CALCULUS UP TO HOMOTOPY ON LEAF SPACES

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Abstract. These are lecture notes of a 4 hour mini-course course which I held at IME-USP, on the occasion of the workshop "Geometry in Algebra and Algebra in Geometry", November 05 – 09, 2018.

Contents

1. Foliations and leaf spaces

Let M be an n-dimensional manifold. A d-dimensional foliation of M is a rank d subbundle $\mathcal{F}$ of the tangent bundle TM satisfying the following involutivity condition: for all $X, Y \in \Gamma(\mathcal{F})$

$$[X, Y] \in \Gamma(\mathcal{F}).$$

An *integral manifold* of $\mathcal{F}$ is a connected, immersed d-dimensional submanifold S such that $T_pS = \mathcal{F}_p$ for all $p \in S$, and a *leaf* is a maximal integral manifold.

Foliations possess local normal forms according to the following

Theorem 1.1 (Local Frobenius)**.** *Let M be an n-dimensional manifold equipped with a d-dimensional foliation $\mathcal{F}$. Then $\mathcal{F}$ locally looks like the foliation on $\mathbb{R}^n$ spanned by the first d coordinate vector fields*

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^d},$$

i.e., for every point $p_0 \in M$, there is a chart $(U, (x^1, \dots, x^n))$ around p_0, such that

$$\mathcal{F}_p = \left\langle \frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^d}|_p \right\rangle$$

for all $p \in U$.

The terminology "foliation" is motivated by the following

Theorem 1.2 (Global Frobenius)**.** *Let M be an n-dimensional manifold equipped with a d-dimensional foliation $\mathcal{F}$. Then the leaves of $\mathcal{F}$ form a partition of M (by immersed, connected, d-dimensional submanifolds). For every point $p_0 \in M$, there is a chart $(U, (x^1, \dots, x^n))$ around p_0, such that, for every leaf $\mathcal{L}$, the intersection $\mathcal{L} \cap U$ is a countable disjoint union of submanifolds of the form $(x^{d+1}, \dots, x^n) = \mathrm{const}$.*

Example 1.3. Let $\pi : M \to B$ be a surjective submersion, with connected fibers, let $\dim M = n$ and let $\dim B = n - d$. The vertical bundle $VM := \ker d\pi \subset TM$ is a foliation whose leaves are the fibers of π.

Example 1.4. Consider the 2-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates (x, y). The (rank 1) subbundle $\mathcal{K}$ spanned by the vector field

$$\frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y}, \quad \alpha \in \mathbb{R},$$

is a foliation. If α is irrational, the leaves of $\mathcal{K}$ are immersed copies of the line wrapping around the torus infinitely many times, and each leaf is dense. In this case, $\mathcal{K}$ is called the *Kronecker foliation*.

Remark 1.5. One can generalize the above definition of a foliation in several directions. The following two are of a particular interest and motivate the importance of foliations in Differential Geometry.

- Every Lie groupoid $\mathcal{G} \rightrightarrows M$ (resp. Lie algebroid $A \Rightarrow M$) determines a, possibly singular, foliation $\mathcal{F}$ on M. The leaves of $\mathcal{F}$ are (connected components of) isomorphism classes of objects of the groupoid. By *singular*, we roughly mean that the dimension of $\mathcal{F}_p$ (hence the dimension of the leaves) may jump, as p ranges over M.
- Every system of PDEs in d independent variables determines a (usually) infinite dimensional manifold $\mathcal{E}$, naturally equipped with a d-dimensional foliation $\mathcal{C}$. Solutions of $\mathcal{E}$ are in one-to-one correspondence with d-dimensional integral submanifolds (however talking about leaves, and using the terminology "foliation" is slightly inappropriate, as the Frobenius Theorem fails in infinite dimensions).

Both Lie groupoids and PDEs are ubiquitous in Differential Geometry, and this explains our interest in foliations (as simple instances of both).

Let M be a smooth manifold equipped with a foliation $\mathcal{F}$. The space of leaves of $\mathcal{F}$ is denoted $M/\mathcal{F}$, or simply $\mathbb{M}$. Topologically, it is a quotient of M under the obvious surjection $\pi : M \to \mathbb{M}$. When $\mathbb{M}$ is a manifold and $\pi : M \to \mathbb{M}$ is a surjective submersion, we say that $\mathcal{F}$ is simple. In other words, a simple foliation is a foliation arising as in Example 1.3. Not all foliations are simple. For instance, the leaf space $\mathbb{T}^2/\mathcal{K}$ of the Kronecker foliation $\mathcal{K}$ on the torus $\mathbb{T}^2$, is not even Hausdorff. Actually, every point in $\mathbb{T}^2/\mathcal{K}$ is dense!

There is a natural question: how should we understand the leaf space $\mathbb{M}$? Just as a (usually very pathological) topological space? Is there a way to do differential geometry on $\mathbb{M}$? For instance, can we define smooth functions, vector fields, differential forms, etc. on $\mathbb{M}$ in any reasonable way?

There are various models for $\mathbb{M}$, all based on the *holonomy groupoid* of $\mathcal{F}$. The holonomy groupoid is a certain global construction attached to every foliation. The models for $\mathbb{M}$ based on the holonomy groupoid are rather strong conceptually, but they are not very handy, and it's not obvious how to make standard differential geometric computations with them. Additionally, it is not clear how to construct them in a setting where the Frobenius Theorem fails (notice that the Frobenius Theorem may fail in both the existence and the uniqueness of leaves on infinite dimensional manifolds, e.g. PDEs).

In this mini-course, I will propose a purely infinitesimal approach, which does not rely on the holonomy groupoid, nor on the Frobenius Theorem and works in infinite dimensions. "Our" approach is algebraic and homological/homotopical. Additionally, it is manageable in the sense that, in principle, it allows for concrete computations. Finally, it is natural in the sense that it provides a receipt: from any given construction in differential geometry (smooth functions, vector fields, differential forms, etc.) it says how to construct its analogue on a leaf space.

This approach is not really new, nor original: it is based on ideas of Alexandre Vinogradov coming from the infinite jet space approach to PDEs, and, at the end of the day, it is a super-simplified version of Homotopic/Derived Geometry.

We stress from the beginning that "our" calculus will not only provide information on the leaf space $\mathbb{M}$ as a topological space, but will also (partly) remember about the *internal structure* of points of $\mathbb{M}$ (as leaves of a foliation).

2. Smooth functions on a leaf space I: leaf-wise constant functions

We will present "our" calculus up to homotopy on leaf spaces in a pedagogical way. We begin with *smooth functions*. We will use the *simple foliation* case as a toy example, and as a source of inspiration on how to give the appropriate definition in the general case.

Remark 2.1. Notice that beginning with functions on $\mathbb{M}$ is a rather natural choice according to a standard *meta-mathematical geometry/algebra duality principle* stating that: *the full information on a space is contained in an appropriate algebra of admissible functions on it.* In the following, in honour of the GAAG workshop, I will refer to this principle as the *GAAG principle.* In Differential Geometry, admissible functions are smooth functions.

So let's start with a manifold M equipped with a simple foliation $\mathcal{F}$. Recall that this means that $\mathcal{F}$ is exactly the vertical bundle with respect to a surjective submersion $\pi : M \to B$ with connected fibers: $\mathcal{F} = VM$. In this case the leaf space $\mathbb{M}$ is just B. In particular, it is equipped with a canonical smooth manifold structure, and there is an obvious definition of smooth functions $C^\infty(\mathbb{M})$ on $\mathbb{M}$:

$$C^\infty(\mathbb{M}) = C^\infty(B). \tag{1}$$

Of course, Definition (1) does not make sense for a generic foliation. To get a hint how to give a definition in the general case, it is useful to express the right hand side $C^\infty(B)$ of (1) purely in terms of $(M, \mathcal{F})$. This is easy, according to the following obvious

Proposition 2.2. *Let $\pi : M \to B$ be a surjective submersion with connected fibers, and let $\mathcal{F} = VM$ be the vertical bundle with respect to π. Then the pull-back*

$$\pi^* : C^\infty(B) \to C^\infty(M)$$

is an injection identifying $C^\infty(B)$ with the subalgebra

$$\{f \in C^\infty(M) : X(f) = 0 \textit{ for all } X \in \Gamma(\mathcal{F})\} \subset C^\infty(M). \tag{2}$$

Now notice that, given a manifold M with a generic foliation $\mathcal{F}$, the subalgebra (2) makes sense, and consists of leaf-wise constant functions on M. Proposition 2.2 now suggests to define

$$C^\infty(\mathbb{M}) := \{f \in C^\infty(M) : X(f) = 0 \textit{ for all } X \in \Gamma(\mathcal{F})\}. \tag{3}$$

In particular, every function $f \in C^\infty(\mathbb{M})$ descends to a honest continuous function on $\mathbb{M}$.

Definition (3) has both positive and negative features. Among positive features, it boils down to the natural definition in the simple case. Additionally, as $C^\infty(\mathbb{M})$ is equipped with an algebra structure, it is in the spirit of the GAAG principle. However, in general, $C^\infty(\mathbb{M})$ is far to small to provide useful information on $\mathbb{M}$. For instance, when $M = \mathbb{T}^2$ in the 2-torus, and $\mathcal{F} = \mathcal{K}$ is the Kronecker foliation on it, then $C^\infty(\mathbb{M}) = C^\infty(\mathbb{T}^2/\mathcal{K})$ consists of constant functions only. For this reason, we consider (3) as a temporary definition, to be improved later on, during the mini-course.

3. Vector field on a leaf space I: infinitesimal symmetries

We now begin discussing calculus on a leaf space. In this section we take care of vector fields.

First notice that, whenever a notion of *smoothness* is available, the GAAG principle can be extended to include the following addendum: *calculus on a smooth space is part of the Commutative Algebra of the algebra of admissible functions*. We call this addendum the 2^{nd} *GAAG principle*. For instance, vector fields on a manifold M are the same as derivations of the algebra $C^\infty(M)$. Notice that the 2^{nd} GAAG principle is not very useful for generic leaf spaces, in this form. For instance, smooth functions on the leaf space of a Kronecker foliation are just constants, hence their derivations are all trivial. In order to improve our point of view, we look again at the simple case. So, let $\mathcal{F}$ be the vertical bundle on M with respect to a surjective submersion $\pi : M \to B$ with connected fibers. In this case, there is an obvious definition of vector fields $\chi(\mathbb{M})$ on $\mathbb{M}$:

$$\chi(\mathbb{M}) = \chi(B).$$

Unfortunately, this definition does not make sense for a generic foliation. However, we can proceed, as for functions, expressing $\chi(B)$ purely in terms of $(M, \mathcal{F})$. To do this we need a

Lemma 3.1. *Let $\pi : M \to B$ be a surjective submersion with connected fibers, and let $\mathcal{F} = VM$ be the vertical bundle with respect to π. Then a vector field Y on M is π-projectable if and only if it is an infinitesimal symmetry of $\mathcal{F}$.*

This statement needs some explanations. First of all, for a surjective submersion $\pi : M \to B$, a vector field Y on M is called π-projectable, if there exists a vector field Y_B on B such that $\pi_* Y = Y_B$. The assignment $Y \mapsto Y_B$ is a surjection from projectable vector fields to vector fields on B.

