ON POISSON MEASURES, GAUSSIAN MEASURES AND THE CENTRAL LIMIT THEOREM IN BANACH SPACES*

by

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INTRODUCTION

In the last few years considerable progress has been made in the central limit theorem for independent identically distributed Banach space valued random vectors. Three aspects of the subject might be singled out: (I) Hoffmann-Jorgensen and Pisier's [22] result characterizing the Banach spaces in which the central limit theorem (i.i.d. case) holds under a second moment assumption as the class of type 2 spaces; (II) the work on the central limit theorem (i.i.d. case) for continuous processes by Dudley and Strassen [14], Giné [17], Dudley [13], Jain and Marcus [25], Araujo [5], Heinkel [20] and others; an elegant argument of Zinn [39] has shown that the results (II) are a corollary of (I) and basic theorems on path continuity of Gaussian processes ([12], [15]); (III) the characterization of Banach spaces in which random vectors with Gaussian covariances satisfy the central limit theorem (i.i.d. case) as the class of spaces of cotype 2 (Jain [24], Aldous [3], Chobanjan and Tarieladze [10]).

However, the general central limit theorem, non-Gaussian case included, has rarely been studied. An important exception is Le Cam [28], which establishes basic results on necessary conditions for the relative shift compactness of the laws of row sums of triangular arrays and on the relation between relative compactness of the laws of the row sums and the associated Poisson exponentials.

In this work we study the general central limit theorem in Banach spaces taking as our point of departure Le Cam's work. We prove a general converse central limit theorem (necessary conditions for the convergence of the laws of the row sums of infinitesimal triangular arrays) and a general direct central limit theorem. We also obtain generalizations of the direct central limit theorems in the special situations (I), (II), (III) depicted above; these results cover non-identically distributed and non-Gaussian cases.

Next we describe the contents of each section.

Section 1 presents a systematic study of Poisson measures. We examine thoroughly their basic properties and set up the background required for an understanding of their role as one of the building blocks in the general central limit theorem. Necessary conditions for relative compactness and convergence of Poisson measures are proved; a first form of the result on relative compactness appears in Araujo [5]. As a corollary, the Lévy-Khinchine representation of infinitely divisible measures on Banach spaces is obtained.

Section 2 deals with the central limit theorem. We start by giving a new and fairly elementary proof of one of the basic results of Le Cam: if $\{X_{nj}\}$ is a triangular array and $\{L(S_n)\}$ is relatively shift compact, then for every $\varepsilon>0$ there exist a compact convex symmetric set K_ε with $\operatorname{diam}(K_\varepsilon)\leq\varepsilon$ and points $\{x_{nj}\}$ such that $\{\Sigma_j L(X_{nj}-x_{nj})|K_\varepsilon^c\}$ is relatively compact. A refinement for infinitesimal triangular arrays follows. Next we obtain some useful integrability and centering results for triangular arrays. The main new result in this section is what appears to be a complete form of the general converse central limit theorem (Theorem 2.10); we deduce from it a general direct central limit theorem. It is of interest to remark that we do not rely on any one-dimensional central limit theorem; our approach rests on some basic inequalities and compactness arguments. A generalization of the classical conditions for Gaussian limits is also obtained. (Araujo and Giné [7] have very recently applied Theorem 2.2 to the study of the tail behavior of measures in the domain of attraction of a stable measure; we point out that it is also possible to apply directly Theorem 2.10).

In Section 3 we present an improvement of the result of Le Cam [28] stating that the relative shift compactness of $\{\mathrm{Pois}(\Sigma_j L(X_{nj}))\}$ implies that of $\{L(S_n)\}$; as a consequence of the results on centering in §2, we are able to specify shifts $\{a_n\}$ such that $\{L(S_n-a_n)\}$ is relatively compact. Some partial converses of Le Cam's result are given. The results of this section are potentially useful in proving direct central limit theorems; in fact, the sufficient conditions for the various forms of the central limit theorem for triangular arrays are often strong enough to imply the relative compactness of $\{\mathrm{Pois}(\Sigma_j L(X_{nj}))\}$. This idea has been exploited by Araujo [5] and is applied in §4 of this paper.

Section 4 contains central limit theorems in C(S) and type 2 spaces. As an immediate consequence of the results in §2 we prove a general direct central limit theorem in type 2 spaces and a corollary for the Gaussian case which contain the theorem of Hoffmann-Jorgensen and Pisier (direct part). Two different sufficient conditions for the relative compactness of a family of Poisson measures in the type 2 framework are given. From the second one we obtain via Zinn's approach a central limit theorem for C(S)-valued random vectors in the Gaussian non-identically distributed case; this theorem generalizes the results (II).

In Section 5 we give several characterizations of a special class of Gaussian measures in a locally convex space: measures which are continuous linear images of Gaussian measures on Hilbert space. We apply these results to the characterization of cotype 2 spaces by the structure of their Gaussian measures. Similar results for the cotype 2 case have previously appeared in Garling [16] and specially in Chobanjan-Tarieladze [10].

Section 6 contains some weak compactness criteria for families of measures in a locally convex space. We obtain sufficient conditions for the weak compactness of a family of Poisson measures and for the weak convergence of the row sums of a triangular array to a Gaussian limit. The direct central limit theorem (i.i.d. case) for cotype 2 spaces ([24], [3], [10]) follows as a corollary. We also show that in certain cotype 2 spaces more specific information can be added to the necessary integrability conditions in the general converse central limit theorem of Section 2.

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NOTATION

Throughout the paper the following notation will be used. B will denote a separable Banach space, B' its dual space, $B_\varepsilon=\{x\in B: \|x\|\le\varepsilon\}$, $B'_\varepsilon=\{f\in B': \|f\|\le\varepsilon\}$ $(\varepsilon>0)$. $\mathcal{B}$ will denote the Borel σ-algebra of B; by "a measure on B" we shall mean a measure defined on $\mathcal{B}$. F is the family of finite-dimensional subspaces of B.

Given a σ-finite measure μ on $(B,\mathcal{B})$, $C(\mu)=\{r>0: \mu\{x: \|x\|=r\}=0\}$; observe that $C(\mu)^c\cap R^+$ is countable. The measure $\bar\mu$ is defined by $\bar\mu(A)=\mu(-A)$ $(A\in\mathcal{B})$; μ is <u>symmetric</u> if $\bar\mu=\mu$. For a finite measure μ, $\hat\mu$ will denote its characteristic function (ch.f.). For $A\in\mathcal{B}$, $\mu|A$ is the measure defined by $(\mu|A)(E)=\mu(A\cap E)$. If $\{\mu_n\}$ converges weakly to μ we write $\mu_n \to_w \mu$.

Given a B-valued random vector (r.v.) X, $L(X)$ will denote its distribution, $X_\delta=XI_{\{\|X\|\le\delta\}}$, $X^\delta=XI_{\{\|X\|>\delta\}}$ $(\delta>0)$. By a triangular array we shall mean a doubly indexed family $\{X_{nj}: j=1,\dots,k_n, n\in N\}$ (N=the set of natural numbers) of B-valued random vectors; unless explicitly stated otherwise, the r.v.'s in each fixed row are assumed to be independent. We will write $S_n=\Sigma_{j=1}^{k_n}X_{nj}$; also $S_{n,\delta}=\Sigma_{j=1}^{k_n}X_{nj\delta}$, $S_n^{(\delta)}=\Sigma_{j=1}^{k_n}X_{nj}^\delta$.

1. POISSON PROBABILITY MEASURES

For τ>0, let $h_\tau(f,x)=e^{if(x)}-1-if(x)I_{B_\tau}(x)$, $f\in B'$, $x\in B$.

<u>1.1</u> <u>Definition</u>. A σ-finite positive measure μ on B is a <u>Lévy measure</u> if for some τ>0,

(1) $\int|h_\tau(f,\cdot)|d\mu<\infty$ for all $f\in B'$,

(2) there exists a probability measure $c_\tau\mathrm{Pois}\mu$ on B such that $(c_\tau\mathrm{Pois}\mu)^\wedge(f)=\exp\int h_\tau(f,\cdot)d\mu$ for every $f\in B'$.

The measure $c_\tau\mathrm{Pois}\mu$ will be called the <u>τ-centered Poisson measure associated to the Lévy measure μ</u>.

<u>Remarks.</u> (1) Every finite measure μ is a Lévy measure. In fact if we define $\mathrm{Pois}\mu=\exp(\mu-\mu(B)\delta_0)$ (where the exponential is defined by

the usual expansion) then $c_\tau \mathrm{Pois}\mu = (\mathrm{Pois}\mu) * \delta_{b_\tau}$, where $b_\tau = -\int_{B_\tau} x\,d\mu(x)$ for every $\tau>0$.

(2) If μ is a Lévy measure, then $\mu(B_\delta^c)<\infty$ for every $\delta>0$ (see Theorem 1.4). It follows that in Definition 1.1 τ may be replaced by any $\tau'>0$, and for a Lévy measure μ $c_\tau \mathrm{Pois}\mu = (c_{\tau'} \mathrm{Pois}\mu) * \delta_b$, with $b = \int_{B_\tau^c \cap B_{\tau'}} x\,d\mu(x)$ $(0<\tau<\tau')$.

(3) If μ is a symmetric Lévy measure, then $(c_\tau \mathrm{Pois}\mu)\hat{}(f) = \exp\int(\cos f - 1)\,d\mu$ $(f \in B')$; in this case we also write $\mathrm{Pois}\mu$ instead of $c_\tau \mathrm{Pois}\mu$.

(4) The classical kernel $K(f,x) = e^{if(x)} - 1 - if(x)/(1+\|x\|^2)$ is not adequate for the definition of Lévy measure in general Banach spaces (for a relevant example, see [5]). It may be argued that the kernels h_τ are "natural" even in R; as we shall see in §2, the centering implicit in h_τ permits simple statements in the general central limit theorem. The discontinuity of $h_\tau(f,\cdot)$ on ∂B_τ may be inconvenient in certain situations; the kernel

$$h(f,x) = e^{if(x)} - 1 - if(\alpha(x)), \text{ where } \alpha(x) = \begin{cases} x & \text{if } x \in B_1 \\ x/\|x\| & \text{if } x \in B_1^c \end{cases}$$

is a continuous substitute which may be used to develop the theory of Lévy and Poisson measures.

We record some elementary properties of Lévy measures. If μ is a Lévy measure, then so is $\bar{\mu}$ and $c_\tau \mathrm{Pois}\bar{\mu} = \overline{c_\tau \mathrm{Pois}\mu}$; if μ and ν are Lévy measures, then so is $\mu+\nu$ and $c_\tau \mathrm{Pois}(\mu+\nu) = (c_\tau \mathrm{Pois}\mu) * (c_\tau \mathrm{Pois}\nu)$.

If μ is a Lévy measure and $T: B \to F$ (a Banach space) is a continuous linear map, then $\mu \circ T^{-1}$ is a Lévy measure; in fact, $c_\tau \mathrm{Pois}(\mu \circ T^{-1})$ is a shift of $(c_\tau \mathrm{Pois}\mu) \circ T^{-1}$. If μ, ν are Lévy measures and $c_\tau \mathrm{Pois}\mu = c_\tau \mathrm{Pois}\nu$, then $\mu|\{0\}^c = \nu|\{0\}^c$ (see Proposition 1.9). Also, if μ_n and μ are finite measures and $\mu_n \to_w \mu$, then $\mathrm{Pois}\mu_n \to_w \mathrm{Pois}\mu$.

It is known that a σ-finite measure μ on Hilbert space satisfying (1) of Definition 1.1 is a Lévy measure if and only if

$\int \min(1, \|x\|^2)\,d\mu(x) < \infty$ (See [33] Ch.6). No such characterization in terms of an integrability property is possible in general Banach spaces; for information on this question, see [4] and [6]. Theorem 1.6 below characterizes Lévy measures on general Banach spaces in terms of approximation by finite measures.

A problem of interest is that of finding conditions on a family $\{\mu_\alpha\}$ of Lévy measures which are necessary and sufficient for the relative compactness of $\{c_\tau \mathrm{Pois}\mu_\alpha\}$. The answer is known in the Hilbert space case (see [33] Ch.6); however, the conditions involve integrability properties of $\{\mu_\alpha\}$ and do not extend to general Banach spaces. In §4 we shall give sufficient conditions for the relative compactness of $\{c_\tau \mathrm{Pois}\mu_\alpha\}$ on spaces of type 2. Theorems 1.4 and 1.10 give necessary conditions for relative compactness and convergence, respectively, which are valid in general Banach spaces.

We start with a basic lemma. Given measures $\mu, \{\mu_n\}$ on B, we will write $\mu_n \uparrow \mu$ if $\mu_n(A) \uparrow \mu(A)$ for each $A \in \mathcal{B}$.

<u>1.2 Lemma</u>. Let μ be a symmetric Lévy measure.

(1) If $\{\mu_n\}$ is a sequence of finite positive symmetric measures such that $\mu_n \uparrow \mu$, then $\mathrm{Pois}\mu_n \to_w \mathrm{Pois}\mu$.

(2) If λ is a positive symmetric measure and $\lambda \le \mu$, then λ is a Lévy measure and $\mathrm{Pois}\lambda$ is a factor of $\mathrm{Pois}\mu$.

<u>Proof.</u> (1) Let $\nu_1 = \mu_1$, $\nu_n = \mu_n - \mu_{n-1}$ for $n \ge 2$. Then $\mu_n = \Sigma_{k=1}^n \nu_k$ and $\mathrm{Pois}\mu_n = \Pi_{k=1}^n \mathrm{Pois}\nu_k$. For each $f \in B'$

$(\mathrm{Pois}\mu_n)\hat{}(f) = \exp\int(\cos f - 1)\,d\mu_n \to \exp\int(\cos f - 1)\,d\mu = (\mathrm{Pois}\mu)\hat{}(f)$. By ([23], Th.4.1), it follows that $\mathrm{Pois}\mu_n \to_w \mathrm{Pois}\mu$.

(2) Obviously λ is σ-finite and $\int|h_\tau(f,\cdot)|\,d\lambda < \infty$ for each $f \in B'$. Let $A_n \uparrow B$, A_n symmetric, $\mu(A_n) < \infty$, and define $\mu_n = \mu|A_n$, $\lambda_n = \lambda|A_n$. Then $\mathrm{Pois}\mu_n = (\mathrm{Pois}\lambda_n) * (\mathrm{Pois}(\mu_n - \lambda_n))$. Since $\mathrm{Pois}\mu_n \to_w \mathrm{Pois}\mu$ by (1), it follows by ([33], III 2.2) and the symmetry of the measures that $\{\mathrm{Pois}\lambda_n\}$ is relatively compact. But $(\mathrm{Pois}\lambda_n)\hat{}(f) \to \exp\int(\cos f - 1)\,d\lambda = \phi(f)$ $(f \in B')$. Therefore $\{\mathrm{Pois}\lambda_n\}$ converges and its limit has ch.f. ϕ. This

shows that λ is a Lévy measure. It is obvious that $\text{Pois}\lambda$ is a factor of $\text{Pois}\mu$. □

The next lemma gives a useful criterion for the relative compactness of a family of probability measures. Let us recall that for any $r>0$, B'_r equipped with the w*-topology is a compact metric space and on B'_r the w*-topology coincides with the topology $\kappa(B',B)$ of uniform convergence on the compact subsets of B. The space of continuous functions on (B'_r, w^*) with the supremum norm will be denoted $C(B'_r)$.

<u>1.3 Lemma.</u> Let $\{\mu_\alpha\}$ be a family of probability measures on B. The following conditions are equivalent:

(1) $\{\mu_\alpha\}$ is relatively compact,

(2) $\{\mu_\alpha\}$ is relatively shift compact and for some (resp. for all) $r>0$ $\{\hat{\mu}_\alpha|B'_r\}$ is w*-equicontinuous at 0.

<u>Proof.</u> Assume that $\{\mu_\alpha\}$ is relatively compact. Given $\varepsilon>0$, let K be a compact set such that $\mu_\alpha(K^c)<\varepsilon/3$ for all α and let $U=\{f\in B'_r: |f(x)|<\varepsilon/3 \text{ for all } x\in K\}$. Then $f\in U$ implies

$$|\hat{\mu}_\alpha(f)-1| \le \int_K |e^{if}-1|\,d\mu_\alpha + \int_{K^c} |e^{if}-1|\,d\mu_\alpha$$
$$< \int_K |f|\,d\mu_\alpha + 2(\varepsilon/3) < \varepsilon$$

and therefore $\{\hat{\mu}_\alpha|B'_r\}$ is $\kappa(B',B)$-equicontinuous at 0, hence w*-equicontinuous at 0.

Suppose now that $\{\hat{\mu}_\alpha|B'_r\}$ is w*-equicontinuous at 0. Then the elementary inequality for ch.f.'s

$$|\hat{\mu}(f)-\hat{\mu}(g)|^2 \le 2(1-\text{Re}\,\hat{\mu}(f-g))$$

implies that $\{\hat{\mu}_\alpha|B'_{r/2}\}$ is w*-uniformly equicontinuous. By the Arzelà-Ascoli theorem, $\{\hat{\mu}_\alpha|B'_{r/2}\}$ is a relatively compact subset of $C(B'_{r/2})$. The proof is completed by invoking the following fact, which is proved in ([33], VI.4.5): if $\{\mu_\alpha\}$ is relatively shift compact and for some $s>0$ $\{\hat{\mu}_\alpha|B'_s\}$ is a relatively compact subset of $C(B'_s)$, then $\{\mu_\alpha\}$ is relatively compact. □

<u>1.4 Theorem.</u> If $\{\mu_\alpha\}$ is a family of Lévy measures on B such that $\{c_\tau \text{Pois}\mu_\alpha\}$ is relatively shift compact, then

(1) the family $\{\mu_\alpha|B^c_\delta\}$ is relatively compact for every $\delta>0$,

(2) for each $\delta>0, r>0$, $\{\psi^{(\delta)}_\alpha|B'_r\}$ is a relatively compact subset of $C(B'_r)$, where $\psi^{(\delta)}_\alpha(f)=\int_{B_\delta} f^2\,d\mu_\alpha$ $(f\in B')$.

<u>Proof.</u> By taking $\{\mu_\alpha+\bar{\mu}_\alpha\}$ if necessary, we may assume that $\{\mu_\alpha\}$ is symmetric. Lemma 1.2 and the first part of the proof of Theorem IV.4.3 of [33] imply that $\mu_\alpha(B^c_\delta)<\infty$ for each α and each $\delta>0$. This in turn makes it possible to apply the same argument to get the relative compactness of $\{\mu_\alpha|B^c_\delta\}$ for each $\delta>0$. (Observe that by Lemma 1.2 (2), $\text{Pois}(\mu|B^c_\delta)$ is a factor of $\text{Pois}\mu$ and hence by ([33], III.2.2) and the symmetry of the measures, $\{\text{Pois}(\mu_\alpha|B^c_\delta)\}$ is relatively compact).

By the relative compactness of $\{\text{Pois}\mu_\alpha\}$, there exists $s>0$ such that $f\in B'_s$ implies that for every α, $\int(1-\cos f)\,d\mu_\alpha\le 1$. Since $1-\cos t\ge ct^2$ for $|t|\le 1$ and some constant c and $1-e^{-t}\ge(1/2)t$ for $0\le t\le 1$, we have for $\|f\|\le \ell=\min(s,1/\delta)$:

$$1-(\text{Pois}\mu_\alpha)^\wedge(f) \ge 1-\exp\left(-\int_{B_\delta}(1-\cos f)\,d\mu_\alpha\right)$$
$$\ge (1/2)c\int_{B_\delta} f^2\,d\mu_\alpha.$$

By Lemma 1.3 it follows that $\{\psi^{(\delta)}_\alpha|B'_\ell\}$ is w*-equicontinuous at 0; using $|\psi(f)-\psi(g)|^2 \le \psi(f+g)\psi(f-g)$ (where $\psi=\psi^{(\delta)}_\alpha$) and arguing as in Lemma 1.3, it follows that $\{\psi^{(\delta)}_\alpha|B'_{\ell/2}\}$ is a relatively compact subset of $C(B'_{\ell/2})$. This implies that (2) is true for any $r>0$. □

<u>1.5 Corollary.</u> If $\{c_\tau \text{Pois}\mu_\alpha\}$ is relatively shift compact, then it is relatively compact.

<u>Proof.</u> By Lemma 1.3, it is enough to show that $\{(c_\tau \text{Pois}\mu_\alpha)^\wedge|B'_1\}$ is $\kappa(B',B)$-equicontinuous at 0. We may write, for any compact set K

$$\left|\int h_\tau(f,\cdot)\,d\mu_\alpha\right| \le \int_{B_\tau}|e^{if}-1-if|\,d\mu_\alpha + \int_{B^c_\tau}|e^{if}-1|\,d\mu_\alpha$$
$$\le \int_{B_\tau} f^2\,d\mu_\alpha + (\sup_{x\in K}|f(x)|)\,\mu_\alpha(B^c_\tau) + 2\mu_\alpha(K^c\cap B^c_\tau).$$

The assertion now follows from Theorem 1.4 and a standard argument.

The following theorem characterizes Lévy measures in terms of approximation by finite measures.

1.6 Theorem. For a σ-finite positive measure μ on B the following conditions are equivalent:

(1) μ is a Lévy measure,

(2) for every sequence of finite positive measures $\mu_n \uparrow \mu$, the sequence $\{c_\tau \mathrm{Pois}\mu_n\}$ is weakly convergent (to $c_\tau \mathrm{Pois}\mu$),

(3) there exists a sequence of finite positive measures $\mu_n \uparrow \mu$ such that $\{c_\tau \mathrm{Pois}\mu_n\}$ is relatively shift compact.

Proof. (1) $\Rightarrow$(2) By Lemma 1.2 $\{\mathrm{Pois}(\mu_n+\bar{\mu}_n)\}$ is relatively compact; by Corollary 1.5, so is $\{c_\tau \mathrm{Pois}\mu_n\}$. Let $f \in B'$. Since $\mu_n \uparrow \mu$ and $\int |h_\tau(f,\cdot)|d\mu<\infty$, it follows that $\int h_\tau(f,\cdot)d\mu_n \to \int h_\tau(f,\cdot)d\mu$ and therefore $(c_\tau \mathrm{Pois}\mu_n)^\wedge \to (c_\tau \mathrm{Pois}\mu)^\wedge$. This implies that $c_\tau \mathrm{Pois}\mu_n \to_w c_\tau \mathrm{Pois}\mu$.

