Perfect simulation and finitary coding for multicolor systems with interactions of infinite range

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Abstract

We consider a particle system on $\mathbb{Z}^d$ with finite state space and interactions of infinite range. Assuming that the rate of change is continuous and decays sufficiently fast, we introduce a perfect simulation algorithm for the stationary process. The basic tool we use is a representation of the infinite range change rates as a mixture of finite range change rates. The perfect simulation scheme provides the basis for the construction of a finitary coding from an i.i.d. finite-value process to the invariant measure of the multicolor system.

Key words : Interacting particle systems, long range interactions, perfect simulation, finitary coding, exponential ergodicity.

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1 Introduction

In this paper we first present a perfect simulation algorithm for a multicolor system on $\mathbb{Z}^d$ with interactions of infinite range. This perfect simulation algorithm is the basis of the construction of a finitary coding from a finite-valued i.i.d. process to the invariant probability measure of the multicolor system.

By a perfect simulation algorithm we mean a simulation which samples precisely from the stationary law of the process. More precisely, for any finite set of sites F and any finite time interval $[0, t]$ we want to sample the stationary time evolution of the coloring of sites in F during $[0, t]$.

By coding we mean a translation invariant deterministic measurable map from the finite-valued i.i.d. process to the invariant probability measure of the system. Finitary means that the value of the map at the origin depends only on a finite subset of the random variables. This finite subset is a function of the realization of the family of independent random variables.

The process we consider is an interacting particle system with finite state space. The elements of this finite state space are called *colors*. To each site in $\mathbb{Z}^d$ is assigned a color. The coloring of the sites changes as time goes by. The rate at which the color of a fixed site i changes from a color a to a new color b is a function of the entire configuration and depends on b.

We do not assume that the system has a dual, or is attractive, or monotone in any sense. Our system is not even spatially homogeneous. The basic assumptions are the continuity of the infinite range change rates together with a fast decay of the long range influence on the change rate. These two properties imply that the change rates can be represented as a countable mixture of local change rates of increasing range. This decomposition (see Theorem 1) extends to the case of interacting particle systems the notion of random Markov chains appearing explicitly in Kalikow (1990) and Bramson and Kalikow (1993) and implicitly in Ferrari et al. (2000) and Comets et al. (2002).

The decomposition of the change rate of infinite range as a countable mixture of finite range change rates suggests the construction of any cylindrical time evolution of the stationary process by the concatenation of two basic algorithms. First we construct a backward black and white sketch of the process. Then in a second forward algorithm we assign colors to the black and white picture.

The proof that the backward black and white algorithm stops after a finite number of steps follows ideas presented in Bertein and Galves (1977) to study dual processes. Using these ideas we prove the existence of our process in a self-contained way. The same ideas appear again in the construction of the finitary coding. This type of construction is similar in spirit to procedures adopted in Ferrari (1990), Ferrari et al. (2002), Garcia and Marić (2006) and Van den Berg and Steif (1999). However all these papers only consider particular models, satisfying restrictive assumptions which are not assumed in the present paper.

Our Theorem 4 shows the existence of a finitary coding from an i.i.d. finite-valued process to the invariant probability measure of the multicolor system. This can be seen as an extension to the infinite range processes of Theorem 3.4 of Van den Berg and Steif (1999).

Häggström and Steif (2000) constructs a finitary coding for Markov fields. This result follows, under slightly stronger assumptions, as a corollary of our Theorem 4 which holds also for non Markov infinite range fields. These authors conclude the above mentioned paper by observing that the extension of their results to "infinite-range Gibbs measures appears to be a more difficult matter". Our Theorem 4 is an attempt in this direction.

This paper is organized as follows. In Section 2 we present the model and state a preliminary result, Theorem 1, which gives the representation of the change rate as a countable mixture of local change rates. In Section 3, we present the perfect simulation algorithm and Theorem 2 which ensures that the algorithm stops after a finite number of steps. Theorem 2 also guarantees the exponential ergodicity of the process. The definitions and results concerning the finitary coding are presented in Section 4. The proofs of the theorems are presented in Sections 5 to 11.

2 Definitions, notation and basic results

In what follows, A will be a finite set of colors, the initial lowercase letters a, b, $c, \ldots$ will denote elements of A. We will call configuration any element of $A^{\mathbb{Z}^d}$. Configurations will be denoted by letters $\eta, \zeta, \xi, \ldots$ A point $i \in \mathbb{Z}^d$ will be called site. As usual, for any $i \in \mathbb{Z}^d$, $\eta(i)$ will denote the value of the configuration η at site i. By extension, for any subset $V \subset \mathbb{Z}^d$, $\eta(V) \in A^V$ will denote the restriction of the configuration η to the set of positions in V. For any η, i and a, we shall denote $\eta^{i,a}$ the modified configuration

$$\eta^{i,a}(j) = \eta(j), \text{ for all } j \neq i, \text{ and } \eta^{i,a}(i) = a.$$

For any $i \in \mathbb{Z}^d$, $\eta \in A^{\mathbb{Z}^d}$ and $a \in A$, $a \neq \eta(i)$, and we denote by $c_i(a, \eta)$ a positive real number. We suppose that there exists a constant $\Gamma_i < +\infty$ such that

$$c_i(a, \eta) \leq \Gamma_i, \tag{2.1}$$

for every a and η such that $a \neq \eta(i)$.

A multicolor system with interactions of infinite range is a Markov process on $A^{\mathbb{Z}^d}$ whose generator is defined on cylinder functions by

$$L\, f(\eta) \;=\; \sum_{i \in \mathbb{Z}^d} \sum_{a \in A, a \neq \eta(i)} c_i(a, \eta)[f(\eta^{i,a}) - f(\eta)]\,. \tag{2.2}$$

Intuitively, this form of the generator means that the site i will be updated to the symbol a, $a \neq \eta(i)$, at a rate $c_i(a, \eta)$ whenever the configuration of the system is η. The choice of $c_i(a, \eta)$ for $a = \eta(i)$ does not affect the generator (2.2) and represents a degree of freedom in our model. In what follows we choose $c_i(\eta(i), \eta)$ in such a way that

$$c_i(a, \eta) = M_i\, p_i(a|\eta). \tag{2.3}$$

In the above formula $M_i < +\infty$ is a suitable constant and for every fixed configuration η, $p_i(\cdot|\eta)$ is a probability measure on A. Condition (2.1) implies that such a choice is always possible, for instance by taking $M_i = |A|\,\Gamma_i$ and defining

$$c_i(\eta(i), \eta) = M_i - \sum_{a: a \neq \eta(i)} c_i(a, \eta). \tag{2.4}$$

We shall call $(c_i)_{i \in \mathbb{Z}^d}$ a family of rate functions for this fixed choice (2.4).

Our first aim is to give sufficient conditions on $c_i(a, \eta)$ implying the existence of a perfect simulation algorithm of the process having generator (2.2). To state these conditions, we need some extra notation. Let $V_i(k) = \{j \in \mathbb{Z}^d; 0 \leq \|j - i\| \leq k\}$, where $\|j\| = \sum_{u=1}^d |j_u|$ is the usual L_1-norm of $\mathbb{Z}^d$. We will impose the following continuity condition on the family of rate functions c.

Continuity condition. For any symbol a, we will assume that

$$\sup_{i \in \mathbb{Z}^d} \sup_{\eta(V_i(k)) = \zeta(V_i(k))} |c_i(a, \eta) - c_i(a, \zeta)| \to 0\,, \tag{2.5}$$

as $k \to \infty$.

Define

$$\alpha_i(-1) \;=\; \sum_{a \in A} \min \left(\inf_{\zeta \in A^{\mathbb{Z}^d}, \zeta(i) \neq a} c_i(a, \zeta),\, M_i - \sup_{\zeta \in A^{\mathbb{Z}^d}, \zeta(i) = a} \sum_{b \neq a} c_i(b, \zeta) \right)\,, \tag{2.6}$$

and for any $k \geq 0$,

$$\alpha_i(k) \;=\; \min_{w \in A^{V_i(k)}} \left((\sum_{a \in A, a \neq w(i)} \inf_{\zeta: \zeta(V_i(k)) = w} c_i(a, \zeta)) + M_i - \sup_{\zeta: \zeta(V_i(k)) = w} \sum_{b \neq w(i)} c_i(b, \zeta) \right)\,. \tag{2.7}$$

In order to clarify the role of $\alpha_i(k)$, we present an example that is a spatial version of the chain that regenerates in 1 : the evolution of a site depends on the random ball around that site where the size of this ball is the smallest radius such that 1 belongs to the ball (without the center itself).

