Bour’s minimal surface in three dimensional Lorentz-Minkowski space

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The origins of *Minimal Surface Theory* can be traced back to...
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The origins of *Minimal Surface Theory* can be traced back to 1744 with the Swedish Mathematician Leonhard Euler’s (1707-1783) paper, and to the 1760 French Mathematician Joseph Louis Lagrange’s (1736-1813) paper.
A *minimal surface* in $\mathbb{E}^3$ is a regular surface for which the mean curvature vanishes identically.
A \textit{minimal surface} in $\mathbb{E}^3$ is a regular surface for which the mean curvature vanishes identically.

This is a definition of Lagrange, who first defined minimal surface in 1760.
Introduction

Brief History of the Classical Minimal Surfaces:
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1. Plane (trivial)
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2. Euler’s (1707-1783) Catenoid (1740)
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4. Scherk’s (1798-1885) surface (1835)
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9. Schwarz’s (1843-1921) surface (1865)
10. Henneberg’s (1850-1922) surface (1875)
11. Richmond’s (1863-1948) surface (?)
Almost a hundred years later...
1980s – 90s.

- Chen-Gackstatter’s surface (1981)
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- Costa’s surface (1982)
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- Jorge-Meeks’s surface (1983)
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- Costa’s surface (1982)
- Jorge-Meeks’s surface (1983)
- Hoffman, Meeks, Karcher, Kusner, Rosenberg, Lopez, Ros, Rossman, Miyaoka, Sato, ...
2000s – ...

- Fujimori, Shoda, Traizet, Weber, ...
In 1862, the French Mathematician Edmond Bour used semigeodesic coordinates and found a number of new cases of deformations of surfaces.
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In 1862, the French Mathematician Edmond Bour used semigeodesic coordinates and found a number of new cases of deformations of surfaces. He gave a well known theorem about the helicoidal and rotational surfaces. And also the Bour-Enneper equation (today called the sine-Gordon wave equation) used in soliton theory and quantum field theories in Physics was first set down by Bour.
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These surfaces have been called \( \mathcal{B}_m \) (following J. Haag) to emphasize the value of \( m \).
papers dealing with the $\mathfrak{B}_m$ in the literature:
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- Demoulin, A. Bulletin des Sciences Mathematiques (2), vol. XXI (1897), pp. 244-252.
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All real minimal surfaces applicable to rotational surfaces setting

$$\mathcal{F}(s) = C \, s^{m-2}$$

in the Weierstrass representation equations, where $s, C \in \mathbb{C}$, $m \in \mathbb{R}$, and $\mathcal{F}(s)$ is an analytic function.
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- $C = i, \ m = 0$, the right Helicoid,
All real minimal surfaces applicable to rotational surfaces setting
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\bar{\mathcal{F}}(s) = C \ s^{m-2}
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- For \( C = 1, m = 0 \) we obtain the Catenoid,
- \( C = i, m = 0 \), the right Helicoid,
- \( C = 1, m = 2 \), Enneper’s surface (see, also [2,4,16]).

Moreover, Bour’s surface has not been studied up till now in three dimensional Minkowski space $\mathbb{L}^3$. 
Ikawa [10, 11] shows that a generalized helicoid is isometric to a rotational surface by Bour’s theorem in the Euclidean and Minkowski 3-spaces. In addition, he determine these surfaces, with the additional conditions that they are minimal and have the same Gauss map.
Güler [5, 7] shows that a generalized helicoid with lightlike profile curve is isometric to a rotational surface with lightlike profile curve, by Bour’s theorem in the Minkowski 3-space.
Güler, Yaylı and Hacısalıhoğlu establish some relations between the Laplace-Beltrami operator and the curvatures of helicoidal surfaces in 3-Euclidean space. In addition, Bour’s theorem on the Gauss map, and some special examples are given in [6]. Some geometric properties of the timelike rotational surfaces with lightlike profile curve of (S,L), (T,L) and (L,L)-types is shown in Minkowski 3-space in [7,8,9].
We will give Bour’s minimal surfaces in $\mathbb{E}^3$ and $\mathbb{L}^3$. 
Euclidean case

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Euclidean case

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- we shall identify a vector $\vec{x} = (u, v, w)$ with its transpose $\vec{x}^t$,
- the surfaces will be smooth,
- and simply connected.
Let $\mathbb{E}^3$ be a three dimensional Euclidean space with natural metric

$$\langle ., . \rangle_0 = dx^2 + dy^2 + dz^2.$$
In 1818, at age 31, C.F. Gauss (1777-1855) contracted to undertake a geodetic survey, for the German state of Hanover, in order to link up with the existing Danish grid.
Euclidean case

- With the help of this surveying, he invented the "heliotrope" (an instrument used in geodetic surveying for making long distance observations by means of the sun’s rays throwing from a mirror).
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Euclidean case

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- Infinitesimal squares were mapped by map $X$ to infinitesimal squares on surface.
Euclidean case

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Infinitesimal squares were mapped by map $X$ to infinitesimal squares on surface.

He obtained a map, and called **conformal** if satisfy

$$\langle X_u, X_u \rangle_0 = \langle X_v, X_v \rangle_0,$$

$$\langle X_u, X_v \rangle_0 = 0,$$

where $u, v$ are local isothermic parameters.
Euclidean case

- A conformal map is a function which preserves the angles.
Euclidean case

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- Conformal map preserves both angles and shape of infinitesimal squares, but not necessarily their size.
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Figure 0  A conformal mapping
An important family of examples of conformal maps comes from complex analysis.
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If $\mathcal{U}$ is an open subset of the complex plane $\mathbb{C}$, then a function $f: \mathcal{U} \rightarrow \mathbb{C}$ is conformal iff it is holomorphic (or complex differentiable) and its derivative is everywhere non-zero on $\mathcal{U}$. 
Euclidean case

Let $\mathcal{U}$ be an open subset of $\mathbb{C}$. A **minimal** (or *isotropic*) **curve** is an analytic function $\Psi : \mathcal{U} \to \mathbb{C}^n$ such that

$$
(\Psi' (z))^2 = 0,
$$

where $z \in \mathcal{U}$, and $\Psi' := \frac{\partial \Psi}{\partial z}$. 

**References**

Applications in $\mathbb{E}^3$
Euclidean case

- Let $\mathcal{U}$ be an open subset of $\mathbb{C}$. A **minimal** (or **isotropic**) curve is an analytic function $\Psi : \mathcal{U} \to \mathbb{C}^n$ such that

$$(\Psi'(z))^2 = 0,$$

where $z \in \mathcal{U}$, and $\Psi' := \frac{\partial \Psi}{\partial z}$.

- If in addition

$$\langle \Psi', \overline{\Psi}' \rangle_0 = |\Psi'|^2 \neq 0,$$

$\Psi$ is a **regular minimal curve**.
Euclidean case

- Let $\mathcal{U}$ be an open subset of $\mathbb{C}$. A minimal (or isotropic) curve is an analytic function $\Psi : \mathcal{U} \to \mathbb{C}^n$ such that

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where $z \in \mathcal{U}$, and $\Psi' := \frac{\partial \Psi}{\partial z}$.

- If in addition

$$\langle \Psi', \overline{\Psi}' \rangle_0 = |\Psi'|^2 \neq 0,$$

$\Psi$ is a regular minimal curve.

- A minimal surface is the associated family of a minimal curve.
Now, we give the *Weierstrass Representation Theorem* for minimal surfaces in $\mathbb{E}^3$ [15], discovered by K. Weierstrass (1815-1897) in 1866 (also see [1, 16], for details).
Euclidean case

Theorem

Let $\mathcal{F}$ and $\mathcal{G}$ be two holomorphic functions defined on a simply connected open subset $U$ of $\mathbb{C}$ such that $\mathcal{F}$ does not vanish on $U$. Then the map

$$
\mathbf{x}(u, v) = \text{Re} \int^z \begin{pmatrix}
\mathcal{F} (1 - \mathcal{G}^2) \\
i \mathcal{F} (1 + \mathcal{G}^2) \\
2\mathcal{F} \mathcal{G}
\end{pmatrix}
dz
$$

is a minimal, conformal immersion of $U$ into $\mathbb{E}^3$, and $\mathbf{x}$ is called the Weierstrass patch, determined by $\mathcal{F}(z)$ and $\mathcal{G}(z)$. 
Lemma

Let $\Psi : U \rightarrow \mathbb{C}^3$ minimal curve and write $\Psi' = (\varphi_1, \varphi_2, \varphi_3)$.

That is

$$\Psi' = \left(\frac{\varphi_1 - i\varphi_2}{2}, \frac{\varphi_1 - i\varphi_2}{2}, 2\varphi_3\right).$$

Then give rise to the Weierstrass representation of $\Psi$. That is

$$s = \frac{\varphi_1 - i\varphi_2}{2}, \quad g = \frac{\varphi_1 - i\varphi_2}{2}, \quad \varphi_3 = \frac{1 - \varphi_2^2}{1 + \varphi_2^2}.$$
Lemma

The Bour’s curve of value $m$

$$
\left( \frac{z^{m-1}}{m-1} - \frac{z^{m+1}}{m+1}, \text{i} \left( \frac{z^{m-1}}{m-1} + \frac{z^{m+1}}{m+1} \right), 2\frac{z^m}{m} \right)
$$

is a minimal curve in $\mathbb{E}^3$, where $m \in \mathbb{R} - \{-1, 0, 1\}$, $z \in \mathcal{U} \subset \mathbb{C}$, $\text{i} = \sqrt{-1}$. 
Euclidean case

Proof.