(2) $\Rightarrow$(3): Trivial.

(3) $\Rightarrow$(1). The measures $\{\mu_n\}$ satisfy (1) and (2) of Theorem 1.4; hence the same is true for the measure μ. It follows that $\int |h_\tau(f,\cdot)|d\mu<\infty (f \in B')$ and then the argument in the first part of the proof shows that $\{c_\tau \mathrm{Pois}\mu_n\}$ converges and its limit has ch.f. $\exp(\int h_\tau(f,\cdot)d\mu)$. This proves that μ is a Lévy measure. □

The following corollary extends Lemma 1.2(2). We omit the proof, which makes use of Theorem 1.6 and is analogous to that of Lemma 1.2 (2).

1.7 Corollary. If μ is a Lévy measure and λ is a positive measure such that $\lambda \le \mu$, then λ is a Lévy measure and $c_\tau \mathrm{Pois}\lambda$ is a factor of $c_\tau \mathrm{Pois}\mu$.

The next result shows how certain scalar moments of $c_\tau \mathrm{Pois}\mu$ are expressed in terms of the corresponding moments of μ.

1.8 Lemma. (1) Let μ be a symmetric Lévy measure. Then for $f \in B'$,

if $\int f^2 d\mu<\infty$, then $\int f^2 d\mathrm{Pois}\mu = \int f^2 d\mu$;

if $\int f^4 d\mu<\infty$, then $\int f^4 d\mathrm{Pois}\mu = \int f^4 d\mu + 3(\int f^2 d\mu)^2$

(2) Let μ be a Lévy measure, $\mu(B_r^c)=0$ for some $r>0$. Then for $\tau \ge r$ and $f \in B'$

$$\int f^2 dc_\tau \mathrm{Pois}\mu = \int f^2 d\mu \quad \text{and}$$
$$\int f^4 dc_\tau \mathrm{Pois}\mu = \int f^4 d\mu + 3(\int f^2 d\mu)^2.$$

Proof. We will only give the details of the first part of (2); the rest is similar. We remark first that for finite symmetric (or centered) measures, the statements are proved by direct computations with the series expansion of $\mathrm{Pois}\mu$. Also, if ν is a finite measure, then $\int |f| d\mathrm{Pois}\nu \le \int |f| d\nu$ and $\int |f| d\nu<\infty$ implies $\int f d\mathrm{Pois}\nu = \int f d\nu (f \in B')$. This implies that if λ is finite and $\lambda(B_r^c)=0$, then $\int f d(c_\tau \mathrm{Pois}\lambda)=0$ for all $f \in B'$ whenever $\tau \ge r$.

Let $\mu_n \uparrow \mu$, $\{\mu_n\}$ a sequence of finite measures. For each n,

$$\int f^2 d(c_\tau \mathrm{Pois}\mu_n) = (1/2)\int f^2 d\mathrm{Pois}(\mu_n+\bar{\mu}_n)$$
$$= (1/2)\int f^2 d(\mu_n+\bar{\mu}_n)$$
$$= \int f^2 d\mu_n.$$

Since $\int f^2 d\mu_n \to \int f^2 d\mu$, it follows that $\lim_n \int f^2 d(c_\tau \mathrm{Pois}\mu_n) = \int f^2 d\mu$. By Theorem 1.6, $c_\tau \mathrm{Pois}\mu_n \to_w c_\tau \mathrm{Pois}\mu$, and consequently

$$\int f^2 d(c_\tau \mathrm{Pois}\mu) \le \liminf_n \int f^2 d(c_\tau \mathrm{Pois}\mu_n).$$

On the other hand, $c_\tau \mathrm{Pois}\mu_n$ is a factor of $c_\tau \mathrm{Pois}\mu$ and $\int f d(c_\tau \mathrm{Pois}\mu_n) = 0$; therefore for all n

$$\int f^2 d(c_\tau \mathrm{Pois}\mu_n) \le \int f^2 d(c_\tau \mathrm{Pois}\mu).$$

It follows that $\int f^2 d(c_\tau \mathrm{Pois}\mu) = \lim_n \int f^2 d(c_\tau \mathrm{Pois}\mu_n) = \int f^2 d\mu$. □

Remark. It is well known that if $\{\mu_\alpha\}$ is a family of Lévy measures on a finite-dimensional Banach space, then $\{c_\tau \mathrm{Pois}\mu_\alpha\}$ is relatively compact if and only if for some (for all) $\delta>0$, $\{\min(\delta, \|\cdot\|^2)d\mu_\alpha\}$ is relatively compact. Since we will need this result later on, we will sketch a proof. We may assume that the norm is Euclidean (if not, we may replace the given norm by an equivalent Euclidean norm). Necessity follows easily from Theorem 1.4. For the proof of the sufficiency, we may assume that each μ_α is symmetric. The relative compactness of $\{\mu_\alpha|B_\delta^c\}$ implies that of $\{\mathrm{Pois}(\mu_\alpha|B_\delta^c)\}$; the proof will be completed by showing that $\{\mathrm{Pois}(\mu_\alpha|B_\delta)\}$ is tight. By Chebyshev's inequality, for every α

$$\mathrm{Pois}(\mu_\alpha|B_\delta)(B_r^c) \le r^{-2}\int \|x\|^2 d\mathrm{Pois}(\mu_\alpha|B_\delta)(x)$$

$$= r^{-2}\int \|x\|^2 d(\mu_\alpha|B_\delta)(x)$$
$$\le Mr^{-2}$$

, where $M=\sup_\alpha \int_{B_\delta} \|x\|^2 d\mu_\alpha(x)$; the equality assertion is a consequence of Lemma 1.8. □

In the next theorem and in the general central limit theorem in §2 we shall need the following uniqueness result.

1.9 Proposition. Let x_i be a point in B, γ_i a centered Gaussian measure, μ_i a Lévy measure (i=1,2). If

$$\delta_{x_1} * \gamma_1 * c_\tau \mathrm{Pois}\mu_1 = \delta_{x_2} * \gamma_2 * c_\tau \mathrm{Pois}\mu_2$$

then $x_1=x_2$, $\gamma_1=\gamma_2$ and $\mu_1|\{0\}^c=\mu_2|\{0\}^c$.

For the proof we refer to ([33] p.110).

If one assumes convergence of $\{c_\tau \mathrm{Pois}\mu_n\}$ instead of relative (shift) compactness, then the necessary conditions of Theorem 1.4 can be considerably sharpened. For a p.m. λ on B, Φ_λ will denote its covariance.

1.10 Theorem. Let $\{\mu_n\}$ be a sequence of Lévy measures. Assume that $c_\tau \mathrm{Pois}\mu_n \to_w \nu$. Then

(1) there exists a Lévy measure μ such that $\mu_n|B_\delta^c \to_w \mu|B_\delta^c$ for every $\delta \in C(\mu)$,

(2) there exists a centered Gaussian measure γ such that

$$\lim_{\delta\downarrow 0} \left\{\begin{matrix}\limsup \\ \liminf\end{matrix}\right\}_n \int_{B_\delta} f^2 d\mu_n = \Phi_\gamma(f,f),$$

(3) $\nu=\gamma * c_\tau \mathrm{Pois}\mu$.

Proof. By Theorem 1.4, $\{\mu_n|B_\delta^c\}$ is relatively compact for every $\delta>0$. Given a subsequence $\{n\}$ of N, by a diagonal procedure there exists a subsequence $\{n'\}$ and a σ-finite measure μ with $\mu\{0\}=0$, such that

$$\mu_{n'}|B_\delta^c \to_w \mu|B_\delta^c \quad \text{for every } \delta\in C(\mu).$$

Therefore $c_\tau \mathrm{Pois}(\mu_{n'}|B_\delta^c) \to_w c_\tau \mathrm{Pois}(\mu|B_\delta^c)$ for $\delta \in C(\mu)$, and since

$$c_\tau \mathrm{Pois}(\mu_n|B_\delta) * c_\tau \mathrm{Pois}(\mu_n|B_\delta^c) = c_\tau \mathrm{Pois}\mu_n$$

and $(c_\tau \mathrm{Pois}(\mu|B_\delta^c))^\wedge(f) \ne 0$ $(f\in B')$ it follows that $c_\tau \mathrm{Pois}(\mu_{n'}|B_\delta) \to_w \gamma_\delta$ for every $\delta\in C(\mu)$, and

(1.1) $\quad \gamma_\delta * c_\tau \mathrm{Pois}(\mu|B_\delta^c) = \nu.$

Let $\{\delta_k\}\subset C(\mu)$, $\tau>\delta_k\downarrow 0$. By (1.1) and ([33] , III.2.2), $\{c_\tau \mathrm{Pois}(\mu|B_{\delta_k}^c)\}$ is relatively shift compact; since $\mu|B_{\delta_k}^c \uparrow \mu$, it follows from Theorem 1.6 that μ is a Lévy measure and $c_\tau \mathrm{Pois}(\mu|B_{\delta_k}^c) \to_w c_\tau \mathrm{Pois}\mu$. Therefore $\gamma_{\delta_k} \to_w \gamma$ and

(1.2) $\quad \gamma * c_\tau \mathrm{Pois}\mu = \nu.$

We shall prove the limit formula in (2) along the subsequence $\{n'\}$ and then show that γ is a centered Gaussian measure. Observe first that (1.1) and (1.2) imply that $\gamma_{\delta_k} = \gamma * c_\tau \mathrm{Pois}(\mu|B_{\delta_k})$.

Since $c_\tau \mathrm{Pois}(\mu_{n'}|B_{\delta_k}) \to_w \gamma_{\delta_k}$, it follows that

$$\lim_{n'} \int f^2 d(c_\tau \mathrm{Pois}(\mu_{n'}|B_{\delta_k})) = \int f^2 d\gamma_{\delta_k};$$

the passage to the limit is justified by Theorem 1.4 and the second statement of Lemma 1.8 (2). But Lemma 1.8 also implies

$$\int f^2 d(c_\tau \mathrm{Pois}(\mu_{n'}|B_{\delta_k})) = \int f^2 d(\mu_{n'}|B_{\delta_k}) \quad \text{and}$$

$$\int f^2 d\gamma_{\delta_k} = \int f^2 d\gamma + \int f^2 d(\mu|B_{\delta_k}).$$

Hence $\lim_k \lim_{n'} \int f^2 d(\mu_{n'}|B_{\delta_k}) = \lim_k (\int f^2 d\gamma + \int f^2 d(\mu|B_{\delta_k})) = \Phi_\gamma(f,f)$.

Since lim sup (lim inf) $\int f^2 d(\mu_{n'}|B_\delta)$ is an increasing function of δ, (2) holds along the subsequence $\{n'\}$.

Now choose $\{n_k\}\subset\{n'\}$ such that $\rho_k = c_\tau \mathrm{Pois}(\mu_{n_k}|B_{\delta_k}) \to_w \gamma$. We shall prove that $\hat\rho_k(f) \to \exp\{-(1/2)\Phi_\gamma(f,f)\}$; this shows that γ is a centered Gaussian measure. Since $\int f^2 d(\mu_{n_k}|B_{\delta_k}) = \int f^2 d\rho_k \to \int f^2 d\gamma$, it is enough to show that

$$|\hat{\rho}_k(f)-\exp\{-(1/2)\int f^2 d(\mu_{n_k}|B_{\delta_k})\}|\to 0 \quad \text{for each } f\in B'.$$

But $|\hat{\rho}_k(f)-\exp\{-(1/2)\int f^2 d(\mu_{n_k}|B_{\delta_k})\}| \le$

$$\le |1-\exp\{\int(h_\tau(f,\cdot)+(1/2)f^2)d(\mu_{n_k}|B_{\delta_k})\}|.$$

Now $|\int(h_\tau(f,\cdot)+(1/2)f^2)d(\mu_{n_k}|B_{\delta_k})| \le \int_{B_{\delta_k}} |(1/2)f^2-(1-\cos f)|d\mu_{n_k} +$

$$+\int_{B_{\delta_k}} |\sin f - f|d\mu_{n_k}$$

$$\le \|f\|^2\delta_k^2\int_{B_{\delta_k}} f^2 d\mu_{n_k} + \|f\|\delta_k\int_{B_{\delta_k}} f^2 d\mu_{n_k} \to 0 \quad \text{as } k\to\infty.$$

Proposition 1.9 and (1.2) imply now that the Lévy measure μ with $\mu\{0\}=0$ and the centered Gaussian measure γ are uniquely determined by ν and do not depend on the subsequence $\{n\}$. It easily follows that (1) holds for the whole sequence $\{\mu_n\}$ and then the above argument shows that (2) as well is true for the whole sequence. □

Theorem 1.10 can also be obtained as a corollary of the general converse central limit theorem of §2. In fact, the proof of Theorem 1.10 parallels that of the converse central limit theorem but is considerably simpler.

As a corollary, we obtain the Lévy-Khinchine representation. This result was proved by Araujo [5] and Dettweiler [11]; previous work was done by Tortrat [41]. Their proofs rely on the finite-dimensional Lévy-Khinchine representation. A direct proof involving ideas close to the present ones may be found in [7]. We point out that our statement contains more information.

Let us recall that a probability measure ν on B is <u>infinitely divisible</u> if for every $n\in N$ there exists a probability measure ν_n on B such that $\nu_n^n=\nu$ (n-fold convolution). It is easily seen that the measures $\{\nu_n\}$ are well determined.

<u>1.11 Corollary</u>. A probability measure ν on B is infinitely divisible if and only if there exist z_τ in B, a centered Gaussian measure γ and a Lévy measure μ such that

$$\nu = \delta_{z_\tau} * \gamma * c_\tau \text{Pois}\mu.$$

Then: (1) for every $\delta\in C(\mu)$, $n\nu_n|B_\delta^c \to_w \mu|B_\delta^c$,

(2) $\lim_{\delta\downarrow 0} \left\{\begin{matrix}\limsup\\ \liminf\end{matrix}\right\}_n \int_{B_\delta} f^2 d(n\nu_n) = \Phi_\gamma(f,f)$,

(3) $\lim_n \int_{B_\tau} x\, d(n\nu_n)(x) = z_\tau$.

<u>Proof.</u> Let ν be infinitely divisible. By ([2] , Theorem 2.1), $\text{Pois}(n\nu_n)\to_w \nu$. Let $x_n=\int_{B_\tau} x\, d(n\nu_n)(x)$; then

$$c_\tau\text{Pois}(n\nu_n)*\delta_{x_n}=\text{Pois}(n\nu_n).$$

Then $\{c_\tau\text{Pois}(n\nu_n)\}$ is relatively compact by Corollary 1.5 and the necessity and assertions (1)-(3) follow from Theorem 1.10 and Proposition 1.9 by an easy argument. On the other hand, if γ is a centered Gaussian measure and μ is a Lévy measure,

$$(\delta_{(z/n)}*\gamma(n^{1/2}(\cdot))*c_\tau\text{Pois}(\mu/n))^n = \delta_z*\gamma*c_\tau\text{Pois}\mu. \; □$$

The Lévy-Khinchine representation can also be obtained as an immediate corollary of the general converse central limit theorem.

2. THE GENERAL CENTRAL LIMIT THEOREM FOR TRIANGULAR ARRAYS.

We begin by giving a new proof of an important necessary condition for relative compactness of the row sums of a triangular array due to Le Cam [28]. It is more elementary than the original one, which was based on Le Cam's [28] concentration inequality.

<u>2.1 Theorem</u>. Let $\{X_{nj}\}$ be a triangular array of B-valued random vectors. Assume that $\{L(S_n)\}$ is relatively shift compact. Then for every $\varepsilon>0$ there exist a compact convex symmetric set $K_\varepsilon\subset B_\varepsilon$ and points $\{x_{nj}\}\subset B$ such that $\{\Sigma_j L(X_{nj}-x_{nj})|K_\varepsilon^c\}$ is relatively compact.

<u>Proof</u>. Let $\{X'_{nj}\}$ be independent of $\{X_{nj}\}$ and $L(X'_{nj})=L(X_{nj})$, $\tilde{X}_{nj}=X_{nj}-X'_{nj}$, $\tilde{S}_n=\Sigma_j\tilde{X}_{nj}$. By ([2] Lemma 2.3), applied to the Minkowski

functional of a compact convex symmetric set K, we have

(2.1) $\sup_n \Sigma_j P\{\widetilde{X}_{nj} \notin K\} \le -\log(1-2\sup_n P\{\widetilde{S}_n \notin K\})$.

Since $\{L(\widetilde{S}_n)\}$ is relatively compact, one may choose K such that $P\{\widetilde{S}_n \notin K\} < \alpha < 1/2$ for all n. Then

(2.2) $\sup_n \Sigma_j P\{\widetilde{X}_{nj} \notin K\} < -\log(1-2\alpha) = T < \infty$.

Let $K_\varepsilon = B_\varepsilon \cap K$. We show next that for every $\varepsilon > 0$,

(2.3) $\sup_n \Sigma_j P\{\widetilde{X}_{nj} \notin K_\varepsilon\} < M < \infty$.

Let $r>0$ be such that $K \subset B_r$. Since $P\{\widetilde{X}_{nj} \notin K_\varepsilon\} = P\{\widetilde{X}_{njr} \notin K_\varepsilon\}$

$$+ \ P\{\|\widetilde{X}_{nj}\| > r\},$$

by (2.2) we may assume that $\sup_{n,j} \|\widetilde{X}_{nj}\| \le r < \infty$ a.s. For every $f \in B_1'$ let $V_f = \{x : |f(x)| > \varepsilon/2\}$. Then $\{V_f\}$ is an open cover of $\overline{B}_\varepsilon^c \cap K$ and hence there exists a finite subset F of B_1' such that $\overline{B}_\varepsilon^c \cap K \subset \cup_{f \in F} V_f$. Then

$$\begin{aligned} \Sigma_j P\{\widetilde{X}_{nj} \notin K_\varepsilon\} &\le \Sigma_j P\{\widetilde{X}_{nj} \notin K\} + \Sigma_j P\{\widetilde{X}_{nj} \in \overline{B}_\varepsilon^c \cap K\} \\ &\le T + \Sigma_j \Sigma_{f \in F} P\{|f(\widetilde{X}_{nj})| > \varepsilon/2\} \\ &\le T + \Sigma_{f \in F} (\varepsilon/2)^{-2} \Sigma_j E(f(\widetilde{X}_{nj}))^2 \\ &= T + (\varepsilon/2)^{-2} \Sigma_{f \in F} E(f(\widetilde{S}_n))^2. \end{aligned}$$

For each f, $\{L(f(\widetilde{S}_n))\}$ is relatively compact; since $|f(\widetilde{X}_{nj})| \le r$ a.s., the converse Kolmogorov inequality (see e.g. [29]) implies $\sup_n E(f(\widetilde{S}_n))^2 < \infty$. This proves (2.3).

For each n, let $J_n = \{j \in \{1,\dots,k_n\} : P\{\widetilde{X}_{nj} \in K_\varepsilon\} < 3/4\}$. Then (2.3) implies that $\sup_n \#(J_n) \le 4M$. By ([33] III.2.2), there exist $x_{nj} \in B$ $(n \in N, j \in J_n)$ such that $\{L(X_{nj} - x_{nj}) : n \in N, j \in J_n\}$ is relatively compact; consequently, so is $\{\Sigma_{j \in J_n} L(X_{nj} - x_{nj})\}$. To complete the proof, we must show that there exist x_{nj} $(n \in N, j \notin J_n)$ such that $\{\Sigma_{j \notin J_n} L(X_{nj} - x_{nj}) | K_\varepsilon^c\}$ is relatively compact.

By Fubini's theorem, there exist points $x_{nj} \in B (n \in N,\ j \notin J_n)$ such that $P\{X_{nj} - x_{nj} \in K_\varepsilon\} \ge 1/2$ and $\sup_n \Sigma_{j \notin J_n} P\{X_{nj} - x_{nj} \notin K_\varepsilon\} < M$.

If Q is a compact convex symmetric set and $G = Q + K_\varepsilon$, then

$$(1/2)\Sigma_{j \notin J_n} P\{X_{nj} - x_{nj} \notin G\} \le \Sigma_{j \notin J_n} P\{X_{nj} - x_{nj} \notin G\} P\{X'_{nj} - x_{nj} \in K_\varepsilon\}$$

$$\le \Sigma_{j \notin J_n} P\{\widetilde{X}_{nj} \notin Q\}.$$

The relative compactness of $\{L(\widetilde{S}_n)\}$ and (2.1) imply now that given $\delta > 0$ one may choose a compact convex symmetric set G so that

$$\sup_n \Sigma_{j \notin J_n} P\{X_{nj} - x_{nj} \notin G\} < \delta.$$

This completes the proof. □

A triangular array $\{X_{nj}\}$ is <u>infinitesimal</u> if for every $\varepsilon > 0$, $\lim_n \max_j P\{\|X_{nj}\| > \varepsilon\} = 0$. The last theorem admits the following refinement for infinitesimal arrays (mentioned in [28]):

<u>2.2 Theorem</u>. If the triangular array $\{X_{nj}\}$ is infinitesimal and $\{L(S_n)\}$ is relatively shift compact, then for every $\varepsilon > 0$ there exists a compact convex symmetric set $K_\varepsilon \subseteq B_\varepsilon$ such that

(1) $\sup_j P\{X_{nj} \in K_\varepsilon^c\} \to 0$,

(2) $\{\Sigma_j L(X_{nj}) | K_\varepsilon^c\}$ is relatively compact.

<u>Proof</u>. By infinitesimality, the set $\{X_{nj}\}$ ordered lexicographically is a sequence which converges to zero in probability; therefore $\{\mu_{nj}\}$ is tight by Prokhorov's theorem, where $\mu_{nj} = L(X_{nj})$. For every $k \in N$, let H_k be a compact set such that $\sup_{n,j} \mu_{nj}(H_k^c) < 1/k$. For $h \in N$, let $F_h = \cup_{k \ge h} (B_{1/k} \cap H_k)$. Then F_h is relatively compact: in fact, if $\{x_j\} \subset F_h$, then either $\{x_j\}$ is included in the compact set $\cup_{k=h}^m (B_{1/k} \cap H_k)$ for some $m \in N$, or $\{x_j\}$ has a subsequence which converges to 0. If Q_h is the closed convex symmetric hull of F_h, then Q_h is a compact convex symmetric set, $Q_h \subset B_{1/h}$ and Q_h satisfies (1) for $\varepsilon = 1/h$: for all $k \ge h, n \in N$,

$$\sup_j \mu_{nj}(Q_h^c) \le \sup_j \mu_{nj}(B_{1/k}^c) + \sup_j \mu_{nj}(H_k^c)$$

$$\le \sup_j \mu_{nj}(B_{1/k}^c) + (1/k)$$

and therefore $\lim_n \sup_j \mu_{nj}(Q_h^c) = 0$.

