New Calabi-Bernstein results for some elliptic non-linear equations

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Joint work with A. Romero (University of Granada)
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**Our goal**

Obtaining uniqueness and non-existence results of entire solutions to:

\[
\text{div} \left( \frac{Du}{f(u) \sqrt{f(u)^2 - |Du|^2}} \right) = -2H - \frac{f'(u)}{\sqrt{f(u)^2 - |Du|^2}} \left( 2 + \frac{|Du|^2}{f(u)^2} \right) \tag{E1}
\]

\[|Du| < \lambda f(u) \tag{E2}\]

where \(H\) is a fixed real number, \(f\) a smooth real-valued function, \(u\) is defined on a non-compact complete Riemannian surface \((F, g)\), \(D\) and \(\text{div}\) denote the gradient and the divergence of \((F, g)\), \(|Du|^2 := g(Du, Du)\), and \(0 < \lambda \leq 1\).

In two particular cases:

1. \(H = 0\) and \(\lambda = 1\) **The maximal surface equation.**
   Quasi-linear elliptic eq.

2. \(H \neq 0\) and \(0 < \lambda < 1\) **The CMC spacelike surface equation.**
   Quasi-linear uniform elliptic eq.
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   Quasi-linear elliptic eq.

2. $H \neq 0$ and $0 < \lambda < 1$ The CMC spacelike surface equation. 
   Quasi-linear uniform elliptic eq.
Brief Introduction
Generalized Robertson-Walker spacetimes (GRW)

We consider $M := I \times F$ with the Lorentzian metric

$$\langle \cdot, \cdot \rangle = -dt^2 + f^2 g,$$

where $I$ is an interval, $(F, g)$ a Riemannian surface and $f > 0$ a smooth function on $I$.

Given a function $u : F \rightarrow \mathbb{R}$:

Its graph $\Sigma_u = \{(p, u(p)) : p \in F\}$ is spacelike if and only if $|Du| < f(u)$,

$D$ being the gradient of $F$. 

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WHERE DOES IT COME FROM?

MAXIMAL CASE

\( u \) is a solution of the equation ...

... iff is an extremal under compact support variations for the area of \( \Sigma_u \)

... iff \( \Sigma_u \) is maximal (zero mean curvature).

CMC CASE

\( u \) is a solution of the equation ...

... iff is an extremal of the area under compact support variations with constant volume\(^a\) with respect to a fixed \( \Sigma_{u=t_0} \).

... with condition (E2) iff \( \Sigma_u \) is a CMC spacelike surface with bounded hyperbolic angle.

\(^a\)Other volume type constraints can be considered, such as balance of volume zero.
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The condition (E2) with $0 < \lambda < 1$

Hyperbolic angle in a GRW

Where $N$ is the unitary normal chosen such that $\langle N, \partial_t \rangle \geq 1$ on $\Sigma_u$, the hyperbolic angle $\theta$ is such that $\langle N, \partial_t \rangle = \cosh \theta = 1 / \sqrt{1 - \frac{|Du|^2}{f(u)^2}}$. 
A little bit of history

In 1968 Calabi proposed to study the maximal surface equation in $\mathbb{L}^{n+1}$. He proved that the only entire solutions are the linear functions for $2 \leq n \leq 4$.

Chen and Yau extended it to $\mathbb{L}^{n+1}$ in 1976 $\sim$ Calabi-Bernstein’s theorem in $\mathbb{L}^{n+1}$.


There exists entire CMC graphs in $\mathbb{L}^{n+1}$ different from hyperplanes: $\mathbb{H}^n$.

Under which conditions can we get uniqueness?

The only entire bounded solutions are the constant functions, J. A. Aledo and L. J. Alías (2000).
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The only entire solutions whose graph has bounded hyperbolic angle are the linear functions, Aiyama (1992) and Xin (1991), independently.

The only entire solutions whose graph has a bounded from one side Gauss map, are the linear functions, H-D. Cao, Y. Shen, S. Zhu (1998).

Many authors have obtained Calabi-Berstein type results in other ambient settings: A.L. Albuje, L.J. Alías, F. Camargo, A. Caminha, A. G. Colares, G. Li, H. F. de Lima, S. Montiel, A. Romero, R. M. Rubio, M. Sánchez, I. M. C. Salavessa, ...