Example 1 *Let* $A = \{0, 1\}$,

$$l_i(\eta) = l, \text{ if } \max_{j \in V_i(l), j \neq i} \eta(j) = 0 \text{ and } \max_{j \in V_i(l+1), j \neq i} \eta(j) = 1$$

and define

$$c_i(1, \eta) = q_{l_i(\eta)}, \; c_i(0, \eta) = 1 - c_i(1, \eta),$$

where $0 < q_k < 1$ *for all* k.

Note that in this case,

$$\sup_i \sup_{\eta(V_i(k)) = \zeta(V_i(k))} |c_i(1, \eta) - c_i(1, \zeta)| = \sup_{l, m \geq k} |q_l - q_m|,$$

and thus the process is continuous, if and only if $\lim_k q_k$ *exists.*

Observe that if $q_k \downarrow q_\infty$, *as* $k \to \infty$, *then* $M_i = 1$ *and*

$$\begin{aligned}
\alpha_i(-1) &= \min\left(\inf_{\zeta \in A^{\mathbb{Z}^d}, \zeta(i)=1} c_i(0, \zeta), \, 1 - \sup_{\zeta \in A^{\mathbb{Z}^d}, \zeta(i)=0} c_i(1, \zeta) \right) \\
&\quad + \min\left(\inf_{\zeta \in A^{\mathbb{Z}^d}, \zeta(i)=0} c_i(1, \zeta), \, 1 - \sup_{\zeta \in A^{\mathbb{Z}^d}, \zeta(i)=1} c_i(0, \zeta) \right) \\
&= \inf_{\zeta \in A^{\mathbb{Z}^d}} c_i(0, \zeta) + \inf_{\zeta \in A^{\mathbb{Z}^d}} c_i(1, \zeta) \\
&= 1 - q_0 + q_\infty.
\end{aligned}$$

Also,

$$\begin{aligned}
\alpha_i(0) &= \min\Big(\inf_{\zeta \in A^{\mathbb{Z}^d}, \zeta(i)=0} c_i(1, \zeta) + 1 - \sup_{\zeta \in A^{\mathbb{Z}^d}, \zeta(i)=0} c_i(1, \zeta), \\
&\qquad \inf_{\zeta \in A^{\mathbb{Z}^d}, \zeta(i)=1} c_i(0, \zeta) + 1 - \sup_{\zeta \in A^{\mathbb{Z}^d}, \zeta(i)=1} c_i(0, \zeta) \Big) \\
&= \min(q_\infty + 1 - q_0, 1 - q_0 + q_\infty) \\
&= \alpha_i(-1).
\end{aligned}$$

Finally,

$$\begin{aligned}
\alpha_i(k) &= \min_{w \in A^{V_i(k)}} \left(\inf_{\zeta : \zeta(V_i(k)) = w} c_i(1 - \zeta(i), \zeta) + 1 - \sup_{\zeta : \zeta(V_i(k)) = w} c_i(\zeta(i), \zeta) \right) \\
&= \min_{w \in A^{V_i(k)}} \left(\inf_{\zeta : \zeta(V_i(k)) = w} c_i(0, \zeta) + \inf_{\zeta : \zeta(V_i(k)) = w} c_i(1, \zeta) \right) \\
&= \min_{w \in A^{V_i(k)}} \left(\inf_{\zeta : \zeta(V_i(k)) = w} q_{l_i(\zeta)} + \inf_{\zeta : \zeta(V_i(k)) = w} (1 - q_{l_i(\zeta)}) \right) \\
&= (1 - q_k) + q_\infty.
\end{aligned}$$

Let us introduce some more notation. Note that for each site i,

$$M_i = \lim_{k\to\infty} \alpha_i(k). \tag{2.8}$$

Hence to each site i we can associate a probability distribution λ_i by

$$\lambda_i(-1) \;=\; \frac{\alpha_i(-1)}{M_i}, \tag{2.9}$$

and for $k \geq 0$

$$\lambda_i(k) \;=\; \frac{\alpha_i(k) \;-\; \alpha_i(k-1)}{M_i}. \tag{2.10}$$

We will see that for each i the family of rate functions $c_i(.,.)$ can be represented as a mixture of local rate functions weighted by $(\lambda_i(k))_{k\geq -1}$. More formally, we have the following theorem.

Theorem 1 *Let $(c_i)_{i\in\mathbb{Z}^d}$ be a family of rate functions satisfying the conditions (2.1), (2.3), the continuity condition (2.5) and the summability condition (3.13). Then for any site i there exists a family of conditional probabilities $p_i^{[k]}$ on A depending on the local configurations $\eta(V_i(k))$ such that*

$$c_i(a,\eta) \;=\; M_i\, p_i(a|\eta), \;\; \textit{where } p_i(a|\eta) = \sum_{k\geq -1} \lambda_i(k) p_i^{[k]}(a|\eta(V_i(k))). \tag{2.11}$$

As a consequence, the infinitesimal generator L given by (2.2) can be rewritten as

$$L\, f(\eta) \;=\; \sum_{i\in\mathbb{Z}^d}\sum_{a\in A}\sum_{k\geq -1} M_i\lambda_i(k)p_i^{[k]}(a|\eta(V_i(k)))[f(\eta^{i,a}) - f(\eta)]\,. \tag{2.12}$$

Remark 1 *Note that for $k = -1$, $V_i(k) = \emptyset$ and hence $p_i^{[-1]}(a|\eta(V_i(k))) = p_i^{[-1]}(a)$ does not depend on the configuration. Therefore, $\lambda_i(-1)$ represents the spontaneous self-coloring rate of site i in the process.*

The representation given by (2.12) provides a clearer description of the time evolution of the process. We start with an initial configuration η at time zero. This configuration is updated in a càdlàg way as follows. For each site $i \in \mathbb{Z}^d$, consider a rate M_i Poisson point process N^i. The Poisson processes corresponding to distinct sites are all independent. If at time t, the Poisson clock associated to site i rings, we choose a range k with probability $\lambda_i(k)$ independently of everything else. And then, update the value of the configuration at this site by choosing a symbol a with probability $p_i^{[k]}(a|\xi_t^\eta(V_i(k)))$.

3 Perfect simulation of the stationary process

The decomposition (2.11) provided by Theorem 1 suggests an algorithm of perfect simulation for the multicolor long range interacting system. This is the main result of this article. The goal is to sample under equilibrium the time evolution of any finite set of sites F during any fixed finite time interval.

We first introduce a simulation procedure to sample the time evolution of any finite set of sites F during any fixed finite time interval $[0,t]$, when starting from a fixed initial configuration η. This simulation procedure has two stages. First, we draw a backward black and white sketch in order to determine the set of sites and the succession of choices affecting the configuration of the set of sites F at time t. Then, in the second stage, a forward coloring procedure assigns colors to every site involved in the black and white sketch. This will be formally described in Algorithms 1 and 2 below.

Let us describe the mathematical ideas behind this algorithm. Our goal is to simulate the configuration of the fixed set of sites F during the time interval $[0,t]$ when the process starts from an initial configuration η. We climb up the rate M_j Poisson processes $N^j, j \in F$, until we find the last occurrence time before time t where the Poisson clock rang. Note that the probability that the clock of site i rings first among all these clocks is given by

$$\frac{M_i}{\sum_{j\in F} M_j}.$$

Then we have to inspect the configuration at the sites belonging to the finite set $V_i(k)$ which is chosen at that time. $V_i(k)$ is chosen with probability $\lambda_i(k), k \geq -1$. If $k=-1$ is chosen, this means that the value of $\xi(i)$ at that time is chosen according to $p_i^{[-1]}$, independently of the other sites, and thus site i can be removed from the set F.

Otherwise, if $k \geq 0$, we have to include all the sites in $V_i(k)$ to the set of sites F and to continue the algorithm. The reverse-time checking continues for each point reached previously until we find an occurrence time before time 0. In this case the algorithm stops.