Additionally, for a foliation $\mathcal{F}$ on M, a vector field $Y \in \chi(M)$ is an *infinitesimal symmetry* of $\mathcal{F}$ if it generates a flow $\{\Phi_t\}$ by symmetries of $\mathcal{F}$, i.e. $d\Phi_t(\mathcal{F}_p) = \mathcal{F}_{\Phi_t(p)}$, for every t, and every point p in the domain of Φ_t. Equivalently,

$$[Y, \Gamma(\mathcal{F})] \subset \Gamma(\mathcal{F}).$$

As usual in Differential Geometry, infinitesimal symmetries of $\mathcal{F}$ form a Lie subalgebra in $\chi(M)$ that we denote $\chi(M, \mathcal{F})$.

We are now ready to state the main proposition of this section:

Proposition 3.2. *Let $\pi : M \to B$ be a surjective submersion with connected fibers, and let $\mathcal{F} = VM$ be the vertical bundle with respect to π. The projection*

$$\chi(M, \mathcal{F}) \to \chi(B), \quad Y \mapsto Y_B$$

is a Lie algebra map, with kernel consisting of vertical vector fields $\Gamma(\mathcal{F})$. In other words, there is a canonical short exact sequence of Lie algebras

$$0 \to \Gamma(\mathcal{F}) \to \chi(M, \mathcal{F}) \to \chi(B) \to 0$$

identifying $\chi(B)$ with the quotient

$$\chi(M, \mathcal{F})/\Gamma(\mathcal{F}). \tag{4}$$

Now notice that, given a manifold M with a generic foliation $\mathcal{F}$, the quotient (4) makes sense. Indeed, by involutivity, sections of $\mathcal{F}$ form an ideal in $\chi(M, \mathcal{F})$. Proposition 3.2 now suggests to define

$$\chi(\mathbb{M}) := \chi(M, \mathcal{F})/\Gamma(\mathcal{F}). \tag{5}$$

In the following, we denote by

$$Y_{\mathbb{M}}$$

the class in $\chi(\mathbb{M})$ of an infinitesimal symmetry $Y \in \chi(M, \mathcal{F})$.

Definition (5) has some positive features. First of all, again by construction, it boils down to the obvious definition in the simple case. Second, $\chi(\mathbb{M})$ is equipped with a Lie algebra structure as we expect from vector fields. Actually, in this respect, not only vector fields on a manifold form a Lie algebra, they are actually sections of a Lie algebroid, or, more algebraically, they form a Lie-Rinehart algebra, and we expect $\chi(\mathbb{M})$ to possess the same structure. This is indeed the case as we now discuss. But first let's recall what is a Lie-Rinehart algebra.

A Lie-Rinehart algebra is a pair (A, L) where A is an associative commutative algebra with unit, and L is a Lie algebra. Additionally,

- L is an A-module,
- A is an L-module,

and there are the following compatibilities

(1) L acts on A by derivations;
(2) the action $L \to \operatorname{Der} A$ is A-linear;
(3) for every $Y, Z \in L$, and every $a \in A$

$$[Y, aZ] = (Y.a)Z + a[Y, Z].$$

Sometimes we refer to L as a Lie-Rinehart algebra (over A).

A Lie algebroid structure $E \Rightarrow M$ on a vector bundle $E \to M$ is then the same as a Lie-Rinehart algebra structure on the pair $(C^\infty(M), \Gamma(E))$ (extending the obvious pre-existing structures).

Proposition 3.3. *Let M be a manifold equipped with a foliation $\mathcal{F}$. Then the pair*

$$(C^\infty(\mathbb{M}), \chi(\mathbb{M}))$$

is a Lie-Rinehart algebra in a canonical way. The algebra structure (resp. Lie algebra structure) on $C^\infty(\mathbb{M})$ (resp. $\chi(\mathbb{M})$) are the preexisting ones. The $C^\infty(\mathbb{M})$-module structure on $\chi(\mathbb{M})$ is given by

$$f(Y_{\mathbb{M}}) = (fY)_{\mathbb{M}}, \quad f \in C^\infty(\mathbb{M}), \quad Y \in \chi(M, \mathcal{F}).$$

Finally, the $\chi(\mathbb{M})$-module structure on $C^\infty(\mathbb{M})$ is given by

$$Y_{\mathbb{M}}.f = Y(f), \quad f \in C^\infty(\mathbb{M}), \quad Y \in \chi(M, \mathcal{F}).$$

However, Definition (5) has also negative features. For instance, for the Kronecker foliation, while $C^\infty(\mathbb{T}^2/\mathcal{K})$ is trivial, $\chi(\mathbb{T}^2/\mathcal{K})$ is not, hence it cannot consist of derivations of $C^\infty(\mathbb{T}^2/\mathcal{K})$ unlike somehow prescribed by the 2^{nd} GAAG principle. For this reason, we consider (5) just a temporary definition.

4. Differential forms on a leaf space I: basic forms

Next we take care of differential forms. We adopt our now customary strategy. When $\mathcal{F}$ is the vertical bundle with respect to a surjective submersion $\pi : M \to B$ with connected fibers, the obvious definition of differential forms $\Omega(\mathbb{M})$ on $\mathbb{M}$ is

$$\Omega(\mathbb{M}) = \Omega(B).$$

Next, we express $\Omega(B)$ purely in terms of $(M, \mathcal{F})$, with the following

Proposition 4.1. *Let $\pi : M \to B$ be a surjective submersion with connected fibers, and let $\mathcal{F} = VM$ be the vertical bundle with respect to π. Then the pull-back*

$$\pi^* : \Omega(B) \to \Omega(M)$$

is an injection identifying $\Omega(B)$ with the graded subalgebra

$$\{\omega \in \Omega(M) : i_X \omega = \mathcal{L}_X \omega = 0 \text{ for all } X \in \Gamma(\mathcal{F})\} \subset \Omega(M). \tag{6}$$

Now, given a manifold M with a generic foliation $\mathcal{F}$, the subalgebra (6) makes sense and, for obvious reasons, its elements are usually called *basic forms.* So we put

$$\Omega(\mathbb{M}) := \{\omega \in \Omega(M) : i_X \omega = \mathcal{L}_X \omega = 0 \textit{ for all } X \in \Gamma(\mathcal{F})\} \subset \Omega(M). \tag{7}$$

Notice that the de Rham differential preserves basic forms. Hence, $\Omega(\mathbb{M})$, equipped with the restricted differential, is not just a graded algebra, but a differential graded algebra (DGA). This is precisely what we expect from differential forms on a space.

Additionally, $\chi(\mathbb{M})$ and $\Omega(\mathbb{M})$ define a *Cartan calculus* on the leaf-space $\mathbb{M}$. For instance, the interior product of a vector field $Y_{\mathbb{M}} \in \chi(\mathbb{M})$ with a differential form $\omega \in \Omega(\mathbb{M})$ is defined as

$$i_{Y_{\mathbb{M}}} \omega := i_Y \omega,$$

and one can check that it is a well-defined differential form in $\Omega(\mathbb{M})$. Similarly, the Lie derivative of ω along $Y_{\mathbb{M}}$ is defined as

$$\mathcal{L}_{Y_{\mathbb{M}}} \omega := \mathcal{L}_Y \omega,$$

and it is a well-defined form in $\Omega(\mathbb{M})$. Finally, one can easily check that all classical Cartan identities hold true in this setting. However, for similar reasons as for functions and vector fields, we consider (7) as a temporary definition.

5. Smooth functions on a leaf space II: leaf-wise cohomology

We have preliminary definitions of smooth functions, vector fields, and differential forms on a leaf space. Unfortunately, for the reasons that we already mentioned, they are not optimal definitions. Our next aim is improving them.

As usual, let M be a manifold equipped with a foliation $\mathcal{F}$. We begin noticing that smooth functions $C^\infty(\mathbb{M})$ on the leaf space $\mathbb{M}$ are the 0-cohomology of a suitable cochain complex: the *leaf-wise de Rham complex.*

Leaf-wise differential forms are defined as sections of the vector bundle $\wedge^\bullet \mathcal{F}^*$. Hence they are skew-symmetric $C^\infty(M)$-multilinear maps

$$\Gamma(\mathcal{F}) \times \cdots \times \Gamma(\mathcal{F}) \to C^\infty(M).$$

We denote by $\Omega_{\mathcal{F}} = \Gamma(\wedge^\bullet \mathcal{F}^*)$ the graded algebra of leaf-wise differential forms, and, for every $q \geq 0$, by $\Omega^q_{\mathcal{F}} = \Gamma(\wedge^q \mathcal{F}^*)$ the module of *leaf-wise q-forms*.

The algebra $\Omega_{\mathcal{F}}$ is not just an algebra. It is a differential graded algebra (DGA) with the *leaf-wise de Rham differential* $d_{\mathcal{F}}$ defined by the usual Chevalley-Eilenberg formula: for all $\omega \in \Omega^q_{\mathcal{F}}$, and all $X_1, \ldots, X_{q+1} \in \Gamma(\mathcal{F})$,

$$\begin{aligned} d_{\mathcal{F}}\omega(X_1, \ldots, X_{q+1}) = & \sum_i (-)^{i+1} X_i(\omega(\ldots, \widehat{X_i}, \ldots)) \\ & + \sum_{i<j} (-)^{i+j} \omega([X_i, X_j], \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots). \end{aligned}$$

and the involutivity of $\mathcal{F}$ is equivalent to $d^2_{\mathcal{F}} = 0$. The cohomology of $(\Omega_{\mathcal{F}}, d_{\mathcal{F}})$ is called the *leaf-wise de Rham cohomology*, and it is denoted by $H_{\mathcal{F}}$. Let us compute the 0-th leaf-wise de Rham cohomology. It is the kernel of the map

$$d_{\mathcal{F}} : C^\infty(M) \to \Omega^1(\mathcal{F}),$$

so it consists of smooth functions $f \in C^\infty(M)$, such that $d_{\mathcal{F}} f = 0$, i.e.

$$0 = d_{\mathcal{F}} f(X) = X(f),$$

for all $X \in \Gamma(\mathcal{F})$. In other words, $H^0_{\mathcal{F}}$ consists exactly of leaf-wise constant functions:

$$H^0_{\mathcal{F}} = C^\infty(\mathbb{M}).$$

Remark 5.1. There are good reasons to interpret to full cohomology $H_{\mathcal{F}}$ as *smooth functions on* $\mathbb{M}$, not just its 0 degree piece. In our opinion, one of the most interesting originates in the (geometric) theory of PDEs. We already mentioned that, given a system of PDEs, say $\mathcal{E}_0$, one can naturally construct an infinite dimensional manifold $\mathcal{E}$ and a d-dimensional foliation $\mathcal{C}$ on it. Here d is the number of independent variables, and d-dimensional integral submanifolds of $\mathcal{C}$ identify with solutions of $\mathcal{E}_0$. In this case, one can show that

$$\begin{aligned} H^d_{\mathcal{C}} & \cong \{\text{variational principles constrained by } \mathcal{E}_0\} \\ H^{d-1}_{\mathcal{C}} & \cong \{\text{conservation laws for } \mathcal{E}_0\} \\ H^{d-2}_{\mathcal{C}} & \cong \{\text{gauge charges for } \mathcal{E}_0\}. \end{aligned}$$

In all these cases, the right hand side consists of functionals on the space of solutions.

