x, y) := 1/2(d(x, y) + d(y, x)).$$

The symmetrized Cauchy completion is

$$M^s_C := M \cup \partial^s_CM$$

endowed with $d^s$. 

\[\text{Symmetrized Cauchy Completion}\]
3.2 BUSEMANN COMPLETION
Busemann Completion

(Forward) Busemann Function

Given any curve $c : [\alpha, \Omega) \to M$, $\Omega \leq \infty$, with $F(\dot{c}) \leq 1$, the (forward) Busemann function associated to $c$ is defined as:

$$b_c^+(\cdot) := \lim_{t \nearrow \Omega} (t - d(\cdot, c(t))) \in \mathcal{L}_1^+(M, d) \cup \{+\infty\}$$

(Forward) Busemann Completion

- $B^+(M) \equiv$ space of finite (forward) Busemann functions.
- The (forward) Busemann completion and boundary are defined as follows:

$$M_B^+ := B^+(M)/\mathbb{R}, \quad \partial_B^+ M := M_B^+ \setminus M.$$
Busemann Completion

The topology adopted here is inspired by the topology of the future c-boundary.

**Limit Operator**

\[ f \in \hat{\mathcal{L}}(\{f_n\}) \iff \begin{cases} & f \leq \liminf f_n f_n \text{ and} \\ & \forall g \in B^+(M) : f \leq g \leq \limsup_n f_n, \text{ it is } g = f. \end{cases} \]

**(Forward) Busemann Topology**

- The topology on \( B^+(M) \) is the one whose closed sets are those sets \( C \) satisfying \( \hat{\mathcal{L}}(\sigma) \in C \) for any sequence \( \sigma \subset C \).
- The \((forward) Busemann topology\) is the induced quotient topology on the Busemann completion \( M_B^+ = B^+(M)/\mathbb{R} \).
Busemann Completion

Backward Busemann Function

Given any curve \( c : [\alpha, \Omega) \to M, \Omega \leq \infty \), with \( F(\dot{c}) \leq 1 \), the \textit{backward Busemann function} associated to \( c \) is defined as:

\[
b_c^- (\cdot) := \lim_{t \to \Omega} (-t + d(c(t), \cdot)) \in \mathcal{L}_1^-(M, d) \cup \{-\infty\}.
\]

Backward Busemann Completion

- \( B^- (M) \equiv \text{space of finite backward Busemann functions.} \)
- The \textit{backward Busemann completion} and \textit{boundary} are defined as follows:

\[
M_B^- := B^- (M)/\mathbb{R}, \quad \partial_B^- M := M_B^- \setminus M.
\]
## Properties

The Busemann completion $M^\pm_B$ satisfies the following properties:

1. $M^\pm_B$ is sequentially compact.
2. $M$ is naturally embedded as an open dense subset in $M^\pm_B$.
3. The points in $\partial^\pm_B M$ can be reached as limits of curves in $M$.
4. $M^\pm_B$ is $T_1$, and non-$T_2$ related points must lie in $\partial^\pm_B M$.
5. The inclusion $M^\pm_C \subset M^\pm_B$ is continuous if $d_Q$ is a generalized distance, and is a topological embedding if $M^\pm_C$ is locally compact.
3.3 RESULT ON THE C-BOUNDARY
C-boundary of Standard Stationary Spacetimes

**Theorem**

Let \((V, g)\) be a standard stationary spacetime such that \(d_Q^+\) is a generalized distance, \(M_C^s\) is locally compact and \(M_B^\pm\) are Hausdorff. Then:

\[
\begin{align*}
\hat{\partial} V & \equiv \text{cone with base } \partial_B^+ M \text{ and apex } i^+ \\
\check{\partial} V & \equiv \text{cone with base } \partial_B^- M \text{ and apex } i^-.
\end{align*}
\]

- Points in \(\partial_B^\pm M \setminus \partial_C^s M\) yield horismotic lines starting at \(i^\pm\).
- Points in \(\partial_C^s M\) yield timelike lines from \(i^-\) to \(i^+\).

\[\partial V \cong \left( \hat{\partial} V \cup \check{\partial} V \right) / \sim_S.\]
C-BOUNDARY OF SPACETIMES
ISOCAUSAL COMPARISON
STANDARD STATIONARY FRAMEWORK
MAIN RESULTS

CAUCHY COMPLETION
BUSEMANN COMPLETION
RESULT ON THE C-BOUNDARY

\[ \partial^+_B M \setminus \partial^s_C M \]

Future cone

\[ \partial^-_B M \setminus \partial^s_C M \]

Past cone

\[ \partial^+_s M \]

\[ \partial^-_s M \]
4. MAIN RESULTS
Spacetimes of Interest

We will consider spacetimes \((V, g)\) of the form

\[
V = \mathbb{R} \times M \quad \text{and} \quad g = -dt^2 + \omega_t \otimes dt + dt \otimes \omega_t + h_t,
\]

where now \(\omega_t\) and \(h_t\) also depend on \(t\).