In the second stage, the algorithm assigns colors to all the sites that have been involved in the first stage. To begin with, all sites that have not yet chosen a range -1, will be colored according to the initial configuration η at time 0. And then successively, going forwards in time, we assign colors to the remaining sites according to $p_i^{[k]}(\cdot|V_i(k))$, where all sites in $V_i(k)$ have already been colored in a previous step of the algorithm. Finally we finish with the colors of the set of sites F at time t.

The finite time simulation Algorithm 1 and 2 uses the following variables.

- N is an auxiliary variables taking values in the set of non-negative integers $\{0,1,2,\ldots\}$
- N_{STOP} is a counter taking values in the set of non-negative integers $\{0,1,2,\ldots\}$
- T_{STOP} is an element of $(0,+\infty)$
- I is variable taking values in $\mathbb{Z}^d$
- K is a variable taking values in $\{-1,0,1,\ldots\}$
- T is an element of $(0,+\infty)$
- $B=(B_1,B_2,B_3)$ where
  - B_1 is an array of elements of $\mathbb{Z}^d$
  - B_2 is an array of elements of $\{-1,0,1,\ldots\}$
  - B_3 is an array of elements of $(0,+\infty)$

- C is variable taking values in the set of finite subsets of $\mathbb{Z}^d$
- W is an auxiliary variable taking values in A
- V is an array of elements of A
- ζ is a function from $\mathbb{Z}^d$ to $A \cup \{\Delta\}$, where Δ is some extra symbol that does not belong to A

Algorithm 1 Backward black and white sketch without deaths

Input: F; *Output:* N_{STOP}, B, C, T_{STOP}
$N \leftarrow 0$, $N_{STOP} \leftarrow 0$, $B \leftarrow \emptyset$, $C \leftarrow F$, $T_{STOP} \leftarrow 0$
while $T_{STOP} < t$ and $C \neq \emptyset$ **do**
  Choose a time $T \in (0, +\infty)$ randomly according to the exponential distribution with parameter $\sum_{j \in C} M_j$. Update

$$T_{STOP} \leftarrow T_{STOP} + T.$$

  $N \leftarrow N + 1$.
  Choose a site $I \in C$ randomly according to the distribution

$$P(I = i) = \frac{M_i}{\sum_{j \in C} M_j}$$

  Choose $K \in \{-1, 0, 1, \ldots\}$ randomly according to the distribution

$$P(K = k) = \lambda_I(k)$$

  $C \leftarrow C \cup V_I(K)$
  $B(N) \leftarrow (I, K, T_{STOP})$
end while
$N_{STOP} \leftarrow N$
return N_{STOP}, B, C, T_{STOP}

Using output V of Algorithm 2 and output B of Algorithm 1 we can construct the time evolution $(\xi_s(F), 0 \leq s \leq t)$ of the process. This is done as follows.

Denote $I(N), T(N)$ the first and the third coordinate of the array $B(N)$ respectively. Introduce the following random times for any $1 \leq n \leq N_{STOP}$,

$$S_n = t - T(N_{STOP} - n + 1).$$

- For $0 \leq s < S_1$ define $\xi_s(F) = \zeta(F)$.
- For $1 \leq n \leq N_{STOP}$, for $S_n \leq s < S_{n+1} \wedge t$, we put
  - for all $i \in F$ such that $i \neq I(N_{STOP} - n + 1)$, $\xi_s(i) = \xi_{S_n}(i)$;
  - for $i = I(N_{STOP} - n + 1)$, $\xi_s(i) = V(n)$.

We summarize the above discussion in the following proposition.

Algorithm 2 Forward coloring procedure

Input: N_{STOP}, B, C, $\eta(C)$; *Output:* V
$N \leftarrow N_{STOP}$
$\zeta(j) \leftarrow \eta(j)$ for all $j \in C$; $\zeta(j) \leftarrow \Delta$ for all $j \in \mathbb{Z}^d \setminus C$
while $N \geq 1$ **do**
  $(I, K, T) \leftarrow B(N)$.
  if $K = -1$ **then**
    Choose W randomly in A according to the probability distribution

$$P(W = v) = p_I^{[-1]}(v)$$

  else
    Choose W randomly in A according to the probability distribution

$$P(W = v) = p_I^{[K]}(v|\zeta(V_I(K))$$

  end if
  $\zeta(I) \leftarrow W$
  $V(N) \leftarrow W$
  $N \leftarrow N - 1$
end while
return V

Proposition 1 *Let $(c_i)_{i\in\mathbb{Z}^d}$ be a family of continuous rate functions satisfying the conditions of Theorem 1. If*

$$\sup_{i\in\mathbb{Z}^d} \sum_{k\geq 0} |V_i(k)|\lambda_i(k) \;<\; +\infty\,, \tag{3.13}$$

then Algorithm 1 stops almost surely after a finite number of steps, i.e.

$$P(N_{STOP} < +\infty) = 1.$$

Moreover, for any initial configuration η, there exists a unique Markov process $(\xi_t^\eta)_{t\geq 0}$ such that $\xi_0^\eta = \eta$ and with infinitesimal generator

$$L\,f(\eta) \;=\; \sum_{i\in\mathbb{Z}^d} \sum_{a\in A} c_i(a,\eta)[f(\eta^{i,a}) - f(\eta)]\,. \tag{3.14}$$

The cylindrical time evolution $(\xi_s(F), 0 \leq s \leq t)$ simulated in Algorithms 1 and 2 is a sample from this process ξ^η.

We now turn to the main object of this paper, the perfect simulation of the multicolor long range interacting system under equilibrium. The goal is to sample under equilibrium the time evolution of any finite set of sites F during any fixed finite time interval.

We first introduce a simulation procedure to sample from equilibrium the cylindrical configuration at a fixed time. As before, this simulation procedure has two stages : First, we draw a backward black and white sketch in order to determine the set of sites and the succession of choices affecting the configuration of the set of sites at equilibrium. Then, in the second stage, a forward coloring procedure assigns colors to every site involved in the black and white sketch. This will be formally described in Algorithms 3 and 4 below.

The following variables will be used.

- N is an auxiliary variables taking values in the set of non-negative integers $\{0, 1, 2, \dots\}$
- N_{STOP} is a counter taking values in the set of non-negative integers $\{0, 1, 2, \dots\}$
- I is variable taking values in $\mathbb{Z}^d$
- K is a variable taking values in $\{-1, 0, 1, \dots\}$
- B is an array of elements of $\mathbb{Z}^d \times \{-1, 0, 1, \dots\}$
- C is variable taking values in the set of finite subsets of $\mathbb{Z}^d$
- W is an auxiliary variable taking values in A
- η is a function from $\mathbb{Z}^d$ to $A \cup \{\Delta\}$, where Δ is some extra symbol that does not belong to A

Algorithm 3 Backward black and white sketch

Input: F; *Output:* N_{STOP}, B
$N \leftarrow 0$, $N_{STOP} \leftarrow 0$, $B \leftarrow \emptyset$, $C \leftarrow F$,
while $C \neq \emptyset$ **do**
  $N \leftarrow N + 1$.
  Choose a site $I \in C$ randomly according to the distribution

$$P(I = i) = \frac{M_i}{\sum_{j \in C} M_j}$$

  Choose $K \in \{-1, 0, 1, \dots\}$ randomly according to the distribution

$$P(K = k) = \lambda_I(k)$$

  if $K = -1$, **then**
    $C \leftarrow C \setminus \{I\}$
  else
    $C \leftarrow C \cup V_I(K)$
  end if
  $B(N) \leftarrow (I, K)$
end while
$N_{STOP} \leftarrow N$
return N_{STOP}, B

Let us call μ the distribution on $A^{\mathbb{Z}^d}$ whose projection on A^F is the law of $\eta(F)$ printed at the end of Algorithm 4. The following theorem gives a sufficient condition ensuring that Algorithm 3 stops after a finite number of steps and shows that μ is actually the invariant measure of the process.