Remark 5.1 strongly suggests that we interpret the full cohomology space $H_{\mathcal{F}}$ as functions on the leaf-space. Now on we adopt this interpretation and denote

$$\mathbb{C}^\infty(\mathbb{M}) := H_{\mathcal{F}}.$$

Being a space of functions, we expect $\mathbb{C}^\infty(\mathbb{M})$ to possess an algebra structure. This is indeed the case: as $\mathbb{C}^\infty(\mathbb{M})$ is the cohomology of a DGA, it possesses a natural graded algebra structure (extending that of $C^\infty(\mathbb{M})$).

Even when $C^\infty(\mathbb{M})$ is trivial, $\mathbb{C}^\infty(\mathbb{M})$ needs not be so, because it may have non-trivial contributions in higher degrees.

Remark 5.2. Let $\mathcal{F}$ be the vertical bundle with respect to a surjective submersion $\pi : M \to B$. We assume for simplicity that (M, π) is a locally trivial fiber bundle. In this case, one can show that $\mathbb{C}^\infty(\mathbb{M})$ is the space of sections of a vector bundle over B whose fiber over $b \in B$ is the cohomology of the fiber $\pi^{-1}(b)$.

Last remark shows that $\mathbb{C}^\infty(\mathbb{M})$ contains some information on the *internal structure* of points of $\mathbb{M}$.

Remark 5.3. The graded algebra $\mathbb{C}^\infty(\mathbb{M})$ is not yet our ultimate definition of the space of functions on $\mathbb{M}$. Actually, $\mathbb{C}^\infty(\mathbb{M})$ is equipped with additional structure encoding the full quasi-isomorphism class of the DGA $(\Omega_{\mathcal{F}}, d_{\mathcal{F}})$ that we wish to take into account.

6. Vector fields on a leaf space II: the Bott connection

Next we want to improve our definition of vector fields on the leaf-space $\mathbb{M}$. The first step is noticing that vector fields $\chi(\mathbb{M})$ on the leaf space $\mathbb{M}$ are (isomorphic to) the 0-cohomology of a suitable cochain complex.

To do this we need a new notion of *representation of a foliation*. So let M be a manifold equipped with a foliation. A *representation of* $\mathcal{F}$ is a vector bundle $E \to M$ equipped with a *flat* $\mathcal{F}$*-connection*, i.e. an $\mathbb{R}$-linear operator

$$\nabla : \Gamma(\mathcal{F}) \to \mathrm{End}_{\mathbb{R}}\Gamma(E), \quad X \mapsto \nabla_X$$

such that

(1) $\nabla_{fX} = f\nabla_X$,
(2) $\nabla_X(fe) = X(f)e + f\nabla_X e$,
(3) $\nabla_{[X_1,X_2]} = [\nabla_{X_1}, \nabla_{X_2}]$

for all $X, X_1, X_2 \in \Gamma(\mathcal{F})$, all $f \in C^\infty(M)$, and all $e \in \Gamma(E)$.

Every representation (E, ∇) of $\mathcal{F}$ gives rise to a cochain complex: the *leaf-wise de Rham complex with coefficients in* E. E*-valued leaf-wise differential forms* are sections of the vector bundle $\wedge^\bullet \mathcal{F}^* \otimes E$. In other words, they are skew-symmetric $C^\infty(M)$-multilinear maps

$$\Gamma(\mathcal{F}) \times \cdots \times \Gamma(\mathcal{F}) \to \Gamma(E).$$

We denote by $\Omega_{\mathcal{F}}(E) = \Gamma(\wedge^\bullet \mathcal{F}^* \otimes E)$ the graded $\Omega_{\mathcal{F}}$-module of E-valued leaf-wise differential forms, and, for every $q \geq 0$, by $\Omega^q_{\mathcal{F}}(E) = \Gamma(\wedge^q \mathcal{F}^* \otimes E)$ the module of E*-valued leaf-wise* q*-forms*.

As E is not just a vector bundle but a representation, $\Omega_{\mathcal{F}}(E)$ is not just a module, but a DG module over the DGA $(\Omega_{\mathcal{F}}, d_{\mathcal{F}})$. This means, first of all, that $\Omega_{\mathcal{F}}(E)$ is equipped with a differential, also called *leaf-wise de Rham differential*, denoted by $d_{\mathcal{F}}$ again, and defined by the obvious formula: for all $\varepsilon \in \Omega^q_{\mathcal{F}}(E)$, and all $X_1, \ldots, X_{q+1} \in \Gamma(\mathcal{F})$:

$$\begin{aligned} d_{\mathcal{F}}\varepsilon(X_1, \ldots, X_{q+1}) = & \sum_i (-)^{i+1} \nabla_{X_i}(\varepsilon(\ldots, \widehat{X_i}, \ldots)) \\ & + \sum_{i<j} (-)^{i+j} \varepsilon([X_i, X_j], \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots). \end{aligned}$$

Now, $(\Omega_{\mathcal{F}}(E), d_{\mathcal{F}})$ is a DG module over $(\Omega_{\mathcal{F}}, d_{\mathcal{F}})$ in the sense that

$$d_{\mathcal{F}}(\omega \wedge \varepsilon) = d_{\mathcal{F}}\omega \wedge \varepsilon + (-)^{|\omega|}\omega \wedge d_{\mathcal{F}}\varepsilon,$$

for all $\omega \in \Omega_{\mathcal{F}}$, and all $\varepsilon \in \Omega_{\mathcal{F}}(E)$, where, as usual, $|\omega|$ denotes the degree of ω. The cohomology of $(\Omega_{\mathcal{F}}(E), d_{\mathcal{F}})$ is called the *leaf-wise de Rham cohomology with coefficients in* E, and it is denoted by $H_{\mathcal{F}}(E)$. As $(\Omega_{\mathcal{F}}(E), d_{\mathcal{F}})$ is a DG module, $H_{\mathcal{F}}(E)$ is a graded $H_{\mathcal{F}}$-module.

Given a foliation, the *normal bundle* provides a canonical representation. The normal bundle is defined as the quotient bundle

$$T_{\perp}M = TM/\mathcal{F}.$$

In particular, there is a short exact sequence of vector bundles:

$$0 \to \mathcal{F} \to TM \to T_{\perp}M \to 0.$$

We denote by $\chi_{\perp}(M)$ the module of sections of $T_{\perp}M$, and, given a vector field $Y \in \chi(M)$, we denote by $Y_{\perp} \in \chi_{\perp}(M)$ its class modulo $\Gamma(\mathcal{F})$.

The normal bundle $T_{\perp}M$ is canonically equipped with a flat $\mathcal{F}$-connection ∇^{Bott} called the *Bott connection* and defined by

$$\nabla^{\mathrm{Bott}}_X Y_{\perp} = [X, Y]_{\perp}$$

for all $X \in \Gamma(\mathcal{F})$, and all $Y \in \chi(M)$. In particular we can take the leaf-wise de Rham cohomology $H_{\mathcal{F}}(T_{\perp}M)$ with coefficients in $T_{\perp}M$. Let us compute the 0-th cohomology. It is the kernel of the map

$$d_{\mathcal{F}} : \chi_{\perp}(M) \to \Omega^1_{\mathcal{F}}(T_{\perp}M),$$

so it consists of normal vector fields $Y_{\perp} \in \chi_{\perp}(M)$, such that $d_{\mathcal{F}}Y_{\perp} = 0$, i.e.

$$0 = d_{\mathcal{F}}Y_{\perp}(X) = \nabla^{\mathrm{Bott}}_X Y_{\perp} = [X, Y]_{\perp},$$

for all $X \in \Gamma(\mathcal{F})$. This shows that Y is an infinitesimal symmetry of $\mathcal{F}$. Hence $H^0_{\mathcal{F}}(T_{\perp}M)$ consists of infinitesimal symmetries modulo sections of $\mathcal{F}$, i.e.

$$H^0_{\mathcal{F}}(T_{\perp}M) \cong \chi(\mathbb{M}).$$

This isomorphism identify the class $Y_{\mathbb{M}}$ of an infinitesimal symmetry $Y \in \chi(M, \mathcal{F})$ with the class $Y_{\perp}$ in normal vector fields $\chi_{\perp}(M)$.

It is now natural to adopt a similar point of view as for functions, and interpret the full cohomology space $H_{\mathcal{F}}(T_{\perp}M)$ as vector fields on the leaf-space. So we denote

$$\mathbb{X}(\mathbb{M}) := H_{\mathcal{F}}(T_{\perp}M).$$

Being a space of vector fields, we expect $\mathbb{X}(\mathbb{M})$ (more precisely the pair $(\mathbb{C}^{\infty}(\mathbb{M}), \mathbb{X}(\mathbb{M}))$) to possess a Lie-Rinehart algebra structure. We know already that the 0-degree piece $(C^{\infty}(\mathbb{M}), \chi(\mathbb{M}))$ is indeed a Lie-Rinehart algebra, and we would like to extend this structure to higher degrees. This is indeed possible. However it is not as easy as in the case of functions. The reason why it's not so easy is that $T_{\perp}M$-valued leaf-wise differential forms are *not* a DG Lie-Rinehart algebra.

We will show that $(\mathbb{C}^\infty(\mathbb{M}), \chi(\mathbb{M}))$ is a graded Lie-Rinehart algebra in 2 different ways:

(1) defining by hands the structure maps;
(2) more conceptually, presenting $\chi(\mathbb{M})$ as the cohomology of a canonical DG-Lie-Rinehart algebra.