We will focus on finding uniqueness and non-existence results of maximal and CMC spacelike graphs in certain 3-dimensional GRW spacetimes.
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Preliminaries
• **Lemma** (A. Romero y R. M. Rubio):

Let $S$ be a Riemannian manifold and $v \in C^2(S)$ such that

$$v \Delta v \geq 0.$$  

If $B_R$ is a geodesic ball in $R$ around $p \in S$, then for each $r > 0$ such that $r < R$ we have

$$\int_{B_r} |\nabla v|^2 \, dV \leq \frac{4 \sup_{B_R} v^2}{\mu_{r,R}},$$

$B_r$ being a geodesic ball of radius $r$ around $p$ in $S$ and

$$\frac{1}{\mu_{r,R}} := \int_{A_{r,R}} |\nabla \omega_{r,R}|^2 \, dV$$

is the capacity of the ring $A_{r,R} = B_R \setminus \bar{B}_r$, $\omega_{r,R}$ being the harmonic measure of the boundary $\partial B_R$ respect to $A_{r,R}$.

• **Theorem** (Ahlfors and Blanc-Fiala-Huber): any complete Riemannian surface with non-negative Gaussian curvature is parabolic.
Our main tools

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  $\omega_{r,R}$ being the harmonic measure of the boundary $\partial B_R$ respect to $A_{r,R}$.

- **Theorem** (Ahlfors and Blanc-Fiala-Huber): any complete Riemmanian surface with non-negative Gaussian curvature is parabolic.
**Our main tools**

- **Null Convergence Condition (NCC):** \( \bar{\text{Ric}}(Z, Z) \geq 0 \) for all \( Z \) null.

\[
NCC \text{ in a 3-dim. GRW: } \frac{K^F(\pi_F)}{f^2} - \left(\log f\right)'' \geq 0
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- **Lemma (—, A. Romero y R. M. Rubio):**

Let \( S \) be a spacelike surface in a 3-dim GRW

\[
K = \frac{f''(t)^2}{f(t)^2} + \left\{ \frac{K^F(\pi_F)}{f(t)^2} - (\log f)''(t) \right\} | \partial_t |^2 + \frac{K^F(\pi_F)}{f(t)^2} - 2H^2 + \frac{1}{2} \text{trace}(A^2).
\]

If NCC holds,

\[
K \geq \frac{f''(t)^2}{f(t)^2} + \frac{K^F(\pi_F)}{f(t)^2} - H^2,
\]

and if \( H = \text{cte}, \)

\(" =" \iff S \text{ is totally umbilic}.\)
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Our main tools

- Generalized maximum principle due to Omori-Yau:

Let $S$ be a complete Riemannian manifold whose Ricci curvature is bounded from below and let $u : S \rightarrow \mathbb{R}$ be a smooth function bounded from below (resp. from above).

Then for each $\varepsilon > 0 \exists p_\varepsilon \in S$ such that

- $|\nabla u(p_\varepsilon)| < \varepsilon$
- $\triangle u(p_\varepsilon) > -\varepsilon$ (resp. $< \varepsilon$).
- $\inf u \leq u(p_\varepsilon) < \inf u + \varepsilon$ (resp. $\sup u - \varepsilon < u(p_\varepsilon) \leq \sup u$).
Our results
The maximal case

Theorem (—, A. Romero and R. M. Rubio)

Let \((F, g)\) be a non-compact complete Riemannian surface with \(K^F \geq 0\) and let \(f : I \rightarrow \mathbb{R}\) satisfy \(\inf(f) > 0\), \(\sup(f) < \infty\) and \((\log f)'' \leq 0\).

If there exists \(p \in F\) such that \(K^F(p) > 0\), then

the only entire solutions to the maximal surface equation are the constant functions

\[ u = u_0 / f'(u_0) = 0. \]
The maximal case - proof

\( u : F \rightarrow \mathbb{R} \) smooth with spacelike graph \( \Sigma_u \) in \( I \times F \) and induced metric \( g_u \).

We identify \((\Sigma_u, \langle \cdot, \cdot \rangle) \equiv (F, g_u)\)

We consider \( \xi := f(\pi_1) \partial_t \) \( \rightarrow \) timelike and conformal.

If \( N \) is the normal to \( \Sigma_u \) in the same orientation as \(-\partial_t\), then \( \langle N, \xi \rangle \) will be our key function on \( \Sigma_u \).
Making a conformal change to get parabolicity:

If \( \inf(f) > 0 \) and \( \sup(f) < \infty \), define \( g^* := (\langle N, \xi \rangle + \sup(f))^2 g_\nu \).

\[
(F, g) \text{ complete} \implies (F, g^*) \text{ complete}.
\]

Using \( \Sigma_\nu \) maximal, \( f \leq \sup(f), K^F \geq 0 \) and \( (\log f)^{''} \leq 0 \), from the lemma on \( K \) we get \( K^* \geq 0 \).

Completeness + \( K^* \geq 0 \implies \text{Parabolic} \)
**Making a conformal change to get parabolicity:**

If $\inf(f) > 0$ and $\sup(f) < \infty$, define $g^* := (\langle N, \xi \rangle + \sup(f))^2 g_u$.