Theorem 2 *Let $(c_i)_{i \in \mathbb{Z}^d}$ be a family of rate functions satisfying the conditions of Theorem 1. If*

$$\sup_{i \in \mathbb{Z}^d} \sum_{k \geq 0} |V_i(k)| \lambda_i(k) \; < \; 1 \,, \tag{3.15}$$

Algorithm 4 Forward coloring procedure

Input: N_{STOP}, B; *Output:* $\{(i,\eta(i)), i \in F\}$
$N \leftarrow N_{STOP}$
$\eta(j) \leftarrow \Delta$ for all $j \in \mathbb{Z}^d$
while $N \geq 1$ **do**
  $(I, K) \leftarrow B(N)$.
  if $K = -1$ **then**
    Choose W randomly in A according to the probability distribution

$$P(W = v) = p_I^{[-1]}(v)$$

  else
    Choose W randomly in A according to the probability distribution

$$P(W = v) = p_I^{[K]}(v|\eta(V_I(K))$$

  end if
  $\eta(I) \leftarrow W$
  $N \leftarrow N - 1$
end while
return $\{(i,\eta(i)) : i \in F\}$

then

$$P(N_{STOP} < +\infty) = 1.$$

The law of the set $\{(i,\eta(i)) : i \in F\}$ *printed at the end of Algorithms 3 and 4 is the projection on* A^F *of the unique invariant probability measure* μ *of the process. Moreover, the law of the process starting from any initial configuration converges weakly to* μ *and this convergence takes place exponentially fast.*

Remark 2 *In the literature, we say that the process is ergodic, if it admits a unique invariant measure which is the weak limit of the law of the process starting from any initial configuration. If this convergence takes place exponentially fast, we say that the process is exponentially ergodic. Therefore, Theorem 2 says that the multicolor system is exponentially ergodic.*

Algorithms 3 and 4 show how to sample the invariant probability measure of the process. We now pursuit a more ambitious goal : how to sample the stationary time evolution of any fixed finite set of sites F during any fixed interval of time $[0, t]$. This is done using Algorithms 1 and 2 as well.

Algorithm 1 produces a backward black and white sketch without removing the spontaneously coloring sites. We start at time t with the set of sites F and run backward in time until time 0. This produces as part of its output the set of sites C whose coloring at time 0 will affect the coloring of the sites in F during $[0, t]$. We then use the output set C of Algorithm 1 as input set of positions in Algorithms 3 and 4. Algorithms 3 and 4 will give us as output the configuration $\eta(C)$ that will be used as input configuration for Algorithm 2.

Theorem 3 *Under the conditions of Theorem 1, Algorithm 3 stops almost surely after a finite number of steps*

$$P(N_{STOP} < +\infty) = 1.$$

Moreover, under the conditions of Theorem 2, for any $t > 0$, the cylindrical time evolution $(\xi_s(F), 0 \leq s \leq t)$ simulated in Algorithms 1, 2, 3 and 4 is a sample from the stationary process.

4 Finitary coding

The perfect simulation procedure described in Algorithms 1–4 gives the basis for the construction a finitary coding for the invariant probability measure of the multicolor system ξ_t. By this we mean the following. Let $(Y(i), i \in \mathbb{Z}^d)$ be a family of i.i.d. random variables assuming values on a finite set S. Let $(\xi_0(i), i \in \mathbb{Z}^d)$ be the configuration sampled according to the invariant probability measure μ obtained as output of Algorithm 2.

Definition 1 *We say that there exists a finitary coding from $(Y(i), i \in \mathbb{Z}^d)$ to $(\xi_0(i), i \in \mathbb{Z}^d)$ if there exists a deterministic function $f : S^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ such that almost surely the following holds:*

- *f commutes with the shift operator, that is, $f(T_i(y)) = T_i(f(y))$ for any $i \in \mathbb{Z}^d$;*
- *$\xi_0 = f((Y(j)), j \in \mathbb{Z}^d)$; and*
- *there exists a a finite subset $\bar{F}$ of $\mathbb{Z}^d$ satisfying*

$$f((Y(j)), j \in \mathbb{Z}^d) = f((Y'(j)), j \in \mathbb{Z}^d)$$

whenever

$$Y'(j) = Y(j) \text{ for all } j \in \bar{F}.$$

In the first condition of the definition, the notation T_i denotes the translation by i steps, in $S^{\mathbb{Z}^d}$, or in $A^{\mathbb{Z}^d}$. More precisely, for any $i \in \mathbb{Z}^d$, if $y \in \{0,1\}^{\mathbb{Z}^d}$ then $T_i(y)$ is the element of $S^{\mathbb{Z}^d}$ such that $T_i(y)(j) = y(j - i)$ with equivalent definition for $\xi \in A^{\mathbb{Z}^d}$.

Theorem 4 *Under the conditions of Theorem 2 there exists a finitary coding from an independent and identically distributed family of a sequence of finite-valued variables $(Y(i), i \in \mathbb{Z}^d)$ to $(\xi_0(i), i \in \mathbb{Z}^d)$.*

Theorem 1.1 of Häggström and Steif (2000) follows as a corollary of Theorem 4 under a slightly stronger condition. In order to state this corollary, we need to introduce the notion of Markov random field.

Definition 2 *A Markov random field X on $\mathbb{Z}^d$ with values in a finite alphabet A has distribution μ if μ admits a consistent set of conditional probabilities*

$$\mu(X(\Lambda) = \xi(\Lambda)|X(\mathbb{Z}^d \setminus \Lambda) = \xi(\mathbb{Z}^d \setminus \Lambda)) = \mu(X(\Lambda) = \xi(\Lambda)|X(\partial\Lambda) = \xi(\partial\Lambda))$$

for all finite $\Lambda \subset \mathbb{Z}^d$, $\xi \in A^{\mathbb{Z}^d}$. Here, $\partial\Lambda = \{j \in \mathbb{Z}^d : \inf_{i\in\Lambda} \|i - j\| = 1\}$. Such a set of conditional probabilities is called the specification of the random field and denoted by Q.

Corollary 1 *For any Markov random field X on $\mathbb{Z}^d$ with specification $\mathcal{Q}$ satisfying*

$$\sum_{a\in A} \min_{\zeta(\partial 0)\in A^{\partial 0}} \mathcal{Q}(X(0)=a|X(\partial 0)=\zeta(\partial 0)) > \frac{2d}{2d+1}, \tag{4.16}$$

there exists an i.i.d. sequence $(Y(i), i\in\mathbb{Z}^d)$ of finite valued random variables such that there exists a finitary coding from $(Y(i), i\in\mathbb{Z}^d)$ to the Markov random field.

Remark 3 *Just for comparison, Condition (4.16) is equivalent to*

$$\alpha_0(-1) > \frac{2d}{2d+1},$$

while Condition HN in Theorem 1.1 of Häggström and Steif (2000) can be rewritten in our notation as

$$\alpha_0(-1) > \frac{2d-1}{2d}.$$

This does not seem to be a too high price to pay in order to be able to treat the general case of long range interactions.

5 Proof of Theorem 1

The countable mixture representation provided by Theorem 1 is the basis of all the other results presented in this paper. Therefore it is just fair that its proof appears in the first place.

Recall that $c_i(a,\eta) = M_i\, p_i(a|\eta)$. Therefore, it is sufficient to provide a decomposition for $p_i(a|\eta)$. Put

$$\begin{aligned}
r_i^{[-1]}(a) &= \inf_{\zeta} p_i(a|\zeta),\\
\Delta_i^{[-1]}(a) &= r_i^{[-1]}(a),\\
r_i^{[0]}(a|\eta(V_i(0))) &= \inf_{\zeta:\zeta(V_i(0))=\eta(V_i(0))} p_i(a|\zeta),\\
\Delta_i^{[0]}(a|\eta(V_i(0))) &= r_i^{[0]}(a|\eta(V_i(0))) - r_i^{[-1]}(a).
\end{aligned}$$

For any $k\geq 1$, define

$$r_i^{[k]}(a|\eta(V_i(k))) = \inf_{\zeta:\zeta(V_i(k))=\eta(V_i(k))} p_i(a|\zeta),$$

$$\Delta_i^{[k]}(a|\eta(V_i(k))) = r_i^{[k]}(a|\eta(V_i(k))) - r_i^{[k-1]}(a|\eta(V_i(k-1))).$$

Then we have that

$$p_i(a|\eta) = \sum_{j=-1}^{k} \Delta_i^{[j]}(a|\eta(V_i(j))) + \left[p_i(a|\eta) - r_i^{[k]}(a|\eta(V_i(k)))\right].$$

By continuity of $c_i(a,\eta)$, hence of $p_i(a|\eta)$,

$$r_i^{[k]}(a|\eta(V_i(k))) \to p_i(a|\eta) \text{ as } k\to\infty.$$

Hence by monotone convergence, we conclude that

$$\sum_{j=-1}^{\infty} \Delta_i^{[j]}(a|\eta(V_i(j))) = p_i(a|\eta).$$

Now, put

$$\lambda_i(k,\eta(V_i(k))) = \sum_a \Delta_i^{[k]}(a|\eta(V_i(k)))$$

and for any i,k such that $\lambda_i(k,\eta(V_i(k))) > 0$, we define

$$\tilde{p}_i^{[k]}(a|\eta(V_i(k))) = \frac{\Delta_i^{[k]}(a|\eta(V_i(k)))}{\lambda_i(k,\eta(V_i(k)))}.$$

For i,k such that $\lambda_i(k,\eta(V_i(k))) = 0$, define $\tilde{p}_i^{[k]}(a|\eta(V_i(k)))$ in an arbitrary fixed way.