Using differential $z$ of the Bour’s curve of value $m$, we have

$$\Omega(z) = (z^{m-2} - z^m, i(z^{m-2} + z^m), 2z^{m-1}). \quad (2)$$

Hence we get

$$(\Omega)^2 = 0.$$
Euclidean case

The Bour’s minimal curve of value 3 (see Fig. 0.1) is intersects itself three times along three straight rays, which meet an angle $2\pi/3$ at the origin in $\mathbb{E}^3$. 
Euclidean case

Figure 0.1  Bour’s minimal curve and its shadows
Euclidean case

- Bour’s minimal surface of value $m$ is the associated family of Bour’s minimal curve.
Lemma

The Weierstrass patch determined by the functions

\[ \mathcal{F}(z) = z^{m-2} \quad \text{and} \quad \mathcal{G}(z) = z \]

is a representation of the Bour’s minimal surface of value \( m \in \mathbb{R} \) in \( \mathbb{R}^3 \).
The Weierstrass representation of the Bour’s surface is

\[ B_m(u, v) = \text{Re}\left( z \Phi(z) \right), \]

where \( m \) is an integer, \( u + iv \) is the corresponding complex coordinate, \( \Phi(z) = z^m + m^{-1}z^{m+1} + \text{other terms} \), and \( \Phi \) is an analytic function.
The Weierstrass representation of the Bour’s surface is

\[ \mathcal{B}_m(u, v) = \text{Re} \int \Phi(z) \, dz, \quad (3) \]
Euclidean case

- The Weierstrass representation of the Bour’s surface is

\[ \mathcal{B}_m(u, v) = \text{Re} \int \Phi(z) \, dz, \quad (3) \]

- where \( m \in \mathbb{R} \), \((u, v)\) are coordinates on the surface, \( z = u + iv \) is the corresponding complex coordinate,

\[ \Phi(z) = \left( \frac{z^{m-1}}{m-1} - \frac{z^{m+1}}{m+1}, i \left( \frac{z^{m-1}}{m-1} + \frac{z^{m+1}}{m+1} \right), 2z^m \right), \]

\( (\Phi)^2 = 0 \), and \( \Phi \) is an analytic function.
Euclidean case

- For $z = re^{i\theta}$, Im part of the $\mathcal{B}_m(r, \theta)$ is a conjugate surface, where $(r, \theta)$ is polar coordinates.
Euclidean case

- For $z = re^{i\theta}$, Im part of the $\mathcal{B}_m(r, \theta)$ is a conjugate surface, where $(r, \theta)$ is polar coordinates.
- The conjugate surface of the Bour’s surface of value $m$ is

$$\mathcal{B}^*_m(r, \theta) = -\text{Re} \int i\Phi$$

$$= \text{Re} \int e^{-i\pi/2}\Phi.$$
Euclidean case

The associated family is thus described by

\[ \mathcal{B}_m (r, \theta; \alpha) = \text{Re} \int e^{-i\alpha \Phi} \]

\[ = \cos(\alpha) \text{Re} \int \Phi + \sin(\alpha) \text{Im} \int \Phi \]

\[ = \cos(\alpha) \mathcal{B}_m (r, \theta) + \sin(\alpha) \mathcal{B}^*_m (r, \theta). \]
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\[ = \cos (\alpha) \mathcal{B}_m (r, \theta) + \sin (\alpha) \mathcal{B}_m^* (r, \theta). \]

When \( \alpha = 0 \) (resp., \( \alpha = \pi/2 \)) we have the Bour’s surface of value \( m \) (resp., the conjugate surface).
Euclidean case

Theorem

Bour’s surface of value $m$

$$\mathcal{B}_m (r, \theta) = \left( \begin{array}{ccc} r^{m-1} \frac{\cos((m-1)\theta)}{m-1} & -r^{m+1} \frac{\cos((m+1)\theta)}{m+1} \\ -r^{m-1} \frac{\sin((m-1)\theta)}{m-1} & -r^{m+1} \frac{\sin((m+1)\theta)}{m+1} \\ 2r^m \frac{\cos(m\theta)}{m} \end{array} \right)$$

is a minimal surface in $\mathbb{E}^3$, where $m \in \mathbb{R} - \{-1, 0, 1\}$, in $(r, \theta)$ coordinates.
Euclidean case

Proof.

The coefficients of the first fundamental form of the Bour’s surface are

\[ E = r^{2m-4} \left(1 + r^2\right)^2, \]
\[ F = 0, \]
\[ G = r^{2m-2} \left(1 + r^2\right)^2. \]
Euclidean case

Proof.

- The coefficients of the first fundamental form of the Bour’s surface are

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\[ F = 0, \]
\[ G = r^{2m-2} (1 + r^2)^2, \]

- So, we have

\[ \det I = r^{4m-6} (1 + r^2)^4. \]
Euclidean case

Proof. (Cont.)
The Gauss map of the surface is

\[ e = \frac{1}{1 + r^2} \begin{pmatrix} 2r \cos(\theta) \\ 2r \sin(\theta) \\ r^2 - 1 \end{pmatrix}. \]
Euclidean case

Proof. (Cont.)

- The coefficients of the second fundamental form of the Bour’s surface are
Euclidean case

Proof. (Cont.)

The coefficients of the second fundamental form of the Bour’s surface are

\[ L = -2r^{m-2} \cos (m\theta) , \]
\[ M = 2r^{m-1} \sin (m\theta) , \]
\[ N = 2r^m \cos (m\theta) . \]
Euclidean case

Proof. (Cont.)

- The coefficients of the second fundamental form of the Bour’s surface are

\[ L = -2r^{m-2} \cos(m\theta), \]
\[ M = 2r^{m-1} \sin(m\theta), \]
\[ N = 2r^m \cos(m\theta). \]

- We have

\[ \det II = -4r^{2m-2}. \]
Proof. (Cont.)

Hence, the mean and the Gaussian curvatures of the Bour’s surface of value $m$, respectively, are
Euclidean case

Proof. (Cont.)

Hence, the mean and the Gaussian curvatures of the Bour’s surface of value $m$, respectively, are

$$H = 0, \ K = - \left( \frac{2r^{2-m}}{(1 + r^2)^2} \right)^2.$$
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Euclidean Bour’s surfaces
Minkowskian Bour’s surfaces
References

Applications in \( \mathbb{E}^3 \)

Euclidean case

Example

If we take \( m = 3 \) in \( \mathcal{B}_m (r, \theta) \), then we have the Bour’s minimal surface (see Fig. 1)

\[
\mathcal{B}_3 (r, \theta) = \left( \begin{array}{c}
\frac{r^2}{2} \cos (2\theta) - \frac{r^4}{4} \cos (4\theta) \\
-\frac{r^2}{2} \sin (2\theta) - \frac{r^4}{4} \sin (4\theta) \\
\frac{2}{3} r^3 \cos (3\theta)
\end{array} \right),
\]

where \( r \in [-1, 1], \ \theta \in [0, \pi] \). When \( r = 1 \), and \( z = 0 \), we have deltoid curve, which is a 3-cusped hypocycloid (Steiner’s hypocycloid (1856)), also called tricuspoid, discovered by Euler in 1745, on plane xy in Fig. 0.1.
Euclidean case

Figure 1  Bour's minimal surface of value 3, $\mathcal{B}_3 (r, \theta)$
Euclidean case

(c)

Figure 1  Bour’s minimal surface of value 3, $\mathcal{B}_3 (r, \theta)$
The coefficients of the first fundamental form of the Bour’s surface of value 3 are

\[ E = r^2 (1 + r^2)^2, \quad F = 0, \quad G = r^4 (1 + r^2)^2. \]

So,

\[ \det I = r^6 (1 + r^2)^4. \]
Euclidean case

The Gauss map of the surface $B_3$ is

$$e = \frac{1}{1 + r^2} \left( 2r \cos(\theta), 2r \sin(\theta), r^2 - 1 \right).$$
Euclidean case

The coefficients of the second fundamental form of the surface are

\[ L = -2r \cos(3\theta), \quad M = 2r^2 \sin(3\theta), \quad N = 2r^3 \cos(3\theta). \]

Then,

\[ \det II = -4r^4. \]
The mean and the Gaussian curvatures of the Bour’s minimal surface of value 3 are, respectively,

\[ H = 0, \quad K = -\frac{4}{r^2 (1 + r^2)^4}. \]
Euclidean case

The Weierstrass patch determined by the functions

$$(\mathcal{F}, \mathcal{G}) = (z, z)$$

is a representation of the Bour’s minimal surface of value 3.
The parametric form of the surface (see Fig. 2) is

\[
\mathcal{B}_3(u, v) = \begin{pmatrix}
-\frac{u^4}{4} - \frac{v^4}{4} + \frac{3}{2} u^2 v^2 + \frac{u^2}{2} - \frac{v^2}{2} \\
-u^3 v - uv^3 - uv \\
\frac{2}{3} u^3 - 2uv^2
\end{pmatrix},
\]  

(6)

where \( u, v \in \mathbb{R} \).
Euclidean case

Figure 2  Surface of $\mathcal{B}_3(u, v), \ u, v \in [-1, 1]$
Euclidean case

The coefficients of the first fundamental form of the Bour’s surface of value 3 in $u, v$ coordinates are

$$E = (u^2 + v^2) \left(1 + u^2 + v^2\right)^2 = G, \quad F = 0,$$

So,

$$\det l = (u^2 + v^2)^2 \left(1 + u^2 + v^2\right)^4.$$
Euclidean case

The Gauss map of the surface $\mathcal{B}_3$ is

$$e = \frac{1}{1 + u^2 + v^2} (2u, 2v, u^2 + v^2 - 1).$$
The coefficients of the second fundamental form of the surface are

\[ L = -2u, \quad M = 2v, \quad N = 2u. \]

Then,

\[ \det II = -4 \left( u^2 + v^2 \right). \]
The mean and the Gaussian curvatures of the Bour’s minimal surface of value 3 are, respectively,

\[ H = 0, \quad K = -\frac{4}{(u^2 + v^2)(1 + u^2 + v^2)^4}. \]
In some literature, however, the Weierstrass representation of the Bour’s minimal surface is known as $(\mathcal{F}, \mathcal{G}) = (1, \zeta^{1/2})$. That is, in polar coordinates, the surface is described by (see Figure 2.1)

\begin{align*}
x &= r \cos (\theta) - \frac{1}{2} r^2 \cos (2\theta), \\
y &= -r \sin (\theta) - \frac{1}{2} r^2 \sin (2\theta), \\
z &= \frac{4}{3} r^{3/2} \cos \left( \frac{3\theta}{2} \right),
\end{align*}

where $r \in [-1/2, 1/2]$, $\theta \in [0, 4\pi]$. 
Euclidean case, some remarks

Figure 2.1 Minimal surface, $(\mathcal{F}, \mathcal{G}) = (1, \zeta^{1/2})$
But this is not Bour’s surface, and these equations are incorrect. Since Enneper’s family $\mathcal{E}_m$ is defined by $(\mathcal{F}, \mathcal{G}) = (1, \zeta^m)$, then the surface belongs to Enneper’s family, and it is the surface $\mathcal{E}_{1/2}$ (see Figure 2.2).
Euclidean case, some remarks

Figure 2.2 The surface $\mathfrak{C}_{1/2}$
(K. Weierstrass, 1903) Assume that the function \( w = f(\zeta) \), where \( \zeta = \xi + i\eta \) and \( w = u + iv \), is analytic in \(|\zeta - \zeta_0| < r\) and satisfies a real algebraic relation \( P(\xi, \eta, u) = 0\). Then \( f(\zeta) \) is an algebraic function of its argument.
An **algebraic curve** over a field $K$ is an equation $f(x, y) = 0$, where $f(x, y)$ is a polynomial in $x$ and $y$ with coefficients in $K$. 
An algebraic curve over a field \( K \) is an equation \( f(x, y) = 0 \), where \( f(x, y) \) is a polynomial in \( x \) and \( y \) with coefficients in \( K \).