General Hypotheses

We will assume that \(g\) satisfies

\[
g_{cl} \prec_0 g \prec_0 g_{op}, \quad \text{thus} \quad g_{op} \text{ conformal to } g_{cl},
\]

being

\[
\begin{aligned}
g_{cl} &= -dt^2 + \omega \otimes dt + dt \otimes \omega + h \\
g_{op} &= -dt^2 + \alpha(t)\omega \otimes dt + \alpha(t)dt \otimes \omega + \alpha^2(t)h.
\end{aligned}
\]
INITIAL OBJECTIVE:

- To relate the future c-boundary of \((V, g)\) with that of \((V, g_{cl})\).

MAIN IDEA:

- Since \(g_{cl} \prec_0 g\), if \(P_{cl} = l_{cl}^-(\gamma)\) is a TIP for \(g_{cl}\) then \(l^-(P_{cl}) = l^-(\gamma)\) is also a TIP for \(g\).

- This suggests to compare the two c-boundaries by defining the following map:

\[
\hat{j} : \hat{\partial}V_{cl} \rightarrow \hat{\partial}V
\]

\[
P_{cl} \mapsto l^-(P_{cl})
\]

* However, \(\hat{j}\) may not be injective, since there may exist different TIPs \(P_{cl} \neq P'_{cl}\) for \(g_{cl}\) such that \(l^-(P_{cl}) = l^-(P'_{cl})\).
Example

Consider the following metrics on $V = \mathbb{R} \times \mathbb{R}$,

$$g_{cl} = -dt^2 + dx^2, \quad g = -dt^2 + dx^2/2, \quad g_{op} = -dt^2 + dx^2/3,$$

which satisfy $g_{cl} \prec g \prec g_{op}$ and $g_{op}$ is conformal to $g_{cl}$.

Any TIP $P_{cl}$ for $g_{cl}$ satisfies $P_{cl} = I_{cl}^{-}(\gamma)$, $\gamma(t) = (t + k, \pm t)$, and thus

$$\hat{j} : \partial V_{cl} \leftrightarrow \partial V, \quad P_{cl} \mapsto I^{-}(P_{cl}) = I^{-}(\gamma) = V.$$
Example

- Consider the following metrics on $V = \mathbb{R} \times \mathbb{R}$,

\[ g_{cl} = -dt^2 + dx^2, \quad g = -dt^2 + dx^2/2, \quad g_{op} = -dt^2 + dx^2/3, \]

which satisfy $g_{cl} \prec g \prec g_{op}$ and $g_{op}$ is conformal to $g_{cl}$.

- Any TIP $P_{cl}$ for $g_{cl}$ satisfies $P_{cl} = I_{cl}^{-}(\gamma), \gamma(t) = (t + k, \pm t)$, and thus

\[ \hat{j} : \hat{\partial}V_{cl} \not\leftrightarrow \hat{\partial}V, \quad P_{cl} \mapsto I^{-}(P_{cl}) = I^{-}(\gamma) = V. \]

Additional Hypothesis

The causal cones of $g$ and $g_{cl}$ must approach at $t$-infinity in this sense:

\[ \int_{0}^{\infty} \left( \frac{1}{\alpha(t)} - 1 \right) dt < \infty. \]
Theorem

Let $V = (\mathbb{R} \times M, g)$ be a spacetime as before such that $g_{op} \prec g \prec g_{op}$ and $g_{cl}$ is conformal to $g_{op}$. If the integral condition

$$\int_0^{\infty} \left( \frac{1}{\alpha(t)} - 1 \right) dt < \infty$$

holds then the map

$$\hat{j} : \hat{V}_{cl} \rightarrow \hat{V}$$

$$P_{cl} \mapsto I^-(P_{cl})$$

is injective. So, $\hat{\partial}V$ contains $\hat{\partial}V_{cl}$ as a point set.

Remark

The map $\hat{j}$ may not be neither surjective nor continuous.
Example

In $V = \mathbb{R} \times \mathbb{R}$ consider the metrics

$$g_{cl} = -dt^2 + dx^2, \quad g = -dt^2 + h_t, \quad g_{op} = -dt^2 + \alpha(t)dx^2,$$

where

$$\alpha(t)dx^2 \leq h_t \leq dx^2 \quad \text{and} \quad \alpha(t) = (e^{-t} + 1)^{-1}.$$
\* \( \gamma(t) = (t, t) \) is lightlike for \( g_{cl} \) \( \Rightarrow P_{cl} = l_{cl}^-(\gamma) \) TIP for \( g_{cl} \).
* \( \rho(t) = (l(t), t) \) is lightlike for \( g_{op} \).
* \( \gamma_k(t) = (l_k(t), t) \) is lightlike for \( g \) \( \Rightarrow P_k = l^-(\gamma_k) \) TIP for \( g \).
They satisfy: $I^-(P_{cl}) \subsetneq P^k$ and $I_{op}^{-1}(P_{cl}) = I_{op}^{-1}(P^k)$.