Hence

$$p_i(a|\eta) = \sum_{k=-1}^{\infty} \lambda_i(k,\eta(V_i(k)))\tilde{p}_i^{[k]}(a|\eta(V_i(k))). \tag{5.17}$$

In (5.17) the factors $\lambda_i(k,\eta(V_i(k)))$ still depend on $\eta(V_i(k))$. To obtain the decomposition as in the theorem, we must rewrite it as follows.

For any i, take M_i as in (2.8) and the sequences $\alpha_i(k), \lambda_i(k), k \geq -1$, as defined in (2.7) and (2.10), respectively. Define the new quantities

$$\alpha_i(k,\eta(V_i(k))) = M_i \sum_{l\leq k} \lambda_i(l,\eta(V_i(l))).$$

Finally put $p_i^{[-1]}(a) = \tilde{p}_i^{[-1]}(a)$, and for any $k \geq 0$,

$$\begin{aligned}
&p_i^{[k]}(a|\eta(V_i(k))) = \\
&\sum_{-1=l'\leq l}^{k-1} 1_{\{\alpha_i(l'-1,\eta(V_i(l'-1)))<\alpha_i(k-1)\leq\alpha_i(l',\eta(V_i(l')))\}} 1_{\{\alpha_i(l,\eta(V_i(l)))<\alpha_i(k)\leq\alpha_i(l+1,\eta(V_i(l+1)))\}} \\
&\qquad \left[\frac{\alpha_i(l',\eta(V_i(l'))) - \alpha_i(k-1)}{M_i\,\lambda_i(k)}\tilde{p}_i^{[l']}(a|\eta(V_i(l')))\right. \\
&\qquad + \sum_{m=l'+1}^{l} \frac{\lambda_i(m,\eta(V_i(m))}{M_i\lambda_i(k)}\tilde{p}_i^{[m]}(a|\eta(V_i(m))) \\
&\qquad \left.+ \frac{\alpha_i(k) - \alpha_i(l,\eta(V_i(l)))}{M_i\,\lambda_i(k)}\tilde{p}_i^{[l+1]}(a|\eta(V_i(l+1)))\right].
\end{aligned}$$

This concludes our proof. □

6 The black and white time-reverse sketch process

The *black and white time-reverse sketch process* gives the mathematically precise description of the backward black and white Algorithm 1 given above. We start by introducing some more notation. For each $i \in \mathbb{Z}^d$, denote by $\ldots T_{-2}^i < T_{-1}^i < T_0^i < 0 < T_1^i < T_2^i < \ldots$

the occurrence times of the rate M_i Poisson point process N^i on the real line. The Poisson point processes associated to different sites are independent. To each point T_n^i associate an independent mark K_n^i according to the probability distribution $(\lambda_i(k))_{k\geq -1}$. As usual, we identify the Poisson point processes and the counting measures through the formula

$$N^i[s,t] \;=\; \sum_{n\in\mathbb{Z}} \mathbf{1}_{\{s\leq T_n^i\leq t\}}.$$

It follows from this identification that for any $t>0$ we have $T^i_{N^i(0,t]} \leq t < T^i_{N^i(0,t]+1}$, and for any $t \leq 0$, $T^i_{-N^i(t,0]} \leq t < T^i_{-N^i(t,0]+1}$.

For each $i \in \mathbb{Z}^d$ and $t \in \mathbb{R}$ we define the time-reverse point process starting at time t, associated to site i,

$$\begin{aligned} \tilde{T}_n^{(i,t)} &= t - T^i_{N^i(0,t]-n+1}, \quad t\geq 0, \\ \tilde{T}_n^{(i,t)} &= t - T^i_{-N^i(t,0]-n+1}, \quad t< 0. \end{aligned} \tag{6.18}$$

We also define the associated marks

$$\begin{aligned} \tilde{K}_n^{(i,t)} &= K^i_{N^i(0,t]-n+1}, \quad t\geq 0, \\ \tilde{K}_n^{(i,t)} &= K^i_{-N^i(t,0]-n+1}, \quad t< 0. \end{aligned} \tag{6.19}$$

For each site $i \in \mathbb{Z}^d$, $k\geq -1$, the reversed k-marked Poisson point process returning from time t is defined as

$$\tilde{N}^{(i,t,k)}[s,u] \;=\; \sum_n \mathbf{1}_{\{s\leq \tilde{T}_n^{(i,t)}\leq u\}} \mathbf{1}_{\{\tilde{K}_n^{(i,t)}=k\}}. \tag{6.20}$$

To define the black and white time-reverse sketch process we need to introduce a family of transformations $\{\pi^{(i,k)}, i\in\mathbb{Z}^d, k\geq 0\}$ on the set of finite subsets of $\mathbb{Z}^d$, $\mathcal{F}(\mathbb{Z}^d)$, defined as follows. For any unitary set $\{j\}$,

$$\pi^{(i,k)}(\{j\}) \;=\; \left\{ \begin{array}{ll} V_i(k), & \text{if } j=i \\ \{j\}, & \text{otherwise} \end{array} \right\}. \tag{6.21}$$

Notice that for $k=-1$, $\pi^{(i,k)}(\{i\})=\emptyset$. For any set finite set $F\subset\mathbb{Z}^d$, we define similarly

$$\pi^{(i,k)}(F) \;=\; \cup_{j\in F}\pi^{(i,k)}(\{j\}). \tag{6.22}$$

The black and white time-reverse sketch process starting at site i at time t will be denoted by $(C_s^{(i,t)})_{s\geq 0}$. $C_s^{(i,t)}$ is the set of sites at time s whose colors affect the color of site i at time t. The evolution of this process is defined through the following equation: $C_0^{(i,t)} := \{i\}$, and

$$f(C_s^{(i,t)}) \;=\; f(C_0^{(i,t)}) + \sum_{k\geq -1}\sum_{j\in\mathbb{Z}^d}\int_0^s [f(\pi^{(j,k)}(C_{u-}^{(i,t)})) - f(C_{u-}^{(i,t)})]\, \tilde{N}^{(j,t,k)}(du), \tag{6.23}$$

where $f:\mathcal{F}(\mathbb{Z}^d)\to\mathbb{R}$ is any bounded cylindrical function. This family of equations characterizes completely the time evolution $\{C_s^{(i,t)}, s\geq 0\}$. For any finite set $F\subset\mathbb{Z}^d$ define

$$C_s^{(F,t)} \;=\; \cup_{i\in F} C_s^{(i,t)}.$$

The following proposition summarizes the properties of the family of processes defined above.

Proposition 2 *For any finite set $F \subset \mathbb{Z}^d$, $C_s^{(F,t)}$ is a Markov jump process having as infinitesimal generator*

$$Lf(C) = \sum_{i \in C} \sum_{k \geq 0} \lambda_i(k)[f(C \cup V_i(k)) - f(C)] + \lambda_i(-1)[f(C \setminus \{i\}) - f(C)], \tag{6.24}$$

where f is any bounded function.

Proof: The proof follows in a standard way from the construction (6.23).