Given $\varepsilon>0$, let Q_ε be a compact convex symmetric set satisfying $Q_\varepsilon \subset B_\varepsilon$ and (1), and let n_o be such that $n \geq n_o$ implies $\sup_j P\{X_{nj} \notin Q_\varepsilon\}<1/2$. Let K_ε be as in (2.3), $G_\varepsilon = K_\varepsilon + Q_\varepsilon$. Then for $n \geq n_o$

$$(1/2)\Sigma_j P\{X_{nj} \notin G_\varepsilon\} \leq \Sigma_j P\{X_{nj} \notin G_\varepsilon\} \cdot P\{X'_{nj} \in Q_\varepsilon\}$$

$$\leq \Sigma_j P\{\widetilde{X}_{nj} \notin K_\varepsilon\}$$

Using (2.1) and arguing as in the proof of Theorem 2.1, we conclude that $\{\Sigma_j \mathcal{L}(X_{nj})|G_\varepsilon^c\}$ is relatively compact. □

The next result gives necessary integrability conditions for uniformly bounded and centered triangular arrays such that $\{\mathcal{L}(S_n)\}$ is relatively shift compact; it also shows that, under the stated conditions, $\{\mathcal{L}(S_n)\}$ is in fact relatively compact.

Let B be a Banach space, $\mathcal{F}$ the family of finite dimensional subspaces of B. An increasing sequence $\{F_k\} \subset \mathcal{F}$ will be called <u>full</u> if $\overline{\cup_k F_k} = B$. For any subspace $G \subset B$, q_G will denote the seminorm $q_G(x) = \inf\{\|x-u\| : u \in G\}$.

<u>2.3 Theorem.</u> Let $\{X_{nj}\}$ be a triangular array such that $\|X_{nj}\| \leq C < \infty$ a.s. and $EX_{nj}=0$ for all n,j. Assume that $\{\mathcal{L}(S_n)\}$ is relatively shift compact. Then for every $p>0$

(1) $\sup_n E\|S_n\|^p < \infty$,

(2) for every full sequence $\{F_k\} \subset \mathcal{F}$, $\lim_k \sup_n Eq^p_{F_k}(S_n)=0$.

Moreover, $\{\mathcal{L}(S_n)\}$ is relatively compact.

<u>Proof.</u> Obviously it is enough to prove the statement for $p \geq 1$. If q is a continuous seminorm on B and $\{Y_j : j=1,\dots,m\}$ are independent symmetric B-valued random vectors such that $q(Y_j) \leq \beta < \infty$ a.s. $(j=1,\dots,m)$ then by ([2], Section 3) we have: for every $p \geq 1$ and every $\alpha>0$ such that $P\{q(T_m)>\alpha\}<2^{-p}$,

(2.4) $Eq^p(T_m) \leq ((\beta+\alpha)^p + \alpha^p(1-2^{1-p}))(1-2^p P\{q(T_m)>\alpha\})^{-1}$

where $T_m = \Sigma_{j=1}^m Y_j$.

Assume first that the random variables $\{X_{nj}\}$ are symmetric and $\{\mathcal{L}(S_n)\}$ is relatively compact. Choose $\alpha>0$ so that $P\{\|S_n\|>\alpha\}<2^{-p-1}$ for all $n \in N$; then (2.4) implies assertion (1).

For $\delta>0$ and $k \in N$, let $V_n(\delta,k) = \Sigma_j X_{nj} I_{\{q_{F_k}(X_{nj}) \leq \delta\}}$, and $A_n = \{\sup_j q_{F_k}(X_{nj}) \leq \delta\}$. Then

(2.5) $Eq^p_{F_k}(S_n) = Eq^p_{F_k}(S_n) I_{A_n} + Eq^p_{F_k}(S_n) I_{A_n^c}$

$$\leq Eq^p_{F_k}(V_n(\delta,k)) + (E\|S_n\|^{2p})^{1/2}(P(A_n^c))^{1/2}.$$

Using the argument in Lemma 2.3 of [2], we have:

$$P(A_n^c) \leq 2P\{q_{F_k}(S_n)>\delta\}.$$

Now

$$P\{q_{F_k}(V_n(\delta,k))>\delta\} = P\{q_{F_k}(V_n(\delta,k))>\delta\} \cap A_n + P\{q_{F_k}(V_n(\delta,k))>\delta\} \cap A_n^c$$

$$\leq P\{q_{F_k}(S_n)>\delta\} + P(A_n^c)$$

$$\leq 3P\{q_{F_k}(S_n)>\delta\}.$$

The relative compactness of $\{\mathcal{L}(S_n)\}$ and the fact that $\{F_k\}$ is a full sequence easily imply (Dini's lemma) that $\lim_k \sup_n P\{q_{F_k}(S_n)>\delta\}=0$ for every $\delta>0$. Given $\delta>0$, choose k so that

$$\sup_n P\{q_{F_k}(S_n)>\delta\} < \min\{(1/3)2^{-p-1}, \delta^2/2M\},$$

where $M = \sup_n E\|S_n\|^{2p}$. Then (2.4) implies: for all $n \in N$,

$$Eq^p_{F_k}(V_n(\delta,k)) \leq 2((2\delta)^p + \delta^p(1-2^{1-p})) = C(\delta)$$

and we obtain from (2.5):

$$\sup_n Eq^p_{F_k}(S_n) \leq C(\delta)+\delta.$$

This proves assertion (2) of the theorem in the symmetric case.

For the general case, let $\widetilde{S}_n$ be a symmetrization of S_n $(n \in N)$. Then $\{\mathcal{L}(\widetilde{S}_n)\}$ is relatively compact; also, since $ES_n=0$, we get $Eq^p(S_n) \leq Eq^p(\widetilde{S}_n)$ for every continuous seminorm q and every $p \geq 1$, by a well-known inequality. It is easy to complete the proof of (1) and (2) from here. In order to prove that $\{\mathcal{L}(S_n)\}$ is relatively compact it is enough to show that $\{L(S_n)\}$ is both spherically and flatly concentrated ([1], Theorem 2.3); but this follows at once from (1) and (2) and Chebyshev's inequality. □

If $\{L(S_n)\}$ is relatively shift compact, it is of interest to be able to find "canonical centerings" $\{a_n\}$ such that $\{L(S_n-a_n)\}$ is relatively compact. A natural choice appears to be $a_n=ES_{n,\delta}$; however, trivial examples with point masses show that in general $\{L(S_n-ES_{n,\delta})\}$ may fail to be relatively compact. We prove next a general result showing that the centerings $\{ES_{n,\delta}\}$ are indeed adequate in many interesting situations, covering in particular the case of infinitesimal triangular arrays.

<u>2.4 Lemma</u>. Let $\{X_{nj}\}$ be a (not necessarily row-wise independent) triangular array. Assume that $\{\Sigma_j L(X_{nj})|B_\delta^c\}$ is relatively compact for some $\delta>0$. Then $\{L(S_n^{(\tau)})\}$ is relatively compact for every $\tau\geq\delta$.

<u>Proof</u>. By ([1], Theorem 2.3), it is enough to show that $\{L(S_n^{(\tau)})\}$ is both spherically and flatly concentrated. Define $\phi_n=\Sigma_j I_{\{X_{nj}^\tau\neq 0\}}$.

Then $E\phi_n=\Sigma_j P\{\|X_{nj}\|>\tau\}$ and therefore

$$\sup_n E\phi_n\leq\sup_n\Sigma_j P\{\|X_{nj}\|>\delta\}<\infty.$$

Also,
$$\|S_n^{(\tau)}\|\leq\Sigma_j\|X_{nj}^\tau\|\leq\phi_n\max_j\|X_{nj}^\tau\|.$$

Now
$$P\{\|S_n^{(\tau)}\|>t\}=P\{\|S_n^{(\tau)}\|>t,\phi_n>m\}+P\{\|S_n^{(\tau)}\|>t,\phi_n\leq m\}$$

$$\leq P\{\phi_n>m\}+P\{\max_j\|X_{nj}^\tau\|>t/m\}$$

$$\leq m^{-1}E\phi_n+\Sigma_j P\{\|X_{nj}^\tau\|>t/m\}.$$

Given $\varepsilon>0$, choose m so that $m^{-1}\sup_n E\phi_n<\varepsilon/2$ and then $t>0$ such that $\Sigma_j P\{\|X_{nj}^\tau\|>t/m\}<\varepsilon/2$ (possible by hypothesis). Then for all n $P\{\|S_n^{(\tau)}\|>t\}<\varepsilon$, i.e. $\{L(S_n^{(\tau)})\}$ is spherically concentrated.

Next, observe that for any subspace $F\subset B$, $q_F(S_n^{(\tau)})\leq\phi_n\max_j q_F(X_{nj}^\tau)$; so, in analogy with the previous computation, given $\varepsilon>0$,

$$P\{q_F(S_n^{(\tau)})>\varepsilon\}\leq m^{-1}E\phi_n+P\{\max_j q_F(X_{nj}^\tau)>\varepsilon/m\}.$$

Now choose a compact set K containing 0 and such that $\sup_n\Sigma_j P\{X_{nj}\in B_\delta^c\cap K^c\}<\varepsilon/2$, a number m such that $m^{-1}\sup_n E\phi_n<\varepsilon/2$ and finally a finite dimensional subspace F such that $\sup_{x\in K}q_F(x)<\varepsilon/m$. Since $\{\max_j q_F(X_{nj}^\tau)>\varepsilon/m\}\subset\bigcup_j\{X_{nj}^\tau\in K^c\}$, we have

$$P\{\max_j q_F(X_{nj}^\tau)>\varepsilon/m\}\leq\Sigma_j P\{X_{nj}^\tau\in K^c\}$$

$$\leq\Sigma_j P\{X_{nj}\in B_\delta^c\cap K^c\}<\varepsilon/2$$

and therefore $\sup_n P\{q_F(S_n^{(\tau)})>\varepsilon\}<\varepsilon$. □

<u>2.5 Theorem</u>. Let $\{X_{nj}\}$ be a (not necessarily row-wise independent) triangular array. Assume that $\{L(S_n)\}$ is relatively shift compact and that $\{\Sigma_j L(X_{nj})|B_\delta^c\}$ is relatively compact for some $\delta>0$. Then for every $\tau\geq\delta$, $\{L(S_n-ES_{n,\tau})\}$ and $\{L(S_{n,\tau}-ES_{n,\tau})\}$ are relatively compact.

<u>Proof.</u> Let $\{x_n\}$ be such that $\{L(S_n-x_n)\}$ is relatively compact. Since $S_n-x_n=(S_{n,\tau}-x_n)+S_n^{(\tau)}$ and $\{L(S_n^{(\tau)})\}$ is relatively compact by Lemma 2.4, it follows that $\{L(S_{n,\tau}-x_n)\}$ is also relatively compact; in fact, if K is a measurable convex symmetric set, then

$$P\{S_{n,\tau}-x_n\in K^c\}\leq P\{S_n-x_n\in\tfrac{1}{2}K^c\}+P\{S_n^{(\tau)}\in\tfrac{1}{2}K^c\}.$$

Therefore Theorem 2.3 implies that $\{L(S_{n,\tau}-ES_{n,\tau})\}$ is relatively compact, and so is $\{L(S_n-ES_{n,\tau})\}=\{L((S_{n,\tau}-ES_{n,\tau})+S_n^{(\tau)})\}$. □

<u>Remark</u>. In the classical approach to the general central limit theorem for infinitesimal triangular arrays of real valued random variables (Gnedenko-Kolmogorov [19], Loève [29]), a fundamental result is (I) the equivalence of the following conditions:

(1) $\{L(S_n)\}$ is relatively shift compact,

(1a) for some (for all) $\tau>0$, $\{L(S_n-ES_{n,\tau})\}$ is relatively compact,

(2) for some (for all) $\tau>0,\delta>0$, $\{\min(\delta^2,\|\cdot\|^2)d(\Sigma_j L(X_{nj}-EX_{nj\tau}))\}$ is relatively compact.

(3) for some (for all) $\tau>0$ $\{\mathrm{Pois}(\Sigma_j L(X_{nj}-EX_{nj\tau}))\}$ is relatively compact.

It follows that (II) $L(S_n-x_n)\to_w \nu$ if and only if for some (for every) $\tau>0$ $\mathrm{Pois}(\Sigma_j L(X_{nj}-EX_{nj\tau}))*\delta_{z_n}\to_w \nu$, where $z_n=ES_{n,\tau}-x_n$; this is proved using (I) and a computation with characteristic functions. Statement (II) together with (III) the classical necessary and sufficient conditions for weak convergence of infinitely divisible laws in terms of their Lévy-Khinchine representations yield both the direct and converse central limit theorem.

Let us observe that assertion (I) for infinitesimal triangular arrays of random vectors taking values in a finite dimensional Banach space can be proved easily from the initial results of this section (the classical proofs for the one-dimensional case involve computations with characteristic functions and centering at medians). The equivalence of (2) and (3) follows from the remark following Lemma 1.8 and infinitesimality. Theorem 2.5 shows that $(1)\Longleftrightarrow$(1a). The equivalence of (1a) and (2) follows from Theorems 2.2, 2.3 and 2.5 and Lemma 2.4; the relative compactness of $\{L(S_n^{(\tau)})\}$ and $\{L(S_{n,\tau}-ES_{n,\tau})\}$ is equivalent to that of $\{L(S_n-ES_{n,\tau})\}$ and we may assume that the norm is Euclidean, in which case $E\|S_{n,\tau}-ES_{n,\tau}\|^2=\Sigma_j E\|X_{nj\tau}-EX_{nj\tau}\|^2$. (If the triangular array $\{X_{nj}\}$ is symmetric, then the same proof shows that (1)-(3) are equivalent even without the assumption of infinitesimality).

Varadhan [38] (see [33], Ch.6) has generalized the classical methods to the Hilbert space case. It is proved in [38] that (1) and (3) are equivalent and consequently (II) holds. The generalization of (III) to the Hilbert space case involves conditions which are expressed in terms of trace-class operators and which depend on the form of Gaussian covariances in Hilbert space and on the integrability property of Lévy measures on Hilbert space.

The equivalence of (1)-(3) breaks down in the infinite dimensional Banach space case. The universal validity of $(1)\Rightarrow(2)$ is equivalent to the cotype 2 character of the space (the fact that in cotype 2 spaces (1) implies (2) may be proved by a modification of Theorem 4.2 of [2]; for the other direction, see [6]). Only $(1)\Leftrightarrow$ (1a) and $(3)\Rightarrow(1)$ ($(3)\Rightarrow$(1a)) are true in any separable Banach space. An example of Le Cam [28] shows that in general (1) does not imply (3) and the "only if" implication in (II) is false (for the relation between (1) and (3), see §3).

Thus the case of triangular arrays in a general Banach space requires a different approach, which we have developed here taking as our point of departure the work of Le Cam [28]; a key idea is the Lévy decomposition. The proof of the general converse central limit theorem (Theorem 2.10) does not require a previous theory of representation and convergence of infinitely divisible laws, but only elementary properties of Lévy and Poisson measures. The general direct central limit theorem (Theorem 2.14) follows as a corollary.

We proceed next to describe the Lévy decomposition (découpage de Lévy) of the law of a random vector (Le Cam [27], [28]).

Given a random vector X and a measurable set $A\in\mathcal{B}$, we define

$$L(X|A) = \begin{cases} (L(X)|A)/P\{X\in A\} & \text{if } P\{X\in A\}\neq 0 \\ \delta_0 & \text{if } P\{X\in A\}=0; \end{cases}$$

also, we write $X_A=XI_{\{X\in A\}}$.

Let U_A, V_A be independent random vectors with $L(U_A)=L(X|A)$, $L(V_A)=L(X|A^c)$, and let ξ_A be a Bernoulli random variable independent of $\{U_A,V_A\}$ and such that $E(\xi_A)=P\{X\in A\}$. Then it is easily verified that $L(\xi_A U_A,(1-\xi_A)V_A)=L(X_A,X_{A^c})$, and consequently

$$L(X_A) = L(\xi_A U_A)$$
$$L(X_{A^c}) = L((1-\xi_A)V_A)$$
$$L(X) = L(\xi_A U_A+(1-\xi_A)V_A).$$

Furthermore, if $L(\eta_A)=L(\xi_A)$ and η_A is independent $\{\xi_A,U_A,V_A\}$ then

(2.6) $\qquad L(X)=L(\eta_A U_A+(1-\xi_A)V_A+(\xi_A-\eta_A)U_A)$.

This is the <u>Lévy decomposition</u> of $L(X)$ (associated with the set A). Since $\eta_A U_A$ and $(1-\xi_A)V_A$ are independent, if $(\xi_A-\eta_A)U_A$ is "small in probability" then (2.6) says that $L(X)$ is approximately equal to the convolution of $L(X_A)$ and $L(X_{A^c})$. If $A=B_\delta$, we will write $X_A=X_\delta, X_{A^c}=X^\delta, \xi_A=\xi_\delta$, etc.

If $\{X_{nj}\}$ is a triangular array and $\{U_{nj\delta}, V_{nj\delta}, \xi_{nj\delta}, \eta_{nj\delta}\}_{n,j}$ are independent, then (2.6) implies

(2.7) $\quad L(S_n)=L(\Sigma_j \eta_{nj\delta}U_{nj\delta}+\Sigma_j(1-\xi_{nj\delta})V_{nj\delta}+\Sigma_j(\xi_{nj\delta}-\eta_{nj\delta})U_{nj\delta})$.

Note that $\Sigma_j\eta_{nj\delta}U_{nj\delta}$ and $\Sigma_j(1-\xi_{nj\delta})V_{nj\delta}$ are independent and $L(\Sigma_j\eta_{nj\delta}U_{nj\delta})=L(S_{n,\delta})$, $L(\Sigma_j(1-\xi_{nj\delta})V_{nj\delta})=L(S_n^{(\delta)})$. If $\{X_{nj}\}$ is infinitesimal and $\{L(S_n)\}$ is relatively shift compact, it turns out that $\Sigma_j(\xi_{nj\delta}-\eta_{nj\delta})U_{nj\delta}\to_P 0$ for every $\delta>0$ (Lemma 2.6); then (2.7) says that $L(S_n)$ is approximately equal to $L(S_{n,\delta})*L(S_n^{(\delta)})$.

We shall see below that if $\{L(S_n)\}$ converges, then $\{L(S_{n,\delta}-a_{n,\delta})\}$ converges to a law γ_δ which is "almost Gaussian" (for appropriate centerings $\{a_{n,\delta}\}$) and $\{L(S_n^{(\delta)})\}$ converges to a Poisson law ρ_δ; from this one obtains the form of the limit of $\{L(S_n)\}$ (for precise statements, see Theorem 2.10). Corollary 2.8 and Lemma 2.9 are preparatory results which, roughly speaking, deal with convergence of the two "components" of $L(S_n)$ to Poisson and Gaussian limits, respectively; in particular, Corollary 2.8 shows that under appropriate conditions $L(S_n^{(\delta)})$ (the law of the sum of independent random vectors "rarely different from zero") is approximated in a very strong sense by a Poisson measure. Lemma 2.6 gives a precise meaning to the idea of approximating $L(S_n)$ by $L(S_{n,\delta})*L(S_n^{(\delta)})$; it is through this lemma that we use the Lévy decomposition.

<u>2.6 Lemma</u>. Let $\{X_{nj}\}$ be an infinitesimal triangular array such that $\sup_n\Sigma_j P\{\|X_{nj}\|>\delta\}<\infty$ for some $\delta>0$. Then for every $\tau\ge\delta$, $\{a_n\}\subset B$,

$\{b_n\}\subset B$, and every bounded uniformly continuous function f,

$$\lim_n |\int f dL(S_n-c_n)-\int fd(L(S_{n,\tau}-a_n)*L(S_n^{(\tau)}-b_n))|=0, \text{ where } c_n=a_n+b_n.$$

<u>Proof.</u> By (2.7) it is enough to show that $\Sigma_j(\xi_{nj\tau}-\eta_{nj\tau})U_{nj\tau}\to_P 0$ as $n\to\infty$. Now

$$E\|\Sigma_j(\xi_{nj\tau}-\eta_{nj\tau})U_{nj\tau}\|\le\Sigma_j E|\xi_{nj\tau}-\eta_{nj\tau}|E\|U_{nj\tau}\|$$
$$=2\Sigma_j P\{\|X_{nj}\|>\tau\}E\|X_{nj\tau}\|$$
$$\le 2\max_j E\|X_{nj\tau}\|\,\Sigma_j P\{\|X_{nj}\|>\tau\}$$
$$\to 0 \text{ as } n\to\infty,$$

since by infinitesimality $\max_j E\|X_{nj\tau}\|\to 0$. □

<u>Remark.</u> Lemma 2.6 may also be formulated in terms of distances. For instance, if $d_{BL}(\mu,\nu)=\sup\{|\int fd(\mu-\nu)|: \|f\|_{BL}\le 1\}$ is the dual bounded Lipschitz distance on the space of probability measures on B, then under the assumptions of Lemma 2.6

$$\lim_n d_{BL}(L(S_n-c_n), L(S_{n,\tau}-a_n)*L(S_n^{(\tau)}-b_n))=0.$$

In the next lemma, $\|\cdot\|$ denotes the total variation norm on the space of finite signed measures on B.

<u>2.7 Lemma</u>. (Khinchine-Le Cam). Let $\{X_i\}_{i=1,\dots,n}$ be independent random vectors. Then

$$\|L(S_n)-\text{Pois}(\Sigma_{j=1}^n L(X_j))\|\le 2\Sigma_{j=1}^n(P\{X_j\ne 0\})^2.$$

A proof may be found in Le Cam ([27], p.186). A very simple argument of Banach algebra type gives a somewhat coarser inequality in which the number 2 is replaced by a larger constant; a slightly more involved proof of the same kind yields the inequality in the stated form.