The set of roots of a polynomial \( f(x, y, z) = 0 \). An algebraic surface is said to be of degree (order) \( n = \max(i + j + k) \), where \( n \) is the maximum sum of powers of all terms \( a_m x^i y^j z^k \).
Integral free form of the Weierstrass representation (obtained by K. Weierstrass, 1903) is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \text{Re} \left( \begin{pmatrix} (1 - w^2) \phi''(w) + 2w\phi'(w) - 2\phi(w) \\ i \left[ (1 + w^2) \phi''(w) - 2w\phi'(w) + 2\phi(w) \right] \\ 2 \left[ w\phi''(w) - \phi'(w) \right] \end{pmatrix} \right)$$

$$\equiv \text{Re} \left( \begin{pmatrix} f_1(w) \\ f_2(w) \\ f_3(w) \end{pmatrix} \right),$$
where \( \phi(w) \) (algebraic function) and the functions \( f_i(w) \) are connected by the relation

\[
\phi(w) = \frac{1}{4} (w^2 - 1) f_1(w) - \frac{i}{4} (w^2 + 1) f_2(w) - \frac{1}{2} w f_3(w).
\]
Euclidean case

- Integral free form formulas are suitable for algebraic minimal surfaces.
Euclidean case

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- For instance, $\phi(w) = \frac{1}{6} w^3$ give rise to Enneper’s minimal surface $E := B_2$ (see, also [16]).
Euclidean case

- Integral free form formulas are suitable for algebraic minimal surfaces.
- For instance, \( \phi(w) = \frac{1}{6}w^3 \) give rise to Enneper’s minimal surface \( \mathcal{E} := \mathcal{B}_2 \) (see, also [16]).
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\[
\phi(w) = \frac{1}{24}w^4
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leads to Bour’s minimal surface \( \mathcal{B} := \mathcal{B}_3 \).
Euclidean case

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\phi(w) = \frac{1}{24}w^4
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leads to Bour’s minimal surface \( \mathcal{B} := \mathcal{B}_3 \).
- And also, it is clear that

\[
\phi'_{\mathcal{B}} = \phi_{\mathcal{E}}.
\]
Euclidean case

- In 1882, Ribaucour shows that
Euclidean case

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In 1882, Ribaucour shows that

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- if \( m \in \mathbb{Z} \) \( \Rightarrow \) \( \text{degree} (\mathcal{B}_m) = (m + 1)^2 \).
Euclidean case

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Euclidean case

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In 1882, Ribaucour shows that

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He also shows that

\[ \text{if } m < 1 \Rightarrow \text{class } (\mathcal{B}_m) = \text{degree } (\mathcal{B}_m), \]

\[ \text{if } m > 1 \Rightarrow \text{class } (\mathcal{B}_m) < \text{degree } (\mathcal{B}_m). \]
Euclidean case

That is,

- $\text{cl}(\mathcal{B}_2) = 6$ (Enneper),
Euclidean case

That is,

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Euclidean case

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- $cl(B_3) = 8$, (Bour),
- $cl(B_4) = 10$, 
- $cl(B_m) = 2q(p+q)$.

$deg(B_2) = 9$ (Enneper),
$deg(B_3) = 16$, (Bour),
$deg(B_4) = 25$, 
$deg(B_m) = (m+1)2$. 
Euclidean case

That is,

- \( cl(\mathcal{B}_2) = 6 \) (Enneper),
- \( cl(\mathcal{B}_3) = 8 \), (Bour),
- \( cl(\mathcal{B}_4) = 10 \),
- \( cl(\mathcal{B}_5) = 12 \),
Euclidean case

That is,

\begin{itemize}
  \item $cl(B_2) = 6$ (Enneper),
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  \item $\ldots$
\end{itemize}
Euclidean case

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- $cl(B_2) = 6$ (Enneper),
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- \( \text{cl} (\mathcal{B}_m) = 2q(p + q) \).
- \( \text{deg} (\mathcal{B}_2) = 9 \) (Enneper),
- \( \text{deg} (\mathcal{B}_3) = 16 \) (Bour),
- \( \text{deg} (\mathcal{B}_4) = 25 \),
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Euclidean case

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- \( cl(\mathcal{B}_3) = 8 \text{, (Bour)} \),
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- \( cl(\mathcal{B}_5) = 12 \),
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- $\ldots$
- $\text{deg} (\mathcal{B}_m) = (m + 1)^2$. 
Euclidean case

We calculate the implicit equations, classes, and degrees of the surfaces $B_2, B_3, B_4, B_5, B_6$ using Sylvester and Gr"obner eliminate methods by the help of Maple programme.
Euclidean case

- We calculate the implicit equations, classes, and degrees of the surfaces $\mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6$ using Sylvester and Gröbner eliminate methods by the help of Maple programme.
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Euclidean case

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- Our findings agree with Ribaucour’s.
- And we give the following table
Euclidean case

<table>
<thead>
<tr>
<th>( \mathcal{B}_m(u, v) )</th>
<th>( \deg(x) )</th>
<th>( \deg(y) )</th>
<th>( \deg(z) )</th>
<th>( cl(\mathcal{B}_m) )</th>
<th>( \deg(\mathcal{B}_m) )</th>
<th>( Syl(x, y, u) )</th>
<th>( Syl(F, G, v) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{B}_2 )</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>9</td>
<td>( 5 \times 5 )</td>
<td>( 11 \times 11 )</td>
</tr>
<tr>
<td>( \mathcal{B}_3 )</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>8</td>
<td>16</td>
<td>( 7 \times 7 )</td>
<td>( 18 \times 18 )</td>
</tr>
<tr>
<td>( \mathcal{B}_4 )</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>10</td>
<td>25</td>
<td>( 9 \times 9 )</td>
<td>( 29 \times 29 )</td>
</tr>
<tr>
<td>( \mathcal{B}_5 )</td>
<td>6</td>
<td>6</td>
<td>5</td>
<td>12</td>
<td>36</td>
<td>( 11 \times 11 )</td>
<td>( 40 \times 40 )</td>
</tr>
<tr>
<td>( \mathcal{B}_6 )</td>
<td>7</td>
<td>7</td>
<td>6</td>
<td>14</td>
<td>49</td>
<td>( 13 \times 13 )</td>
<td>( 55 \times 55 )</td>
</tr>
<tr>
<td>( \mathcal{B}_7 )</td>
<td>8</td>
<td>8</td>
<td>7</td>
<td>16</td>
<td>64</td>
<td>( 15 \times 15 )</td>
<td>( 70 \times 70 )</td>
</tr>
<tr>
<td>( \mathcal{B}_8 )</td>
<td>9</td>
<td>9</td>
<td>8</td>
<td>18</td>
<td>81</td>
<td>( 17 \times 17 )</td>
<td>( 89 \times 89 )</td>
</tr>
<tr>
<td>( \mathcal{B}_9 )</td>
<td>10</td>
<td>10</td>
<td>9</td>
<td>20</td>
<td>100</td>
<td>( 19 \times 19 )</td>
<td>( 108 \times 108 )</td>
</tr>
<tr>
<td>( \mathcal{B}_{10} )</td>
<td>11</td>
<td>11</td>
<td>10</td>
<td>22</td>
<td>121</td>
<td>( 21 \times 21 )</td>
<td>( 131 \times 131 )</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( \mathcal{B}_m )</td>
<td>( m+1 )</td>
<td>( m+1 )</td>
<td>( m )</td>
<td>( 2m+2 )</td>
<td>( (m+1)^2 )</td>
<td>( \frac{(m+1)^2 + m}{(m+1)^2 + m - 1} )</td>
<td>( \ldots \text{if } m \text{ even} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( (m+1)^2 \times (2m+1) )</td>
<td>( \ldots \text{if } m \text{ odd} )</td>
<td></td>
</tr>
</tbody>
</table>

degree and class of \( \mathcal{B}_m(u, v) \)
Euclidean case

We also show the relations between the degree of the algebraic function \( \phi^2(w) \) in the integral free form formulas and the class of the surfaces. We know...
Euclidean case

- We also show the relations between the degree of the algebraic function $\phi^2(w)$ in the integral free form formulas and the class of the surfaces. We know
- $\mathcal{B}_2 : \phi_{\mathcal{B}_2}(w) = \frac{1}{6}w^3$ (Enneper’s minimal surface),
We also show the relations between the \textbf{degree of the algebraic function} $\phi^2(w)$ in the integral free form formulas and the \textbf{class of the surfaces}. We know

- $\mathcal{B}_2 : \phi_{\mathcal{B}_2}(w) = \frac{1}{6}w^3$ (Enneper’s minimal surface),
- $\mathcal{B}_3 : \phi_{\mathcal{B}_3}(w) = \frac{1}{24}z^4$ (Bour’s minimal surface),
Euclidean case

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- ...
- $\mathcal{B}_m : \phi_{\mathcal{B}_m}(w) = \frac{1}{(m+1)!} w^{m+1}$. 
Euclidean case

- Then we can see
We can see
\[
\deg \left( \phi_{\mathcal{B}_2}^2 \right) = 6 = cl \left( \mathcal{B}_2 \right),
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\]

\[
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\]
Then we can see

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\[ \deg \left( \phi_{\mathcal{B}_4}^2 \right) = 10 = \text{cl} (\mathcal{B}_4), \]
Then we can see