We begin with the first (not very conceptual) method. We need to recall the *Frölicher-Njenhuis calculus* of vector valued differential forms. So, let M be a manifold, and denote by $\Omega(M, TM)$, the graded $\Omega(M)$-module of TM-valued differential forms, i.e. skew-symmetric, $C^\infty(M)$-multilinear maps

$$\chi(M) \times \cdots \times \chi(M) \to \chi(M).$$

Equivalently, TM-valued differential forms can be viewed as differential form valued vector fields, i.e. $\mathbb{R}$-linear operators

$$Z : C^\infty(M) \to \Omega(M)$$

satisfying the obvious Leibniz rule. We will often take this point of view.

Proposition 6.1. *For every $Z \in \Omega(M, TM)$ there exist*

(1) *a unique graded derivation*

$$i_Z : \Omega(M) \to \Omega(M)$$

of degree $|Z| - 1$, called the interior product with Z, *such that, for all $f \in C^\infty(M)$,*
- $i_Z f = 0$, *and*
- $i_Z df = Z(f)$;

(2) *a unique graded derivation*

$$\mathcal{L}_Z : \Omega(M) \to \Omega(M)$$

of degree $|Z|$, called the Lie derivative along Z, *such that, for all $f \in C^\infty(M)$,*
- $\mathcal{L}_Z f = Z(f)$, *and*
- $\mathcal{L}_Z df = dZ(f)$.

Every graded derivation $\Delta : \Omega(M) \to \Omega(M)$ is of the form

$$\Delta = i_J + \mathcal{L}_K$$

for some unique $J, K \in \Omega(M, TM)$.

Proposition 6.2. *For every $Z_1, Z_2 \in \Omega(M, TM)$, there exist*

(1) *a unique $[Z_1, Z_2]^{\mathrm{nr}} \in \Omega(M, TM)$ of degree $|Z_1| + |Z_2| - 1$, called the* Nijenhuis-Richardson bracket *of Z_1 and Z_2, such that*

$$[i_{Z_1}, i_{Z_2}] = i_{[Z_1, Z_2]^{\mathrm{nr}}};$$

(2) *a unique $[Z_1, Z_2]^{\mathrm{fn}} \in \Omega(M, TM)$ of degree $|Z_1| + |Z_2|$, called the* Frölicher-Nijenhuis bracket *of Z_1 and Z_2, such that*

$$[\mathcal{L}_{Z_1}, \mathcal{L}_{Z_2}] = \mathcal{L}_{[Z_1, Z_2]^{\mathrm{fn}}}.$$

The Frölicher-Nijenhuis bracket $[-,-]^{\mathrm{fn}}$ gives $\Omega(M,TM)$ the structure of a graded Lie algebra, extending the Lie algebra of vector fields.

Next we want to interpret $T_\perp M$-valued leaf-wise differential forms as honest TM-valued differential forms, so to be able to apply the Frölicher-Nijenhuis bracket. This can only be done by choosing some extra data.

We begin noticing that there are obvious projections:

$$P : \Omega(M) \to \Omega_{\mathcal{F}}, \quad \text{and} \quad P : \Omega(M,TM) \to \Omega_{\mathcal{F}}(T_\perp M) \tag{8}$$

that consist in restricting a form to sections of $\mathcal{F}$ and then projecting to $T_\perp M$ (in the second case). Now, consider again the short exact sequence

$$0 \to \mathcal{F} \to TM \to T_\perp M \to 0$$

A splitting $T_\perp M \to TM$ is equivalent to the choice of a subbundle $N \subset TM$ complementary to $\mathcal{F}$, i.e. $TM = \mathcal{F} \oplus N$. Given such a splitting, sections of $T_\perp M$ identify with sections of N, i.e. certain honest vector fields. Additionally, we have a splitting $\mathcal{F}^* \to T^*M$ of the dual short exact sequence

$$0 \leftarrow \mathcal{F}^* \leftarrow T^*M \leftarrow (T_\perp M)^* \leftarrow 0,$$

allowing us to identify leaf-wise differential forms with certain honest differential forms. Overall, (together with the projections (8)), we get two inclusions of $C^\infty(M)$-modules

$$\Omega_{\mathcal{F}} \hookrightarrow \Omega(M), \quad \text{and} \quad \Omega_{\mathcal{F}}(T_\perp M) \hookrightarrow \Omega(M,TM).$$

both denoted I, depending of N, and inverting the P on the right. With the inclusions I at hand, we are ready to present the structure maps of the graded Lie-Rinehart algebra

$$(\mathbb{C}^\infty(\mathbb{M}), \mathbb{\chi}(\mathbb{M})).$$

Recall that $\mathbb{C}^\infty(\mathbb{M}) = H_{\mathcal{F}}$, and $\mathbb{\chi}(\mathbb{M}) = H_{\mathcal{F}}(T_\perp M)$. In particular, $\mathbb{C}^\infty(\mathbb{M})$ is already a graded algebra and $\mathbb{\chi}(\mathbb{M})$ is already a graded $\mathbb{C}^\infty(\mathbb{M})$-module. It remains to describe the Lie algebra structure on $\mathbb{\chi}(\mathbb{M})$ and the action of this Lie algebra on $\mathbb{C}^\infty(\mathbb{M})$. So, take $\mathbb{Y}, \mathbb{Y}_1, \mathbb{Y}_2 \in \mathbb{\chi}(\mathbb{M})$, and $\mathbb{f} \in \mathbb{C}^\infty(\mathbb{M})$. They are cohomology classes of $d_{\mathcal{F}}$-coboundaries $Z, Z_1, Z_2 \in \Omega_{\mathcal{F}}(T_\perp M)$, and $\omega \in \Omega_{\mathcal{F}}$. We define

$$[\![\mathbb{Y}_1, \mathbb{Y}_2]\!] := \text{the cohomology class of } P[IZ_1, IZ_2]^{\mathrm{fn}},$$

and

$$\mathbb{Y}(\mathbb{f}) := \text{the cohomology class of } P(\mathcal{L}_{IZ} I\omega).$$

Theorem 6.3. *With the structure maps just defined,* $(\mathbb{C}^\infty(\mathbb{M}), \mathbb{\chi}(\mathbb{M}))$ *is a well-defined graded Lie-Rinehart algebra.*

Notice that there is still something weird about the definition of $\mathbb{\chi}(\mathbb{M})$: it does not yet consist of derivations of $\mathbb{C}^\infty(\mathbb{M})$. We will discuss this specific point a little bit later.

7. Differential forms on the leaf space II: the $\mathcal{F}$-spectral sequence

We proceed as for functions and vector fields. Differential forms $\Omega(\mathbb{M})$ on the leaf space $\mathbb{M}$ are the 0-cohomology of the *leaf-wise de Rham complex* with coefficients in *normal differential forms*.

A *normal differential* 1*-form* is a section of the dual bundle $(T_\perp M)^*$ that we also denote by $T^*_\perp M$. Equivalently, it is a honest 1-form $\eta \in \Omega^1(M)$ vanishing on $\mathcal{F}$:

$$\eta(X) = 0 \quad \text{for all } X \in \Gamma(\mathcal{F}).$$

Normal differential form are sections of the vector bundle $\wedge^\bullet T^*_\perp M$. Equivalently, they are honest differential forms in the graded subalgebra spanned by normal 1-forms. Yet in other words, differential forms $\eta \in \Omega(M)$ such that

$$i_X \eta = 0 \quad \text{for all } X \in \Gamma(\mathcal{F}).$$

We denote by $\Omega_\perp(M) = \Gamma(\wedge^\bullet T^*_\perp M) \subset \Omega(M)$ the graded subalgebra of normal forms, and, for every $k \geq 0$, by $\Omega^p_\perp(M) = \Gamma(\wedge^p T^*_\perp M) \subset \Omega^p(M)$ the submodule of *normal k forms*.

The vector bundle $\wedge^\bullet T^*_\perp M$ is canonically equipped with a flat $\mathcal{F}$-connection, "dual" to the Bott connection, and also denoted ∇^{Bott}. By definition

$$\nabla^{\mathrm{Bott}}_X \eta = \mathcal{L}_X \eta$$

for all $X \in \Gamma(\mathcal{F})$, and all $\eta \in \Omega^\perp(M)$. In particular we can take the leaf-wise de Rham cohomology $H_{\mathcal{F}}(\wedge^\bullet T^*_\perp M)$ with coefficients in $\wedge^\bullet T^*_\perp M$. The 0-th cohomology consists of normal forms $\eta \in \Omega^\perp(M)$ such that

$$0 = d_{\mathcal{F}} \eta(X) = \nabla^{\mathrm{Bott}}_X \eta = \mathcal{L}_X \eta,$$

for all $X \in \Gamma(\mathcal{F})$, equivalently, differential forms $\eta \in \Omega(M)$ such that

$$i_X \eta = \mathcal{L}_X \eta = 0 \quad \text{for all } X \in \Gamma(\mathcal{F}),$$

i.e. basic differential forms. So

$$H^0_{\mathcal{F}}(\wedge^\bullet T^*_\perp M) = \Omega(\mathbb{M}).$$

We adopt the same point of view as for functions and vector field and denote

$$\Omega(\mathbb{M}) := H_{\mathcal{F}}(\wedge^\bullet T^*_\perp M).$$

Notice that $\Omega(\mathbb{M})$ is bi-graded by the *leaf-wise degree* q and the *normal degree* p:

$$\Omega(\mathbb{M}) = \bigoplus_{q,p} H^q_{\mathcal{F}}(\wedge^p T^*_\perp M).$$

We also write

$$\Omega^p(\mathbb{M}) = H_{\mathcal{F}}(\wedge^p T^*_\perp M).$$

so that

$$\Omega(\mathbb{M}) = \bigoplus_p \Omega^p(\mathbb{M})$$

Being a space of differential forms, we expect $\mathbb{\Omega}(\mathbb{M})$ to be a DGA. For the algebra structure, it is easy to see that $\mathbb{\Omega}(\mathbb{M})$ inherits that of

$$\Omega_{\mathcal{F}}(\wedge^\bullet T_\perp^* M) \cong \Omega_{\mathcal{F}} \otimes \Omega_\perp(M),$$

where the tensor product is over smooth functions on M. What about the differential? We expect to have a cochain complex

$$\cdots \longrightarrow \mathbb{\Omega}^p(\mathbb{M}) \xrightarrow{\mathbb{d}_{\mathrm{dR}}} \mathbb{\Omega}^{p+1}(\mathbb{M}) \xrightarrow{\mathbb{d}_{\mathrm{dR}}} \cdots .$$

We know already that the 0-degree piece $\Omega(\mathbb{M})$ is a DGA, and we would like to extend the differential to higher leaf-wise degrees. This is indeed possible: the differential $\mathbb{d}_{\mathrm{dR}}$ exists. Notice, however, that cochains $\Omega_{\mathcal{F}}(\wedge^\bullet T_\perp^* M)$ possess just one differential $d_{\mathcal{F}}$: there is no other canonical differential

$$\cdots \longrightarrow \Omega_{\mathcal{F}}(\wedge^p T_\perp^* M) \longrightarrow \Omega_{\mathcal{F}}(\wedge^{p+1} T_\perp^* M) \longrightarrow \cdots$$

responsible for the appearance of $\mathbb{d}_{\mathrm{dR}}$. So, in order to construct $\mathbb{d}_{\mathrm{dR}}$, we need a different strategy. Actually, $\mathbb{d}_{\mathrm{dR}}$ can be constructed in at least three different ways:

(1) by hands,
(2) via a(n easy) spectral sequence argument,
(3) presenting $\Omega(\mathbb{M})$ as the (horizontal) cohomology of a canonical double differential algebra.