<u>2.8 Corollary</u>. Let $\{X_{nj}\}$ be an infinitesimal triangular array such that $\sup_n\Sigma_j P\{\|X_{nj}\|>\delta\}<\infty$ for some $\delta>0$. Then for every $\tau\ge\delta$

$$\lim_n \|L(S_n^{(\tau)})-\text{Pois}(\Sigma_j L(X_{nj})|B_\tau^c)\|=0.$$

<u>Proof.</u> Since $\Sigma_j(P\{X_{nj}^\tau\ne 0\})^2=\Sigma_j(P\{\|X_{nj}\|>\tau\})^2$

$$\le \max_j P\{\|X_{nj}\|>\tau\}\ \Sigma_j P\{\|X_{nj}\|>\tau\}$$
$$\to 0 \quad \text{as } n\to\infty,$$

it follows that $\|L(S_n^{(\tau)})-\mathrm{Pois}(\Sigma_j L(X_{nj}^{\tau}))\| \to 0.$

But $L(X_{nj}^{\tau})=P\{\|X_{nj}\|\le\tau\}\delta_0+L(X_{nj})|B_\tau^c$; hence $\mathrm{Pois}(\Sigma_j L(X_{nj}^{\tau}))=\mathrm{Pois}(\Sigma_j L(X_{nj})|B_\tau^c)$. □

2.9 Lemma. Let $\{X_{nj}\}$ be a triangular array of uniformly bounded and centered random vectors, and assume

(1) $L(S_n)\to_w \mu$,

(2) $\max_j \|X_{nj}\|\le\delta_n$ a.s. and $\delta_n\downarrow 0$.

Then μ is a centered Gaussian measure.

Proof. Let $\psi(f)=\int f^2 d\mu(f\in B')$. Then for every $f\in B'$, $\psi(f)$ is finite and $\lim_n Ef^2(S_n)=\psi(f)$ (observe that Theorem 2.3 provides the uniform integrability needed for the validity of the passage to the limit). We shall prove that $\hat{\mu}=\exp(-(1/2)\psi)$ by showing that $\lim_n E(\exp if(S_n))=\exp(-(1/2)\psi(f))$; it is a simple classical argument with ch.f.'s. We have

$$|E(\exp if(S_n))-\exp(-(1/2)Ef^2(S_n))|$$
$$=|\Pi_j E(\exp if(X_{nj}))-\Pi_j \exp(-(1/2)Ef^2(X_{nj}))|$$
$$\le\Sigma_j|E(\exp if(X_{nj}))-\exp(-(1/2)Ef^2(X_{nj}))|$$
$$\le\Sigma_j(E|f^3(X_{nj})|+(Ef^2(X_{nj}))^2\exp(\|f\|\delta_1)^2)$$
$$\le\|f\|\delta_n Ef^2(S_n)+\|f\|^2\delta_n^2\exp(\|f\|\delta_1)^2 Ef^2(S_n)$$
$$\to 0 \quad \text{as } n\to\infty. \quad □$$

Remark. Lemma 2.9 is of an auxiliary nature. It will be considerably improved in Corollary 2.11.

The next result is the general converse central limit theorem. Given a triangular array $\{X_{nj}\}$, $\psi_n(\delta,f)(\delta>0,f\in B')$ is defined by

$$\psi_n(\delta,f)=\Sigma_j Ef^2(X_{nj\delta}-EX_{nj\delta}).$$

2.10 Theorem. Let $\{X_{nj}\}$ be an infinitesimal triangular array. Suppose $L(S_n-x_n)\to_w\nu$ for some sequence $\{x_n\}\subset B$. Then

(1) there exists a Lévy measure μ such that for every $\tau\in C(\mu)$

$$\Sigma_j L(X_{nj})|B_\tau^c\to_w \mu|B_\tau^c,$$

(2) there exists a centered Gaussian measure γ such that for every $f\in B'$

$$\lim_{\delta\downarrow 0}\left\{\begin{matrix}\limsup\\ \liminf\end{matrix}\right\}_n \psi_n(\delta,f)=\lim_{\tau\downarrow 0,\tau\in C(\mu)}\lim_n\psi_n(\tau,f)=\phi_\gamma(f,f),$$

(3) for every $\tau\in C(\mu)$,

$$L(S_n-ES_{n,\tau})\to_w \gamma*c_\tau\mathrm{Pois}\mu,$$
$$L(S_n^{(\tau)})\to_w\mathrm{Pois}(\mu|B_\tau^c),$$
$$L(S_{n,\tau}-ES_{n,\tau})\to_w\gamma*c_\tau\mathrm{Pois}(\mu|B_\tau),$$

and there exists $z_\tau\in B$ such that $E(S_{n,\tau})-x_n\to z_\tau$ in B and

$$\nu=\delta_{z_\tau}*\gamma*c_\tau\mathrm{Pois}\mu.$$

Proof. By Theorem 2.2, $\{\Sigma_j L(X_{nj})|B_\delta^c\}$ is relatively compact for every $\delta>0$. Given a subsequence $\{n\}$ of N, it is possible by a diagonal procedure to find a subsequence $\{n'\}$ of $\{n\}$ and a σ-finite measure μ with $\mu\{0\}=0$ such that for every $\delta\in C(\mu)$,

$$\Sigma_j L(X_{n'j})|B_\delta^c\to_w\mu|B_\delta^c. \tag{2.8}$$

Fix $\tau\in C(\mu)$ and define $b_{n\tau}^{\delta}=\Sigma_j\int_{B_\tau}x\,dL(X_{nj}^{\delta})(x)$ for $0<\delta<\tau$. Since $\int_{B_\tau}x\,dL(X_{nj}^{\delta})(x)=\int_{B_\tau}x\,d(L(X_{nj})|B_\delta^c)(x)$ it follows from (2.8) that $b_{n',\tau}^{\delta}\to\int_{B_\tau}x\,d(\mu|B_\delta^c)$ whenever $\delta\in C(\mu)$. By Corollary 2.8 and (2.8), $L(S_{n'}^{(\delta)})\to_w\mathrm{Pois}(\mu|B_\delta^c)(\delta\in C(\mu))$. Therefore, for $\delta\in C(\mu)$,

$$L(S_{n'}^{(\delta)}-b_{n',\tau}^{\delta})\to_w c_\tau\mathrm{Pois}(\mu|B_\delta^c).$$

Choose and fix a sequence $\{\delta_k\}\subset C(\mu)$ such that $\tau>\delta_k\downarrow 0$. Since $\{L(S_n-ES_{n,\delta})\}$ and $\{L(S_{n,\delta}-ES_{n,\delta})\}$ are relatively compact for every $\delta>0$ by Theorem 2.5, it is possible by a diagonal procedure to choose a subsequence $\{n''\}\subset\{n'\}$ such that

$$L(S_{n''}-ES_{n'',\tau})\to_w \rho_\tau$$

$$L(S_{n'',\delta_k}-ES_{n'',\delta_k})\to_w \gamma_k \quad \text{for each } k\in N.$$

Since $S_n-ES_{n,\tau}=(S_{n,\delta}-ES_{n,\delta})+(S_n^{(\delta)}-b_{n\tau}^{\delta})(0<\delta<\tau)$ it follows from Lemma 2.6 that

$$\rho_\tau=\gamma_k * c_\tau \mathrm{Pois}(\mu|B_{\delta_k}^c) \quad \text{for each } k\in N.$$

By ([33], III.2.2), $\{c_\tau \mathrm{Pois}(\mu|B_{\delta_k}^c)\}$ is relatively shift compact; by Theorem 1.6, μ is a Lévy measure and $c_\tau \mathrm{Pois}(\mu|B_{\delta_k}^c)\to_w c_\tau \mathrm{Pois}\mu$. It follows that $\gamma_k\to_w\gamma$ ($\{\gamma_k\}$ is relatively compact and $(c_\tau\mathrm{Pois}\mu)\hat{}(f)\neq 0$ for all $f\in B'$) and

$$\rho_\tau=\gamma * c_\tau \mathrm{Pois}\mu.$$

It is possible to find a sequence $n_k\uparrow\infty$ such that $L(S_{n_k,\delta_k}-ES_{n_k,\delta_k})\to_w\gamma$; therefore, by Lemma 2.9, γ is a centered Gaussian measure. Since obviously $\rho_\tau=\nu*\delta_{-z_\tau}$ for some $z_\tau\in B$, it follows that

$$\nu=\delta_{z_\tau}*\gamma*c_\tau\mathrm{Pois}\mu.$$

By Proposition 1.9, γ and μ are unique; therefore the whole sequence $L(S_n-ES_{n,\tau})$ converges to $\rho_\tau=\gamma*c_\tau\mathrm{Pois}\mu$. This proves the first and fourth statements of (3); (1) follows analogously.

Now (1) and Corollary 2.8 imply that $L(S_n^{(\tau)})\to_w \mathrm{Pois}(\mu|B_\tau^c)$. If γ_τ is any subsequential limit of $\{L(S_{n,\tau}-ES_{n,\tau})\}$ it follows from Lemma 2.6 that

$$\gamma_\tau*\mathrm{Pois}(\mu|B_\tau^c)=\gamma*c_\tau\mathrm{Pois}\mu$$

and therefore $\gamma_\tau=\gamma*c_\tau\mathrm{Pois}(\mu|B_\tau)$. This proves the third statement of (3).

It remains to prove (2). The integrability conditions given by Theorem 2.3 imply that for $\delta\in C(\mu)$

$$\lim_n\psi_n(\delta,f)=\int f^2 d(\gamma*c_\delta\mathrm{Pois}(\mu|B_\delta))$$
$$=\int f^2 d\gamma+\int f^2 d(c_\delta\mathrm{Pois}(\mu|B_\delta))=\Phi_\gamma(f,f)+\int f^2 d(\mu|B_\delta)$$

in the last step we have applied Lemma 1.8. But the scalar integrability property of Lévy measures (Theorem 1.4(2)) implies $\lim_{\delta\downarrow 0,\delta\in C(\mu)}\int f^2 d(\mu|B_\delta)=0$; this proves one of the statements of (2).

In order to prove the lim sup statement in (2), it is enough to show that for every $f\in B'$, $\psi(\cdot,f)=\limsup_n\psi_n(\cdot,f)$ is an increasing function (analogously for lim inf). Let $0<\delta<\beta$. Since obviously $Ef^2(X_{nj\delta})\leq Ef^2(X_{nj\beta})$, an elementary computation yields

$$\psi_n(\delta,f)\leq\psi_n(\beta,f)+\Sigma_j|(Ef(X_{nj\beta}))^2-(Ef(X_{nj\delta}))^2|.$$

We will prove that $\limsup_n$ of the last sum vanishes. In fact, let $\xi_{nj}=f(X_{nj\beta})$ and $\eta_{nj}=f(X_{nj\delta})$. Then

$$\Sigma_j|(E\xi_{nj})^2-(E\eta_{nj})^2|=\Sigma_j|E(\xi_{nj}+\eta_{nj})E(\xi_{nj}-\eta_{nj})|$$
$$\leq\max_j|E(\xi_{nj}+\eta_{nj})|\,\Sigma_j|E(\xi_{nj}-\eta_{nj})|$$
$$\leq 2\,\|f\|^2(\max_j E\,\|X_{nj\beta}\|)\beta\Sigma_j P\{\|X_{nj}\|>\delta\}$$
$$\to 0 \quad \text{as} \quad n\to\infty.$$

It follows that $\psi(\delta,f)\leq\psi(\beta,f)$. This completes the proof of (2). ☐

<u>2.11 Corollary</u>. Let $\{X_{nj}\}$ be an infinitesimal triangular array. Assume $L(S_n-x_n)\to_w\nu$ for some $\{x_n\}\subset B$. Then the following conditions are equivalent:

(1) ν is Gaussian,

(2) for every $f\in B'$ and every $\delta>0$, $\lim_n\Sigma_j P\{|f(X_{nj})|>\delta\}=0$,

(3) for every $\delta>0$, $\lim_n\Sigma_j P\{\|X_{nj}\|>\delta\}=0$.

<u>Proof.</u> (1) implies (3) by Theorem 2.10; the fact that (3) implies (2) is obvious. It remains to show that (2) implies (1). Let $f\in B'$; the triangular array $\{f(X_{nj})\}$ is infinitesimal and

$$L(\Sigma_j f(X_{nj})-f(x_n))=L(f(S_n-x_n))\to_w\nu\circ f^{-1},$$

By Theorem 2.10 (applied for B=R) (2) implies that $\nu\circ f^{-1}$ is Gaussian. Since this is true for every $f\in B'$, it follows that ν is Gaussian. ☐

The next proposition gives necessary conditions for convergence to a Gaussian limit.

2.12 Corollary. Let $\{X_{nj}\}$ be an infinitesimal triangular array. Suppose $L(S_n - x_n) \to_w \gamma$, a centered Gaussian measure. Then for every $\delta > 0$,

(1) $\lim_n \Sigma_j P\{ \|X_{nj}\| > \delta\} = 0$,

(2) $\lim_n \psi_n(\delta, f) = \Phi_\gamma(f,f)$,

(3) $L(S_n - ES_{n,\delta}) \to_w \gamma$,

$L(S_n^{(\delta)}) \to_w \delta_0$,

$L(S_{n,\delta} - ES_{n,\delta}) \to_w \gamma$

and $E(S_{n,\delta}) - x_n \to 0$ in B.

Proof. Only (2) requires some comment; but (2) follows at once by examining the proof of Theorem 2.10 (2), since in this case $\mu|\{0\}^c = 0$. □

2.13 Corollary. Let $\{X_j\}$ be a sequence of independent identically distributed random vectors. Suppose $L(T_n/n^{1/2}) \to_w \nu$, where $T_n = \Sigma_{j=1}^n X_j$. Then

(1) ν is a centered Gaussian measure,

(2) for every $f \in B'$, $Ef^2(X_1) < \infty$ and $Ef(X_1) = 0$,

(3) $\lim_{\lambda\to\infty} \lambda^2 P\{ \|X_1\| > \lambda\} = 0$.

Proof. (1) and (2) are classical one dimensional statements; we remark that they can be proved either using characteristic functions or using Theorem 2.10 for B=R, applied to $X_{nj} = X_j/n^{1/2}$ $(j=1,\dots,n; n\in N)$ which is obviously an infinitesimal array. The proofs are omitted. Now, by (1) and Corollary 2.12 (1), $\lim_{n\to\infty} nP\{ \|X_1\| > n^{1/2}\} = 0$ and (3) follows by interpolation. □

Conclusion (3) has also been recently proved by Pisier and Zinn [35].

The following proposition is a general direct central limit theorem.

2.14 Theorem. Let $\{X_{nj}\}$ be an infinitesimal triangular array. Assume

(1) there exists a σ-finite measure μ such that for every $\delta \in C(\mu)$,

$$\Sigma_j L(X_{nj})|B_\delta^c \to_w \mu|B_\delta^c ,$$

(2) there exist a sequentially w*-dense subset W of B' and a sequence $\delta_k \downarrow 0$ such that

$$\psi(f) = \lim_k \begin{Bmatrix} \lim\sup \\ \lim\inf \end{Bmatrix}_n \psi_n(\delta_k, f) \text{ exists for every } f \in W,$$

(3) there exist $\beta > 0$, $p > 0$ and a sequence $\{F_k\} \subset F$ such that

$$\lim_k \sup_n Eq_{F_k}^p (S_{n,\beta} - ES_{n,\beta}) = 0.$$

Then

(a) μ is a Lévy measure,

(b) there exists a centered Gaussian measure γ such that $\Phi_\gamma(f,f) = \psi(f)$ for every $f \in W$,

(c) for every $\tau \in C(\mu)$, $L(S_n - ES_{n,\tau}) \to_w \gamma * c_\tau \text{Pois}\,\mu$.

Proof. We first prove that $\{L(S_n - ES_{n,\beta})\}$ is relatively compact. Let $0 < \delta < \beta$. Since $S_n^{(\delta)} = S_{n,\beta}^{(\delta)} + S_n^{(\beta)}$ and $\{L(S_n^{(\delta)})\}$, $\{L(S_n^{(\beta)})\}$ are relatively compact by (1) and Lemma 2.4, it follows that $\{L(S_{n,\beta}^{(\delta)})\}$ is relatively compact. Consequently, so is $\{L(S_{n,\beta}^{(\delta)} - ES_{n,\beta}^{(\delta)})\}$.

Let $f \in W$. By (1) there exists $k = k(f)$ such that $\lim\sup_n \psi_n(\delta_k, f) < \infty$ and $\delta_k < \beta$. It follows by Chebyshev's inequality that $\{L(f(S_{n,\delta_k} - ES_{n,\delta_k}))\}$ is relatively compact. Since

$$f(S_{n,\beta} - ES_{n,\beta}) = f(S_{n,\delta_k} - ES_{n,\delta_k}) + f(S_{n,\beta}^{(\delta_k)} - ES_{n,\beta}^{(\delta_k)})$$

it follows that $\{L(f(S_{n,\beta} - ES_{n,\beta}))\}$ is relatively compact for every $f \in W$.

Now (3) and ([1], Theorem 2.3) imply (Chebyshev's inequality):

$\{L(S_{n,\beta}-ES_{n,\beta})\}$ is relatively compact.

Finally, since $S_n-ES_{n,\beta}=(S_{n,\beta}-ES_{n,\beta})+S_n^{(\beta)}$ we conclude that $\{L(S_n-ES_{n,\beta})\}$ is relatively compact.

Given a subsequence $\{n\}$ of N, let $\{n'\}\subset\{n\}$ be a subsequence such that $L(S_{n'}-ES_{n',\beta})\to_w \nu$. By Theorem 2.10, there exists a centered Gaussian measure γ and a Lévy measure λ such that for every $\tau\in C(\lambda)$,

$$(2.10) \qquad L(S_{n'}-ES_{n',\tau})\to_w \gamma * c_\tau \mathrm{Pois}\lambda$$

and (1) and (2) of Theorem 2.10 hold. But this implies, by (1) and (2) of the present theorem, that $\Phi_\gamma(f,f)=\psi(f)$ for every $f\in W$ and $\mu|\{0\}^c=\lambda|\{0\}^c$. Thus γ and λ are unique and (2.10) is true for the whole sequence $\{L(S_n-ES_{n,\tau})\}$. □

The next result gives sufficient conditions for convergence to a Gaussian limit.

<u>2.15 Corollary</u>. Let $\{X_{nj}\}$ be an infinitesimal triangular array. Assume

(1) there exist $\delta>0$ and a sequentially w*-dense subset W of B' such that $\psi(f)=\lim_n \psi_n(\delta,f)$ exists for every $f\in W$,

(2) for every $\varepsilon>0$

$$\lim_n \Sigma_j P\{\|X_{nj}\|>\varepsilon\}=0,$$

(3) there exist $\beta>0$, $p>0$ and a sequence $\{F_k\}\subset F$ such that

$$\lim_k \sup_n Eq^p_{F_k}(S_{n,\beta}-ES_{n,\beta})=0.$$

Then (a) there exists a centered Gaussian measure γ such that $\Phi_\gamma(f,f)=\psi(f)$ for every $f\in W$,

(b) for every $\tau>0$, $L(S_n-ES_{n,\tau})\to_w \gamma$.

<u>Proof</u>. By a computation analogous to the one in the proof of Theorem 2.10, it follows that for every $\tau>0$, $\lim_n \psi_n(\tau,f)=\psi(f)$. The conclusion follows now from Theorem 2.14. □

<u>Remark</u>. Assumption (3) in the two previous theorems is rather strong; however, some assumption of this nature is unavoidable in general results such as Theorems 2.14 and 2.15. Under special hypotheses on the space -or on the triangular array- it is possible to replace (3) by simpler conditions, in particular, conditions on the individual r.v's in the array rather than on the row sums (See e.g. Theorems 4.2, 4.3).

3. TRIANGULAR ARRAYS AND THEIR ASSOCIATED POISSON MEASURES

Let $\{X_{nj}\}$ be a triangular array. The relation between the relative compactness and convergence of $\{L(S_n)\}$ and those of $\{\mathrm{Pois}(\Sigma_j L(X_{nj}))\}$ (the "accompanying laws") is a cornerstone of the classical approach to the one-dimensional central limit theorem (see the remark following Theorem 2.5). In the approach to the general central limit theorem in Banach spaces presented in §2, one does not need to compare $\{L(S_n)\}$ and $\{\mathrm{Pois}(\Sigma_j L(X_{nj}))\}$, but only $\{L(S_n^{(\delta)})\}$ and $\{\mathrm{Pois}(\Sigma_j L(X_{nj}^{(\delta)}))\}$, which under appropriate conditions are close in a very strong sense for large n (Corollary 2.8). Nevertheless, the relation between $\{L(S_n)\}$ and $\{\mathrm{Pois}(\Sigma_j L(X_{nj}))\}$ is still of considerable interest. Le Cam [28] has proved that in a general Banach space the relative shift compactness of $\{\mathrm{Pois}(\Sigma_j L(X_{nj}))\}$ implies that of $\{L(S_n)\}$; but a counterexample in [28] shows that the converse statement need not be true, even for symmetric infinitesimal triangular arrays. One of the main results of this section (Theorem 3.1) consists in specifying shifts $\{a_n\}$ such that $\{L(S_n-a_n)\}$ is relatively compact in Le Cam's theorem. We also give some partial converses of Theorem 3.1.

<u>3.1 Theorem</u>. Let $\{X_{nj}\}$ be a triangular array.

(1) If $\{\mathrm{Pois}(\Sigma_j L(X_{nj}))\}$ is relatively shift compact, then for every $\tau>0$ $\{L(S_n-ES_{n,\tau})\}$ is relatively compact.

(2) If $\{\mathrm{Pois}(\Sigma_j L(X_{nj}))\}$ is relatively compact, then so is

$\{L(S_n)\}$.

Proof. (1) By ([28], Théorème 3), $\{L(S_n)\}$ is relatively shift compact. The result follows now from Theorems 1.4 (1) and 2.5.

(2) By Corollary 1.5, $\{c_\tau \mathrm{Pois}(\Sigma_j L(X_{nj}))\}$ is relatively compact. Hence $\{ES_{n,\tau}=\int_{B_\tau} x\,d(\Sigma_j L(X_{nj}))(x)\}$ is relatively compact and the result follows from (1). □

Part (2) of the above theorem has been obtained by Araujo [5] with a different method.