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\[ \deg \left( \phi_{\mathcal{B}_4}^2 \right) = 10 = cl \left( \mathcal{B}_4 \right), \]
\[ \ldots \]
Then we can see
\[ \deg \left( \phi_{B_2}^2 \right) = 6 = \text{cl} \left( B_2 \right), \]
\[ \deg \left( \phi_{B_3}^2 \right) = 8 = \text{cl} \left( B_3 \right), \]
\[ \deg \left( \phi_{B_4}^2 \right) = 10 = \text{cl} \left( B_4 \right), \]
\[ \ldots \]
\[ \deg \left( \phi_{B_m}^2 \right) = 2m + 2 = \text{cl} \left( B_m \right). \]
Bour’s minimal surface $\mathcal{B}_3$ and its conjugate are as follows:

\[
\mathcal{B}_3 (u, v) = \begin{pmatrix}
-\frac{u^4}{4} - \frac{v^4}{4} + \frac{3}{2} u^2 v^2 + \frac{u^2}{2} - \frac{v^2}{2} \\
-u^3 v - uv^3 - uv \\
\frac{2}{3} u^3 - 2uv^2
\end{pmatrix},
\]

\[
\mathcal{B}_3^* (u, v) = \begin{pmatrix}
\frac{u^4}{4} + \frac{v^4}{4} - \frac{3}{2} u^2 v^2 + \frac{u^2}{2} - \frac{v^2}{2} \\
\frac{2}{3} u^3 + 2uv^2
\end{pmatrix}.
\]
Bour’s minimal surface $\mathcal{B}_3$ and its conjugate are as follow

$$\mathcal{B}_3 (u, v) = \begin{pmatrix} -\frac{u^4}{4} - \frac{v^4}{4} + \frac{3}{2} u^2 v^2 + \frac{u^2}{2} - \frac{v^2}{2} \\ -u^3 v - uv^3 - uv \\ \frac{2}{3} u^3 - 2uv^2 \end{pmatrix},$$

$$\mathcal{B}^*_3 (u, v) = \begin{pmatrix} \frac{u^4}{4} + \frac{v^4}{4} - \frac{3}{2} u^2 v^2 + \frac{u^2}{2} - \frac{v^2}{2} \\ -\frac{2}{3} u^3 - 2u^2 v \\ -u^3 v + uv^3 + uv \end{pmatrix},$$

and then we can see the Cauchy-Riemann equations hold

$$(\mathcal{B}_3)_u = (\mathcal{B}^*_3)_v, \quad (\mathcal{B}_3)_v = - (\mathcal{B}^*_3)_u.$$
We know $\mathcal{B}_3 (u, v) = \Re \int \Phi dz$, $X := \mathcal{B}_3$, $X_{uu} + X_{vv} = 0$, (i.e. $\mathcal{B}_3$ minimal),

$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle$,

$\langle X_u, X_v \rangle = 0$, (i.e. $\mathcal{B}_3$ conformal).
Euclidean case

- We know $\mathcal{B}_3(u, v) = \text{Re} \int \Phi dz$, $X := \mathcal{B}_3$,
  
  $$X_{uu} + X_{vv} = 0,$$

  (i.e. $\mathcal{B}_3$ minimal),

  $$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle,$$
  $$\langle X_u, X_v \rangle = 0,$$

  (i.e. $\mathcal{B}_3$ conformal).

- Then we can see

  $$\Phi = X_u - iX_v$$
  $$= (z - z^3, i(z + z^3), 2z^2),$$

  and $(\Phi)^2 = 0$, $\Phi$ analytic (in each component), and also it can be seen for $Y := \mathcal{B}_3^*$. [140x266]
Euclidean case

Problems.

1. Find the \( \mathcal{B}_3 \) algebraic or not,
Problems.

1. Find the $\mathcal{B}_3$ algebraic or not,
2. the cartesian equation of $\mathcal{B}_3$, 
Euclidean case

Problems.

1. Find the $B_3$ algebraic or not,
2. the cartesian equation of $B_3$,
3. degree,
**Problems.**

1. Find the $\mathcal{B}_3$ algebraic or not,
2. the cartesian equation of $\mathcal{B}_3$,
3. degree,
4. and class.
4. **Hint.** The tangent plane at a point \((u, v)\) on Bour’s surface \(\mathcal{B}_3\) is given in terms of running coordinates \(x, y, z\) by

\[
X(u, v)x + Y(u, v)y + Z(u, v)z + P(u, v) = 0.
\]

For the inhomogeneous tangential coordinates \(\bar{u} = X/P\), \(\bar{v} = Y/P\), and \(\bar{w} = Z/P\). By eliminating \(u\) and \(v\), obtain the equation for the surface \(\mathcal{B}_3\) in tangential coordinates. Maximum degree of the equation gives **class** of Bour’s surface \(\mathcal{B}_3\).
Euclidean case

We compute the irreducible implicit equation of surface $\mathcal{B}_3$ using Sylvester and Gröbner eliminate methods by software programmes:
Euclidean case

\[-859963392\, x^4\, z^6 - 764411904\, y^2\, x^4\, z^4 - 1719926784\, y^2\, x^2\, z^6 + 509607936\, y^4\, x^2\, z^4 - 1934917632\, z^{10} - 2579890176\, x^2\, z^8
\]
\[-859963392\, z^6\, y^4 - 84934656\, z^4\, y^6 - 2579890176\, z^8\, y^2 + 1632586752\, z^{12} + 268435456\, y^{12} - 28991029248\, x^6\, y^6
\]
\[+ 31340888064\, x^6\, z^6 - 3877393536\, z^{12}\, y^2 + 37650272256\, z^8\, y^4 - 3654844416\, z^6\, y^6 + 38985007104\, z^6\, y^8
\]
\[+ 1451182240\, z^{10}\, y^2 - 7255941120\, z^{10}\, y^4 + 3623878656\, z^8\, y^8 + 17836277760\, z^4\, y^8 - 14834368512\, z^8\, x^4\, y^2
\]
\[+ 6115295232\, x^7\, z^4\, y^2 - 56396611584\, y^6\, x^3\, z^4 - 10192158720\, x^5\, y^4\, z^4 + 5435817984\, x^9\, z^4 - 3009871872\, x^6\, z^6
\]
\[+ 21743271936\, y^4\, x^8 - 22932357120\, x^5\, z^6\, y^2 + 119757864960\, x^6\, y^2\, z^4 + 3057647616\, x^7\, z^4 + 945957312\, x^5\, z^6
\]
\[+ 7309688832\, x^3\, z^8 + 272097792\, z^{12}\, x^3 + 37650272256\, x^4\, z^8 + 1451182240\, z^{10}\, x^2 + 10037385216\, x^3\, z^{10}
\]
\[+ 29023764480\, x^5\, z^8 + 8153726976\, x^8\, z^4 - 9965666304\, x^5\, z^4\, y^4 - 58047528960\, z^8\, x^3\, y^2 - 18919194624\, x^3\, y^2\, z^6
\]
\[+ 43486543872\, x^6\, y^4\, z^2 - 7255941120\, x^4\, z^{10} + 22932357120\, x^7\, z^6 + 77970014208\, x^4\, y^4 - 3057647616\, x^5\, z^4\, y^2
\]
\[-3877393536\, x^2\, z^{12} - 459165024\, z^{14} + 43046721\, z^{16} + 14495514624\, y^8\, x^4 - 3221225472\, x^2\, y^{10} - 21929066496\, x\, z^8\, y^2
\]
\[+ 75300544512\, x^2\, z^8\, y^2 - 9059696664\, y^8\, x\, z^2 - 14495514624\, y^8\, x^2\, z^2 - 15288238080\, y^4\, x^4 + 48157949952\, y^4\, z^2\, x^6
\]
\[-28378791936\, y^4\, z^6\, x - 14269022208\, y^8\, z^4\, x + 162819735552\, y^4\, z^6\, x^2 + 2717908992\, y^2\, x^2\, z^2 + 32614907904\, x^8\, z^2\, y^2
\]
\[+ 5737807872\, x^3\, y^6\, z^2 - 114661785600\, x^3\, y^6\, z^2 - 7247757312\, x^4\, y^6\, z^2 - 15797846016\, y^6\, x^2\, z^4 - 14511882240\, x^2\, z^{10}
\]
\[-68797071360\, x\, z^6\, y^6 - 30112155648\, x\, z^6\, y^6 - 9172942848\, x^4\, y^6 - 87071293440\, x^8\, y^4 - 5159780352\, x^2\, z^8\, y^4
\]
\[-816293376\, x\, z^{12}\, y^2 = 0
\]

Implicit equation of $\mathcal{B}_3$, degree($\mathcal{B}_3$)=16
Euclidean case

**Answers.**

1. $B_3$ is an algebraic minimal surface.
Answers.

1. $B_3$ is an algebraic minimal surface.
2. We find the irreducible implicit equation of $B_3$. 
Euclidean case

Answers.

1. $\mathcal{B}_3$ is an algebraic minimal surface.
2. We find the irreducible implicit equation of $\mathcal{B}_3$.
3. Degree $(\mathcal{B}_3) = 16.$
Euclidean case

Answers.

1. $\mathcal{B}_3$ is an algebraic minimal surface.
2. We find the irreducible implicit equation of $\mathcal{B}_3$.
3. Degree $(\mathcal{B}_3) = 16$.
4. Class $(\mathcal{B}_3) = 8$. 
Answer (4). We find \( P(u, v) = \frac{(u^2 + v^2 + 2)(3uv^2 - u^3)}{6(u^2 + v^2 + 1)} \), and the inhomogeneous tangential coordinates

\[
\bar{u} = \frac{12u}{(u^2 + v^2 + 2)(3uv^2 - u^3)}, \\
\bar{v} = \frac{12v}{(u^2 + v^2 + 2)(3uv^2 - u^3)}, \\
\bar{w} = \frac{6(u^2 + v^2 - 1)}{(u^2 + v^2 + 2)(3uv^2 - u^3)}.
\]

By eliminating \( u \) and \( v \), we obtain the equation for the surface \( \mathcal{B}_3 \) in tangential coordinates. Maximum degree of the equation gives \textbf{class}=8 of Bour’s surface \( \mathcal{B}_3 \).
Euclidean case

\[ 9u^8 + 72u^7 + 144u^6 + 288u^5w^2 + 192u^3w^4 + 8u^6w^2 
   - 48u^4v^2w^2 
   - 576uv^2w^4 
   + 81u^2v^6 
   + 432u^4v^2 
   - 45u^6v^2 
   - 72u^5v^2 
   + 432u^2v^4 
   - 360u^3v^4 
   - 216uv^6 
   + 27u^4v^4 
   + 144v^6 
   - 576u^3v^2w^2 
   + 72u^2v^4w^2 
   - 864uv^4w^2 = 0 \]

Implicit equation of \( \mathcal{B}_3 \) in tangential coordinates, class(\( \mathcal{B}_3 \))=8
Euclidean case

Figure 30 $\mathcal{B}_3 (r, \theta)$ and its curve $\gamma (r)$ on plane $xz$
Euclidean case

5. Find the implicit equation of the curve
\[ \gamma (r) = \left( \frac{r^2}{2} - \frac{r^4}{4}, 0, \frac{2}{3} r^3 \right) \] (see Fig. 30) on plane xz, and its degree.