7 Proof of Theorem 1

The existence issue addressed by Theorem 1 can be reformulated in terms of the black and white time-reverse sketch process described above. The process is well defined if for each site i and each time t, the time-reverse procedure $C^{(i,t)}$ described above is a non-explosive Markov jump process. This means that for each time t, the number of operations needed to determine the value of $\xi_t^\eta(i)$ is finite almost surely. Note that by equation (6.23), the jumps of $C^{(i,t)}$ occur at total rate

$$\sum_{j \in C_s^{(i,t)}} M_j \sum_{k \geq -1} \lambda_j(k) \leq (\sup_j M_j)\, |C_s^{(i,t)}|,$$

where $|\cdot|$ denotes the cardinal of a set. Hence it suffices to show that the cardinal of $C^{(i,t)}$ remains finite.

More precisely, fix some $N \in \mathbb{N}$. Let $L_s = |C_s^{(i,t)}|$ and

$$T_N = \inf\{t : L_t \geq N\}.$$

Then by (6.23),

$$\begin{aligned} L_{s \wedge T_N} \quad &\leq \quad 1 + \sum_{k \geq 1} \sum_{j \in \mathbb{Z}^d} \int_0^{s \wedge T_N} [|V_j(k)| - 1] 1_{\{j \in C_{u-}^{(i,t)}\}} \tilde{N}^{(j,t,k)}(du) \\ &\qquad - \sum_{j \in \mathbb{Z}^d} \int_0^{s \wedge T_N} 1_{\{j \in C_{u-}^{(i,t)}\}} \tilde{N}^{(j,t,0)}(du). \end{aligned} \tag{7.25}$$

Passing to expectation and using that by condition (3.13),

$$m = \sup_i \sum_{k \geq 1} M_i\, \lambda_i(k) |V_i(k)| < +\infty,$$

this yields

$$\begin{aligned} E(L_{s \wedge T_N}) \quad &\leq \quad 1 + \sum_{j \in \mathbb{Z}^d} M_j \left((\sum_{k \geq 1} \lambda_j(k)[|V_j(k)| - 1]) - \lambda_j(-1) \right) \\ &\qquad \times E \int_0^{s \wedge T_N} 1_{\{j \in C_{u-}^{(i,t)}\}} du \\ &\leq \quad 1 + m\, E \int_0^{s \wedge T_N} L_u du. \end{aligned} \tag{7.26}$$

Letting $N \to \infty$, we thus get that

$$E(L_s) \leq 1 + m \int_0^s E(L_u) du,$$

and Gronwall's lemma yields

$$E(L_s) \leq e^{ms}. \tag{7.27}$$

This implies that the number of sites that have to be determined in order to know the value of site i at time t is finite almost surely. This means that the process $C_s^{(F,t)}$ admits only a finite number of jumps on any finite time interval. Hence, we have necessarily $N_{STOP} < +\infty$ almost surely which means that the algorithm stops almost surely after a finite time. This concludes the proof of Theorem 1.

8 Proof of Theorem 2

We show that under condition (3.15), Algorithm 3 stops after a finite time almost surely. Write L_s^i for the cardinal of $C_s^{(i,t)}$. Using once more the upper-bound (7.26) and the fact that under condition (3.15),

$$M_j \left((\sum_{k \geq 1} \lambda_j(k)[|V_j(k)| - 1]) - \lambda_j(-1) \right) \leq -\varepsilon < 0,$$

Gronwall's lemma yields that

$$E(L_s^i) \leq e^{-\varepsilon s}.$$

Hence, since $|C_s^{(F,t)}| \leq \sum_{i \in F} |C_s^{(i,t)}| = \sum_{i \in F} L_s^i$,

$$E(|C_s^{(F,t)}|) \leq |F| e^{-\varepsilon s.}$$

This implies that $\inf\{s : C_s^{(F,t)} = \emptyset\}$ is finite almost surely. Due to Theorem 1, the process $C_s^{(F,t)}$ is non-explosive, which means that it admits only a finite number of jumps on any finite time interval. Hence, we have necessarily $N_{STOP} < +\infty$ almost surely which means that the algorithm stops almost surely after a finite time.

In order to show that the measure μ that we have simulated in this way is necessarily the unique invariant probability measure of the process, we prove the following lemma.

Lemma 1 *Fix a time $t > 0$, some finite set of sites $F \subset \mathbb{Z}^d$ and two initial configurations η and $\zeta \in A^{\mathbb{Z}^d}$. Then there exists a coupling of the two processes $(\xi_s^\eta)_s$ and $(\xi_s^\zeta)_s$ such that*

$$P(\xi_t^\eta(F) \neq \xi_t^\zeta(F)) \leq |F| e^{-\varepsilon t}.$$

¿From this lemma, it follows immediately that μ is the unique invariant measure of the process and that the convergence towards the invariant measure takes place exponentially fast.

Proof of Lemma 1. We use a slight modification of Algorithm 1 and 2 in order to construct ξ_t^η and ξ_t^ζ. The modification is defined as follows. Replace step 8 of Algorithm 1 by

if $K = -1$, **then**

$$C \leftarrow C \setminus \{I\}$$

else

$$C \leftarrow C \cup V_I(K)$$

We use the same realizations of T, I and K for the construction of ξ_t^η and ξ_t^ζ. Write L_s for the cardinal of $C_s^{(F,t)}$. Clearly, both realizations of ξ_t^η and ξ_t^ζ do not depend on the initial configuration η, ζ respectively if and only if the output C of Algorithm 1 is void. Thus,

$$\begin{aligned} P(\xi_t^\eta(F) \neq \xi_t^\zeta(F)) &\leq P(T_{STOP} \geq t) \\ &= \mathbb{P}(L_t \geq 1) \\ &\leq E(L_t) \leq |F| e^{-\varepsilon t}. \end{aligned}$$

This concludes the proof of lemma 1.

9 Proof of Theorem 3

The proof of Theorem 3 goes according to the following lines.

We start at time t with the sites in F and go back into the past until time 0 following the backward black and white sketch without deaths described in Algorithm 1. The set C of points reached by this procedure at time 0 are the only ones which coloring affects the evolution during the interval of time $[0, t]$ of the sites belonging to F.

We need to know that C is a finite set. This follows from a slight modification of the proof of Theorem 1. Notice that in the construction by Algorithm 1, even if $K = -1$, the corresponding site is not removed from the set C. This implies that in the upper bound (7.25) the negative term on the right hand side disappears. This modification does not affect the upper bound (7.26) which remains true.

Using Theorem 2 we assign colors to the sites belonging to C using the invariant distribution at the origin of the multicolor system. Then we apply Algorithm 2 to describe the time evolution of the coloring of the sites in F. This evolution depends on the colors of the sites in C at time zero as well as on the successive choices of sites and ranges made during the backward steps starting at time t. This concludes the proof.

10 Proof of Theorem 4

The construction of the finitary coding can be better understood if we do it using two intermediate steps based on families of infinite-valued random variables.

Using a slightly abusive terminology, let us introduce the following definitions of a finitary coding from families of piles of i.i.d. random variables.

Definition 3 *We say that there exists a finitary coding from a family of i.i.d. uniform random variables* $(U_n(i), i \in \mathbb{Z}^d, n \in \mathbb{N})$ *to the configuration* $(\xi_0(i), i \in \mathbb{Z}^d)$ *sampled with respect to the invariant probability measure* μ*, if there exists a deterministic function* $f : [0,1]^{\mathbb{Z}^d \times \mathbb{N}} \to A^{\mathbb{Z}^d}$ *such that, almost surely, the following holds*

- f *commutes with the shift operator in* $\mathbb{Z}^d$*;*

- $\xi_0 = f((U_n(j)), j \in \mathbb{Z}^d, n \in \mathbb{N})$; *and*
- *for each site* $i \in \mathbb{Z}^d$, *there exists a a finite subset* $\bar{F}_i$ *of* $\mathbb{Z}^d$ *and* $\bar{n}_i \geq 1$ *such that if*

$$U'_n(j) = U_n(j) \text{ for all } j \in \bar{F}_i, n \leq \bar{n}_i$$

then

$$f((U_n(j)), j \in \mathbb{Z}^d, n \in \mathbb{N})(i) = f((U'_n(j)), j \in \mathbb{Z}^d, n \in \mathbb{N})(i).$$