One can prove further results for (properly centered) infinitesimal triangular arrays; in particular, Theorem 3.4 generalizes one direction of assertion (II) in the remark following Theorem 2.5.

3.2 Theorem. Let $\{X_{nj}\}$ be an infinitesimal triangular array, $\tau>0$. If $\{\mathrm{Pois}(\Sigma_j L(X_{nj}-EX_{nj\tau}))\}$ is relatively shift compact, then it is relatively compact and so is $\{L(S_n-ES_{n,\tau})\}$.

Proof. Let $Y_{nj}=X_{nj}-EX_{nj\tau}$, $z_{nj}=\int_{\|Y_{nj}\|\le\tau} Y_{nj}\,dP$, $z_n=\Sigma_j z_{nj}$. By Corollary 1.5, $\{\mathrm{Pois}(\Sigma_j L(Y_{nj}))*\delta_{-z_n}\}=\{c_\tau \mathrm{Pois}(\Sigma_j L(Y_{nj}))\}$ is relatively compact. Thus the result will be proved if we show that $\{z_n\}$ is relatively compact. Now

$$z_{nj}=\int_{\|Y_{nj}\|\le\tau} X_{nj}\,dP-EX_{nj\tau}+EX_{nj\tau}-(EX_{nj\tau})P\{\|Y_{nj}\|\le\tau\}$$
$$=u_{nj}-v_{nj}+w_{nj},$$

where $u_{nj}=\int_{\|Y_{nj}\|\le\tau,\ \|X_{nj}\|>\tau} X_{nj}\,dP$, $v_{nj}=\int_{\|Y_{nj}\|>\tau,\ \|X_{nj}\|\le\tau} X_{nj}\,dP$,

$w_{nj}=(EX_{nj\tau})P\{\|Y_{nj}\|>\tau\}$. To prove that $\{z_n\}$ is relatively compact it is enough to prove that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are relatively compact, where $u_n=\Sigma_j u_{nj}$, $v_n=\Sigma_j v_{nj}$, $w_n=\Sigma_j w_{nj}$.

By infinitesimality, $\lim_n \sup_j E\|X_{nj\tau}\|=0$; by Theorem 1.4 (1), $\sup_n \Sigma_j P\{\|Y_{nj}\|>\tau\}<\infty$. Therefore, as $n\to\infty$

$$\|w_n\|\le\Sigma_j\|w_{nj}\|\le\sup_j E\|X_{nj\tau}\|\,\Sigma_j P\{\|Y_{nj}\|>\tau\}\to 0.$$

We prove next that $\{u_n\}$ is relatively compact. Let $\varepsilon>0$. By Theorem 1.4(1), $a=\sup_n\Sigma_j P\{\|Y_{nj}\|>\tau/2\}<\infty$ and there exists a compact convex symmetric set K such that $\sup_n\Sigma_j P\{Y_{nj}\in B^c_{\tau/2}\cap K^c\}<\varepsilon/4\tau$. Let $\delta=\min\{\tau,\varepsilon/a\}$ and let $n_\circ$ be such that $\max_j E\|X_{nj\tau}\|<\delta/2$ for $n\ge n_\circ$; then $\{\|X_{nj}\|>\tau\}\subset\{\|Y_{nj}\|>\tau/2\}$ for $n\ge n_\circ$. Now if

$$A_{nj}=\{\|Y_{nj}\|\le\tau,\ \|X_{nj}\|>\tau, Y_{nj}\in K\},\quad u'_{nj}=\int_{A_{nj}} Y_{nj}\,dP,\quad u'_n=\Sigma_j u'_{nj},$$

it follows that $\|u_n-u'_n\|<\varepsilon$ for all $n\ge n_\circ$. Let K° be the polar set of K, $K^\circ=\{f\in B':\sup_{x\in K}|f(x)|\le 1\}$. For $f\in K^\circ$, $n\ge n_\circ$ we have

$$|f(u'_n)|\le\Sigma_j|f(u'_{nj})|\le\Sigma_j\int_{\|Y_{nj}\|>\tau/2,\ Y_{nj}\in K}|f(Y_{nj})|\,dP\le a.$$

Therefore $u'_n\in aK^{\circ\circ}=aK$ for $n\ge n_\circ$. We have shown: for every $\varepsilon>0$, there exists a compact set K_ε such that $\{u_n\}\subset K_\varepsilon+B_\varepsilon$. It follows that $\{u_n\}$ is relatively compact. Similarly for $\{v_n\}$.

The second statement is a consequence of Theorem 3.1. □

3.3 Lemma. Let $\{X_{nj}\}$ be an infinitesimal triangular array. Assume that $\{L(S_n)\}$ is relatively shift compact. Then for every $\tau>0$, $r>0$

$$\lim_n\sup_{f\in B'_r}|L(S_n-ES_{n,\tau})\hat{}(f)-\mathrm{Pois}(\Sigma_j L(X_{nj}-EX_{nj\tau}))\hat{}(f)|=0.$$

Proof. We will only sketch the proof; it is a classical argument with characteristic functions. Let $Y_{nj}=X_{nj}-EX_{nj\tau}$. For $f\in B'$,

$$|L(S_n-ES_{n,\tau})\hat{}(f)-\mathrm{Pois}(\Sigma_j L(Y_{nj}))\hat{}(f)|=$$
$$=|\Pi_j L(Y_{nj})\hat{}(f)-\Pi_j\mathrm{Pois}(L(Y_{nj}))\hat{}(f)|$$
$$\le\Sigma_j|L(Y_{nj})\hat{}(f)-\exp\{L(Y_{nj})\hat{}(f)-1\}|$$
$$\le e^2\Sigma_j|L(Y_{nj})\hat{}(f)-1|^2$$
$$\le e^2\sup_j|L(Y_{nj})\hat{}(f)-1|\,\Sigma_j|L(Y_{nj})\hat{}(f)-1|.$$

By infinitesimality, $\sup_j|L(Y_{nj})\hat{}(f)-1|\to 0$ as $n\to\infty$ uniformly on B'_r. Using the inequality $|\cos t-1|\le t^2/2$ $(t\in R)$, one can show that

$$\Sigma_j|\mathrm{Re}L(Y_{nj})\hat{}(f)-1|\le(1/2)Ef^2(T_n)+2\Sigma_j P\{\|X_{nj}\|>\tau\},$$

where $T_n=S_{n,\tau}-ES_{n,\tau}$. Using $|\sin t-t|\le|t|^3/6$ $(t\in R)$ one obtains

$$\Sigma_j |\mathrm{Im}\,L(Y_{nj})^\wedge(f)| \le (1/3)\tau \|f\| Ef^2(T_n) + (1+\tau\|f\|)\Sigma_j P\{\|X_{nj}\| > \tau\}.$$

Now Theorems 2.2, 2.3 and 2.5 imply: $\sup_n \sup_{f\in B'_r} \Sigma_j |L(Y_{nj})^\wedge(f) - 1|$ is finite. □

<u>3.4 Theorem.</u> Let $\{X_{nj}\}$ be an infinitesimal triangular array, $\tau>0$. If $\mathrm{Pois}(\Sigma_j L(X_{nj}-EX_{nj\tau}))*\delta_{x_n} \to_w \nu$ for some sequence $\{x_n\}\subset B$, then $L(S_n - ES_{n,\tau})*\delta_{x_n} \to_w \nu$.

<u>Proof.</u> Follows from Theorem 3.2, Lemma 1.3 (or [33], VI.4.5) and Lemma 3.3. □

We show next that if in a Banach space the converse of Theorem 3.1 is true for symmetric infinitesimal triangular arrays, then it is true for all (properly centered) infinitesimal triangular arrays. We will denote by $\{\varepsilon_{nj}\}$ a triangular array of random variables with $P\{\varepsilon_{nj}=1\}=P\{\varepsilon_{nj}=-1\}=1/2$ for all n,j and independent of $\{X_{nj}\}$.

<u>3.5 Lemma.</u> Let $\{X_{nj}\}$ be a triangular array. Assume that $\{L(S_n)\}$ is relatively shift compact and $\{\Sigma_j L(X_{nj})|B^c_\delta\}$ is relatively compact for some $\delta>0$. Then for every $\tau\ge\delta$, $\{L(\Sigma_j \varepsilon_{nj}(X_{nj}-EX_{nj\tau}))\}$ is relatively compact.

<u>Proof.</u> Let $Y_{nj}=X_{nj\tau}-EX_{nj\tau}$, $\tau\ge\delta$. Since

$$\Sigma_j \varepsilon_{nj}(X_{nj}-EX_{nj\tau}) = \Sigma_j \varepsilon_{nj} Y_{nj} + \Sigma_j \varepsilon_{nj} X^\tau_{nj},$$

it is enough to prove that both $\{L(\Sigma_j \varepsilon_{nj} Y_{nj})\}$ and $\{L(\Sigma_j \varepsilon_{nj} X^\tau_{nj})\}$ are relatively compact.

Since $\Sigma_j L(\varepsilon_{nj}X_{nj})|B^c_\tau = \Sigma_j (1/2)(L(X_{nj})+L(-X_{nj}))|B^c_\tau$, it follows that $\{\Sigma_j L(\varepsilon_{nj}X_{nj})|B^c_\tau\}$ is relatively compact. By Lemma 2.4 and the obvious equality $\varepsilon_{nj}X^\tau_{nj}=(\varepsilon_{nj}X_{nj})^\tau$, we conclude that $\{L(\Sigma_j \varepsilon_{nj} X^\tau_{nj})\}$ is relatively compact.

Let F be a closed subspace of B. Then

$$\mathrm{Eq}_F(\Sigma_j \varepsilon_{nj} Y_{nj}) \le 2\mathrm{Eq}_F(\Sigma_j Y_{nj}) \tag{3.1}$$

In fact, if $\lambda_n = L(\{\varepsilon_{nj}: 1\le j\le k_n\})$, $t=\{t_j: 1\le j\le k_n\}\in\{-1,1\}^{k_n}$, $S(t)=\{j: t_j=1\}$, then

$$\mathrm{Eq}_F(\Sigma_j \varepsilon_{nj} Y_{nj}) = \int \mathrm{Eq}_F(\Sigma_j t_j Y_{nj})\, d\lambda_n(t)$$

$$= \int \mathrm{Eq}_F(\Sigma_{j\in S(t)} Y_{nj} - \Sigma_{j\notin S(t)} Y_{nj})\, d\lambda_n(t)$$

$$\le \int \{\mathrm{Eq}_F(\Sigma_{j\in S(t)} Y_{nj}) + \mathrm{Eq}_F(\Sigma_{j\notin S(t)} Y_{nj})\}\, d\lambda_n(t)$$

$$\le 2\int \mathrm{Eq}_F(\Sigma_j Y_{nj})\, d\lambda_n(t)$$

$$= 2\mathrm{Eq}_F(\Sigma_j Y_{nj})$$

; in the fourth step we have used a well-known elementary inequality.

By Theorem 2.5, $\{L(S_{n,\tau} - ES_{n,\tau})\}$ is relatively compact. Hence Theorem 2.3 and (3.1) imply

$$\sup_n E\|\Sigma_j \varepsilon_{nj} Y_{nj}\| < \infty \quad \text{and}$$

$$\lim_k \sup_n \mathrm{Eq}_{F_k}(\Sigma_j \varepsilon_{nj} Y_{nj}) = 0 \text{ for any full sequence } \{F_k\}\subset F.$$

Now Chebyshev's inequality implies that $\{L(\Sigma_j \varepsilon_{nj} Y_{nj})\}$ is both spherically and flatly concentrated and it follows from ([1], Theorem 2.3) that the sequence is relatively compact. □

<u>Remark.</u> Inequality (3.1) is, of course, a simple case of a comparison principle.

<u>3.6 Theorem.</u> Let B be a separable Banach space with the following property: if $\{Y_{nj}\}$ is a symmetric infinitesimal triangular array and $\{L(\Sigma_j Y_{nj})\}$ is relatively compact, then $\{\mathrm{Pois}(\Sigma_j L(Y_{nj}))\}$ is relatively compact.

Let $\{X_{nj}\}$ be an infinitesimal triangular array and assume that $\{L(S_n)\}$ is relatively shift compact. Then $\{\mathrm{Pois}(\Sigma_j L(X_{nj}-EX_{nj\tau}))\}$ is relatively compact for every $\tau>0$.

<u>Proof.</u> By Theorem 2.2 and Lemma 3.5, $\{L(\Sigma_j \varepsilon_{nj}(X_{nj}-EX_{nj\tau}))\}$ is relatively compact. Since $\{\varepsilon_{nj}(X_{nj}-EX_{nj\tau})\}$ is symmetric and infinites-

imal, it follows from the assumption on B that $\{\mathrm{Pois}(\Sigma_j L(\varepsilon_{nj}(X_{nj}-EX_{nj\tau})))\}$ is relatively compact.

Let $\mu_{nj}=L(X_{nj}-EX_{nj\tau})$, $\nu_{nj}=L(\varepsilon_{nj}(X_{nj}-EX_{nj\tau}))$; then $\nu_{nj}=(1/2)\mu_{nj}+(1/2)\bar{\mu}_{nj}$. Therefore $\mathrm{Pois}((1/2)\Sigma_j\mu_{nj})$ is a factor of $\mathrm{Pois}(\Sigma_j\nu_{nj})$. By ([33], III.2.2), $\{\mathrm{Pois}((1/2)\Sigma_j\mu_{nj})\}$ is relatively shift compact, and consequently so is $\{\mathrm{Pois}(\Sigma_j\mu_{nj})\}$. The conclusion follows now from Theorem 3.2. □

The following technical lemma is needed for the proof of Theorem 3.9.

<u>3.7 Lemma</u>. Let $\{\mu_{nj}\},\{\nu_{nj}\}(j=1,\dots,k_n;\ n\in N)$ be p.m.'s on B. Assume

(1) ν_{nj} is symmetric for all n,j

(2) $\{\mathrm{Pois}(\Sigma_j\mu_{nj}*\nu_{nj})\}$ is relatively compact.

Then $\{\mathrm{Pois}(\Sigma_j\mu_{nj})\}$ is relatively compact.

<u>Proof.</u> For each $n\in N$, let $\{X_{nji},Y_{nji},\delta_{nj}:\ j=1,\dots,k_n;\ i=0,1,\dots\}$ be independent r.v.'s such that

$X_{nj0}=0$, $L(X_{nji})=\mu_{nj}$ for $i\geq 1$

$Y_{nj0}=0$, $L(Y_{nji})=\nu_{nj}$ for $i\geq 1$

δ_{nj} is a real random variable with $L(\delta_{nj})=\mathrm{Poisson}$ (1).

Let $V_n=\Sigma_{j=1}^{k_n}\Sigma_{i=0}^{\delta_{nj}}X_{nji}$, $W_n=\Sigma_{j=1}^{k_n}\Sigma_{i=0}^{\delta_{nj}}Y_{nji}$. Then $L(V_n)=\mathrm{Pois}(\Sigma_j\mu_{nj})$ and by the symmetry of ν_{nj},

$$L(V_n-W_n)=L(V_n+W_n)=\lambda_n,$$

where $\lambda_n=\mathrm{Pois}(\Sigma_j\mu_{nj}*\nu_{nj})$.

Let K be a measurable convex symmetric set. Since $2V_n=(V_n+W_n)+(V_n-W_n)$, it follows that

$$P\{V_n\notin K\}\leq P\{V_n+W_n\notin K\}+P\{V_n-W_n\notin K\}=2\lambda_n(K^c).$$

Therefore the tightness of $\{\lambda_n\}$ implies that of $\{\mathrm{Pois}(\Sigma_j\mu_{nj})\}$. □

As shown in [28], Theorem 3.1 has no converse in general

Banach spaces. We shall give next two partial converses in the infinite dimensional situation. We need some definitions for the first one. A Banach space B has the <u>approximation property</u> if there exists a net $\{\Pi_\lambda\}_{\lambda\in D}$ of continuous operators of B into B with finite dimensional range such that Π_λ converges to the identity operator uniformly on compact subsets of B. Let Φ be a real continuous positive definite function on B with $\Phi(0)=1$; we say that the norm $\|\cdot\|$ is <u>accessible with respect to Φ</u> if for every $\varepsilon>0$ there exists $c(\varepsilon)>0$ such that $\|x\|>\varepsilon$ implies $1-\Phi(x)>c(\varepsilon)$. The class of spaces possessing the approximation property and a norm accessible in the above sense includes the separable L^p spaces, $1\leq p\leq 2$ (here $\Phi(x)=\exp(-\|x\|^p)$).

The sufficiency part of the following lemma is proved in Kuelbs [26]; the proof of necessity follows easily from the arguments in Kuelbs' paper and is omitted. We denote by Π_λ^t the transpose of Π_λ; it is an operator of B' into B' with finite-dimensional range.

<u>3.8 Lemma</u>. Let B be a separable Banach space with the approximation property via $\{\Pi_\lambda\}_{\lambda\in D}$ and with its norm accessible with respect to Φ in the above sense, and let ρ be the cylinder measure on B' with $\hat{\rho}=\Phi$. Then a family $\{\mu_\alpha\}$ of p.m.'s on B is relatively compact if and only if

(1) $\{\mu_\alpha\circ\Pi_\lambda^{-1}\}_\alpha$ is tight for each $\lambda\in D$, and

(2) $\lim_{\lambda'}\sup_\alpha\lim_\lambda\int_{B'}\{1-\hat{\mu}_\alpha(\Pi_\lambda^t f-\Pi_{\lambda'}^t f)\}d\rho(f)=0$.

The first assertion of the following proposition is mentioned in Le Cam ([28], p.248). (Observe, however, that the inequality stated there is not correct).

<u>3.9 Theorem.</u> Let B be as in Lemma 3.8 and $\{X_{nj}\}$ a triangular array of B-valued r.v.'s. Assume that $\{L(S_n)\}$ is relatively shift compact.

(1) If $\{X_{nj}\}$ is symmetric, then $\{\mathrm{Pois}(\Sigma_j L(X_{nj}))\}$ is relatively compact.

(2) If $\{X_{nj}\}$ is infinitesimal, then $\{\mathrm{Pois}(\Sigma_j L(X_{nj}-EX_{nj\tau}))\}$ is relatively compact for every $\tau>0$.

Proof. (1) Let $\mu_{nj}=L(X_{nj})$, $\nu_{nj}=\mu_{nj}*\tilde{\mu}_{nj}$, $\nu_n=\Pi_j\nu_{nj}$. We prove first that $\{Pois(\Sigma_j\nu_{nj})\}$ is relatively compact by verifying (1) and (2) of Lemma 3.8. For each $\lambda\in D$, $\{\nu_n\circ\Pi_\lambda^{-1}\}_n$ is relatively compact; therefore $\{(Pois\Sigma_j\nu_{nj})\circ\Pi_\lambda^{-1}\}=\{Pois\Sigma_j(\nu_{nj}\circ\Pi_\lambda^{-1})\}$ is relatively compact by the remark following Theorem 2.5. Now the inequality $\exp(x-1)\geq x$ for $0\leq x\leq 1$ and the fact that $\hat{\nu}_{nj}(f)\geq 0$ $(f\in B')$ imply that $\hat{\nu}_{nj}(f)\leq\exp(\hat{\nu}_{nj}(f)-1)$ and consequently

$$1-Pois(\Sigma_j\nu_{nj})\hat{}(f)\leq 1-\hat{\nu}_n(f)\quad (f\in B').$$

Therefore

$$\int_{B'}\{1-(Pois\Sigma_j\nu_{nj})\hat{}(\Pi_\lambda^t f-\Pi_\lambda^t,f)\}d\rho(f)\leq\int_{B'}\{1-\hat{\nu}_n(\Pi_\lambda^t f-\Pi_\lambda^t,f)\}d\rho(f).$$

Applying Lemma 3.8 in both directions, it follows that $\{Pois(\Sigma_j\nu_{nj})\}$ is relatively compact; by Lemma 3.7, so is $\{Pois(\Sigma_n\mu_{nj})\}$.

(2) follows from (1) and Theorem 3.6. □

3.10 Theorem. Let B be as in Lemma 3.8 and $\{X_{nj}\}$ an infinitesimal triangular array of B-valued r.v.'s. If $L(S_n-x_n)\to_w\nu$ for some sequence $\{x_n\}\subset B$, then for every $\tau>0$ $Pois(\Sigma_j L(X_{nj}-EX_{nj\tau}))*\delta_{z_n}\to_w\nu$, where $z_n=ES_{n,\tau}-x_n$.

Proof. Follows from Theorem 3.9, Lemma 1.3 (or [33], VI.4.5) and Lemma 3.3. □

The second partial converse of Theorem 3.1 is valid in any separable Banach space. The symmetric case appears in de Acosta and Samur [2].

3.11 Theorem. Let $\{X_{nj}\}$ be a triangular array such that $L(X_{nj})$ does not depend on j. Assume that $\{L(S_n)\}$ is relatively shift compact.

(1) If $\{X_{nj}\}$ is symmetric, then $\{Pois(\Sigma_j L(X_{nj}))\}$ is relatively compact.

(2) If $\{X_{nj}\}$ is infinitesimal, then $\{Pois(\Sigma_j L(X_{nj}-EX_{nj\tau}))\}$ is relatively compact for every $\tau>0$.

Proof. (1) is proved as indicated in ([2], Theorem 2.2)(It is easily

seen that the hypothesis "$k_n\to\infty$" stated there is not needed).

(2) follows from (1) by arguing as in the proof of Theorem 3.6. □

Remark. Theorems 3.4 and 3.10 together extend Varadhan's result on the comparison of the laws of row sums of infinitesimal triangular arrays in Hilbert space and their associated "accompanying laws" (see [33], Theorem VI.6.2) to the class of spaces considered in Lemma 3.8. A different kind of application of Theorem 3.1 will be given in §4.

4. THE DIRECT CENTRAL LIMIT THEOREM IN C(S) AND IN TYPE 2 SPACES

The first result is a sufficient condition for the relative compactness of a family of Poisson measures. Theorem 4.4 gives a different sufficient condition.