Theorem

(L. Henneberg, 1876) A plane intersects an algebraic minimal surface in an algebraic curve [16].
5. Find the implicit equation of the curve

\[ \gamma(r) = \left( \frac{r^2}{2} - \frac{r^4}{4}, 0, \frac{2}{3} r^3 \right) \] (see Fig. 30) on plane xz, and its degree.

- **Answer.** The implicit equation of \( \gamma \) is

\[
1024 x^2 + 864 x z^2 = z^2 (288 - 81 z^2), \quad \text{degree} (\gamma) = 4.
\]

**Theorem**

(L. Henneberg, 1876) A plane intersects an algebraic minimal surface in an algebraic curve [16].
Euclidean case

5. Find the implicit equation of the curve
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- **Answer.** The implicit equation of \(\gamma\) is
  \[ 1024x^2 + 864xz^2 = z^2(288 - 81z^2) \], degree(\(\gamma\)) = 4.

**Theorem**

*(L. Henneberg, 1876)* A plane intersects an algebraic minimal surface in an algebraic curve [16].
Total curvature of $\mathcal{B}_m$ is

$$\mathcal{C}(\mathcal{B}_m) = \iint K dA$$

$$= \iint -\frac{4}{(1 + u^2 + v^2)^2} dudv$$

$$= -4\pi.$$
Example

If take \( m = 2 \), we have **Enneper’s minimal surface** (see Fig. 3)

\[
\mathcal{B}_2(r, \theta) = \begin{pmatrix}
    r \cos(\theta) - \frac{r^3}{3} \cos(3\theta) \\
    -r \sin(\theta) - \frac{r^3}{3} \sin(3\theta) \\
    r^2 \cos(2\theta)
\end{pmatrix},
\]

where \( r \in [-1, 1] \), \( \theta \in [0, \pi] \).
Applications

Figure 3  Bour’s minimal surface of value 2
If $m = 2$, we have **Enneper’s minimal surface** (see Fig. 4) $\mathcal{B}_2 (r, \theta)$, where $r \in [-3, 3]$, $\theta \in [0, \pi]$. 
Applications

Figure 4 Bour’s minimal surface of value 2

(a)  
(b)
Applications

Example

If take $m = \frac{1}{2}$, we have **Richmond's-like minimal surface** (see Fig. 5)

\[
\begin{pmatrix}
-2r^{-1/2} \cos \left( \frac{\theta}{2} \right) & -\frac{2}{3} r^{3/2} \cos \left( \frac{3\theta}{2} \right) \\
-2r^{-1/2} \sin \left( \frac{\theta}{2} \right) & -\frac{2}{3} r^{3/2} \sin \left( \frac{3\theta}{2} \right) \\
4r^{1/2} \cos \left( \frac{\theta}{2} \right)
\end{pmatrix},
\]

where $r \in [-1, 1]$, $\theta \in [-2\pi, 2\pi]$. 
Applications

(a) (b)

Figure 5 Bour’s minimal surface of value 1/2
Applications

Example

If \( m = \frac{3}{2} \), we have (see Fig. 6)

\[
\begin{pmatrix}
2r^{1/2} \cos \left( \frac{\theta}{2} \right) - \frac{2}{5} r^{5/2} \cos \left( \frac{5\theta}{2} \right) \\
-2r^{1/2} \sin \left( \frac{\theta}{2} \right) - \frac{2}{5} r^{5/2} \sin \left( \frac{5\theta}{2} \right) \\
\frac{4}{3} r^{3/2} \cos \left( \frac{3\theta}{2} \right)
\end{pmatrix},
\]

where \( r \in [-3, 3] \), \( \theta \in [-2\pi, 2\pi] \).
Applications

Figure 6  Bour’s minimal surface of value $3/2$
Applications

Example

If \( m = \frac{3}{2} \), we have (see Fig. 7)

\[
\begin{pmatrix}
2r^{1/2} \cos \left( \frac{\theta}{2} \right) - \frac{2}{5} r^{5/2} \cos \left( \frac{5\theta}{2} \right) \\
-2r^{1/2} \sin \left( \frac{\theta}{2} \right) - \frac{2}{5} r^{5/2} \sin \left( \frac{5\theta}{2} \right) \\
\frac{4}{3} r^{3/2} \cos \left( \frac{3\theta}{2} \right)
\end{pmatrix},
\]

where \( r \in [-1, 1], \theta \in [-2\pi, 2\pi] \).
Applications

Figure 7  Bour’s minimal surface of value $3/2$
Applications

Example

If $m = \frac{2}{3}$, we have (see Fig. 8)

$$
\begin{pmatrix}
-3r^{-1/3} \cos \left( \frac{\theta}{3} \right) - \frac{3}{5} r^{5/3} \cos \left( \frac{5\theta}{3} \right) \\
-3r^{-1/3} \sin \left( \frac{\theta}{3} \right) - \frac{3}{5} r^{5/3} \sin \left( \frac{5\theta}{3} \right) \\
3r^{2/3} \cos \left( \frac{2\theta}{3} \right)
\end{pmatrix},
$$

where $r \in [-1, 1]$, $\theta \in [-3\pi, 3\pi]$. 

Applications

(a) Figure 8 Bour’s minimal surface of value 2/3

(b)
Applications

Example

If \( m = \frac{4}{3} \), we have (see Fig. 9)

\[
\begin{pmatrix}
3r^{1/3} \cos \left( \frac{\theta}{3} \right) - \frac{3}{7} r^{7/3} \cos \left( \frac{7\theta}{3} \right) \\
-3r^{1/3} \sin \left( \frac{\theta}{3} \right) - \frac{3}{7} r^{7/3} \sin \left( \frac{7\theta}{3} \right) \\
\frac{3}{2} r^{4/3} \cos \left( \frac{4\theta}{3} \right)
\end{pmatrix},
\]

where \( r \in [-2, 2], \ \theta \in [-3\pi, 3\pi] \).
Applications

Figure 9  Bour’s minimal surface of value $4/3$
If $m = \frac{5}{2}$, we have (see Fig. 10)

$$
\begin{pmatrix}
\frac{2}{3} r^{3/2} \cos \left( \frac{3\theta}{2} \right) - \frac{2}{7} r^{7/2} \cos \left( \frac{7\theta}{2} \right) \\
-\frac{2}{3} r^{3/2} \sin \left( \frac{3\theta}{2} \right) - \frac{2}{7} r^{7/2} \sin \left( \frac{7\theta}{2} \right) \\
\frac{4}{5} r^{5/2} \cos \left( \frac{5\theta}{2} \right)
\end{pmatrix},
$$

where $r \in [-1, 1]$, $\theta \in [-2\pi, 2\pi]$. 
Applications

Figure 10 Bour’s minimal surface of value $5/2$
Applications

Example

If \( m = 4 \), we have (see Fig. 11)

\[
\left( \begin{array}{c} \frac{1}{3} r^3 \cos (3\theta) - \frac{1}{5} r^5 \cos (5\theta) \\ -\frac{1}{3} r^3 \sin (3\theta) - \frac{1}{5} r^5 \sin (5\theta) \\ \frac{1}{2} r^4 \cos (4\theta) \end{array} \right),
\]

where \( r \in [-1, 1], \theta \in [0, 2\pi] \).
Applications

(a) Figure 11 Bour’s minimal surface of value 4
Now, we will see the definite and indefinite cases of the Bour’s minimal surface.
Let $\mathbb{L}^3$ be a 3-dimensional Minkowski space with natural Lorentzian metric

$$\langle ., . \rangle_1 = dx^2 + dy^2 - dz^2.$$
Definite case

- A vector \( w \) in \( \mathbb{L}^3 \) is called
A vector $w$ in $\mathbb{L}^3$ is called

*spacelike* if $\langle w, w \rangle_1 > 0$ or $w = 0$, 

*timelike* if $\langle w, w \rangle_1 < 0$,

*lightlike* if $w \neq 0$ satisfies $\langle w, w \rangle_1 = 0$. 

A vector $w$ in $\mathbb{L}^3$ is called

- **spacelike** if $\langle w, w \rangle_1 > 0$ or $w = 0$,
- **timelike** if $\langle w, w \rangle_1 < 0$,
Definite case

A *vector* \( w \) in \( \mathbb{L}^3 \) is called

- **spacelike** if \( \langle w, w \rangle_1 > 0 \) or \( w = 0 \),
- **timelike** if \( \langle w, w \rangle_1 < 0 \),
- **lightlike** if \( w \neq 0 \) satisfies \( \langle w, w \rangle_1 = 0 \).
Definite case

A surface in $\mathbb{L}^3$ is called a spacelike (resp. timelike, degenerate (lightlike)) if the induced metric on the surface is a positive definite Riemannian (resp. Lorentzian, degenerate) metric.
Definite case

- A *surface* in $\mathbb{L}^3$ is called a **spacelike** (resp. **timelike**, **degenere** (lightlike)) if the induced metric on the surface is a **positive definite Riemannian** (resp. Lorentzian, degenere) metric.

- A space-like surface with vanishing mean curvature is called a **maximal surface**.
Theorem

(Weierstrass representation for maximal surfaces in $\mathbb{L}^3$). Let $\mathfrak{F}$ and $\mathcal{G}$ be two holomorphic functions defined on a simply connected open subset $U$ of $\mathbb{C}$ such that $\mathfrak{F}$ does not vanish and $|\mathcal{G}| \neq 1$ on $U$. Then the map

$$x(u, v) = \text{Re} \int^z \begin{pmatrix} \mathfrak{F}(1 + \mathcal{G}^2) \\ i \mathfrak{F}(1 - \mathcal{G}^2) \\ 2\mathfrak{F}\mathcal{G} \end{pmatrix} \, dz$$

is a conformal immersion of $U$ into $\mathbb{L}^3$ whose image is a maximal surface [1, 13, 15].
Definite case

Lemma

The Weierstrass patch determined by the functions

\[(F(z), G(z)) = (z^{m-2}, z)\]

is a representation of the Bour’s surface of value \(m \in \mathbb{R}\) in \(\mathbb{L}^3\).
Definite case

Theorem

Bour’s surface of value $m$

\[
\mathcal{V}_m (r, \theta) = \left( \begin{array}{c}
\frac{r^{m-1}}{m-1} \cos [(m-1) \theta] + \frac{r^{m+1}}{m+1} \cos [(m+1) \theta] \\
-\frac{r^{m-1}}{m-1} \sin [(m-1) \theta] + \frac{r^{m+1}}{m+1} \sin [(m+1) \theta] \\
2 \frac{r^m}{m} \cos (m\theta)
\end{array} \right)
\]

is a maximal surface in $\mathbb{L}^3$, where $m \in \mathbb{R} - \{-1, 0, 1\}$. 
Definite case

Proof.