* The map $\hat{j}$ is not surjective, because $P^k$ are not the image of $P_{cl}$.
* The map $\hat{j}$ is not continuous, since $\{I_{cl}^-(\gamma_k(n))\} \to P_{cl}$ but

$$\{I^-(\gamma_k(n))\} \to P^k \neq I^-(P_{cl}).$$
They satisfy: $I^{-}(P_{cl}) \subsetneq P^{k}$ and $I^{-}_{op}(P_{cl}) = I^{-}_{op}(P^{k})$.

* The map $\hat{j}$ is not surjective, because $P^{k}$ are not the image of $P_{cl}$.

* The map $\hat{j}$ is not continuous, since $\{I^{-}_{cl}(\gamma_{k}(n))\} \rightarrow P_{cl}$ but

$$\{I^{-}(\gamma_{k}(n))\} \rightarrow P^{k} \neq I^{-}(P_{cl})$$
They satisfy: $I^{-}(P_{cl}) \subsetneq P^{k}$ and $I_{op}^{-}(P_{cl}) = I_{op}^{-}(P^{k})$.

* The map $\hat{j}$ is not surjective, because $P^{k}$ are not the image of $P_{cl}$.

* The map $\hat{j}$ is not continuous, since $\{I^{-}(\gamma_{k}(n))\} \to P_{cl}$ but

$$\{I^{-}(\gamma_{k}(n))\} \to P^{k} \neq I^{-}(P_{cl})$$.
st-Relation

$P^1, P^2 \in \partial V$ are $st$-related, $P^1 \sim_{st} P^2$, if there exists a TIP $P_{cl}$ for $g_{cl}$ such that $P_{cl} \subset P^1 \cap P^2$ and $I_{op}^-(P^1) = I_{op}^-(P_{cl}) = I_{op}^-(P^2)$.

Theorem

Let $V = (\mathbb{R} \times M, g)$ be a spacetime as before such that $g_{cl} \prec_0 g \prec_0 g_{op}$ and $g_{cl}$ is conformal to $g_{op}$. If the integral condition holds, $d_Q$ is a generalized distance and $M_C^+$ is locally compact then

$$
\hat{J} = \hat{\Pi} \circ \hat{j} : \hat{V}_{cl} \to \hat{V} / \sim_{st}, \quad \text{with} \quad \hat{\Pi} : \hat{V} \to \hat{V} / \sim_{st}
$$

is bijective and continuous.

If, in addition, $\hat{V} / \sim_{st}$ is Hausdorff, then $\hat{J}$ is an homeomorphism.
st-Relation

\( F^1, F^2 \in \partial V \) are st-related, \( F^1 \sim_{st} F^2 \), if there exists a TIF \( F_{cl} \) for \( g_{cl} \) such that \( F_{cl} \subset F^1 \cap F^2 \) and \( I_{op}^+(F^1) = I_{op}^+(F_{cl}) = I_{op}^+(F^2) \).

Theorem

Let \( V = (\mathbb{R} \times M, g) \) be a spacetime as before such that \( g_{cl} \preceq_0 g \preceq_0 g_{op} \) and \( g_{cl} \) is conformal to \( g_{op} \). If the integral condition holds, \( d_Q^- \) is a generalized distance and \( M^-_C \) is locally compact then

\[ \tilde{\mathcal{J}} = \tilde{\Pi} \circ \tilde{j} : \tilde{V}_{cl} \to \tilde{V} / \sim_{st}, \quad \text{with} \quad \tilde{\Pi} : \tilde{V} \to \tilde{V} / \sim_{st} \]

is bijective and continuous.

If, in addition, \( \tilde{V} / \sim_{st} \) is Hausdorff, then \( \tilde{\mathcal{J}} \) is an homeomorphism.
**FINAL OBJECTIVE:**

- To relate the c-boundary of \((V, g)\) with that of \((V, g_{cl})\).

**MAIN IDEA:**

- Try to define a map of the form
  \[
  \bar{j} : \partial V_{cl} \longrightarrow \partial V, \quad (P_{cl}, F_{cl}) \mapsto \bar{j}((P_{cl}, F_{cl}))
  \]
  by using the maps \(\hat{j}, \check{j}\) previously defined.