We illustrate the first two methods in this section. Let's begin with the first one. The idea is interpreting $\wedge^\bullet T_\perp^* M$-valued leaf-wise forms as honest differential forms, so to be able to apply the standard de Rham differential. Again, this can only be done by choosing the extra datum of a splitting $T_\perp M \to TM$ of the short exact sequence

$$0 \to \mathcal{F} \to TM \to T_\perp M \to 0.$$

Such a splitting induces a dual splitting of the dual short exact sequence

$$0 \leftarrow \mathcal{F}^* \leftarrow T^* M \leftarrow T_\perp^* M \leftarrow 0,$$

and, in turn, a direct sum decomposition $T^*M \cong \mathcal{F}^* \oplus T_\perp^* M$. From the latter we immediately get a factorization

$$\Omega(M) \cong \Omega_{\mathcal{F}} \otimes \Omega_\perp(M) = \Omega_{\mathcal{F}}(\wedge^\bullet T_\perp^* M),$$

where the tensor product is over $C^\infty(M)$. In particular, we have a module isomorphism

$$I : \Omega_{\mathcal{F}} \otimes \Omega_\perp(M) \to \Omega(M)$$

and, for any $p, q \in \mathbb{N}_0$, projections

$$P_{p,q} : \Omega(M) \to \Omega_{\mathcal{F}}^q \otimes \Omega_\perp^p(M).$$

We are now ready to define a differential

$$\mathbb{d}_{\mathrm{dR}} : \mathbb{\Omega}^p(\mathbb{M}) \to \mathbb{\Omega}^{p+1}(\mathbb{M})$$

for all p. So, take $\mathbb{\omega} \in \mathbb{\Omega}^p(\mathbb{M})$. It is the cohomology class of a $d_{\mathcal{F}}$-coboundary $\omega \in \Omega_{\mathcal{F}}(\wedge^p T_\perp^* M)$. We define

$$\mathbb{d}_{\mathrm{dR}}\mathbb{\omega} := \text{the cohomogly class of } P_{p+1,\bullet}(dI\omega).$$

Theorem 7.1. *With the differential just defined, $\mathbb{\Omega}(\mathbb{M})$ is a well-defined DGA.*

We now come to the spectral sequence argument. For every $p \in \mathbb{N}$, let

$$F^p\Omega \subset \Omega(M)$$

be the ideal spanned by

$$\Omega_\perp^p(M) \subset \Omega^p(M) \subset \Omega(M).$$

One can easily check using the involutivity of $\mathcal{F}$ that

$$\Omega(M) \supset F^1\Omega \supset \cdots \supset F^p\Omega \supset \cdots$$

is a filtration of the de Rham complex (by ideals). Hence there is a spectral sequence $\{(E_r, d_r)\}$ computing the de Rham cohomology of M. A closer inspection reveals that, for all p, q, there is a canonical identification

$$E_0^{p,q} \cong \Omega_{\mathcal{F}}^q(\wedge^p T_\perp^* M),$$

and, under this identification, the differential

$$d_0 : E_0^{p,\bullet} \to E_0^{p,\bullet+1}$$

identifies with the leaf-wise differential

$$d_{\mathcal{F}} : \Omega_{\mathcal{F}}^\bullet(\wedge^p T_\perp^* M) \to \Omega_{\mathcal{F}}^{\bullet+1}(\wedge^p T_\perp^* M).$$

It follows that

$$E_1^{p,\bullet} = H^{p,\bullet}(E_0, d_0) \cong H_{\mathcal{F}}(\wedge^p T_\perp^* M) = \mathbb{\Omega}^p(\mathbb{M}).$$

So, the differential

$$d_1 : E_1^{p,\bullet} \to E_1^{p+1,\bullet}$$

induces a differential

$$\mathbb{d}_{\mathrm{dR}} : \mathbb{\Omega}^p(\mathbb{M}) \to \mathbb{\Omega}^{p+1}(\mathbb{M}).$$

One can check that the latter is the same as the differential defined above.

One can also define interior products and Lie derivatives. Namely, let $\mathbb{Y} \in \mathbb{\chi}(\mathbb{M})$ be the cohomology class of a cocycle $Z \in \Omega_{\mathcal{F}}(T_\perp M)$, and let $\mathbb{\omega} \in \mathbb{\Omega}^p(\mathbb{M})$ be the cohomology class of a cocycle $\eta \in \Omega_{\mathcal{F}}(\wedge^p T_\perp^* M)$. We put

$$i_{\mathbb{Y}}\mathbb{\omega} := \text{the cohomology class of } P_{p-1,\bullet}(i_{IZ} I\eta)$$

and

$$\mathcal{L}_{\mathbb{Y}}\mathbb{\omega} := \text{the cohomology class } P_{p,\bullet}(\mathcal{L}_{IZ} I\eta).$$

The latter forms are well-defined and one can check that appropriate graded versions of the Cartan identities hold true in this setting.

Remark 7.2. It is very nice that $\mathbb{\Omega}(\mathbb{M})$ possesses the "correct" algebraic structure for differential forms (on a space). However, one may still wonder whether or not there are more facts supporting our interpretation. I can provide one more evidence from the theory of PDEs. So, let $\mathcal{E}_0$, $\mathcal{E}$, $\mathcal{C}$, and d be as in Remark 5.1. In this case, one can show that

$$H^d_{\mathcal{C}} \xrightarrow{\mathbb{d}_{\mathrm{dR}}} H^d_{\mathcal{C}}(\Omega^1_\perp)$$

associates to a (constrained) variational principle, the associated Euler-Lagrange equations (with Lagrange multipliers). In other words, in this case, (in leaf-wise degree d) the first de Rham differential $\mathbb{d}_{\mathrm{dR}} : \mathbb{C}^\infty(\mathbb{M}) \to \mathbb{\Omega}^1(\mathbb{M})$ is exactly the *functional differential*.

The case of functions, vector fields, and differential forms, suggests the following recipe to find the analogue $\mathbb{\Phi}(\mathbb{M})$ on the leaf-space $\mathbb{M}$ of any (natural) construction Φ in differential geometry:

(1) Find the *normal analogue* $\Phi_\perp(M)$ of Φ;
(2) Recognize that $\Phi_\perp(M)$ carries a representation of $\mathcal{F}$;
(3) Define $\mathbb{\Phi}(\mathbb{M})$ as $H_{\mathcal{F}}(\Phi_\perp(M))$;
(4) Check that $\mathbb{\Phi}(\mathbb{M})$ possesses the "correct" algebraic structure.

This recipe has been successfully tested on other constructions like *multivector fields* and *scalar differential operators*.

8. Vector fields on a leaf space III: $\chi(\mathcal{F}[1])$

We announced that $\mathbb{\chi}(\mathbb{M})$ can be presented as cohomology of a DG Lie-Rinehart algebra, and that the latter structure is responsible for the graded Lie-Rinehart algebra structure on $\mathbb{\chi}(\mathbb{M})$. In this section we explain this in some details.

First notice that the DGA $\Omega_{\mathcal{F}}$ can be thought of as the DGA of functions on a *DG manifold*. Informally, a *graded manifold* is a manifold whose coordinates carry (integer) degrees. Coordinates of even degree commute (with all other coordinates), while coordinates of odd degrees anticommute. A *DG manifold* is a graded manifold equipped with a degree 1 vector field Q squaring to zero: $Q^2 = 0$. More rigorously, a (real) graded manifold $\mathcal{M}$ is a honest manifold M, equipped with a graded $C^\infty(M)$-algebra, denoted $C^\infty(\mathcal{M})$, which is isomorphic to a graded algebra of the form

$$\Gamma(S^\bullet \mathcal{V}),$$

where $\mathcal{V} = \bigoplus_k \mathcal{V}^k \to M$ is a graded bundle over M (concentrated in non-zero degrees), and $S^\bullet\mathcal{V}$ is its graded symmetric algebra (notice that, in order to accommodate for complex graded manifolds, the definition must be slightly improved). According to the definition, coordinates on M, together with a local frame of $\Gamma(\mathcal{V})$, serve as graded coordinates on $\mathcal{M}$. We stress, however, that the isomorphism $C^\infty(\mathcal{M}) \cong \Gamma(S^\bullet\mathcal{V})$ is not part of the data. The simplest, non trivial instance of a graded manifold $(M, C^\infty(\mathcal{M}))$ is obtained by putting

$$C^\infty(\mathcal{M}) = \Gamma(\wedge^\bullet V^*)$$

where $V \to M$ is a non-graded vector bundle. In other words $C^\infty(\mathcal{M}) = \Gamma(S^\bullet \mathcal{V})$, where $\mathcal{V} \to M$ is the graded vector bundle given by

$$\mathcal{V}^k = \begin{cases} 0 & \text{if } k \neq 1 \\ V^* & \text{if } k = 1 \end{cases} .$$

In this case, we denote $\mathcal{M} =: V[1]$, and we think of sections of V^* as degree 1 coordinates on $V[1]$. In particular, $V[1]$ does only possess degree 0 and degree 1 coordinates.

Calculus on a graded manifold $\mathcal{M}$ can be constructed algebraically applying the (1^{st} and 2^{nd}) GAAG principle(s) to the graded algebra $C^\infty(\mathcal{M})$. For instance, vector fields on $\mathcal{M}$ are just derivations of $C^\infty(\mathcal{M})$.

A *DG manifold* is a graded manifold $\mathcal{M}$ equipped with a a *homological vector field*, i.e. a degree 1 vector field $Q \in \chi(\mathcal{M})$ such that $Q^2 = 0$. A foliation $\mathcal{F}$ on a manifold M provides an example of a DG manifold. Indeed, consider the graded manifold $\mathcal{F}[1]$. By definition,

$$C^\infty(\mathcal{F}[1]) = \Gamma(\wedge^\bullet \mathcal{F}^*) = \Omega_{\mathcal{F}}$$

which is canonically equipped with the homological derivation $d_{\mathcal{F}}$. So $(\mathcal{F}[1], d_{\mathcal{F}})$ is a DG manifold.