Definition 4 *We say that there exists a finitary coding from a family of i.i.d. fair Bernoulli random variables* $(Y_{n,r}(i), i \in \mathbb{Z}^d, (n,r) \in \mathbb{N}^2)$ *to the configuration* $(\xi_0(i), i \in \mathbb{Z}^d)$ *sampled with respect to the invariant probability measure* μ*, if there exists a deterministic function* $f : \{0,1\}^{\mathbb{Z}^d \times \mathbb{N}^2} \to A^{\mathbb{Z}^d}$ *such that, almost surely, the following holds*

- f *commutes with the shift operator in* $\mathbb{Z}^d$;
- $\xi_0 = f((Y_{n,r}(j)), j \in \mathbb{Z}^d, (n,r) \in \mathbb{N}^2)$; *and*
- *for each site* $i \in \mathbb{Z}^d$, *there exists a a finite subset* $\bar{F}_i$ *of* $\mathbb{Z}^d$ *and* $\bar{n}_i \geq 1$ *such that if*

$$Y'_{n,r}(j) = Y_{n,r}(j) \text{ for all } j \in \bar{F}_i, 1 \leq n \leq \bar{n}_i, 1 \leq r \leq \bar{n}_i$$

then

$$f((Y_{n,r}(j)), j \in \mathbb{Z}^d, (n,r) \in \mathbb{N}^2)(i) = f((Y'_{n,r}(j)), j \in \mathbb{Z}^d, (n,r) \in \mathbb{N}^2)(i).$$

We will first prove the existence of a finitary coding in this extended definition from the sequence $(U_n(i), i \in \mathbb{Z}^d, n \in \mathbb{N})$ to the configuration $(\xi_0(i), i \in \mathbb{Z}^d)$.

Proposition 3 *Under the conditions of Theorem 2, there exists a finitary coding from a family of i.i.d. uniform random variables* $(U_n(i), i \in \mathbb{Z}^d, n \in \mathbb{N})$ *to the configuration* $(\xi_0(i), i \in \mathbb{Z}^d)$.

Proof Our goal is to choose the color of site i at time 0 using Algorithms 3 and 4. For notational convenience, we will represent the sequence $U_n(j)$ as

$$U_n^v(j),\, v \in \mathcal{V}\,,$$

where $\mathcal{V} = \{I, K, W\}$.

The first sequence of uniform variables $U_n^I(j)$ is used for the choice of successive points I at Step 5 of Algorithm 3. The second sequence $U_n^K(j)$ will be used to construct the sequence of ranges K of Step 6 of Algorithm 4. Finally, the third sequence $U_n^W(j)$ will be used to construct the corresponding colors W at Steps 7 and 9 of the forward procedure described in Algorithm 4.

The fact that N_{STOP} is finite almost surely implies that the backward Algorithm 1 must run only a finite number of steps for any fixed i. Thus only a finite set of sites $\bar{F}_i$ is involved in this procedure. This also implies that the number of uniform random variables we must use is finite, thus $\bar{n}_i$ is finite. The definition of the function f is explicitly given by Algorithms 3 and 4.

This concludes the proof of Proposition 3.

Proposition 3 can be rewritten using piles of piles of Bernoulli random variables instead of piles of uniform random variables.

Proposition 4 *Under the conditions of Theorem 2, there exists a finitary coding from a family of i.i.d. fair Bernoulli random variables* $(Y_{n,r}(i), i \in \mathbb{Z}^d, (n,r) \in \mathbb{N}^2)$ *to the configuration* $(\xi_0(i), i \in \mathbb{Z}^d)$.

Proof As before, for notional convenience, we will represent the sequence $Y_{n,r}(j)$ as

$$Y^v_{n,r}(j),\ v \in \mathcal{V}.$$

All we need to prove is that the successive uniform random variables $\{U^v_n(j), j \in \bar{F}_i, 1 \leq n \leq \bar{n}_i\}$ used in Proposition 3 can be generated using a finite number of Bernoulli random variables from the pile $Y^v_{n,r}(j), r \in \mathbb{N}$.

In all the steps, the uniform random variables were used to generate random variables taking values in a countable set. Let us identify this countable set with $\mathbb{N}$. In the successive steps of Algorithms 3 and 4, this selection could generate either K, I or W. The selection is made by defining in each case a suitable partition of $[0,1] = \cup_{l=1}^{\infty}[\theta(l), \theta(l+1))$ and then choosing the index l whenever $U^v_n(j) \in [\theta(l), \theta(l+1))$. It is easy to see that $U^v_n(j)$ has the same law as $\sum_{r=1}^{\infty} 2^{-r} Y^v_{n,r}(j)$.

We borrow from Knuth and Yao (1976) the following algorithm to generate the discrete random variables I, K and W which appear in the perfect sampling procedure (see also Harvey *et al.*, 2005). For any $m \geq 1$, we define

$$S_m(j,n,v) = \sum_{r=1}^{m} 2^{-r} Y^v_{n,r}(j).$$

Now, put

$$J(S_m(j,n,v)) = \sup\{k \geq 1 : \theta(k) \leq S_m(j,n,v)\},$$

and finally define

$$N^v_n(j) = \inf\{m \geq 1 : J(S_m(j,n,v)) = J(S_{m'}(j,n,v))\, \forall m' \geq m\}. \tag{10.28}$$

Notice that $N^v_n(j)$ is a finite stopping time with respect to the σ-algebra generated by $\{Y^v_{n,r}(j), r \geq 1\}$. Therefore, the total number of piles $N(j)$ equals $N(j) = \sum_{v\in\mathcal{V}} \sum_{n=1}^{\bar{n}_j} N^v_n(j)$, where $\bar{n}_j$ is defined in Proposition 3, used at site j is finite and the event $[N(j) = \ell]$ is measurable with respect to the σ-algebra generated by $\{Y^v_{n,r}(j), v \in \mathcal{V}, 1 \leq n \leq \ell, 1 \leq r \leq \ell\}$. Since the set of sites used is $\bar{F}_i$ (the same one as Proposition 3), the proof is complete.

We are finally ready to prove Theorem 4. The difficulty is to show that the construction achieved in Proposition 4 using the random sized piles $(Y^v_{n,r}(i), v \in \mathcal{V}, i \in \mathbb{Z}^d, n \leq N(i), r \leq N(i))$ can actually be done using finite piles of fair Bernoulli random variables. Specifically, for $i \in \mathbb{Z}^d$, let us call

$$Z(i) = (Y^v_{n,r}(i), v \in \mathcal{V}, n \in \{0,\dots,M\}, r \in \{0,\dots,M\}),$$

where M is a suitable fixed positive integer.

The proof that there exists a finitary coding from the family of finite-valued i.i.d. random variables $\{Z(i), i \in \mathbb{Z}^d\}$ to $\{\xi_0(i), i \in \mathbb{Z}^d\}$ follows from the construction in Van den Berg and Steif (1999) if we can take

$$M > \sup_{j\in\mathbb{Z}^d} \mathbb{E}[N(j)].$$

It follows from the definition of $N_n^v(j)$ given by (10.28) that

$$\mathbb{P}[N_n^v(j) > k] \leq \mathbb{P}\left(\cup_{i=1}^{m_k}\left[\theta(i) - \frac{1}{2^k} < S_k(j,n,v) \leq \theta(i)\right]\right) + \mathbb{P}\left(1 - \frac{1}{2^k} < S_k(j,n,v) \leq 1\right) \tag{10.29}$$

where

$$m_k = \sup\{i \geq 1; \theta(i) < 1 - \frac{1}{2^k}\}.$$

In the above formula, the partition of $[0,1] = \cup_{i=1}^{\infty}[\theta(i), \theta(i+1))$ was used to simulate the countable-valued random variable at stake at that level (either I, K or W).

Since $S_k(j,n,v)$ converges in law to a uniform random variable as $k \to \infty$, the right hand side of (10.29) is bounded above by

$$\frac{m_k + 1}{2^{k-1}}.$$

For piles choosing colors, the result is obvious since the set of possible colors is finite and therefore all the corresponding $m_k = |A|$ for all $k \geq |A|$.

For piles choosing sites in the backward black and whitesketch, the result follows from inequality (7.27) as inthe conclusion of the proof of Proposition 1.

Finally, for piles choosing ranges, the result follows from the following two lemmas and Wald's inequality observing that $\bar{n}_j \leq N_{STOP}$.

Lemma 2 $\sup_j \mathbb{E}[N_n^K(j)] < \infty$.