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Using $\Sigma_u$ maximal, $f \leq \sup(f)$, $K^F \geq 0$ and $(\log f)'' \leq 0$, from the lemma on $K$ we get $K^* \geq 0$.

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THE MAXIMAL CASE - PROOF

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Completeness + \( K^* \geq 0 \implies \text{Parabolic} \]
Applying the lemma by Romero and Rubio to get \( \text{trace}(A^2) = 0 \):

We replace \( \langle N, \xi \rangle \) by \( h := \arctan(\langle N, \xi \rangle) \) because it is bounded.

We prove \( \triangle^* h \geq 0 \), from the lemma and the boundedness of \( h \)

\[
\int_{D^*_r} |\nabla^* \langle N, \xi \rangle|^2 dA^* \leq \frac{C}{\mu^*_{r, R}}
\]

where \( C = C(p, r) > 0 \) is a constant.

\((F, g^*)\) is parabolic \( \Longrightarrow \lim_{R \to \infty} \frac{1}{\mu^*_{r, R}} = 0 \Longrightarrow \langle N, \xi \rangle = \text{cte} \Longrightarrow \text{trace}(A^2) = 0.\)
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The maximal case - proof

We finish using the lemma on $K$:

If there exists $p \in F$ such that $K^F(p) > 0$, from the lemma on $K$ we get

\[ N(p) = -\partial_t(p). \]

Since $\Sigma_u$ is totally geodesic, $u$ is constant.
The CMC case

We define $\mathcal{B}_f = \{-f'(t)/f(t) : t \in I\} \subset \mathbb{R}$.

Theorem (—, A. Romero and R. M. Rubio)

Let $(F, g)$ be a non-compact complete Riemannian surface with $K^F \geq 0$. Assume $f : I \to \mathbb{R}^+$ satisfies $(\log f)'' \leq 0$ and $\inf(f) > 0$.

1. If $H \notin \mathcal{B}_f \cup \{0\}$, there exists no bounded entire solution to the H-CMC spacelike surface equation.

2. If $H \in \mathcal{B}_f \setminus \{0\}$, then $u \equiv t_0$, where $H = -\frac{f'(t_0)}{f(t_0)}$, is the only entire bounded solution to the H-CMC spacelike surface equation.
The CMC case - proof

\[ u : F \rightarrow \mathbb{R} \text{ smooth with spacelike graph } \Sigma_u \text{ in } I \times F \text{ and induced metric } g_u. \]

We identify \((\Sigma_u, \langle , \rangle) \equiv (F, g_u)\)

Our distinguished function is \(\text{ch}\theta = \langle N, \partial_t \rangle\).

Completeness of the graph:

\[ L_u \geq \sqrt{1 - \lambda^2 \inf(f)^2 L}, \]

being \(L_u\) and \(L\) the lengths of a curve in \((F, g_u)\) and \((F, g)\), resp.

Then, \(\inf(f) > 0\) and \((F, g)\) complete \(\implies (F, g_u)\) complete.
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Then, \( \inf(f) > 0 \) and \((F, g)\) complete \( \implies \) \((F, g_u)\) complete.
We need to prove $H^2 \leq \frac{f'(u)^2}{f(u)^2}$.

From the expression of $\triangle u$, we get

$$H = \frac{-f'(u) \{2 + |\nabla u|^2\} - \Delta u}{2ch\theta}.$$

Since $u$ is bounded, we can apply Omori-Yau twice.

For the supremum:

$$\frac{-f'(\sup u)}{f(\sup u)} \leq H \leq \frac{-f'(\inf u)}{f(\inf u)}.$$

We finish the proof using $(\log f)'' \leq 0 \implies -\frac{f'}{f}$ increasing.
We get parabolicity:

Since $K^F \geq 0$ and $(\log f)'' \leq 0$, $M$ satisfies NCC, and so

$$K \geq \frac{f'(t)^2}{f(t)^2} + \frac{K^F(\pi_F)}{f(t)^2} - H^2 \geq 0.$$ 

The result of Ahlfors and Blanc-Fiala-Huber gives us: $\Sigma_u$ is parabolic.

A distinguished function, $\text{ch}\theta$:

Using the expression of the Hessian of $u$ we obtain $\text{ch}\theta \Delta \text{ch}\theta \geq 0$.

Lemma by Romero and Rubio + $\theta$ bounded + parabolicity $\implies \theta = \text{cte}$.

From the expression of $\text{ch}\theta \Delta \text{ch}\theta$ and $H \neq 0$, we get $\theta = 0 \implies u = \text{cte}$. 
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From the expression of $\text{ch} \theta \Delta \text{ch} \theta$ and $H \neq 0$, we get $\theta = 0 \implies u = \text{cte}$.