- The coefficients of the first fundamental form of the surface $B_m$ are

\[
E = r^{2m-4} (1 - r^2)^2, \\
F = 0, \\
G = r^{2m-2} (1 - r^2)^2.
\]
Definite case

Proof.

- The coefficients of the first fundamental form of the surface $B_m$ are

$$
E = r^{2m-4} (1 - r^2)^2 ,
F = 0,
G = r^{2m-2} (1 - r^2)^2 .
$$

- We have

$$
\det I = \left[ r^{2m-3} (1 - r^2)^2 \right]^2 .
$$
Definite case

Proof.

• The coefficients of the first fundamental form of the surface \( \mathcal{B}_m \) are

\[
\begin{align*}
E &= r^{2m-4} (1 - r^2)^2, \\
F &= 0, \\
G &= r^{2m-2} (1 - r^2)^2.
\end{align*}
\]

• We have

\[
\det I = \left[ r^{2m-3} (1 - r^2)^2 \right]^2.
\]

• So, \( \mathcal{B}_m \) is a spacelike surface.
Proof. (Cont.)
The Gauss map of the surface is

\[ e = \frac{1}{r^2 - 1} \begin{pmatrix} 2r \cos(\theta) \\ 2r \sin(\theta) \\ r^2 + 1 \end{pmatrix}. \]
Proof. (Cont.)

- The coefficients of the second fundamental form of the Bour’s surface are

\[
\begin{align*}
L & = 2r^{m-2} \cos (m\theta), \\
M & = -2r^{m-1} \sin (m\theta), \\
N & = -2r^m \cos (m\theta).
\end{align*}
\]
Definite case

Proof. (Cont.)

- The coefficients of the second fundamental form of the Bour’s surface are

\[ L = 2r^{m-2} \cos (m\theta) , \]
\[ M = -2r^{m-1} \sin (m\theta) , \]
\[ N = -2r^m \cos (m\theta) . \]

- Then, we have

\[ \det II = -4r^{2m-2} . \]
Proof. (Cont.)

In spacelike case, the Gaussian curvature is defined by

\[ K = \epsilon \frac{\det II}{|\det I|}, \]

where \( \epsilon := \langle e, e \rangle_1 = -1 \) in \( \mathbb{L}^3 \).
Definite case

Proof. (Cont.)

- In spacelike case, the Gaussian curvature is defined by

\[ K = \epsilon \frac{\det II}{|\det I|}, \]

where \( \epsilon := \langle e, e \rangle_1 = -1 \) in \( \mathbb{I}^3 \).

- Hence, the Gaussian curvature and the mean curvature of the Bour’s surface of value \( m \), respectively, are

\[ K = \left( \frac{2r^2 - m}{(1 - r^2)^2} \right)^2, \quad H = 0. \]
Definite case

Proof. (Cont.)

- In spacelike case, the Gaussian curvature is defined by

\[ K = \epsilon \frac{\det II}{|\det I|}, \]

where \( \epsilon := \langle e, e \rangle_1 = -1 \) in \( \mathbb{L}^3 \).

- Hence, the Gaussian curvature and the mean curvature of the Bour’s surface of value \( m \), respectively, are

\[ K = \left( \frac{2r^2 - m}{(1 - r^2)^2} \right)^2, \quad H = 0. \]

- So, the \( \mathcal{B}_m \) is a maximal surface in \( \mathbb{L}^3 \).
Definite case

Example

If take $m = 3$ in $\mathcal{B}_m (r, \theta)$, we have Bour’s maximal surface (see Fig. 12)

$$
\mathcal{B}_3 (r, \theta) = \begin{pmatrix}
\frac{r^2}{2} \cos (2\theta) + \frac{r^4}{4} \cos (4\theta) \\
-\frac{r^2}{2} \sin (2\theta) + \frac{r^4}{4} \sin (4\theta) \\
\frac{2}{3} r^3 \cos (3\theta)
\end{pmatrix}
$$

(9)

in Minkowski 3-space, where $r \in [-1, 1]$, $\theta \in [0, \pi]$. 
Figure 12  Bour’s maximal surface \( \mathcal{B}_3 (r, \theta) \), \((\mathcal{F}, \mathcal{G}) = (z, z)\)
Definite case

The coefficients of the first fundamental form of the Bour’s maximal surface of value 3 are

\[ E = r^2 (1 - r^2)^2, \quad F = 0, \quad G = r^4 (1 - r^2)^2. \]

So,

\[ \det I = r^6 (1 - r^2)^4. \]
The Gauss map of the surface is

\[ e = \frac{1}{r^2 - 1} \left( 2r \cos(\theta), 2r \sin(\theta), 1 + r^2 \right). \]
The coefficients of the second fundamental form of the surface are

\[ L = 2r \cos (3\theta), \quad M = -2r^2 \sin (3\theta), \quad N = -2r^3 \cos (3\theta). \]

Then,

\[ \det II = -4r^4. \]
The mean and the Gaussian curvatures of the Bour’s maximal surface of value 3 are, respectively,

\[ H = 0, \quad K = \frac{4}{r^2 (1 - r^2)^4}. \]
The Weierstrass patch determined by the functions

\[(\mathcal{F}, \mathcal{G}) = (z, z)\]

is a representation of the Bour’s maximal surface of value 3 in \(\mathbb{L}^3\).
Definite case

The parametric form of the surface (see Fig. 13) is

\[
\mathcal{B}_3(u, v) = \left(\begin{array}{c}
\frac{u^4}{4} + \frac{v^4}{4} - \frac{3}{2} u^2 v^2 + \frac{u^2}{2} - \frac{v^2}{2} \\
u^3 v - uv^3 - uv \\
\frac{2}{3} u^3 - 2uv^2
\end{array}\right), \tag{10}
\]

where \( u, v \in \mathbb{R} \).
Definite case

Figure 13  Maximal surface $\mathcal{B}_3 (u, \nu)$, $u, \nu \in [-1, 1]$
Applications of the definite case

Example

If take \( m = 2 \), we have **Enneper’s maximal surface** (see Fig. 14)

\[
\mathcal{B}_2(r, \theta) = \begin{pmatrix}
    r \cos(\theta) + \frac{r^3}{3} \cos(3\theta) \\
    -r \sin(\theta) + \frac{r^3}{3} \sin(3\theta) \\
    r^2 \cos(2\theta)
\end{pmatrix}
\]

in \( \mathbb{L}^3 \), where \( r \in [-1, 1], \theta \in [0, \pi] \).
Applications of the definite case

Figure 14  Maximal surface $\mathcal{B}_2$, $(\mathcal{F}, \mathcal{G}) = (1, z)$
Applications of the definite case

Example

If take $m = 2$, we have $\mathcal{B}_2 (r, \theta)$ (see Fig. 15) in $\mathbb{L}^3$, where $r \in [-3, 3]$, $\theta \in [0, \pi]$. 
Applications of the definite case

Figure 15  Maximal surface $\mathcal{B}_2$, $(\mathcal{F}, \mathcal{G}) = (1, z)$
Applications of the definite case

Example

If take $m = \frac{1}{2}$, we have (see Fig. 16)

\[
\begin{pmatrix}
-2r^{-1/2} \cos \left( \frac{\theta}{2} \right) + \frac{2}{3} r^{3/2} \cos \left( \frac{3\theta}{2} \right) \\
-2r^{-1/2} \sin \left( \frac{\theta}{2} \right) + \frac{2}{3} r^{3/2} \sin \left( \frac{3\theta}{2} \right) \\
4r^{1/2} \cos \left( \frac{\theta}{2} \right)
\end{pmatrix}
\]

in $\mathbb{L}^3$. 
Applications of the definite case

Figure 16  Maximal surface $\mathcal{B}_{1/2}$, $(\mathcal{F}, \mathcal{G}) = (z^{-3/2}, z)$
Applications of the definite case

Example

If \( m = \frac{3}{2} \), we have (see Fig. 17)

\[
\begin{pmatrix}
2r^{-1/2} \cos \left( \frac{\theta}{2} \right) + \frac{2}{5} r^{5/2} \cos \left( \frac{5\theta}{2} \right) \\
-2r^{-1/2} \sin \left( \frac{\theta}{2} \right) + \frac{2}{5} r^{5/2} \sin \left( \frac{5\theta}{2} \right) \\
\frac{4}{3} r^{3/2} \cos \left( \frac{3\theta}{2} \right)
\end{pmatrix}
\]

in \( \mathbb{L}^3 \).
Applications of the definite case

Figure 17  Maximal surface $\mathcal{B}_{3/2}$, $(\mathcal{F}, \mathcal{G}) = (z^{-1/2}, z)$
Applications of the definite case

Example

If \( m = \frac{3}{2} \), we have \( \mathcal{B}_{3/2}(r, \theta) \) (see Fig. 18) in \( \mathbb{L}^3 \).
Applications of the definite case

Figure 18  Maximal surface $\mathcal{B}_{3/2}$, $(\mathcal{F}, \mathcal{G}) = (z^{-1/2}, z)$
Applications of the definite case

Example

If \( m = \frac{2}{3} \), we have (see Fig. 19)

\[
\begin{pmatrix}
-3r^{-1/3} \cos \left( \frac{\theta}{3} \right) + \frac{3}{5} r^{5/3} \cos \left( \frac{5\theta}{3} \right) \\
-3r^{-1/3} \sin \left( \frac{\theta}{3} \right) + \frac{3}{5} r^{5/3} \sin \left( \frac{5\theta}{3} \right) \\
3r^{2/3} \cos \left( \frac{2\theta}{3} \right)
\end{pmatrix}
\]

in \( \mathbb{L}^3 \).
Applications of the definite case

(a)