Proof It follows from Knuth and Yao (1976) that

$$\mathbb{E}[N_n^K(j)] \leq H(\{\lambda_j(k)\}_{k\geq -1}) + 2, \tag{10.30}$$

where $H(\{\lambda_j(k)\}_{k\geq -1})$ is the entropy of the discrete distribution $\{\lambda_j(k)\}_{k\geq -1}$ of the random variable K. On the other hand, the condition $\sum_k |V_k(j)|\lambda_k(j) < 1$ in Theorem 2 implies that $\sum_k k\lambda_j(k) = m_j < 1$ for all $j \in \mathbb{Z}^d$. We want to compare $\{\lambda_j(k)\}_{k\geq -1}$ to a geometric distribution. For that sake, we introduce a distribution $\nu_j(k)$ on $\{1,2,\ldots\}$ by $\nu_j(k) = \lambda_j(k-2)$. Then

$$\sum_{k\geq 1} k\nu_j(k) = m_j + 2 - \lambda_j(-1) =: \tilde{m}_j.$$

By a direct comparison with the geometric distribution of mean $\tilde{m}_j$ we have that

$$H(\{\lambda_j(k)\}_{k\geq -1}) \leq -\log(p_j) - \log(1-p_j)(\tilde{m}_j - 1) < \infty, \tag{10.31}$$

where $p_j = 1/(\tilde{m}_j)$.

Lemma 3 $\mathbb{E}[N_{STOP}] < \infty$.

Proof Without loss of generality we can consider $F = \{0\}$ to start Algorithm 3. Define

$$L_n := |C_n|,$$

the cardinal of the set C_n after n steps of Algorithm 3. Let $(K_n^i)_{n\geq 0, i\in\mathbb{Z}^d}$ be the i.i.d. marks defined in Section 6, taking values in $\{-1, 0, 1, 2, \ldots\}$ such that

$$P(K_n^i = k) = \lambda_i(k).$$

Define $X_n^i = |V_0(K_n^i)| - 1$. Note that by condition (3.15),

$$\sup_{i\in\mathbb{Z}^d} E(X_1^i) \leq (\bar{\lambda} - 1) < 0,$$

where $\bar{\lambda} = \sup_{i\in\mathbb{Z}^d} \sum_{k\geq 0} |V_i(k)|\lambda_i(k)$.

Consider the sequence I_n which gives the site of the particle chosen at the nth step of Algorithm 3. Put

$$S_n = \sum_{k=0}^{n} X_k^{I_k}.$$

Note that by construction, $S_n + n(1-\bar{\lambda})$ is a super-martingale. Then a very rough upper bound is

$$L_n \leq 1 + S_n \text{ as long as } n \leq V_{STOP},$$

where V_{STOP} is defined as

$$V_{STOP} = \min\{k : S_k = -1\}.$$

By construction

$$N_{STOP} \leq V_{STOP}.$$

Fix a truncation level N. Then by the stopping rule for super-martingales, we have that

$$E(S_{V_{STOP}\wedge N}) + (1-\bar{\lambda})E(V_{STOP}\wedge N) \leq 0.$$

But notice that

$$E(S_{V_{STOP}\wedge N}) = -1 \cdot P(V_{STOP} \leq N) + E(S_N; V_{STOP} > N).$$

On $V_{STOP} > N$, $S_N \geq 0$, hence we have that $E(S_{V_{STOP}\wedge N}) \geq -P(V_{STOP} \leq N)$. We conclude that

$$E(V_{STOP}\wedge N) \leq \frac{1}{(1-\bar{\lambda})} P(V_{STOP} \leq N).$$

Now, letting $N \to \infty$, we get

$$E(V_{STOP}) \leq \frac{1}{(1-\bar{\lambda})},$$

and therefore

$$E(N_{STOP}) \leq \frac{1}{(1-\bar{\lambda})}.$$

Finally for piles choosing sites in the backward black and white sketch, the result follows from inequality (7.27) as in the conclusion of the proof of Theorem 1.

This concludes the proof.

11 Proof of Corollary 1

The strategy of the proof is the following. We will consider a multicolor system having the law of the Markov random field as invariant measure. Then the corollary follows from the second assertion of Theorem 4.

A standard way to obtain a system having the law μ as invariant measure is to ask for reversibility. Usually, in the statistics literature such dynamic is known as Gibbs sampler. In the statistical physics literature, where it first appeared, it is known as Glauber dynamics. We will use a particular case of the Glauber dynamics called the heat bath algorithm. The idea is that at each site there is a Poisson clock which rings independently of all other sites. Each time its clock rings the color of the site is updated according to the specification of the Markov random field $\mathcal{Q}$.

We are only considering the Markov spatially homogeneous case. This means that the rate $c_i(a, T_i\xi) = c_0(a, \xi)$ where $(T_i\xi)(j) = \xi(j - i)$. $c_0(a, \xi)$ only depends on $\xi(\partial 0)$. Moreover, in this case $M = M_i = 1$.

With the heat bath algorithm, the rates are defined as

$$c_0(a, \xi) = \mathcal{Q}(X(0) = a | X(\partial 0) = \xi(\partial 0))$$

where $\mathcal{Q}$ is the *specification* of the random field X.

Since we are considering the homogeneous case, we will drop the subscript from the notation. Therefore, we have

$$\begin{aligned}
\alpha(-1) &= \sum_{a \in A} \min \left(\inf_{\zeta \in A^{\mathbb{Z}^d}, \zeta(i) \neq a} c_0(a, \zeta),\, 1 - \sup_{\zeta \in A^{\mathbb{Z}^d}, \zeta(i) = a} \sum_{b \neq a} c_0(b, \zeta) \right) \\
&= \sum_{a \in A} \min \left(\inf_{\zeta \in A^{\mathbb{Z}^d}, \zeta(i) \neq a} c_0(a, \zeta),\, \inf_{\zeta \in A^{\mathbb{Z}^d}, \zeta(i) = a} c_0(a, \zeta) \right) \\
&= \sum_{a \in A} \min_{\zeta(\partial 0) \in A^{\partial 0}} \mathcal{Q}(X(0) = a | X(\partial 0) = \zeta(\partial 0)).
\end{aligned}$$

Also,

$$\begin{aligned}
\alpha(0) &= \min_{w \in A} \left(\sum_{a \in A, a \neq w} \inf_{\zeta : \zeta(0) = w} c_0(a, \zeta) + 1 - \sup_{\zeta : \zeta(0) = w} \sum_{b \neq w} c_0(b, \zeta) \right) \\
&= \min_{w \in A} \sum_{a \in A} \inf_{\zeta : \zeta(0) = w} c_0(a, \zeta) \\
&= \min_{w \in A} \sum_{a \in A} \inf_{\zeta ; \zeta(0) = w} \mathcal{Q}(X(0) = a | X(\partial 0) = \zeta(\partial 0)) \\
&= \sum_{a \in A} \min_{\zeta(\partial 0) \in A^{\partial 0}} \mathcal{Q}(X(0) = a | X(\partial 0) = \zeta(\partial 0)).
\end{aligned}$$

The last equality follows from the fact that $\inf_{\zeta ; \zeta(0) = w} \mathcal{Q}(X(0) = a | X(\partial 0) = \zeta(\partial 0))$ only depends on the value of the random field at $\partial 0$.

Finally, for all $k \geq 1$

$$
\begin{aligned}
\alpha(k) &= \min_{w \in A^{V_0(k)}} \left(\sum_{a \in A, a \neq w(0)} \inf_{\zeta : \zeta(V_0(k)) = w} c_0(a, \zeta) + 1 - \sup_{\zeta : \zeta(V_0(k)) = w} \sum_{b \neq w(0)} c_0(b, \zeta) \right) \\
&= \min_{w \in A^{V_0(k)}} \left(\sum_{a \in A} \inf_{\zeta : \zeta(V_0(k)) = w} c_0(a, \zeta) \right) \\
&= \min_{w \in A^{\partial 0}} \left(\sum_{a \in A} \mathcal{Q}(X(0) = a | X(\partial 0) = w(\partial 0) \right) = 1.
\end{aligned}
$$

Observe that $\alpha(0) = \alpha(-1)$. Therefore, condition (3.13) reduces to

$$
\alpha(-1) > \frac{2d}{2d+1}.
$$

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