Vector fields on a DG manifold form a *DG Lie-Rinehart algebra*. The definition of a DG Lie-Rinehart algebra should be clear: A DG Lie-Rinehart algebra is a pair $(\mathcal{A}, \mathcal{L})$, where $\mathcal{A}$ is a commutative DGA (with unit), and $\mathcal{L}$ is a DG Lie algebra (DGLA). Additionally,

- $\mathcal{L}$ is a DG $\mathcal{A}$-module,
- $\mathcal{A}$ is a DG L-module,

and, if we forget the differentials, $(\mathcal{A}, \mathcal{L})$ is a (graded) Lie-Rinehart algebra. Sometimes we refer to $\mathcal{L}$ as a DG Lie-Rinehart algebra (over $\mathcal{A}$). It's clear that if $(\mathcal{A}, \mathcal{L})$ is a DG Lie-Rinehart algebra, then $(H(\mathcal{A}), H(\mathcal{L}))$ is a graded Lie-Rinehart algebra.

Now, let $(\mathcal{M}, Q)$ be a DG manifold, then $(C^\infty(\mathcal{M}), \chi(\mathcal{M}))$ is a DG Lie-Rinehart algebra. The differential in $C^\infty(\mathcal{M})$ is Q, and the differential in $\chi(\mathcal{M})$ is the adjoint operator $[Q, -]$: the graded commutator with Q.

We finally come to the example of our interest. Let $\mathcal{F}$ be a foliation on a manifold M.

Theorem 8.1. *There is a canonical quasi-isomorphism*

$$\mathcal{P} : \chi(\mathcal{F}[1]) \to \Omega_{\mathcal{F}}(T_\perp M),$$

where $\chi(\mathcal{F}[1])$ *is equipped with the differential* $[d_{\mathcal{F}}, -]$*, and* $\Omega_{\mathcal{F}}(T_\perp M)$ *is equipped with the leaf-wise differential.*

Corollary 8.2. *The leaf-wise cohomology*

$$\chi(\mathbb{M}) = H_{\mathcal{F}}(T_\perp M)$$

is canonically equipped with a graded Lie-Rinehart algebra structure.

We now sketch the proof of Theorem 8.1. Begin with a vector field $X \in \chi(\mathcal{F}[1])$. It is a derivation of $C^\infty(\mathcal{F}[1]) = \Omega_{\mathcal{F}}$. Restricting X to the degree 0 piece, we get

$$X : C^\infty(M) \to \Omega_{\mathcal{F}}$$

a leaf-wise form valued vector field on M, that we may think of as a leaf-wise form with values in vector field. Composing the latter with the projection

$$\chi(M) \to \chi_\perp(M), \quad Y \mapsto Y_\perp$$

we get a $T_\perp M$-valued leaf-wise form, denoted $\mathcal{P}X$. This defines the map

$$\mathcal{P} : \chi(\mathcal{F}[1]) \to \Omega_{\mathcal{F}}(T_\perp M)$$

in the statement. It is easy to check that $\mathcal{P}$ is a cochain map. To see that it is a quasi-isomorphism we complete it to a *contraction*:

$$\mathcal{H} \circlearrowright \chi(\mathcal{F}[1]) \underset{\mathcal{I}}{\overset{\mathcal{P}}{\rightleftarrows}} \Omega_{\mathcal{F}}(T_\perp M).$$

This can be done, but not in a canonical way. We need to fix a splitting of the exact sequence

$$0 \to \mathcal{F} \to TM \to T_\perp M \to 0. \tag{9}$$

With a splitting at hand, we can define a right inverse

$$\mathcal{I} : \Omega_{\mathcal{F}}(T_\perp M) \to \chi(\mathcal{F}[1])$$

as follows. Let $Z \in \Omega_{\mathcal{F}}(T_\perp M)$, then $\mathcal{I}Z$ is the vector field on $\mathcal{F}[1]$ given by

$$\mathcal{I}Z(\omega) := P(\mathcal{L}_{IZ} I\omega),$$

for all $\omega \in C^\infty(\mathcal{F}[1]) = \Omega_{\mathcal{F}}$. Here P and I are exactly the maps defined in Section 6. Finally, one can construct the homotopy $\mathcal{H}$. I will not provide a formula for $\mathcal{H}$ in these notes.

9. Differential forms on a leaf space III: $\Omega(\mathcal{F}[1])$

A similar situation occurs for differential forms. First of all, we remark that differential forms on a DG manifold form a *double DGA*. Namely, without insisting too much on the technical details, let first $\mathcal{M}$ be a graded manifold. Differential forms on $\mathcal{M}$ are usually defined (algebraically) in such a way that, together with the de Rham differential, they form a DGA, whose grading is given by the *total degree*, i.e. the sum of the *form degree*, and the *internal*, coordinate degree. We will follow this convention (however, other conventions are possible). Now, let $(\mathcal{M}, Q)$ be a DG manifold. Then the de Rham differential, and the Lie derivative along Q are degree 1, homological derivations of $\Omega(\mathcal{M})$ commuting with each other, and we say that

$$(\Omega(\mathcal{M}), \mathcal{L}_Q, d_{\mathrm{dR}})$$

is a double DGA. It follows that d_{dR} induces a differential in the $\mathcal{L}_Q$-cohomology of $\Omega(\mathcal{M})$ (and vice-versa).

Theorem 9.1. *For every p, there is a canonical quasi-isomorphism*

$$\Omega^p(\mathcal{F}[1]) \leftarrow \Omega_{\mathcal{F}}(\wedge^p T_\perp^* M) : \mathcal{P}^*,$$

where $\Omega^p(\mathcal{F}[1])$ *is equipped with the differential* $\mathcal{L}_Q$*, and* $\Omega_{\mathcal{F}}(\wedge^p T_\perp^* M)$ *is equipped with the leaf-wise differential.*

Corollary 9.2. *The leaf-wise cohomology*

$$\mathbb{\Omega}(\mathbb{M}) = H_{\mathcal{F}}(\wedge^\bullet T_\perp^* M)$$

is canonically equipped with a differential:

$$\cdots \longrightarrow \mathbb{\Omega}^p(\mathbb{M}) \xrightarrow{\mathbb{d}_{\mathrm{dR}}} \mathbb{\Omega}^{p+1}(\mathbb{M}) \xrightarrow{\mathbb{d}_{\mathrm{dR}}} \cdots .$$

(induced by the de Rham differential) turning $\mathbb{\Omega}(\mathbb{M})$ *in a DGA.*

We will not provide a proof (not even a sketch) of Theorem 9.1, nor will we describe $\mathcal{P}^*$ explicitly. We only mention that the proof is formally the same as that of Theorem 9.1. Namely, one actually shows that, for every p, there is a contraction

$$\mathcal{H}^* \circlearrowright \Omega^p(\mathcal{F}[1]) \underset{\mathcal{P}^*}{\overset{\mathcal{I}^*}{\rightleftarrows}} \Omega_{\mathcal{F}}(\wedge^p T_\perp M),$$

depending on a splitting of the short exact sequence (9) only.

We also remark that the full Cartan calculus on the leaf-space $\mathbb{M}$. that we mentioned in Section 7, can be induced from the ordinary Cartan calculus on the DG manifold $\mathcal{F}[1]$ via taking the Q-cohomology. Theorems 8.1 and 9.1, and their corollaries, together with the latter remark, strongly suggest a new paradigm on the calculus on $\mathbb{M}$. Namely, it is natural to declare that: *the calculus on* $\mathbb{M}$ *is just the ordinary calculus on* $\mathcal{F}[1]$ *up to taking cohomology*. We formalize this paradigm a little bit better, providing a new recipe to find the analogue $\mathbb{\Phi}(\mathbb{M})$ on the leaf-space $\mathbb{M}$ of any (natural) construction Φ in differential geometry:

(1) Consider $\Phi(\mathcal{F}[1])$;
(2) Recognize that $\Phi(\mathcal{F}[1])$ carries a differential $\mathcal{Q}$, induced by $d_{\mathcal{F}}$ in a natural way;
(3) Define $\mathbb{\Phi}(\mathbb{M})$ as $H(\Phi(\mathcal{F}), \mathcal{Q})$.

This approach has the conceptual advantage of being much closer to the spirit of the GAAG principles: for instance, now, vector fields are honest derivations of the algebra of admissible functions, up to taking cohomology.

For safety reasons, it is wise to add a little bit more information to our *calculus up to taking cohomology* (on the leaf-space). Namely, we know that, whatever the algebra (associative, commutative, Lie, etc.), the cohomology of a DG algebra (with its algebra structure) is a *homotopy invariant*, i.e. two homotopy equivalent DG algebras share the same cohomology (with its algebra structure). However, the cohomology does not contain a full-information on the quasi-isomorphism class of the DG algebra. We prefer to keep this information. To do that, we pass from a *calculus up to taking cohomology*, to the *calculus up to homotopy* in the title.

10. Vector fields on a leaf space IV: a homotopy Lie-Rinehart algebra

Roughly, a (non-curved) *homotopy* (associative, resp. commutative, resp. Lie, etc.) *algebra* is a cochain complex $\mathcal{K}$ equipped with an algebraic structure (compatible with the differential, but) satisfying the axioms of the (associative, resp. commutative, resp. Lie, etc.) algebra only up to (a coherent system of higher) homotop(ies). See below for more precise definitions in the case of differential, Lie, and Lie-Rinehart algebras.

Algebraic structures are not *homotopy invariant* in the following sense. Let $\mathcal{A}$ be any kind of DG algebra (associative, commutative, Lie, etc.), let $\mathcal{K}$ be a cochain complex and let

$$\mathcal{A} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{K}$$

be a pair of (mutually homotopy inverse) homotopy equivalences. We may try using (f, g) to transfer the algebraic structure from $\mathcal{A}$ to $\mathcal{K}$. However, if we do so, we do not get a DG algebra structure on $\mathcal{K}$ but a *homotopy algebra structure*. On the other hand, homotopy algebras are homotopy invariant, in the sense that, if we try to play the same game starting from a homotopy algebra structure on $\mathcal{A}$ we indeed get a homotopy algebra structure on $\mathcal{K}$. This is the main reason why homotopy algebras pop up whenever one deals at the same time with algebraic structures, cohomology and homotopy. More precisely, we have the following

Theorem 10.1 (Homotopy Transfer). *Let $\mathcal{A}$ be a DG algebra, let $\mathcal{K}$ be a cochain complex, and let*

$$\mathcal{H} \circlearrowright \mathcal{A} \underset{\mathcal{I}}{\overset{\mathcal{P}}{\rightleftarrows}} \mathcal{K}.$$

be a contraction. Then $\mathcal{K}$ can be promoted to a homotopy algebra depending only on $\mathcal{A}$ and the contraction. With its homotopy algebra structure, $\mathcal{K}$ contains a full information on the quasi-isomorphism class of $\mathcal{A}$ (more precisely the ∞-quasi-isomorphism class of $\mathcal{K}$ is equivalent to the quasi-isomorphism class of $\mathcal{A}$).