Let us recall ([22], [39]) that if B,E are Banach spaces, a continuous linear map $u:B\to E$ is of type 2 if there exists a constant M such that $E\|\Sigma_{j=1}^n\varepsilon_j u(x_j)\|^2\leq M\Sigma_{j=1}^n\|x_j\|^2$ for every finite sequence $\{x_j:j=1,\ldots,n\}\subset B$, where $\{\varepsilon_j:j\in N\}$ is a symmetric Bernoulli sequence. It is known that the induced maps $u_F:B/u^{-1}(F)\to E/F$, with F a closed subspace of E, are of type 2 with the same constant as u; hence there exists a constant C such that

$$(4.1)\qquad Eq_F^2(\Sigma_{j=1}^n u(X_j))\leq C\Sigma_{j=1}^n Eq^2_{u^{-1}(F)}(X_j)$$

for any closed subspace F of E and independent r.v.'s $\{X_j:j=1,\ldots,n\}$ such that $E\|X_j\|^2<\infty$ and $EX_j=0$ $(j=1,\ldots,n)$.

4.1 Theorem. Let B,E be separable Banach spaces, $u:B\to E$ a continuous linear map of type 2, $\{\mu_\alpha\}$ a family of σ-finite positive measures on B such that

(1) $\mu_\alpha(B_\varepsilon^c)<\infty$ for all α, all ε>0 and $\{\mu_\alpha|B_1^c\}$ is relatively compact,

(2) for every $f\in B'$, $\sup_\alpha\int_{B_1}f^2d\mu_\alpha<\infty$,

(3) there exists a sequence $\{F_n\} \subset F(E)$ satisfying

$$\lim_n \sup_\alpha \int_{B_1 u^{-1}(F_n)} q^2 \, d\mu_\alpha = 0 .$$

Then $\mu_\alpha \circ u^{-1}$ is a Lévy measure for every α and $\{c_\tau \mathrm{Pois}(\mu_\alpha \circ u^{-1})\}$ is relatively compact for every $\tau > 0$.

Proof. Assume first that each μ_α is symmetric. By (1), $\{(\mu_\alpha|B_1^c)\circ u^{-1}\}$ is relatively compact, and therefore so is $\{\mathrm{Pois}(\mu_\alpha|B_1^c)\circ u^{-1}\}$. It follows that it is enough to prove that $(\mu_\alpha|B_1)\circ u^{-1}$ is a Lévy measure for each α and $\{\mathrm{Pois}(\mu_\alpha|B_1)\circ u^{-1}\}$ is relatively compact. Hence we may suppose that $\mu_\alpha(B_1^c)=0$ for each α.

Let $\mu_\alpha^r = \mu_\alpha|B_{1/r}^c$, and for each α, each $r \in N$, let $\{Z_{\alpha j}^r : j \in N\}$ be independent B-valued r.v.'s such that $L(Z_{\alpha j}^r) = \mu_\alpha^r / \|\mu_\alpha^r\|$ if $\|\mu_\alpha^r\| > 0$ and $L(Z_{\alpha j}^r) = \delta_o$ otherwise. Then for $F \in F(E)$, $G = u^{-1}(F)$, we have by (4.1)

$$Eq_F^2(\Sigma_{j=1}^k u(Z_{\alpha j}^r)) \le C\Sigma_{j=1}^k Eq_G^2(Z_{\alpha j}^r) = CkEq_G^2(Z_{\alpha 1}^r),$$

and therefore

$$\begin{aligned}(4.2)\quad \int q_F^2 d\mathrm{Pois}(\mu_\alpha^r \circ u^{-1}) &= \exp(-\|\mu_\alpha^r\|)\Sigma_{k=0}^\infty (k!)^{-1} \int q_F^2 d(\mu_\alpha^r \circ u^{-1})^k \\ &= \exp(-\|\mu_\alpha^r\|)\Sigma_{k=1}^\infty (k!)^{-1} \|\mu_\alpha^r\|^k Eq_F^2(\Sigma_{j=1}^k u(Z_{\alpha j}^r)) \\ &\le \exp(-\|\mu_\alpha^r\|) CEq_G^2(Z_{\alpha 1}^r)\Sigma_{k=1}^\infty \|\mu_\alpha^r\|^k/(k-1)! \\ &= C\int q_G^2 d\mu_\alpha^r .\end{aligned}$$

From (4.2), assumption (3) and Chebyshev's inequality it follows that $\{\mathrm{Pois}(\mu_\alpha^r \circ u^{-1})\}_{\alpha,r}$ is flatly concentrated ([1],§2). On the other hand, if $g \in E'$, then by Lemma 1.8(1)

$$\int g^2 d\mathrm{Pois}(\mu_\alpha^r \circ u^{-1}) = \int g^2 d(\mu_\alpha^r \circ u^{-1}) = \int (g\circ u)^2 d\mu_\alpha ;$$

hence $\{\mathrm{Pois}(\mu_\alpha^r \circ u^{-1})\circ g^{-1}\}_{\alpha,r}$ is tight by assumption (2) and Chebyshev's inequality. Now ([1], Theorem 2.3) implies that for each fixed α, $\{\mathrm{Pois}(\mu_\alpha^r \circ u^{-1})\}_r$ is a tight sequence; by Theorem 1.6, it follows that $\mu_\alpha \circ u^{-1}$ is a Lévy measure. Repeating the argument, we obtain: $\{\mathrm{Pois}(\mu_\alpha \circ u^{-1})\}_\alpha$ is relatively compact. This proves the theorem in the symmetric case.

In the general case, let $\nu_\alpha = \mu_\alpha + \bar{\mu}_\alpha$. Then $\{\nu_\alpha\}$ satisfies assumptions (1)-(3) and the preceding argument proves that $\nu_\alpha \circ u^{-1}$ is a Lévy measure and $\{\mathrm{Pois}(\nu_\alpha \circ u^{-1})\}_\alpha$ is relatively compact. By Corollary 1.7, $\mu_\alpha \circ u^{-1}$ is a Lévy measure for each α; by ([33], III.2.2) and Corollary 1.5, $\{c_\tau \mathrm{Pois}(\mu_\alpha \circ u^{-1})\}_\alpha$ is relatively compact. □

Remarks. (1) The power 2 in Theorem 4.1 may be replaced by $p \in [1,2]$.

(2) Theorem 4.1 together with Theorem 3.1 may be used to prove the theorem of Hoffmann-Jorgensen and Pisier [22], direct part; for a more straightforward proof, see Theorem 4.3 below.

(3) Theorem 4.1 may also be used to prove that if u is of type 2 and $\int \min(1, \|x\|^2) d\mu(x) < \infty$, then $\mu \circ u^{-1}$ is a Lévy measure (Araujo and Giné [6]).

(4) For an application of Theorem 4.1 to the study of domains of attraction of stable laws in Banach spaces we refer to Araujo and Giné [7].

Next we give general results on the convergence of triangular arrays in type 2 spaces. They are immediate consequences of Theorem 2.14 and Corollary 2.15. The present statements differ from those in Section 2 in that, on account of (4.1), the condition for flat concentration (3) can be given here in terms of the individual r.v.'s in the array. Recall that B is of type 2 if Id_B is of type 2.

The first statement is a general direct central limit theorem. We use the notation of Theorem 2.10.

4.2 Theorem. Let $\{X_{nj}\}$ be an infinitesimal triangular array of B-valued r.v.'s, B a type 2 space. Assume

(1) there exists a σ-finite measure μ such that for every $\delta \in C(\mu)$,

$$\Sigma_j L(X_{nj})|B_\delta^c \to_w \mu|B_\delta^c ,$$

(2) there exist a sequentially w*-dense subset W of B' and a sequence $\delta_k \downarrow 0$ such that

$$\psi(f) = \lim_k \left\{ \begin{matrix} \lim\sup \\ \lim\inf \end{matrix} \right\}_n \psi_n(\delta_k, f) \text{ exists for every } f \in W,$$

(3) there exist $\delta>0$ and a sequence $\{F_k\}\subset\mathcal{F}$ such that

$$\lim_k \sup_n \Sigma_j Eq^2_{F_k}(X_{nj\delta}-EX_{nj\delta})=0.$$

Then

(a) μ is a Lévy measure,

(b) there exists a centered Gaussian measure γ such that $\Phi_\gamma(f,f)=\psi(f)$ for every $f\in W$,

(c) for every $\tau\in C(\mu)$, $\mathcal{L}(S_n-ES_{n,\tau})\to_w \gamma * c_\tau \text{Pois}\mu$.

<u>Remarks.</u> (1) Theorem 4.2 may also be derived from Theorems 3.1 and 4.1 and the one-dimensional central limit theorem.

(2) Condition (3) in Theorems 4.2 and 4.3 is a necessary condition in the converse central limit theorem in duals of type 2 spaces and in other cotype 2 spaces, ℓ_1 included (see Theorem 6.7).

(3) The exposition [18] contains a general direct central limit theorem for the Hilbert space case which is somewhat weaker than Theorem 4.2.

The next result gives sufficient conditions for convergence to a Gaussian limit. For the particular case of bounded variances, it has been proved by Garling [16].

<u>4.3 Theorem.</u> Let $\{X_{nj}\}$ be an infinitesimal triangular array of B-valued r.v.'s, B a type 2 space. Assume

(1) there exist $\delta>0$ and a sequentially w*-dense subset W of B' such that $\psi(f)=\lim_n \psi_n(\delta,f)$ exists for every $f\in W$,

(2) for every $\varepsilon>0$,

$$\lim_n \Sigma_j P\{\|X_{nj}\|>\varepsilon\}=0,$$

(3) there exist $\beta>0$ and a sequence $\{F_k\}\subset\mathcal{F}$ such that

$$\lim_k \sup_n \Sigma_j Eq^2_{F_k}(X_{nj\beta}-EX_{nj\beta})=0.$$

Then

(a) there exists a centered Gaussian measure γ such that $\Phi_\gamma(f,f)=\psi(f)$ for every $f\in W$, and

(b) for every $\tau>0$, $\mathcal{L}(S_n-ES_{n,\tau})\to_w \gamma$.

<u>Remarks.</u> (1) Theorem 4.3 contains the direct part of the central limit theorem of Hoffmann-Jorgensen and Pisier [22].

(2) When specialized to the case of Hilbert space, Theorem 4.3 gives an improvement of the direct part of the central limit theorem, Gaussian case, in Varadhan [38] (see [33], VI.6.3).

The special framework in Theorems 4.4 and 4.5 is motivated by the fact that in the application to C(S)-valued r.v's, (Theorem 4.9) the distributions of the random vectors are supported by a space of Lipschitz functions, which in general is not separable.

Given a Banach space E and a compact convex symmetric subset $K\subset E$, we shall denote $(E_K, \|\cdot\|_K)$ the Banach space generated by K ([36], p.97).

<u>4.4 Theorem.</u> Let $(B, \|\cdot\|)$ be a normed linear space, $\mathcal{B}_o$ a σ-algebra of subsets of B such that $\|\cdot\|$ is $\mathcal{B}_o$ measurable. Let E be a separable Banach space, $\mathcal{B}_E$ its Borel σ-algebra, K a compact convex symmetric subset of E. Let $u:B\to E$ be a $\mathcal{B}_o$-$\mathcal{B}_E$ measurable linear map such that $u(B)\subset E_K$ and u is of type 2 as a map from B into E_K. Let $\{\mu_\alpha\}$ be a family of σ-finite positive measures on $(B,\mathcal{B}_o)$ such that for some $\delta>0$,

(1) $\mu_\alpha(B^c_\delta)<\infty$ for all α and $\{(u_\alpha|B^c_\delta)\circ u^{-1}\}$ is relatively compact on E,

(2) $\sup_\alpha \int_{B_\delta} \|x\|^2 d\mu_\alpha(x)<\infty$.

Then $\mu_\alpha\circ u^{-1}$ is a Lévy measure on E for every α and $\{c_\tau \text{Pois}(\mu_\alpha\circ u^{-1})\}$ is relatively compact on E for every $\tau>0$.

<u>Proof</u>. We will only sketch the proof, which is very similar to that of Theorem 4.1. It is enough to prove the statement under the additional assumption: $\mu_\alpha(B^c_\delta)=0$ and the restriction of μ_α to the σ-algebra generated by $u^{-1}(\mathcal{B}_E)$ and $\|\cdot\|$ is symmetric.

The steps in the proof of Theorem 4.1, carried out with $\|\cdot\|$ and $\|\cdot\|_K$ instead of q_G and q_F, lead to

$$\text{Pois}(\mu^r_\alpha\circ u^{-1})\{y\in E: \|y\|_K>\lambda\}\le C\lambda^{-2}\int \|x\|^2 d\mu^r_\alpha(x).$$

(Let us observe that, under the hypotheses of the present theorem

(a) $\|\cdot\|_K$ is $\mathcal{B}_E$- measurable and (b) the inequality $E\,\|\Sigma_{j=1}^n u(X_j)\|_K^2 \le C\Sigma_{j=1}^n E\,\|X_j\|^2$ is true for $\{X_j\}$ independent and symmetric, $E\,\|X_j\|^2<\infty$ for $j=1,\dots,n$.) It follows from assumption (2) that $\{Pois(\mu_\alpha^r \circ u^{-1})\}_{\alpha,r}$ is tight in E and therefore $\mu_\alpha \circ u^{-1}$ is a Lévy measure on E for each α and $\{Pois(\mu_\alpha \circ u^{-1})\}$ is tight in E. □

Next we apply Theorem 4.4 together with Theorem 3.1 to obtain a result on the weak convergence of triangular arrays.

We shall need the following terminology. A σ-algebra $\mathcal{G}$ of subsets of a vector space G is <u>compatible with $\mathcal{G}$</u> if the map $(x,y)\to x+y$ from $G\times G$ into G is $\mathcal{G}\otimes\mathcal{G}$-$\mathcal{G}$ measurable and the map $(\lambda,x)\to\lambda x$ from $R\times G$ into G is $\mathcal{R}\otimes\mathcal{G}$-$\mathcal{G}$ measurable ($\mathcal{R}$ is the Borel σ-algebra of R). If $(\Omega,\mathcal{A},P)$ is a probability space and $(M,\mathcal{M})$ is a measurable space, a map $X:\Omega\to M$ is a <u>$(M,\mathcal{M})$ random variable</u> if X is $\mathcal{A}$-$\mathcal{M}$ measurable. If G is a vector space, $\mathcal{G}$ is compatible with G and X, Y are $(G,\mathcal{G})$ random variables, then X+Y and λX ($\lambda\in R$) are also $(G,\mathcal{G})$ r.v.'s.

<u>4.5 Theorem</u>. Let B,E,K,u be as in Theorem 4.4 and assume furthermore that $\mathcal{B}_o$ is compatible with B. Let $\{X_{nj}\}$ be a triangular array of $(B,\mathcal{B}_o)$ r.v.'s, $Y_{nj}=u(X_{nj})$, $T_n=\Sigma_j Y_{nj}$. Assume

(1) there exists a sequentially w*-dense subset W of E' such that $\{\mathcal{L}(f(T_n))\}$ is shift convergent for every $f\in W$,

(2) for every $\varepsilon>0$, $\lim_n \Sigma_j P\{\|X_{nj}\|>\varepsilon\}=0$,

(3) $\sup_n \Sigma_j \int_{\|X_{nj}\|\le 1} \|X_{nj}\|^2 dP<\infty$.

Then there exists a centered Gaussian measure γ on E such that for every $\tau>0$, $\mathcal{L}(T_n-ET_{n,\tau})\to_w \gamma$ on E.

<u>Proof.</u> Let $\mu_n=\Sigma_j \mathcal{L}(X_{nj})$. Then $\{\mu_n\}$ satisfies the assumptions of Theorem 4.4 and therefore $\{Pois(\Sigma_j \mathcal{L}(Y_{nj}))\}=\{Pois(\mu_n\circ u^{-1})\}$ is a relatively shift compact family of p.m.'s on E. By Theorem 3.1, $\{\mathcal{L}(T_n - ET_{n,\tau})\}$ is relatively compact on E for each $\tau>0$. Now assumptions (1) and (2) and the one-dimensional converse central limit theorem (Theorem 2.10, case B=R) imply that for each $f\in W$, $\{\mathcal{L}(f(T_n)-c_n)\}$ converges to a centered Gaussian measure γ_f on R, where $c_n=\Sigma_j E(f(Y_{nj}))_1$. But assumption (2) and a simple computation show that $c_n-f(ET_{n,\tau})\to 0$ ($n\to\infty$); therefore $\mathcal{L}(f(T_n-ET_{n,\tau}))$ converges to γ_f for every $f\in W$ and every $\tau>0$. If ν_1,ν_2 are two subsequential limits of the relatively compact sequence $\{\mathcal{L}(T_n-ET_{n,\tau})\}$, it follows that $\nu_1\circ f^{-1}=\nu_2\circ f^{-1}=\gamma_f$ for all $f\in W$, which implies $\nu_1=\nu_2$. Therefore there exists a centered Gaussian measure γ on E such that $\mathcal{L}(T_n-ET_{n,\tau})\to_w\gamma$ for all $\tau>0$. □

<u>Remark.</u> This proof illustrates the idea of obtaining limit theorems for triangular arrays by proving first the relative (shift) compactness of the associated Poisson measures. In this instance one may also prove the theorem as follows. The proof of Theorem 4.5 shows that it is enough to prove that $\{\mathcal{L}(T_n)\}$ is relatively shift compact. By assumption (2) and proceeding as in the first part of the proof of Lemma 2.4, one obtains $S_n^{(1)}\to_P 0$ on B, and therefore $u(S_n^{(1)})\to_P 0$ on E. Since $T_n=u(S_{n,1})+u(S_n^{(1)})$, it is enough to prove that $\{\mathcal{L}(u(S_{n,1}))\}$ is relatively shift compact. Thus we may assume that the X_{nj}'s are uniformly bounded. Also, by a standard procedure, we may suppose that each X_{nj} is symmetric. Then, under these additional assumptions on the X_{nj}'s,

$$P\{\|u(S_n)\|_K>\lambda\}\le\lambda^{-2}E\,\|\Sigma_{j=1}^n u(X_{nj})\|_K^2\le C\lambda^{-2}\Sigma_{j=1}^n E\,\|X_{nj}\|^2,$$

and assumption (3) implies that $\{\mathcal{L}(u(S_n))\}$ is relatively compact on E. □

We will now apply Theorem 4.5 to obtain a convergence result for triangular arrays of random variables taking values in C(S,d), where (S,d) is a compact metric space. This approach originates in Zinn [39].

Let (S,d) be a compact metric space; we shall write C(S) for C(S,d). Given a continuous distance e on (S,d), we define for $x\in C(S)$

$$q_e(x)=\sup\{|x(s)-x(t)|/e(s,t):s\in S,t\in S,s\neq t\}.$$

The space of e-Lipschitz functions on S is the set

$$Lip(e)=\{x\in C(S):q_e(x)<\infty\},$$

equipped with the norm $\|x\|_e=|x(a)|+q_e(x)$, where a is a fixed point in S. The set

$$K=\{x\in C(S): \|x\|_e\le 1\}$$

is a compact, convex and symmetric subset of C(S) (compactness follows from the Arzelà-Ascoli Theorem) and $(Lip(e), \|\cdot\|_e)$ is precisely $((C(S))_K, \|\cdot\|_K)$, the Banach space generated by K.

Lemma 4.8 is the main step. We need first a definition along the lines of Zinn [39].

4.6 Definition. Let (S,d) be a compact metric space. A continuos distance e on (S,d) is said to imply Lipschitz paths (L.P.I.) if there exists a continuous distance ρ on (S,d) such that for every centered Gaussian process X on S satisfying

$$E(X(s)-X(t))^2\le C(e(s,t))^2 \quad s,t\in S,$$

for some constant C>0, there exists a version of X with almost all its sample paths in Lip(ρ).

Sufficient conditions under which a distance e is L.P.I. are given by basic results on the path behavior of Gaussian processes.

Examples. (1) If on (S,e) there exists a p.m.λ such that

$$(4.3)\qquad \lim_{\varepsilon\to 0}\sup_{s\in S}\int_0^{\varepsilon}(\log(1+1/\lambda\{t\in S:e(s,t)\le u\}))^{1/2}du=0$$

then by a result of Fernique ([15], Cor.6.2.3) e is L.P.I.. For other possible examples of this type see e.g. [32], [20], [40].

(2) If H(S,e,.) is the metric entropy of (S,e) and

$$(4.3)'\qquad \int_0^{\alpha}H^{1/2}(S,e,x)\,dx<\infty$$

for some (hence for all) α>0, then e is L.P.I. (Dudley [12]). It is shown in [15] that (4.3)' implies (4.3).

4.7 Lemma. Let (T,d) be a compact metric space, $z_j\in C(T), j\in N$. Assume

(1) for every $s\in T, t\in T, K(s,t)=\lim_n \Sigma_{j=1}^n z_j(s)z_j(t)$ exists,

(2) there exists a centered Gaussian process Z on T with covariance K and almost all of its sample paths in C(T).

Let $\{\eta_j\}$ be a standard Gaussian sequence. Then $\Sigma_j\eta_j z_j$ converges almost surely in C(T).

Proof. By ([23], Theorem 4.1) it is enough to prove: for every $\nu\in(C(T))'$, $L(\nu(Z_n))\to_w L(\nu(Z))$, where $Z_n=\Sigma_{j=1}^n\eta_j z_j$. Since the distributions are centered Gaussian, we only have to show:

$$(4.4)\qquad \lim_n E(\nu(Z_n))^2=E(\nu(Z))^2.$$

By the Riesz representation theorem, the elements of (C(T))' are the finite signed measures on T; for $\nu\in(C(T))'$, $|\nu|$ will denote the total variation measure associated with ν. By Fubini's theorem,

$$E(\nu(Z_n))^2=\iint K_n(s,t)d\nu(s)d\nu(t),\quad E(\nu(Z))^2=\iint K(s,t)d\nu(s)d\nu(t),$$

where $K_n(s,t)=\Sigma_{j=1}^n z_j(s)z_j(t)$; thus

$$(4.5)\qquad |E(\nu(Z))^2-E(\nu(Z_n))^2|\le\iint|K(s,t)-K_n(s,t)|d|\nu|(s)d|\nu|(t).$$

Assumption (2) implies that K is bounded on T×T; since $|K_n(s,t)|^2\le K_n(s,s)K_n(t,t)\le K(s,s)K(t,t)$, (4.4) follows from (4.5) and assumption (1) by the dominated convergence theorem. □

4.8 Lemma. Let (S,d) be a compact metric space, e a continuous distance on (S,d). Assume that e is L.P.I. and let ρ be a continuous distance on (S,d) associated to e as in Definition 4.6. Then for any continuous distance ρ' such that

(1) $\rho'\ge e$, $e(s,t)/\rho'(s,t)\to 0$ as $e(s,t)\to 0$,

(2) $\rho'\ge\rho$, $\rho(s,t)/\rho'(s,t)\to 0$ as $\rho(s,t)\to 0$,

the inclusion map $i:Lip(e)\to Lip(\rho')$ is of type 2.