Figure 19  Maximal surface $\mathfrak{H}_{2/3}$, $(\mathcal{F}, \mathcal{G}) = (z^{-4/3}, z)$

(b)
Applications of the definite case

Example

If \( m = \frac{4}{3} \), then we have (see Fig. 20)

\[
\begin{pmatrix}
3r^{1/3} \cos \left( \frac{\theta}{3} \right) + \frac{3}{7} r^{7/3} \cos \left( \frac{7\theta}{3} \right) \\
-3r^{1/3} \sin \left( \frac{\theta}{3} \right) + \frac{3}{7} r^{7/3} \sin \left( \frac{7\theta}{3} \right) \\
\frac{3}{2} r^{4/3} \cos \left( \frac{4\theta}{3} \right)
\end{pmatrix},
\]

in \( \mathbb{L}^3 \).
Applications of the definite case

Figure 20 Maximal surface $\mathcal{B}_{4/3}$, $(\mathcal{F}, \mathcal{G}) = (z^{-2/3}, z)$
Applications of the definite case

Example

If \( m = \frac{5}{2} \), then we have (see Fig. 21)

\[
\begin{pmatrix}
\frac{2}{3} r^{3/2} \cos \left( \frac{3\theta}{2} \right) + \frac{2}{7} r^{7/2} \cos \left( \frac{7\theta}{2} \right) \\
-\frac{2}{3} r^{3/2} \sin \left( \frac{3\theta}{2} \right) + \frac{2}{7} r^{7/2} \sin \left( \frac{7\theta}{2} \right) \\
\frac{4}{5} r^{5/2} \cos \left( \frac{5\theta}{2} \right)
\end{pmatrix},
\]

in \( \mathbb{L}^3 \).
Applications of the definite case

Figure 21  Maximal surface $\mathcal{B}_{5/2}, (\mathcal{F}, \mathcal{G}) = (z^{1/2}, z)$
Applications of the definite case

Example

If $m = 4$, then we have (see Fig. 22)

$$
\begin{pmatrix}
\frac{1}{3} r^3 \cos (3\theta) + \frac{1}{5} r^5 \cos (5\theta) \\
-\frac{1}{3} r^3 \sin (3\theta) + \frac{1}{5} r^5 \sin (5\theta) \\
\frac{1}{2} r^4 \cos (4\theta)
\end{pmatrix}
$$

in $\mathbb{L}^3$. 
Applications of the definite case

(a) Figure 22 Maximal surface $\mathcal{B}_4$, $(\mathcal{F}, \mathcal{G}) = (z^2, z)$
Let $\mathbb{L}^2 = (\mathbb{R}^2, -dx^2 + dy^2)$ be Minkowski plane, and $\mathbb{L}^3$ be a 3-dimensional Minkowski space with natural Lorentzian metric 

$$\langle ., . \rangle_1 = -dx^2 + dy^2 + dz^2.$$
Theorem

(Weierstrass representation for timelike minimal surfaces in $\mathbb{L}^3$)

Let $\mathbf{x} : \mathbf{M} \to \mathbb{L}^3$ be a timelike surface parametrized by null coordinates $(u, v)$, where $u := -x + y$, $v := x + y$. Timelike minimal surface is represented by

$$
\mathbf{x}(u, v) = \int^u \left( \begin{array}{c} -f (1 + g^2) \\ f (1 - g^2) \\ 2fg \end{array} \right) du + \int^v \left( \begin{array}{c} f (1 + g^2) \\ f (1 - g^2) \\ 2fg \end{array} \right) dv.
$$

(11)
The functions \( f(u), g(u), f(v) \) and \( g(v) \) are defined by

\[
\begin{align*}
f &= \frac{-\phi_1 + \phi_2}{2}, \quad g = \frac{\phi_3}{-\phi_1 + \phi_2}, \\
f &= \frac{\mu_1 + \mu_2}{2}, \quad g = \frac{\mu_3}{\mu_1 + \mu_2},
\end{align*}
\]

and \( \phi = (\phi_1, \phi_2, \phi_3), \mu = (\mu_1, \mu_2, \mu_3) \) vector valued functions, \( \phi(u) := \mathbf{x}_u, \mu(v) := \mathbf{x}_v \) satisfy

\[
(\phi)^2 = 0, \quad (\mu)^2 = 0.
\]
Hence, the timelike minimal surface has the form

\[ x(u, v) = \int^u \phi(u) \, du + \int^v \mu(v) \, dv = \Omega(u) + \Psi(v), \]

and its conjugate

\[ x^*(u, v) = \Omega(u) - \Psi(v), \]

where \( \phi(u) \) and \( \mu(v) \) are linearly independent, \( \Omega(u) \) and \( \Psi(v) \) are null curves in \( \mathbb{L}^3 \).
Weierstrass formula for the timelike minimal surfaces obtained by M. Magid [14] in 1991 (see [12], for details).
Indefinite case

Lemma

The Weierstrass patch determined by the functions

\[(f(u), g(u)) = (u^{m-2}, u) \quad \text{and} \quad (f(v), g(v)) = (v^{m-2}, v)\]

is a representation of the Bour’s timelike minimal surface of value \(m\) in \(\mathbb{L}^3\), where \(m \in \mathbb{R}\).
Bour’s timelike minimal surface of value $m$ is
Indefinite case

- Bour’s timelike minimal surface of value $m$ is

$$
\int^u \begin{pmatrix} -u^{m-2} (1 + u^2) \\ u^{m-2} (1 - u^2) \\ 2u^{m-1} \end{pmatrix} \, du + \int^v \begin{pmatrix} v^{m-2} (1 + v^2) \\ v^{m-2} (1 - v^2) \\ 2v^{m-1} \end{pmatrix} \, dv,
$$
Indefinite case

- Bour’s timelike minimal surface of value \( m \) is

\[
\int_{u}^{u} \begin{pmatrix} -u^{m-2}(1+u^2) \\ u^{m-2}(1-u^2) \\ 2u^{m-1} \end{pmatrix} \, du + \int_{v}^{v} \begin{pmatrix} v^{m-2}(1+v^2) \\ v^{m-2}(1-v^2) \\ 2v^{m-1} \end{pmatrix} \, dv,
\]

- and it has the form

\[
\mathcal{B}_m(u, v) = \begin{pmatrix} -\frac{1}{m-1}(u^{m-1} - v^{m-1}) - \frac{1}{m+1}(u^{m+1} - v^{m+1}) \\ \frac{1}{m-1}(u^{m-1} + v^{m-1}) - \frac{1}{m+1}(u^{m+1} + v^{m+1}) \\ 2\frac{1}{m}(u^m + v^m) \end{pmatrix}.
\]

(12)
Indefinite case

Therefore, \( \mathfrak{B}_m (r, \theta) \) is

\[
\begin{align*}
x &= - \frac{r^{m-1}}{m-1} \left( \cos^{(m-1)} (\theta) - \sin^{(m-1)} (\theta) \right) \\
&\quad - \frac{r^{m+1}}{m+1} \left( \cos^{(m+1)} (\theta) - \sin^{(m+1)} (\theta) \right), \\
y &= \frac{r^{m-1}}{m-1} \left( \cos^{(m-1)} (\theta) + \sin^{(m-1)} (\theta) \right) \\
&\quad - \frac{r^{m+1}}{m+1} \left( \cos^{(m+1)} (\theta) + \sin^{(m+1)} (\theta) \right), \\
z &= 2 \frac{r^m}{m} \left( \cos^m (\theta) + \sin^m (\theta) \right).
\end{align*}
\]
Indefinite case

**Theorem**

Bour’s surface $\mathcal{B}_m (r, \theta)$ is a timelike minimal surface in $\mathbb{L}^3$, where $m \in \mathbb{R} - \{-1, 0, 1\}$. 
Proof.

- The coefficients of the first fundamental form of the $\mathcal{B}_m$ are
Indefinite case

Proof.

- The coefficients of the first fundamental form of the $\mathcal{B}_m$ are

\[
E = 4r^{2m-4}(\sin \theta \cos \theta)^{m-1}(1 + r^2 \sin \theta \cos \theta)^2, \\
F = 2r^{2m-3}(\sin \theta \cos \theta)^{m-2}(1 + r^2 \sin \theta \cos \theta)^2 \cos (2\theta), \\
G = -4r^{2m-2}(\sin \theta \cos \theta)^{m-1}(1 + r^2 \sin \theta \cos \theta)^2.
\]
Indefinite case

Proof. (Cont.)

Then we have

\[
\det I_m = h^2 r^2 m^3 \left( \sin \theta \cos \theta \right)^m + r^2 \sin \theta \cos \theta \cdot i^2.
\]

So, \( B_m \) is a timelike surface.
Indefinite case

Proof. (Cont.)

Then we have

\[
\det l = - \left[ 2r^{2m-3} (\sin \theta \cos \theta)^{m-2} \left( 1 + r^2 \sin \theta \cos \theta \right)^2 \right]^2.
\]
Proof. (Cont.)

- Then we have

\[
det I = - \left[ 2r^{2m-3} (\sin \theta \cos \theta)^{m-2} \left(1 + r^2 \sin \theta \cos \theta\right)^2 \right]^2.
\]

- So, $B_m$ is a timelike surface.
Indefinite case

Proof. (Cont.)

- The Gauss map is
Proof. (Cont.)

The Gauss map is

\[
e = \frac{1}{1 + r^2 \sin \theta \cos \theta} \begin{pmatrix} -r (\sin \theta - \cos \theta) \\ r (\sin \theta + \cos \theta) \\ r^2 \sin \theta \cos \theta - 1 \end{pmatrix}.
\]
Indefinite case

Proof. (Cont.)

- The coefficients of the second fundamental form of the surface are

\[
L = 2r^2 m^2 \left( \sin m(\theta) + \cos m(\theta) \right),
\]

\[
M = 2r^2 m \left( \sin(\theta) \cos \theta \right),
\]

\[
N = 2r^2 m \left( \sin 2(\theta) \cos 2(\theta) + \cos 2(\theta) \sin 2(\theta) \right).
\]
Indefinite case

Proof. (Cont.)