This statement needs some explanations. First of all we have to explain more precisely what is a homotopy algebra. We will mainly consider the cases of a homotopy Lie algebra (also called an L_∞-algebra) and of a homotopy differential algebra (not very popular but appearing in the theory of foliations and elsewhere). We will also quickly consider the case of a homotopy Lie-Rinehart algebra (also called an LR_∞-algebra). We begin with L_∞-algebras. There are a couple of equivalent definitions. I will present the most complicated one, which is more intuitive for our purposes.

An L_∞*-algebra* is a graded vector space V equipped with a family of *multi-brackets* $(\mathfrak{l}_k)_{k\in\mathbb{N}}$:

$$\mathfrak{l}_k : S^k V \to V$$

with $\mathfrak{l}_k$ being a degree 1, k-multilinear, graded symmetric map, for all k. Additionally, the $\mathfrak{l}_k$ satisfy the following *coherence conditions*: for all $n \in \mathbb{N}$

$$\sum_{i+j=n} \sum_{\sigma \in S_{i,j}} \varepsilon(\sigma) \mathfrak{l}_{i+1}(\mathfrak{l}_j(v_{\sigma(1)}, \dots, v_{\sigma(j)}), v_{\sigma(j+1)}, \dots, v_{\sigma(i+j)}) = 0, \tag{10}$$

for all $v_1, \dots, v_n \in V$, where $\varepsilon(\sigma)$ is a Koszul sign.

In order to better understand this definition, let's have a look at the identities (10) for low n. For $n = 1$, we get

$$\mathfrak{l}_1 \mathfrak{l}_1(v) = 0,$$

for all v, i.e., $(V, \mathfrak{l}_1)$ is a cochain complex. For $n = 2$, we get

$$\mathfrak{l}_1(\mathfrak{l}_2(v, w)) = \pm \mathfrak{l}_2(\mathfrak{l}_1(v), w) \pm \mathfrak{l}_2(v, \mathfrak{l}_1(w)),$$

for all v, w, i.e. $\mathfrak{l}_1$ is a derivation with respect to the binary bracket $\mathfrak{l}_2$. For $n = 3$, we get

$$\mathfrak{l}_2(v, \mathfrak{l}_2(w, z)) + \circlearrowleft \, = \pm \mathfrak{l}_1(\mathfrak{l}_3(v, w, z)) \pm \mathfrak{l}_3(\mathfrak{l}_1(v), w, z) \pm \mathfrak{l}_3(v, \mathfrak{l}_1(w), z) \pm \mathfrak{l}_3(v, w, \mathfrak{l}_1(z)),$$

for all v, w, z, where $\circlearrowleft$ denotes graded cyclic permutations, and $\pm$ denotes the appropriate Koszul sign. In particular, we see that $\mathfrak{l}_2$ satisfies the Jacobi identity, hence it's a Lie bracket only up to a homotopy encoded by $\mathfrak{l}_3$. Similarly for higher n. Notice that, as $\mathfrak{l}_1$ is a differential, one can take the cohomology $H(V, \mathfrak{l}_1)$, and $\mathfrak{l}_2$ induces a honest Lie bracket on it. So *the cohomology of an L_∞-algebra is a honest graded Lie algebra.* Finally, we remark that every DGLA is an L_∞-algebra such that $\mathfrak{l}_k = 0$ for $k > 2$, up to a shift in the degree.

Now, we want to explain briefly LR_∞-algebras. First of all, given an L_∞-algebra $(V, (\mathfrak{l}_k)_{k \in \mathbb{N}})$ there is a notion of an L_∞-module over it. It is a graded vector space W equipped with a family of *multi-brackets* $(\mathfrak{m}_k)_{k \in \mathbb{N}}$:

$$\mathfrak{m}_k : S^{k-1} V \otimes W \to W$$

with $\mathfrak{m}_k$ being a degree 1, k-multilinear map, graded symmetric in the first $k-1$ entries, for all k. Additionally, the $\mathfrak{m}_k$ satisfy certain coherence conditions.

An *LR_∞-algebra* is a pair $(\mathcal{A}, \mathcal{L})$, where $\mathcal{A}$ is a commutative DGA (with unit), and $(\mathcal{L}, (\mathfrak{l}_k)_{k \in \mathbb{N}})$ is an L_∞-algebra. Additionally,

- $(\mathcal{L}, \mathfrak{l}_1)$ is a DG $\mathcal{A}$-module,
- $\mathcal{A}$ is an L_∞-module over $\mathcal{L}$,

there are more compatibilities, telling, e.g., that the multibrackets

$$\mathfrak{m}_k : S^{k-1} \mathcal{L} \otimes \mathcal{A} \to \mathcal{A}$$

are graded $\mathcal{A}$-linear in the first $k-1$ entries, and they are graded derivations in the last entry. This definition of an LR_∞-algebra is actually a simplification of the "should be" definition of a homotopy Lie-Rinehart algebra, where only the Lie bracket and the Lie algebra action are up to homotopy, while the commutative, associative product is honestly associative and commutative. There is a version of the Homotopy Transfer Theorem for LR_∞-algebras:

Theorem 10.2 (Lie-Rinehart Homotopy Transfer). *Let $(\mathcal{A}, \mathcal{L})$ be a DG Lie-Rinehart algebra, let $\mathcal{K}$ be a DG $\mathcal{A}$-module, and let*

$$\mathcal{H} \circlearrowright \mathcal{L} \underset{\mathcal{I}}{\overset{\mathcal{P}}{\rightleftarrows}} \mathcal{K}$$

be a contraction in the category of graded $\mathcal{A}$-modules, i.e., $\mathcal{P}, \mathcal{I}, \mathcal{H}$ are graded $\mathcal{A}$-linear. Then $(\mathcal{A}, \mathcal{K})$ can be promoted to an LR_∞-algebra depending only on $(\mathcal{A}, \mathcal{L})$ and the contraction. With its LR_∞-algebra structure, $(\mathcal{A}, \mathcal{K})$ contains a full information on the quasi-isomorphism class of $(\mathcal{A}, \mathcal{L})$ (more precisely, the ∞-quasi-isomorphism class of $(\mathcal{A}, \mathcal{K})$ is equivalent to the quasi-isomorphism class of $(\mathcal{A}, \mathcal{L})$).

We are now ready to go back to a manifold M with a foliation $\mathcal{F}$ (and the associated leaf-space $\mathbb{M}$). So far, we defined vector fields on $\mathbb{M}$ in different ways. Let us summarize the latest outcome of our discussion. A first paradigm stated that vector fields on $\mathbb{M}$ are the leaf-wise cohomology

$$\mathbb{X}(\mathbb{M}) = H_{\mathcal{F}}(T_\perp M)$$

with coefficients in the normal bundle (with the Bott representation). This is a graded Lie-Rinehart algebra. Later we realized that $\mathbb{X}(\mathbb{M})$ is also the cohomology of the DG Lie-Rinehart algebra of vector fields on the DG manifold $\mathcal{F}[1]$:

$$\mathbb{X}(\mathbb{M}) = H(\chi(\mathcal{F}[1]), \mathcal{Q}),$$

where $\mathcal{Q} = [d_{\mathcal{F}}, -]$ is the graded commutator with the leaf-wise differential $d_{\mathcal{F}}$. This suggested us to formulate a new principle: *vector fields on $\mathbb{M}$ are just vector fields on $\mathcal{F}[1]$ up to homotopy*. In other words, the relevant object here is the *quasi-isomorphism class of the DG Lie-Rinehart algebra* $(C^\infty(\mathcal{F}[1], \chi(\mathcal{F}[1]))$:

$$\text{vector fields on } \mathbb{M} = \text{quasi-isomorphism class of } \chi(\mathcal{F}[1]).$$

Of course, the quasi-isomorphism class of $(C^\infty(\mathcal{F}[1], \chi(\mathcal{F}[1]))$ is represented by $(C^\infty(\mathcal{F}[1], \chi(\mathcal{F}[1]))$. However, we also have a contraction

$$\mathcal{H} \circlearrowright \chi(\mathcal{F}[1]) \underset{\mathcal{I}}{\overset{\mathcal{P}}{\rightleftarrows}} \Omega_{\mathcal{F}}(T_\perp M), \tag{11}$$

and one can check that $\mathcal{P}, \mathcal{I}, \mathcal{H}$ are $(C^\infty(\mathcal{F}[1]) = \Omega_{\mathcal{F}})$-linear. Hence $(\Omega_{\mathcal{F}}, \Omega_{\mathcal{F}}(T_\perp M))$ is an LR_∞-algebra which also contains a full information on the quasi-isomorphism class of $(C^\infty(\mathcal{F}[1], \chi(\mathcal{F}[1]))$ (more precisely, the ∞-quasi-isomorphism class of $(\Omega_{\mathcal{F}}, \Omega_{\mathcal{F}}(T_\perp M))$ is equivalent to the quasi-isomorphism class of $(C^\infty(\mathcal{F}[1], \chi(\mathcal{F}[1])))$.

We conclude this section, describing explicitly the transferred structure on $(\Omega_{\mathcal{F}}, \Omega_{\mathcal{F}}(T_\perp M))$. Recall, that the contraction (11) is defined via the choice of a splitting $\sigma : T_\perp M \to TM$ of the short exact sequence (9). This gives inclusions

$$I : \Omega_{\mathcal{F}} \to \Omega(M), \quad \text{and} \quad I : \Omega_{\mathcal{F}}(T_\perp M) \to \Omega(M, TM).$$

In the remaining part of this section, we understand the inclusions I, and interpret $\Omega_{\mathcal{F}}$ and $\Omega_{\mathcal{F}}(T_\perp M)$ as subspaces of $\Omega(M), \Omega(M, TM)$ respectively.

Notice that, composing the canonical projection $TM \to T_\perp M$ with the splitting $\sigma : T_\perp M \to TM$ gives a projector $\Theta : TM \to TM$ that could also be regarded as a TM-valued 1-form, i.e., $\Theta \in \Omega^1(M, TM)$. The Frolicher-Nijenhuis bracket

$$R = [\Theta, \Theta]^{\mathrm{fn}} \in \Omega^2(M, TM)$$

measures how far is the subbundle $\operatorname{im} \sigma \subset TM$ from being involutive, and it is called the *curvature* of the splitting σ.