Proof. It is well known (see e.g. [22]) that in order to prove that i is of type 2, it is enough to prove: if $\{x_j\}\subset Lip(e)$ and $\Sigma_j\|x_j\|_e^2<\infty$, then $\Sigma_j\eta_j x_j$ converges a.s. in Lip(ρ'), where $\{\eta_j\}$ is a standard Gaussian sequence.

Assume now that $\{x_j\}\subset Lip(e)$ and $\Sigma_j\|x_j\|_e^2<\infty$. Then for every

$s \in S$,

$$\Sigma_j E(\eta_j x_j(s))^2 = \Sigma_j (x_j(s))^2 \le \Sigma_j (|x_j(a)| + (\mathrm{diam}_e S) q_e(x))^2 < \infty,$$

and therefore $\Sigma_j \eta_j x_j(s)$ converges in L^2, hence almost surely, say to $Y(s)$. Now

$$E(Y(t)-Y(s))^2 = \Sigma_j (x_j(t)-x_j(s))^2 \le (\Sigma_j \|x_j\|_e^2)(e(s,t))^2.$$

It follows that there exists a version Y' of Y with almost all its sample paths in $\mathrm{Lip}(\rho)$.

Let $T=S\times S$ and define the distance d_T on T by $d_T((s,t),(s',t')) = d(s,s')+d(t,t')$. Define $z_j : T \to R$ ($j \in N$) by

$$z_j(s,t) = \begin{cases} (x_j(t)-x_j(s))/\rho'(s,t) & \text{if } s \ne t \\ 0 & \text{if } s=t. \end{cases}$$

Then z_j is continuous on (T,d_T): it is clear that z_j is continuous at $(u,v)\in T$ if $u \ne v$, and if $d_T((s,t),(u,u)) \to 0$, then $d(s,t)\to 0$ and therefore

$$z_j(s,t) = \{(x_j(t)-x_j(s))/e(s,t)\}\{e(s,t)/\rho'(s,t)\} \to 0$$

by assumption (1) and the fact that $x_j \in \mathrm{Lip}(e)$ (obviously we only need to consider points (s,t) with $s \ne t$).

Define $Z : \Omega \times T \to R$ by

$$Z(\cdot)(s,t) = \begin{cases} (Y'(t)-Y'(s))/\rho'(s,t) & \text{if } s \ne t \\ 0 & \text{if } s=t. \end{cases}$$

Since $Y'(\omega) \in \mathrm{Lip}(\rho)$ for almost all ω, using assumption (2) and reasoning as above, it follows that $Z(\omega) \in C(T,d_T)$ for almost all ω.

Let $K_n((s,t),(s',t')) = \Sigma_{j=1}^n z_j(s,t) z_j(s',t')$. As $n\to\infty$, K_n converges pointwise to the covariance of Z; therefore by Lemma 4.7 $\Sigma_j \eta_j z_j$ converges a.s. in $C(T,d_T)$. Thus there exists a null set A such that for $\omega \in A^c$

$$\lim_{m,n} q_{\rho'}(\Sigma_{j=m}^n \eta_j(\omega) x_j) = \lim_{m,n} \|\Sigma_{j=m}^n \eta_j(\omega) z_j\|_{C(T)} = 0.$$

Since also $\lim_{m,n} |\Sigma_{j=m}^n \eta_j(\omega) x_j(a)| = 0$ for ω outside a null set, it follows that $\Sigma_j \eta_j x_j$ is a.s. Cauchy in $\mathrm{Lip}(\rho')$, hence converges a.s. in $\mathrm{Lip}(\rho')$. □

The next result is in the line of those of Dudley and Strassen [14], Giné [17], Dudley [13], Jain-Marcus [25], Araujo [5] and Heinkel [20]. It extends these results for non-identically distributed $C(S)$-valued random vectors.

<u>4.9 Theorem.</u> Let (S,d) be a compact metric space, $\{X_{nj}\}$ a triangular array of $C(S)$-valued r.v.'s such that

(1) the finite dimensional distributions of $\{L(S_n)\}$ are shift convergent,

(2) for every $\varepsilon>0$, $\lim_n \Sigma_j P\{\|X_{nj}\| > \varepsilon\} = 0$,

(3) $\sup_n \Sigma_j \int_{\|X_{nj}\|\le 1} \|X_{nj}\|^2 dP < \infty$,

(4) there exists an L.P.I. distance e on (S,d) such that
$P\{X_{nj} \in \mathrm{Lip}(e)\} = 1$ for all n,j,
for every $\varepsilon>0$, $\lim_n \Sigma_j P\{q_e(X_{nj}) > \varepsilon\} = 0$,
$\sup_n \Sigma_j \int_{q_e(X_{nj})\le 1} q_e^2(X_{nj})\, dP < \infty$.

Then there exists a centered Gaussian measure γ on $C(S)$ such that for every $\tau>0$,

(a) $\lim_n \Sigma_j \mathrm{Cov}\{(X_{nj}(s))_\tau, (X_{nj}(t))_\tau\} = \mathrm{Cov}_\gamma(s,t)$,

(b) $L(S_n - ES_{n,\tau}) \to_w \gamma$ on $C(S)$.

<u>Proof.</u> Let $\mathcal{B}_o$ be the trace of the Borel σ-algebra of $C(S)$ on $\mathrm{Lip}(e)$. Then $\mathcal{B}_o$ is compatible with $\mathrm{Lip}(e)$ and $\|\cdot\|_e$ is $\mathcal{B}_o$ measurable.

Let ρ be a continuous distance on (S,d) associated to e as in Definition 4.6, and let ρ' be a continuous distance on (S,d) satisfying (1) and (2) of Lemma 4.8 (here is one possible construction of ρ': fix $\alpha \in (0,1)$ and define $\rho' = C \max\{e^\alpha, \rho^\alpha\}$ for an appropriate constant $C>0$). By Lemma 4.8, the inclusion map $i: \mathrm{Lip}(e) \to \mathrm{Lip}(\rho')$ is of type 2. It follows that if $B=\mathrm{Lip}(e)$, $E=C(S)$, $K=\{x\in C(S): \|x\|_{\rho'} \le 1\}$, $u: B \to E_K$ is the inclusion map, then the objects $B, \mathcal{B}_o, E, K, u$ satisfy the assumptions of Theorem 4.5.

We show next that the triangular array $\{X_{nj}\}$ satisfies the assumptions of Theorem 4.5. By (4), we may assume that $X_{nj}(\Omega)\subset \mathrm{Lip}(e)$ for all n,j; then each X_{nj} is a $(B,\mathcal{B}_o)$ r.v.. Assumption (1) of Theorem 4.5 follows at once from (1) by taking

$$W=\{\nu\in(C(S))': \nu=\Sigma_{j=1}^{n} a_j\delta_{t_j} \quad \text{for some } a_j\in R,\ t_j\in S,\ j=1,\dots,n\}.$$

Next, (2) and the second condition in (4) imply assumption (2) of Theorem 4.5. Finally assumption (3) of Theorem 4.5 is an immediate consequence of (3) and the last condition in (4). Thus assertion (b) follows from Theorem 4.5.

We turn now to assertion (a). It easily follows from Theorem 2.12(1) that for any $s,t\in S,\tau>0$

$$\lim_n \Sigma_j E(a_{nj}b_{nj})=\Phi_\gamma(\delta_s,\delta_t)=\mathrm{Cov}_\gamma(s,t),$$

where $a_{nj}=\delta_s(X_{nj\tau}-EX_{nj\tau})$, $b_{nj}=\delta_t(X_{nj\tau}-EX_{nj\tau})$. Let $\alpha_{nj}=(X_{nj}(s))_\tau-E(X_{nj}(s))_\tau$, $\beta_{nj}=(X_{nj}(t))_\tau-E(X_{nj}(t))_\tau$. Then (a) will be proved if we show that $\Sigma_j E(a_{nj}b_{nj}-\alpha_{nj}\beta_{nj})\to 0$ $(n\to\infty)$. But

$$E|a_{nj}-\alpha_{nj}|\le 2\tau P\{\,\|X_{nj}\|>\tau\};$$

therefore $\Sigma_j E|a_{nj}-\alpha_{nj}|\to 0$ as $n\to\infty$ by (2), and similarly $\Sigma_j E|b_{nj}-\beta_{nj}|\to 0$. The proof is completed by means of the inequality

$$|a_{nj}b_{nj}-\alpha_{nj}\beta_{nj}|\le 2\tau(|a_{nj}-\alpha_{nj}|+|b_{nj}-\beta_{nj}|). \quad \square$$

<u>Remark.</u> Let $\delta>0$, $\xi_{nj}=X_{nj}(a)-(EX_{nj\delta})(a)$. If $\{L(S_n(a))\}$ is relatively shift compact then Theorems 2.5 and 2.3 followed by a simple computation show that

$$\text{(4.6)} \qquad \sup_n \Sigma_j \int_{|\xi_{nj}|\le 1} |\xi_{nj}|^2 dP<\infty.$$

This implies that if in assumption (4) of Theorem 4.9 X_{nj} is replaced by $Y_{nj}=X_{nj}-EX_{nj\delta}$ then assumption (3) becomes superfluous: the fact that

$$\sup_n \Sigma_j \int_{\|Y_{nj}\|_e\le 1} \|Y_{nj}\|_e^2 dP<\infty$$

follows at once from (4.6) and assumption (4). Theorem 4.9 then shows that (a) and (b) are true for the array $\{Y_{nj}\}$. An additional elementary computation then shows that (a) and (b) are in fact true

for the original array $\{X_{nj}\}$.

Finally we state a consequence of Theorem 4.4 and Lemma 4.8 on the relative compactness of a family of Poisson p.m.'s on C(S). We point out that Theorem 4.9 could also be obtained from this result.

<u>4.10 Theorem</u>. Let (S,d) be a compact metric space, e a L.P.I. distance on (S,d), $\{\mu_\alpha\}$ a family of σ-finite positive measures. Then if for some $\delta>0$

(1) $\mu_\alpha\{x: \|x\|_e>\delta\}<\infty$ for all α and $\{\mu_\alpha|\{x: \|x\|_e>\delta\}\}$ is relatively compact on C(S),

(2) $\sup_\alpha \int \min(\delta, \|x\|_e^2)d\mu_\alpha(x)<\infty$,

then the measures μ_α are Lévy measures and $\{c_\tau \mathrm{Pois}\mu_\alpha\}$ is relatively compact on C(S) for every $\tau>0$.

A first result in the direction of the last theorem appears in Araujo [5].

5. A CLASS OF GAUSSIAN MEASURES AND COTYPE 2 SPACES.

Let us recall that if E is a real locally convex (Hausdorff) topological vector space (l.c.t.v.s.) and H is a Hilbert space, then a linear map T:E→H is <u>Hilbert-Schmidt</u> if there exist a Hilbert space H_1, a continuous linear map $A:E\to H_1$ and a Hilbert-Schmidt map $S:H_1\to H$ such that $T=S\circ A$. The <u>Hilbert-Schmidt topology</u> on E is the coarsest (locally convex) topology which makes continuous all Hilbert-Schmidt maps (into all Hilbert spaces). If τ is the given topology on E, we denote the associated Hilbert-Schmidt topology by τ_{HS}. (See e.g [8], Exposé 10).

<u>5.1 Lemma</u>. Let (E,τ) be a l.c.t.v.s. Suppose Φ is a non-negative definite symmetric bilinear form on E × E. Then the quadratic form $x\to\Phi(x,x)$ $(x\in E)$ is τ_{HS}-continuous if and only if there exists a τ-Hilbert-Schmidt operator T:E→H (a Hilbert space) such that $\Phi(x,y)=(Tx,Ty)_H$ $(x\in E, y\in E)$.

Proof. Suppose $x\mapsto\Phi(x,x)$ is τ_{HS}-continuous, and let $T:E\to H$ be a Hilbert-Schmidt map such that $\|Tx\|_H\le 1$ implies $\Phi(x,x)\le 1$. Let $\psi(x,y)=(Tx,Ty)_H$, and let W be the completion of the pre-Hilbert space E/ψ. We denote by $\tilde{\psi}$ both the bilinear form induced by ψ on E/ψ and its continuous extension to W. Let $\Pi:E\to E/\psi$ be the canonical projection. Then there exists a continuous non-negative definite symmetric bilinear form $\tilde{\Phi}$ on W×W such that $\Phi(x,y)=\tilde{\Phi}(\Pi x,\Pi y)$ $(x\in E, y\in E)$. By standard facts on operators on Hilbert space, there exists a bounded $\tilde{\psi}$-self-adjoint non-negative operator $A:W\to W$ such that $\tilde{\Phi}(u,v)=\tilde{\psi}(Au,Av)$ $(u\in W, v\in W)$. Therefore $\Phi(x,y)=(Sx,Sy)_H$, where $S=\tilde{T}\circ A\circ\Pi$ is clearly a τ-Hilbert-Schmidt map (here $\tilde{T}$ is the map induced by T on W).

The converse assertion is obvious. □

We recall next the construction of the Hilbert space of a Gaussian measure. More generally, let F be a real vector space, F* its algebraic dual, Φ a non-negative definite symmetric bilinear form on F × F, $\Pi:F\to F/\Phi$ the canonical map, and $\tilde{\Phi}$ the bilinear form induced by Φ on F/Φ. We denote by $\widehat{F/\Phi}$ the Hilbert space which is the $\tilde{\Phi}$-completion of F/Φ; the extension of $\tilde{\Phi}$ to $\widehat{F/\Phi}$ will again be denoted $\tilde{\Phi}$. Now let

$$H=\{z\in F^*: \text{ the map } y\mapsto z(y) \text{ is } \Phi\text{-continuous on } F\}.$$

Then H is canonically isomorphic to $\widehat{F/\Phi}'$, the topological dual of $\widehat{F/\Phi}$. The Riesz representation $\phi:H\to\widehat{F/\Phi}$ is characterized by

$$\langle x,y\rangle=\tilde{\Phi}(\phi(x),\Pi(y)) \quad x\in H, y\in F.$$

We will denote H_Φ the Hilbert space consisting of the set H endowed with the inner product

$$(u,v)_H=\tilde{\Phi}(\phi(u),\phi(v)) \quad u\in H, v\in H.$$

Now let E be a l.c.t.v.s.. Let $\kappa(E',E)$ be the topology (on E') of uniform convergence on the compact convex (balanced) subsets of E. If Φ is as described above, and furthermore, $f\mapsto\Phi(f,f)$ $(f\in E')$ is $\kappa(E',E)$ continuous, then $H_\Phi\subset E$. If Φ is the covariance of a centered Gaussian measure μ on the Borel σ-algebra of E which satisfies $\sup\{\mu(K):K\subset E, K \text{ compact and convex}\}=1$, then Φ is $\kappa(E',E)$ continuous and consequently $H_\Phi\subset E$. In this situation we call H_Φ the <u>Hilbert space of μ</u>, and will sometimes write H_μ instead of H_Φ.

<u>5.2 Definition</u>. Let E be a l.c.t.v.s.. A centered Gaussian measure μ on the Borel σ-algebra on E is <u>strongly Gaussian</u> if there exist a Hilbert space H, a continuous linear map $T:H\to E$ and a tight centered Gaussian measure ν on H such that $\mu=\nu\circ T^{-1}$; equivalently, μ is the image under a continuous linear map of a centered Gaussian measure on a separable Hilbert space.

<u>5.3 Theorem.</u> Let E be a l.c.t.v.s.. Let Φ be a non-negative definite symmetric bilinear form on E' × E'. The following statements are equivalent:

(a) Φ is the covariance of a strongly Gaussian measure,

(b) the map $f\mapsto\Phi(f,f)$ is $\kappa(E',E)$ continuous and the inclusion map $i:H_\Phi\to E$ admits a factorization $i=A\circ T$ with $T:H_\Phi\to H$ (a Hilbert space) a Hilbert-Schmidt map and $A:H\to E$ a continuous linear map,

(c) the map $f\mapsto\Phi(f,f)$ is $\kappa(E',E)_{HS}$-continuous on E',

(d) there exists a $\kappa(E',E)$-Hilbert-Schmidt map $S:E'\to H$ such that $\Phi(f,g)=(Sf,Sg)_H$ $(f\in E', g\in E')$.

<u>Proof.</u> We will prove (a)⟹(b)⟹(c)⟹(d)⟹(a).

(a)⟹(b). Let us first observe that if ν is a Gaussian measure on a separable Hilbert space, Ψ is its covariance and $\Pi_\Psi:H\to\widehat{H/\Psi}$ is the canonical map, then Π_Ψ is a Hilbert-Schmidt map. In fact, if $\{e_n\}_{n\in N}$ is an orthonormal basis in H, then

$$\Sigma_n \|\Pi_\Psi(e_n)\|^2_{\widehat{H/\Psi}} = \Sigma_n \Psi(e_n,e_n)<\infty.$$

Now suppose μ is a strongly Gaussian measure on E: explicitly, there is a separable Hilbert space H, a continuous linear map $A:H\to E$ and a Gaussian measure ν on H such that $\mu=\nu\circ A^{-1}$. Let Φ be the covariance of μ, Ψ the covariance of ν. The fact that μ is strongly Gaussian implies that the map $f\mapsto\Phi(f,f)$ $(f\in E')$ is $\kappa(E',E)$

continuous and therefore $H_\Phi \subset E$. We proceed to construct the desired factorization of $i:H_\Phi \to E$. Let $\Pi_\Psi : H \to H/\Psi$, $\Pi_\Phi : E' \to E'/\Phi$ be the canonical maps. Let $\tilde{A}: E'/\Phi \to H/\Psi$ be defined by $\Pi_\Psi \circ A' = \tilde{A} \circ \Pi_\Phi$. Since $\Phi(f,f) = \Psi(A'f, A'f)$ for all $f \in E'$, it follows that $\tilde{A}$ is an isometry from E'/Φ into H/Ψ and consequently extends to an isometry of $\widehat{E'/\Phi}$ into $\widehat{H/\Psi}$. Let $\phi: H_\Phi \to \widehat{E'/\Phi}$, $\psi: H_\Psi \to \widehat{H/\Psi}$ be the Riesz representations, and let $j: H_\Psi \to H$ be the inclusion map. Define $u: H_\Phi \to H_\Psi$ by $u = \psi^{-1} \circ \tilde{A} \circ \phi$; then

$$i = A \circ j \circ u. \tag{5.1}$$

Equality (5.1) is proved in a routine way from the definitions; one first checks it for $x \in \phi^{-1}(E'/\Phi)$. Now $j = \Pi_\Psi'$ is Hilbert-Schmidt and u is an isometry of H_Φ into H_Ψ; therefore $j \circ u : H_\Phi \to H$ is Hilbert-Schmidt.

(b)$\Rightarrow$(c). Assume that $i: H_\Phi \to E$ can be factored

$$\begin{array}{ccc} H_\Phi & \xrightarrow{i} & E \\ A \downarrow & \nearrow B & \\ H & & \end{array}$$

with A Hilbert-Schmidt and B continuous. It follows that the canonical map $i' = \Pi_\Phi : E' \to \widehat{E'/\Phi}$ can be factored in the form $\Pi_\Phi = A' \circ B'$, and consequently

$$\Phi(f,f) = \|A'B'f\|^2$$

(the norm is the Hilbert norm on $\widehat{E'/\Phi}$). Since A' is Hilbert-Schmidt, it may be factored in the form $A' = S \circ C$, with S Hilbert-Schmidt and C compact, $S: H \to \widehat{E'/\Phi}$, $C: H \to H$ ([37], p.217). Hence $\Phi(f,f) = \|S(CB')f\|_H^2$. We claim that CB' is a continuous linear map from $(E', \kappa(E',E))$ into H. In fact, for $f \in E'$

$$\|CB'f\|_H = \sup_{\|u\|_H \le 1} |CB'f(u)| = \sup_{\|u\|_H \le 1} |B'f(C^*u)|$$

$$= \sup_{\|u\|_H \le 1} |f(B(C^*u))| \le \sup_{x \in K} |f(x)|,$$

where K is the compact set $K = B\overline{(C^*(U))}$, U the unit ball in H. This completes the proof of (b)$\Rightarrow$(c).

(c)$\Rightarrow$(d): Lemma 5.1.

(d)$\Rightarrow$(a). Let S be factored in the form $S = B \circ A$, with $A: E' \to H_1$ Hilbert, A $\kappa(E',E)$-continuous and $B: H_1 \to H$ a Hilbert-Schmidt map. Let $A': H_1 \to E$, $B': H \to H_1$ be the adjoint maps, and let γ be the canonical Gaussian cylinder measure on H. Then $\nu = \gamma \circ (B')^{-1}$ is a tight measure on H_1 (by Sazonov's theorem; see e.g. [37], p. 215); if $\mu = \nu \circ (A')^{-1}$, then for $f \in E'$

$$\Phi_\mu(f,f) = \Phi_\nu(Af, Af) = \|BAf\|_H^2 = \Phi(f,f). \quad \square$$

Let us recall that if F and G are Banach spaces, a map $T \in L(F,G)$ is <u>absolutely 2-summing</u> if there exists a constant $C > 0$ such that

$$\Sigma_{j=1}^n \|Tx_j\|_G^2 \le C \sup\{\Sigma_{j=1}^n |f(x_j)|^2 : f \in F', \|f\| \le 1\}.$$

Pietsch [34] has proved the following result. Let F be a Hilbert space, G a Banach space. Then a map $T \in L(F,G)$ is absolutely 2-summing if and only if there exists a Hilbert space H, a Hilbert-Schmidt map $S: F \to H$ and a continuous linear map $A: H \to G$ such that $T = A \circ S$. As a consequence we obtain from Theorem 5.3:

<u>5.4 Theorem</u>. Let B be a Banach space. Let Φ be a non-negative definite symmetric bilinear form on $B' \times B'$. Then the following statements are equivalent: each of conditions (a), (c) and (d) of Theorem 5.3, and (b'): The map $f \to \Phi(f,f)$ is $\kappa(B',B)$ continuous and the inclusion map $i: H_\Phi \to B$ is absolutely 2- summing.