- The coefficients of the second fundamental form of the surface are

\[
\begin{align*}
L &= -2r^{m-2}(\sin^m(\theta) + \cos^m(\theta)), \\
M &= 2r^{m-1}(\sin(\theta)\cos^{m-1}(\theta) - \cos(\theta)\sin^{m-1}(\theta)), \\
N &= -2r^m(\sin^2(\theta)\cos^{m-2}(\theta) + \cos^2(\theta)\sin^{m-2}(\theta)).
\end{align*}
\]
Proof. (Cont.)

- The coefficients of the second fundamental form of the surface are
  
  \[ L = -2r^{m-2}(\sin^m(\theta) + \cos^m(\theta)), \]
  \[ M = 2r^{m-1}(\sin(\theta)\cos^{m-1}(\theta) - \cos(\theta)\sin^{m-1}(\theta)), \]
  \[ N = -2r^m(\sin^2(\theta)\cos^{m-2}(\theta) + \cos^2(\theta)\sin^{m-2}(\theta)). \]

- We have
  \[ \det II = -4r^{2m-2}(\sin \theta \cos \theta)^{m-2}. \]
Indefinite case

Proof. (Cont.)

- Hence, the Gaussian curvature and the mean curvature, respectively, are

\[
K = (\sin \theta \cos \theta)^{2-m} \left( \frac{r^{2-m}}{(1 + r^2 \sin \theta \cos \theta)^2} \right)^2,
\]
Indefinite case

Proof. (Cont.)

- Hence, the Gaussian curvature and the mean curvature, respectively, are

\[ K = (\sin \theta \cos \theta)^{2-m} \left( \frac{r^{2-m}}{(1 + r^2 \sin \theta \cos \theta)^2} \right)^2, \]

- and

\[ H = 0. \]
Proof. (Cont.)

- Hence, the Gaussian curvature and the mean curvature, respectively, are

\[ K = (\sin \theta \cos \theta)^{2-m} \left( \frac{r^{2-m}}{(1 + r^2 \sin \theta \cos \theta)^2} \right)^2, \]

- and

\[ H = 0. \]

- So, the \( \mathcal{B}_m \) is a timelike minimal surface in \( \mathbb{L}^3 \).
Indefinite case

Example

If take $m = 3$ in $\mathcal{B}_m(r, \theta)$, we have Bour’s timelike minimal surface (see Fig. 23)

$$
\mathcal{B}_3(r, \theta) = \begin{pmatrix}
-\frac{r^2}{2} - \frac{r^4}{4} \\
\frac{r^2}{2} - \frac{r^4}{4} \left( \cos^4(\theta) + \sin^4(\theta) \right) \\
2\frac{r^3}{3} \left( \cos^3(\theta) + \sin^3(\theta) \right)
\end{pmatrix}
$$

in Minkowski 3-space, where $r \in [-1, 1], \theta \in [0, \pi]$. 
Indefinite case

\begin{align*}
(a) & \\
(b) & \\
\text{Figure 23} & \text{Bour’s timelike minimal surface } \mathcal{B}_3 (r, \theta) \\
\end{align*}
The coefficients of the first fundamental form of the Bour’s timelike minimal surface of value 3 are

\begin{align*}
E &= 4r^2(\sin \theta \cos \theta)^2 + r^2 \sin \theta \cos \theta^2, \\
F &= r^3 \sin 2\theta \left(1 + r^2 \sin \theta \cos \theta^2 \cos (2\theta)\right), \\
G &= 4r^4(\sin \theta \cos \theta)^2 + r^2 \sin \theta \cos \theta^2^2.
\end{align*}

Then \( \det I = 4r^6(\sin \theta \cos \theta)^2 + r^2 \sin \theta \cos \theta^4. \)
Indefinite case

The coefficients of the first fundamental form of the Bour’s timelike minimal surface of value 3 are

\[
E = 4r^2 (\sin \theta \cos \theta)^2 \left(1 + r^2 \sin \theta \cos \theta\right)^2, \\
F = r^3 \sin 2\theta \left(1 + r^2 \sin \theta \cos \theta\right)^2 \cos (2\theta), \\
G = -4r^4 (\sin \theta \cos \theta)^2 \left(1 + r^2 \sin \theta \cos \theta\right)^2.
\]
Indefinite case

- The coefficients of the first fundamental form of the Bour’s timelike minimal surface of value 3 are

\[
E = 4r^2 (\sin \theta \cos \theta)^2 \left(1 + r^2 \sin \theta \cos \theta\right)^2,
\]
\[
F = r^3 \sin 2\theta \left(1 + r^2 \sin \theta \cos \theta\right)^2 \cos (2\theta),
\]
\[
G = -4r^4 (\sin \theta \cos \theta)^2 \left(1 + r^2 \sin \theta \cos \theta\right)^2.
\]

- Then

\[
\det l = -4r^6 (\sin \theta \cos \theta)^2 \left(1 + r^2 \sin \theta \cos \theta\right)^4.
\]
Indefinite case

- The coefficients of the second fundamental form of the surface are

\[
L = 2r(\sin^3(\theta) + \cos^3(\theta)),
\]

\[
M = 2r^2(\sin(\theta) \cos^2(\theta) \cos(\theta) \sin^2(\theta)),
\]

\[
N = 2r^3(\sin^2(\theta) \cos(\theta) + \cos^2(\theta) \sin(\theta)).
\]

So,

\[
\det II = 4r^4 \sin \theta \cos \theta. 
\]
Indefinite case

- The coefficients of the second fundamental form of the surface are

\[
\begin{align*}
L &= -2r(\sin^3(\theta) + \cos^3(\theta)), \\
M &= 2r^2(\sin(\theta)\cos^2(\theta) - \cos(\theta)\sin^2(\theta)), \\
N &= -2r^3(\sin^2(\theta)\cos(\theta) + \cos^2(\theta)\sin(\theta)).
\end{align*}
\]
The coefficients of the second fundamental form of the surface are

\[
L = -2r(\sin^3(\theta) + \cos^3(\theta)),
\]
\[
M = 2r^2(\sin(\theta) \cos^2(\theta) - \cos(\theta) \sin^2(\theta)),
\]
\[
N = -2r^3(\sin^2(\theta) \cos(\theta) + \cos^2(\theta) \sin(\theta)).
\]

So,

\[
\det II = -4r^4 \sin \theta \cos \theta.
\]
The mean and the Gaussian curvatures of the Bour’s minimal surface of value 3 are, respectively,

\[ H = 0, \quad K = \frac{1}{r^2 \sin \theta \cos \theta \left(1 + r^2 \sin \theta \cos \theta\right)^4}. \]
Indefinite case

- The Weierstrass patch determined by the functions...
The Weierstrass patch determined by the functions

\[(f, g) = (u, u) \quad \text{and} \quad (f, g) = (v, v)\]

in $\mathbb{L}^3$. 
The parametric form of the surface (see Fig. 24) is

\[ \mathcal{B}_3(u, v) = \left( \begin{array}{c} -\frac{1}{2} (u^2 - v^2) - \frac{1}{4} (u^4 - v^4) \\ \frac{1}{2} (u^2 + v^2) - \frac{1}{4} (u^4 + v^4) \\ \frac{2}{3} (u^3 + v^3) \end{array} \right), \tag{14} \]

where \( u, v \in I \subset \mathbb{R} \).
Indefinite case

\[(a) \quad (b)\]

Figure 24  Timelike minimal surface $\mathcal{B}_3(u, v)$, $u, v \in [-1, 1]$
Applications of the indefinite case

Example

If take $m = 2$, we have $\mathcal{B}_2 (r, \theta)$ (see Fig. 25)

$$\begin{pmatrix}
-r (\cos (\theta) - \sin (\theta)) - \frac{r^3}{3} (\cos^3 (\theta) - \sin^3 (\theta)) \\
r (\cos (\theta) + \sin (\theta)) - \frac{r^3}{3} (\cos^3 (\theta) + \sin^3 (\theta)) \\
r^2 (\cos^3 (\theta) + \sin^3 (\theta))
\end{pmatrix}$$

in $\mathbb{L}^3$, where $r \in [-2, 2]$, $\theta \in [-\pi/2, \pi/2]$. 
Applications of the indefinite case

Figure 25  Bour’s timelike minimal surface $\mathcal{B}_2 (r, \theta)$
Applications of the indefinite case

Example

If take \( m = 2 \), we have \( \mathcal{B}_2(r, \theta) \) (see Fig. 26) in \( \mathbb{L}^3 \).
Applications of the indefinite case

Figure 26  Bour’s timelike minimal surface $\mathcal{B}_2(r, \theta)$
Applications of the indefinite case

Example

If take $m = 4$, we have $\mathcal{B}_4 (r, \theta)$ (see Fig. 27)

\[
\begin{pmatrix}
-\frac{r^3}{3} (\cos^3 (\theta) - \sin^3 (\theta)) - \frac{r^5}{5} (\cos^5 (\theta) - \sin^5 (\theta)) \\
\frac{r^3}{3} (\cos^3 (\theta) + \sin^3 (\theta)) - \frac{r^5}{5} (\cos^5 (\theta) + \sin^5 (\theta)) \\
\frac{r^4}{4} (\cos^4 (\theta) + \sin^4 (\theta))
\end{pmatrix}
\]

in $\mathbb{L}^3$. 
Applications of the indefinite case

Figure 27  Bour’s timelike minimal surface $\mathcal{B}_4 (r, \theta)$
Applications of the indefinite case

Example

If take $m = 4$, we have $\mathcal{B}_2 (r, \theta)$ (see Fig. 28) in $\mathbb{L}^3$. 
Applications of the indefinite case

Figure 28  Bour’s timelike minimal surface $\mathcal{B}_4 (r, \theta)$
Applications of the indefinite case

Example

If take $m = 5$, we have $\mathcal{B}_5 (r, \theta)$ (see Fig. 29)

\[
\begin{pmatrix}
- \frac{r^4}{4} (\cos^4 (\theta) - \sin^4 (\theta)) - \frac{r^6}{6} (\cos^6 (\theta) - \sin^6 (\theta)) \\
\frac{r^4}{4} (\cos^4 (\theta) + \sin^4 (\theta)) - \frac{r^6}{6} (\cos^6 (\theta) + \sin^6 (\theta)) \\
\frac{r^5}{5} (\cos^5 (\theta) + \sin^5 (\theta))
\end{pmatrix}
\]

in $\mathbb{L}^3$. 
Applications of the indefinite case

Figure 29  Bour’s timelike minimal surface $\mathcal{B}_5 (r, \theta)$
References


References


References


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Jacques Edmond Emile BOUR
(1832 – 1866)
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