- A natural choice is:
  \[
  \bar{j}((P_{cl}, F_{cl})) := (\hat{j}(P_{cl}), \check{j}(F_{cl})).
  \]

* However

\[
P_{cl} \sim_{S} F_{cl} \quad \nRightarrow \quad \hat{j}(P_{cl})(= I^{-}(P_{cl})) \sim_{S} \check{j}(F_{cl})(= I^{+}(F_{cl})).
\]
Lemma

If \((P_{cl}, F_{cl}) \in V_{cl}, \ P_{cl} \neq \emptyset \neq F_{cl}\) then there exist \(P_0 \in \hat{\Pi}^{-1}(\hat{J}(P_{cl}))\) and \(F_0 \in \hat{\Pi}^{-1}(\hat{J}(P_{cl}))\) such that \(P_0 \sim S F_0\).

Proposition

Let \(V = (\mathbb{R} \times M, g)\) be a spacetime as before such that \(g_{cl} \prec_0 g \prec_0 g_{op}\) and \(g_{cl}\) is conformal to \(g_{op}\). Assume that the integral conditions hold and \(d_Q\) is a generalized distance. Consider any map \(\tilde{j} : \overline{V}_{cl} \rightarrow \overline{V}\) given by

\[
(P_{cl}, F_{cl}) \mapsto \tilde{j}((P_{cl}, F_{cl})) := \begin{cases} 
(I^{-}(P_{cl}), \emptyset) & \text{if } F_{cl} = \emptyset \\
(\emptyset, I^{+}(F_{cl})) & \text{if } P_{cl} = \emptyset \\
(P_0, F_0) & \text{otherwise,}
\end{cases}
\]

where \((P_0, F_0)\) is any choice of a pair provided by previous lemma. Then, \(\tilde{j}\) is well defined and injective.
Strain

Let \( \sim_{st} \) be the relation of equivalence on \( \overline{V} \) defined by:

\[
(P, F) \sim_{st} (P', F') \iff (P, F), (P', F') \in ST((P_{cl}, F_{cl})).
\]

for some \( (P_{cl}, F_{cl}) \in \overline{V}_{cl} \), where

\[
(P, F) \in ST((P_{cl}, F_{cl})) \iff \begin{cases} 
P \neq \emptyset \neq P_{cl} \Rightarrow P \in \hat{\Pi}^{-1}(\hat{J}(P_{cl})) 
F \neq \emptyset \neq F_{cl} \Rightarrow F \in \tilde{\Pi}^{-1}(\tilde{J}(F_{cl})) 
P_{cl} = \emptyset \Rightarrow P = \emptyset 
F_{cl} = \emptyset \Rightarrow F = \emptyset 
\end{cases}
\]

The (non-trivial) classes in \( \overline{V}/\sim_{st} \) are called strains.

We will consider the natural projection \( \Pi : \overline{V} \to \overline{V}/\sim_{st} \).
Theorem

Let $V = (\mathbb{R} \times M, g)$ be a spacetime as before such that $g_{cl} \prec_0 g \prec_0 g_{op}$ and $g_{cl}$ is conformal to $g_{op}$. If the integral conditions

$$
\int_{-\infty}^{0} \left( \frac{1}{\alpha(t)} - 1 \right) dt < \infty, \quad \int_{0}^{\infty} \left( \frac{1}{\alpha(t)} - 1 \right) dt < \infty.
$$

hold, $d_Q$ is a generalized distance and $M_C^s$ is locally compact, then the map

$$
\mathcal{J} = \Pi \circ \bar{j} : \overline{V}_{cl} \to \overline{V} / \sim_{st}, \quad \text{with} \quad \Pi : \overline{V}_{cl} \to \overline{V} / \sim_{st}
$$

is injective and continuous.

If, in addition, $\overline{V} / \sim_{st}$ is Hausdorff, then $\mathcal{J}$ is an homeomorphism.
SUMMARY:

- Isocausality yields the qualitative behavior of the c-boundary for a wide class of spacetimes.
- Such spacetimes include those in the general split form $V = \mathbb{R} \times M$, under the hypothesis that the metric will stabilize for large $|t|$.
- Approach developed in full generality by using stationary spacetimes rather than static ones.
- This broad viewpoint leads to consider some technical conditions, which nonetheless hold trivially in practical cases.
- These results can be extended to the case $V = I \times M$, with $I \subset \mathbb{R}$ interval, which includes Robertson-Walker spacetimes.
- Possible extension to spacetimes isocausal to wave type spacetimes??
References


- **F, Herrera, Sánchez**: Isocausal spacetimes may have different causal boundaries, *Class. Quant. Grav.*, **28** (2011) 175016.