<u>5.5 Corollary</u>. Let B be a Banach space, μ a strongly Gaussian measure and H its Hilbert space. Then there exists a constant $M > 0$ such that for any orthonormal set $\{e_\alpha\} \subset H$, $\Sigma_\alpha \|e_\alpha\|_B^2 \le M$.

<u>Proof</u>. By Theorem 5.4, there exists a constant M such that if $\{x_1, \dots, x_n\} \subset H$ then

$$\Sigma_{i=1}^n \|x_i\|_B^2 \le M \sup\{\Sigma_{i=1}^n (x_i, y)_H^2 : y \in H, \|y\|_H \le 1\}$$

If $\{e_i : i=1,\dots,n\}$ is an orthonormal set, then $\Sigma_{i=1}^n (e_i, y)_H^2 \le \|y\|_H^2$. Therefore $\Sigma_{i=1}^n \|e_i\|_B^2 \le M$. $\square$

One may deduce from this Corollary the fact that the Wiener measure on $C[0,1]$ is not strongly Gaussian (see [9], p.10).

Let us recall that a Banach space B is of <u>cotype 2 Rademacher</u> (resp., <u>stable</u>) if there exists a constant $C>0$ such that for every finite set $\{x_1,\dots,x_n\}\subset B$,

$$\Sigma_{i=1}^n \|x_i\|^2 \le CE\,\|\Sigma_{i=1}^n \varepsilon_i x_i\|^2 \quad (\text{resp., } \le CE\,\|\Sigma_{i=1}^n \eta_i x_i\|^2)$$

where $\{\varepsilon_i : i\in N\}$ is a symmetric Bernoulli sequence of random variables and $\{\eta_i : i\in N\}$ is a standard Gaussian sequence. It follows from results of Maurey and Pisier [31] that the two notions of cotype 2 are equivalent.

The following consequence of Theorem 5.4 characterizes Banach spaces of cotype 2 in terms of the structure of the Gaussian measures on the space. A similar result has been obtained by Chobanjan and Tarieladze [10]; see also Garling [16].

The link is provided by some arguments of Maurey [30]; for the sake of completeness we give a direct (and somewhat simplified) proof of one of the results of Maurey.

<u>5.6 Theorem</u>. Let B be a separable Banach space. The following conditions are equivalent.

(a) B is of cotype 2.

(b) For every centered Gaussian measure μ on B, the inclusion map $i:H_\mu\to B$ is absolutely 2-summing.

(c) Every centered Gaussian measure on B is strongly Gaussian.

(d) A non-negative definite symmetric bilinear form Φ on $B'\times B'$ is the covariance of a centered Gaussian measure if and only if there exists a $\kappa(B',B)$-Hilbert-Schmidt map $T:B'\to H$ such that $\Phi(f,f)=\|Tf\|_H^2\ (f\in B')$.

<u>Proof</u>. In view of Theorem 5.4, it is enough to prove that (a)$\Longleftrightarrow$(b)

(a)$\Rightarrow$(b): Let μ be a centered Gaussian measure, Φ its covariance. Let $\{x_i : i=1,\dots,n\}$ be a finite subset of H_μ. Let Φ_1 be the covariance of the B-valued random vector $\Sigma_{i=1}^n \eta_i x_i$; then

$$\Phi_1(f,f)=\Sigma_{i=1}^n |f(x_i)|^2 \quad (f\in B').$$

For $x\in H=H_\mu$, $f\in B'$, we have $f(x)=\tilde{\Phi}(\phi(x),\Pi(f))=(x,\phi^{-1}\Pi(f))_H$ (recall the construction of H). Therefore

$$\Phi_1(f,f)\le C\,\|\phi^{-1}\Pi(f)\|_H^2=C\Phi(f,f)$$

where $C=\sup_{\|z\|_H\le 1}\Sigma_{i=1}^n (x_i,z)_H^2$. Define $\Phi_2=C\Phi-\Phi_1$. Then Φ_2 is a non-negative definite symmetric bilinear form on B'; if μ_2 is the Gaussian cylinder measure with covariance Φ_2, $\mu_1=L(\Sigma_{i=1}^n \eta_i x_i)$ and $\nu=L(C^{1/2}X)$, where $L(X)=\mu$, then the cylinder measures μ_1,μ_2 and ν satisfy

$$(5.2)\qquad \nu=\mu_1*\mu_2.$$

By a well-known argument based on Prokhorov's theorem on the extension of cylinder measures (see e.g. [8]), we may conclude that μ_2 is in fact a tight measure and equation (5.2) is valid for the measures μ_1,μ_2 and ν.

Then

$$E\,\|\Sigma_{i=1}^n \eta_i x_i\|^2=\int \|x\|^2 d\mu_1(x)\le\int \|x\|^2 d\nu(x)=CE\,\|X\|^2$$

by a well-known elementary inequality. Since $M=E\,\|X\|^2<\infty$ by a result of Fernique [15], we have

$$(5.3)\qquad E\,\|\Sigma_{i=1}^n \eta_i x_i\|^2\le M \sup_{\|z\|_H\le 1}\Sigma_{i=1}^n (x_i,z)_H^2 .$$

Inequality (5.3) and the assumption that B is of cotype 2 yields the assertion (a)$\Rightarrow$(b).

(b)$\Rightarrow$(a): Let $\{x_i : i\in N\}$ be a sequence in B and assume that $\Sigma_i \eta_i x_i$ converges almost surely. We will prove that $\Sigma_i \|x_i\|^2<\infty$; by a well known argument, this implies that B is of cotype 2. Let $\mu=L(\Sigma_i \eta_i x_i)$, and let Φ be the covariance of μ.

Since $\Phi(f,f)=\Sigma_{i=1}^n |f(x_i)|^2\ (f\in B')$, we have: for each $i\in N$, the linear form $f\to f(x_i)$ is Φ-continuous on B' and therefore $x_i\in H=H_\mu$. Then

$$\Sigma_{i=1}^n \|x_i\|^2\le C\sup_{\|z\|_H\le 1}\Sigma_{i=1}^n (x_i,z)_H^2$$

$$=C\sup_{\|z\|_H\le 1}\Sigma_{i=1}^n \Phi(\phi(x_i),\phi(z))$$

$$=C\sup\{\Sigma_{i=1}^{n}\Phi(\phi(x_i),\Pi(f)):f\in B',\Phi(f,f)\le 1\}$$

$$=C\sup\{\Sigma_{i=1}^{n}|f(x_i)|^2:f\in B',\Phi(f,f)\le 1\}=C. \quad \square$$

As a consequence of Theorem 5.6, it is possible to make more precise the Lévy-Khinchine representation of infinitely divisible distributions on spaces of cotype 2, recently obtained by de Acosta and Samur [2] and Araujo and Giné [6].

5.7 Corollary. Let B be a separable Banach space of cotype 2, ν an infinitely divisible probability measure on B, $\tau>0$. Then there exist a point x_τ in B, a $\kappa(B',B)$-Hilbert-Schmidt operator $T:B'\to H$ (a Hilbert space) and a Lévy measure μ satisfying $\int\min(1,\|x\|^2)d\mu(x)<\infty$, such that for every $f\in B'$

$$\hat{\nu}(f)=\exp\{if(x_\tau)-(1/2)\|Tf\|_H^2+\int h_\tau(f,x)d\mu(x)\}$$

The triple (x_τ,Ψ,μ) is unique, where $\Psi(f)=\|Tf\|_H^2$ $(f\in B')$.

Remark. (1) This result refines the general Lévy-Khinchine representation (Corollary 1.11) in the case of cotype 2 spaces by providing additional information on the Gaussian measure (whose covariance must be $\kappa(B',B)$-Hilbert-Schmidt continuous) and on the Lévy measure (which must satisfy the integrability condition).

(2) The fact that $\int\min(1,\|x\|^2)d\mu(x)<\infty$ makes it possible to write the representation formula in terms of the classical kernel $K(f,x)=e^{if(x)}-1-if(x)/(1+\|x\|^2)$ (of course, the point x_τ changes).

6. SOME WEAK COMPACTNESS RESULTS AND THE CENTRAL LIMIT THEOREM IN COTYPE 2 SPACES

6.1 Theorem. Let E be a l.c.t.v.s., H a separable Hilbert space, $T:H\to E$ a continuous linear map. Let $\tilde{\mu}$ be a p.m. on H with $\int\|x\|^2 d\tilde{\mu}(x)<\infty$, $\mu=\tilde{\mu}\circ T^{-1}$. Let K be a family of tight p.m.'s on E, and assume that $\{\hat{\nu}:\nu\in K\}$ is equicontinuous at 0 for the Φ_μ topology on E'. Then there exists a family K of p.m.'s on H which is relatively compact for the weak topology and such that $K=\{\nu\circ T^{-1}:\nu\in K\}$; in particular, K is relatively compact.

Proof. We first show that $\text{support}(\nu)\subset\overline{T(H)}$ for all $\nu\in K$; therefore we may assume without loss of generality that $\overline{T(H)}=E$. To prove the claim, let $G=\{f\in E':\Phi(f,f)=0\}$, where $\Phi=\Phi_\mu$. Then for each $\nu\in K$, $G\subset\{f\in E':\hat{\nu}(tf)=1 \text{ for all } t\in R\}$. It easily follows (see e.g. [1], p. 275) that $\text{support}(\nu)\subset G^\perp$. It remains to show that $G^\perp\subset\overline{T(H)}$. It is enough to prove that $[T(H)]^\perp\subset G$; it will then follow that $G^\perp\subset[T(H)]^{\perp\perp}=\overline{T(H)}$. But $[T(H)]^\perp=\text{Ker}T'$, where $T':E'\to H'$ is the adjoint map of T. Since $\mu=\tilde{\mu}\circ T^{-1}$ one has $\Phi(f,f)=\Phi_{\tilde{\mu}}(T'f,T'f)$ $(f\in E')$; therefore $f\in[T(H)]^\perp$ implies $\Phi(f,f)=0$, so indeed $G^\perp\subset\overline{T(H)}$.

We identify H and H' as usual and call $S=T'$. Since $\overline{T(H)}=E$, S is injective. Let $M=S(E')$. Given $\nu\in K$, we define $f_\nu:M\to C$ by

$$f_\nu(h)=\hat{\nu}(S^{-1}h),\ h\in M.$$

Then $f_\nu(0)=1$ and f_ν is positive definite for each $\nu\in K$, and it follows at once from the assumption that $\{f_\nu:\nu\in K\}$ is $\Phi_{\tilde{\mu}}$-equicontinuous at 0 on M.

Each f_ν is $\Phi_{\tilde{\mu}}$-uniformly continuous on M, hence uniformly continuous for the Hilbert norm, and can be uniquely extended to a uniformly continuous function on $\overline{M}$ (we still denote the extension f_ν). Clearly the set of extended functions on $\overline{M}$ is $\Phi_{\tilde{\mu}}$-equicontinuous at 0. Define $\tilde{f}_\nu:H\to C$ by

$$\tilde{f}_\nu(h)=f_\nu(\Pi h),\ h\in H,$$

where Π is the orthogonal projection with range $\overline{M}$. Then for each $\nu\in K$, $\tilde{f}_\nu(0)=1$, $\tilde{f}_\nu$ is positive definite and $\{\tilde{f}_\nu:\nu\in K\}$ is equicontinuous at 0 for the Ψ-topology, where Ψ is defined by $\Psi(h,h')=\Phi_{\tilde{\mu}}(\Pi h,\Pi h')$.

Since Ψ is nuclear, by Sazonov's theorem (see e.g. [33], p.160) there exists a probability measure $\tilde{\nu}$ on H such that $\hat{\tilde{\nu}}=\tilde{f}_\nu$, and Prokhorov's criterion (see e.g. [33], p.155) implies that $\tilde{K}=\{\tilde{\nu}:\nu\in K\}$ is relatively compact for the weak topology; we further remark for future use that $\text{support}(\tilde{\nu})\subset\overline{M}$ for each $\tilde{\nu}\in\tilde{K}$.

We complete the proof by showing that $\tilde{\nu}\circ T^{-1}=\nu$. In fact, for all $f\in E'$

$$(\tilde{\nu}\circ T^{-1})^\wedge(f)=\hat{\tilde{\nu}}(Sf)=\tilde{f}_\nu(Sf)=\hat{\nu}(S^{-1}(Sf))=\hat{\nu}(f). \quad \square$$

6.2 Theorem. Let E, μ be as in Theorem 6.1. Let

$$K=\{\nu\in P(E):\Phi_\nu(f,f)\le\Phi_\mu(f,f) \text{ for all } f\in E'\}.$$

Then the conclusion of Theorem 6.1 holds for K and, furthermore, for every continuous seminorm p on E,

$$\sup_{\nu\in K}\int p^2 d\nu<\infty.$$

Proof. A standard argument shows that $\{\hat{\nu}:\nu\in K\}$ is Φ_μ-equicontinuous at 0; thus the first claim follows from Theorem 6.1.

To prove the second statement, let C>0 be such that $p(Tx)\le C\|x\|$ $(x\in H)$. Let M, S be as in the proof of Theorem 6.1, and let $\{e_\alpha\}$ be an orthonormal basis of $\bar{M}$ with $e_\alpha\in M$ for each α. Then

$$\int_E p^2 d\nu=\int_H p^2(Tx)d\tilde{\nu}(x)\le C^2\int\|x\|^2 d\tilde{\nu}(x).$$

Since $\text{support}(\tilde{\nu})\subset\bar{M}$ (see the proof of Theorem 6.1),

$$\int\|x\|^2 d\tilde{\nu}(x)=\Sigma_\alpha\int(x,e_\alpha)^2 d\tilde{\nu}(x)=\Sigma_\alpha\Phi_{\tilde{\nu}}(e_\alpha,e_\alpha).$$

Now $\Phi_{\tilde{\nu}}(y,y)=\Phi_\nu(S^{-1}y,S^{-1}y)\le\Phi_{\tilde{\mu}}(y,y)$ for each $y\in M$. Therefore

$$\sup_{\nu\in K}\int p^2 d\nu\le C^2\Sigma_\alpha\Phi_{\tilde{\mu}}(e_\alpha,e_\alpha)<\infty. \quad \Box$$

We derive now some consequences of Theorem 6.2. The first one deals with Poisson measures.

6.3 Theorem. Let B be a separable Banach space, $\{\nu_\alpha\}$ a family of σ-finite positive measures, and let μ be as in Theorem 6.1. Assume that $\int f^2 d\nu_\alpha\le\Phi_\mu(f,f)$ for all α, for all $f\in B'$. Then ν_α is a Lévy measure for each α and $\{c_\tau \text{Pois}\nu_\alpha\}$ is relatively compact for each $\tau>0$.

Proof. Let $\lambda_\alpha=\nu_\alpha+\bar{\nu}_\alpha$; it is enough to prove the statements for $\{\lambda_\alpha\}$. Let $\{A_{\alpha,n}\}$ be symmetric Borel sets such that for each α, $A_{\alpha,n}\uparrow B$ as $n\to\infty$ and $\lambda_\alpha(A_{\alpha,n})<\infty$ for all n. Define $\lambda_{\alpha,n}=\lambda_\alpha|A_{\alpha,n}$. By Lemma 1.8,

$$\int f^2 d\text{Pois}\lambda_{\alpha,n}=\int f^2 d\lambda_{\alpha,n}\le\int f^2 d\lambda_\alpha=2\int f^2 d\nu_\alpha\le2\Phi_\mu(f,f).$$

By Theorem 6.2, $\{\text{Pois}\lambda_{\alpha,n}\}_n$ is relatively compact for each α; it follows from Theorem 1.6 that λ_α is a Lévy measure for each α. Applying Lemma 1.8 again,

$$\int f^2 d\text{Pois}\lambda_\alpha=\int f^2 d\lambda_\alpha\le2\Phi_\mu(f,f).$$

Theorem 6.2 implies now that $\{\text{Pois}\lambda_\alpha\}$ is relatively compact. $\Box$

The next application of Theorem 6.2 is a central limit theorem, Gaussian case; for simplicity, we state the result in the case of bounded covariances.

6.4 Theorem. Let B be a separable Banach space and let μ be as in Theorem 6.1. Let $\{X_{nj}\}$ be a triangular array such that $Ef^2(X_{nj})<\infty$ and $Ef(X_{nj})=0$ for all n,j, for all $f\in B'$. Assume

(1) for every $f\in B'$, $\Psi(f)=\lim_n\Sigma_j Ef^2(X_{nj})$ exists,

(2) for every $\varepsilon>0$ and every $f\in B'$
$\lim_n\Sigma_j E(f^2(X_{nj})I_{\{|f(X_{nj})|>\varepsilon\}})=0$,

(3) for every $f\in B'$, $\Sigma_j Ef^2(X_{nj})\le\Phi_\mu(f,f)$.

Then (a) there exists a centered Gaussian measure γ such that $\Phi_\gamma(f,f)=\Psi(f)$ for every $f\in B'$ and $L(S_n)\to_w\gamma$ and

(b) $\sup_n E\|S_n\|^2<\infty$.

Proof. By well-known classical arguments (Lindeberg's theorem), $L(f(S_n))$ converges weakly to the centered Gaussian measure on R with variance $\Psi(f)$ for each $f\in B'$. Thus it is enough to prove that $\{L(S_n)\}$ is relatively compact. But $Ef^2(S_n)=\Sigma_j Ef^2(X_{nj})$ and therefore the result follows from assumption (3) and Theorem 6.2. $\Box$

We have the following corollary for the i.i.d. case.

6.5 Corollary. Let B be a separable Banach space and let μ be as in Theorem 6.1. Let $\{X_j:j\in N\}$ be independent identically distributed r.v.'s with $Ef^2(X_1)\le\Phi_\mu(f,f)$ and $Ef(X_1)=0$ $(f\in B')$. Then

(a) there exists a centered Gaussian measure γ on B such that $L(S_n/n^{1/2})\to_w\gamma$, and

(b) $\sup_n E\|S_n/n^{1/2}\|^2<\infty$.

Remark. Since on a space of cotype 2 every centered Gaussian measure is strongly Gaussian (Theorem 5.6), Corollary 6.5 contains the direct part of the central limit theorem for cotype 2 spaces in Jain [24], Aldous [3] and Chobanjan-Tarieladze [10].

We will now make some remarks on necessary integrability conditions in the general converse central limit theorem in cotype 2 spaces. Let B be a Banach space which is the dual of a type 2 space E. Since for every closed subspace $F\subset B$, B/F is the dual of $F^{\perp}\subset E$ and $F^{\perp}$ is a type 2 space with the same constant C as E, by a result of Maurey [30] it follows that B/F is a cotype 2 space with the same constant C (in particular, B is of cotype 2). A standard argument (see [22]) then gives: there exists a constant M>0 such that for every closed subspace $F\subset B$ and every finite sequence $\{X_j:j=1,..,n\}$ of independent B-valued random vectors such that $X_j\ L_2(B)$ and $E(X_j)=0$,

(6.1) $$\Sigma_{j=1}^{n} Eq_F^2(X_j) \le MEq_F^2(\Sigma_{j=1}^{n} X_j).$$

Theorem 2.3 implies now:

<u>6.6 Theorem.</u> Let $\{X_{nj}\}$ be a triangular array of B-valued r.v.'s such that $\|X_{nj}\| \le C<\infty$ a.s. and $EX_{nj}=0$ for all n,j. Assume that $\{L(S_n)\}$ is relatively shift compact. Then:

(1) if B is of cotype 2, then $\sup_n \Sigma_j E\|X_{nj}\|^2<\infty$,

(2) if moreover B is the dual of a type 2 space, then for every full sequence $\{F_k\}\subset \mathcal{F}$, $\lim_k \sup_n \Sigma_j Eq_{F_k}^2(X_{nj})=0$.

<u>Remark</u>. The assumption that B is the dual of a type 2 space may be replaced by the following alternative hypothesis. Suppose B is of cotype 2 and $\{F_k\}\subset \mathcal{F}$ is a full sequence such that there exist M>0 and for every k a continuous linear map $\Pi_k:B\to F_k$ such that $q_{F_k}(x)\ge M\|x-\Pi_k(x)\|$. If $\{\varepsilon_j:j\in N\}$ is a symmetric Bernoulli sequence, then

$$Eq_{F_k}^2(\Sigma_{j=1}^{n}\varepsilon_j x_j)\ge ME\|\Sigma_{j=1}^{n}\varepsilon_j(x_j-\Pi_k x_j)\|^2 \ge MC\Sigma_{j=1}^{n}\|x_j-\Pi_k x_j\|^2$$
$$\ge MC\Sigma_{j=1}^{n} q_{F_k}^2(x_j),$$

showing that B/F_k is of cotype 2 with constant MC. Clearly Theorem 6.6 holds with the obvious change in the formulation of (2). It may be shown that the alternative assumption covers the case of cotype 2 spaces with a Schauder basis (in particular, separable L_1 spaces).

For the sake of completeness, we state a general converse central limit theorem in duals of type 2 spaces; it is an obvious consequence of Theorems 2.10 and 6.6.

<u>6.7 Theorem.</u> Let B be the dual of a type 2 space and $\{X_{nj}\}$ an infinitesimal triangular array of B-valued r.v.'s. Suppose $L(S_n-x_n)$ converges weakly for some sequence $\{x_n\}\subset B$. Then the conclusions (1)-(3) of Theorem 2.10 hold, and moreover, for every $\tau>0$

(4) $\sup_n \Sigma_j E\|X_{nj\tau}-EX_{nj\tau}\|^2<\infty$,

(5) for every full sequence $\{F_k\}\subset \mathcal{F}$,

$$\lim_k \sup_n \Sigma_j Eq_{F_k}^2(X_{nj\tau}-EX_{nj\tau})=0.$$

<u>Remarks</u>. (1) The remark following Theorem 6.6 applies also to this theorem.

(2) Theorem 6.7 together with Theorem 4.2 give the general (direct and converse) central limit theorem in Hilbert space.

(3) Theorem 6.7 contains somewhat more information than the general converse central limit theorem for the Hilbert space case in [18].

(4) As an immediate consequence of Theorem 6.7, one may state a converse central limit theorem for the case of Gaussian convergence which in the Hilbert space case contains that in [33], VI.6.3.

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