Theorem 10.3. *The transferred LR_∞-algebra structure on $(\Omega_{\mathcal{F}}, \Omega_{\mathcal{F}}(T_\perp M))$ is the following. The L_∞-algebra $(\Omega_{\mathcal{F}}(T_\perp M), (\mathfrak{l}_k)_{k\in\mathbb{M}})$ is given by*

$$\begin{aligned} \mathfrak{l}_1(Z) &= d_{\mathcal{F}} Z \\ \mathfrak{l}_2(W, Z) &= \pm[W, Z]^{\mathrm{fn}} + [[R, W]^{\mathrm{nr}}, Z]^{\mathrm{nr}} \\ \mathfrak{l}_3(V, W, Z) &= -[[[R, V]^{\mathrm{nr}}, W]^{\mathrm{nr}}, Z]^{\mathrm{nr}} \end{aligned}$$

and

$$\mathfrak{l}_k = 0 \quad \text{for } k > 3.$$

for all $V, W, Z \in \Omega_{\mathcal{F}}(T_\perp M)$. The L_∞-module $(\Omega_{\mathcal{F}}, (\mathfrak{m}_k)_{k\in\mathbb{N}})$ is given by

$$\begin{aligned} \mathfrak{m}_1(Z) &= d_{\mathcal{F}} \omega \\ \mathfrak{m}_2(Z|\omega) &= \pm\mathcal{L}_Z \omega + i_{[R,Z]^{\mathrm{nr}}} \omega \\ \mathfrak{m}_3(W, Z) &= -i_{[[R,W]^{\mathrm{nr}},Z]^{\mathrm{nr}}} \omega \end{aligned}$$

and

$$\mathfrak{m}_k = 0 \quad \text{for } k > 3.$$

for all $W, Z \in \Omega_{\mathcal{F}}(T_\perp M)$ and all $\omega \in \Omega_{\mathcal{F}}$.

11. Differential forms on a leaf space IV: a homotopy differential algebra

As already mentioned quickly, a *double DGA* is a graded (associative, commutative, unital) algebra $\mathcal{B}$ equipped with two commuting homological derivations d_1, d_2. One can transfer a double DGA structure along a contraction to get a *homotopy DGA*. A *homotopy DGA* is a graded (associative, commutative, unital) algebra $\mathcal{K}$ equipped with a family of graded derivations $(\mathfrak{d}_k)_{k\in\mathbb{N}}$ satisfying the following *coherence conditions*: for all $n \in \mathbb{N}$

$$\sum_{i+j=n} [\mathfrak{d}_i, \mathfrak{d}_j] = 0 \tag{12}$$

(in particular $\mathfrak{d}_1 + \mathfrak{d}_2 + \cdots$, if well-defined, is a homological derivation). Let's have a look at the identities (12) for low n. For $n = 1$, we get

$$\mathfrak{d}_1 \mathfrak{d}_1 = 0,$$

i.e., $(\mathcal{K}, \mathfrak{d}_1)$ is a plain DGA. For $n = 2$, we get

$$[\mathfrak{d}_1, \mathfrak{d}_2] = 0.$$

So, in some sense, $\mathfrak{d}_2$ is *compatible* with $\mathfrak{d}_1$. For $n = 3$, we get

$$2\mathfrak{d}_2\mathfrak{d}_2 = [\mathfrak{d}_1, \mathfrak{d}_3],$$

i.e., $\mathfrak{d}_2$ is a homological derivation, hence $(\mathcal{K}, \mathfrak{d}_2)$ is a DGA only up to a homotopy encoded by $\mathfrak{d}_3$. Similarly for higher n. As $\mathfrak{d}_1$ is a differential, one can take the cohomology $H(\mathcal{K}, \mathfrak{d}_1)$, and $\mathfrak{d}_2$ induces a honest differential on it. So *the cohomology of a homotopy DGA is a honest DGA*. Finally, we remark that a double DGA is a homotopy DGA such that $\mathfrak{d}_k = 0$ for $k > 2$.

Notice that the definition of homotopy DGA just provided is a simplification of the "should be" definition where only the differential is up to homotopy, while the commutative, associative product is honestly associative and commutative. There is a version of the Homotopy Transfer Theorem for homotopy DGA:

Theorem 11.1 (DGA Homotopy Transfer)**.** *Let $(\mathcal{B}, d_1, d_2)$ be a double DGA, let $(\mathcal{K}, \mathfrak{d}_1)$ be a DGA, and let*

$$\mathcal{H}^* \circlearrowright (\mathcal{B}, d_1) \underset{\mathcal{P}^*}{\overset{\mathcal{I}^*}{\rightleftarrows}} (\mathcal{K}, \mathfrak{d}_1)$$

be a contraction such that $\mathcal{P}^*, \mathcal{I}^*$ are DGA maps, and $\mathcal{H}^*$ is $\mathcal{K}$-linear (i.e., $\mathcal{H}^*(\mathcal{P}^*(\kappa)\alpha) = \pm\mathcal{P}^*(\kappa)\mathcal{H}^*(\alpha)$ for all $\kappa \in \mathcal{K}$, and all $\alpha \in \mathcal{B}$). Then $(\mathcal{K}, \mathfrak{d}_1)$ can be promoted to a homotopy DGA depending only on $(\mathcal{B}, d_1, d_2)$ and the contraction. With its homotopy DGA structure, $\mathcal{K}$ contains a full information on the quasi-isomorphism class of $(\mathcal{B}, d_1, d_2)$ (more precisely, the ∞-quasi-isomorphism class of $\mathcal{K}$ is equivalent to the quasi-isomorphism class of $(\mathcal{B}, d_1, d_2)$).

Now, let M be a manifold with a foliation $\mathcal{F}$. According to our latest principle: *differential forms on the leaf-space $\mathbb{M}$ are differential forms on $\mathcal{F}[1]$ up to homotopy.* In other words,

differential forms on $\mathbb{M}$ = quasi-isomorphism class of the double DGA $(\Omega(\mathcal{F}[1]), \mathcal{L}_{d_{\mathcal{F}}}, d_{\mathrm{dR}})$.

Now recall that we have a contraction

$$\mathcal{H}^* \circlearrowright (\Omega(\mathcal{F}[1]), \mathcal{L}_{d_{\mathcal{F}}}) \underset{\mathcal{P}^*}{\overset{\mathcal{I}^*}{\rightleftarrows}} (\Omega_{\mathcal{F}}(\wedge^\bullet T^*_\perp M), d_{\mathcal{F}}), \tag{13}$$

and one can choose $\mathcal{P}^*, \mathcal{I}^*, \mathcal{H}^*$ so to satisfy the hypotheses of Theorem 11.1. Hence $\Omega_{\mathcal{F}}(\wedge^\bullet T^*_\perp M)$ is a homotopy DGA which contains a full information on the quasi-isomorphism class of $(\Omega(\mathcal{F}[1]), \mathcal{L}_{d_{\mathcal{F}}}, d_{\mathrm{dR}})$ (more precisely, the ∞-quasi-isomorphism class of $\Omega_{\mathcal{F}}(\wedge^\bullet T^*_\perp M)$ is equivalent to the quasi-isomorphism class of $(\Omega(\mathcal{F}[1]), \mathcal{L}_{d_{\mathcal{F}}}, d_{\mathrm{dR}})$).

We now describe explicitly the transferred structure on $\Omega_{\mathcal{F}}(\wedge^\bullet T^*_\perp M)$. The contraction (13) is defined via the choice of a splitting of (9). This gives an isomorphism

$$\Omega_{\mathcal{F}}(\wedge^\bullet T^*_\perp M) \to \Omega(M)$$

that, in the remaining part of this section, we understand. Let Θ and R be as in the previous section.

Theorem 11.2. *The transferred homotopy DGA* $(\Omega_{\mathcal{F}}(\wedge^\bullet T^*_\perp M), (\mathfrak{d}_k)_{k\in\mathbb{N}})$ *is given by*

$$\mathfrak{d}_1 = d_{\mathcal{F}}$$
$$\mathfrak{d}_2 = d_{\mathrm{dR}} - d_{\mathcal{F}} + i_R$$
$$\mathfrak{d}_3 = -i_R$$

and

$$\mathfrak{d}_k = 0 \quad \textit{for } k > 3.$$

In particular $\mathfrak{d}_1 + \mathfrak{d}_2 + \cdots = d_{\mathrm{dR}}$.

We conclude discussing briefly the relationship between the LR_∞-algebra of Theorem (10.3) and the homotopy DGA of Theorem (11.2). Vector fields and differential forms on a manifold are related by a *Chevalley-Eilenberg construction*. More generally, there is a *Chevalley-Eilenberg construction* defining a (double) DGA out of a (DG) Lie-Rinehart algebra. Similarly, there is a *higher Chevalley-Eilenberg construction* defining a homotopy DGA out of an LR_∞-algebra. The homotopy DGA of Theorem (11.2) and the LR_∞-algebra of Theorem (10.3) are related by this higher Chevalley-Eilenberg construction. Finally, $\Omega_{\mathcal{F}}(T_\perp M)$ and $\Omega_{\mathcal{F}}(\wedge^\bullet T^*_\perp M)$ do also define a *higher Cartan calculus* involving the higher homotopy, which is another evidence that our point of view on the leaf-space is consistent.

The table in the next page quickly summarizes what is known for other constructions on leaf spaces. In that table, given a (graded manifold) $\mathcal{M}$, we denoted by $\chi^{\mathrm{alt}}_{\mathrm{poly}}(\mathcal{M})$, $\chi^{\mathrm{sym}}_{\mathrm{poly}}(\mathcal{M})$, and $D(\mathcal{M})$ the Gertenhaber algebra of multivetor fields on $\mathcal{M}$, the Poisson algebra of fiber-wise polynomial functions on $T^*\mathcal{M}$ (symmetric multi-vector fields), and the associative algebra of scalar differential operators $C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ respectively. Notice that $D(\mathcal{M})$ is also the module of sections of a(n infinite dimensional, filtered) vector bundle over $\mathcal{M}$ that ve denoted $D\mathcal{M}$ (without the brackets). Finally, the symbols ∞-DGA, P_∞, G_∞, and A_∞ in the last column refer to homotopy DGAs, homotopy Poisson algebras, homotopy Gerstenhaber algebras, and homotopy associative algebras (also known as A_∞-algebras) respectively. The rest should be clear.

	deg = 0	**$\mathcal{F}$-representation**	**DG Algebra**	**Homotopy Algebra**	
$\mathbb{C}^\infty(\mathbb{M})$	leaf-wise contant functions	$\mathbb{R}_M$	$C^\infty(\mathcal{F}[1])$	$\Omega_{\mathcal{F}}$	DGA
$\mathbb{X}(\mathbb{M})$	infinitesimal symmetries mod $\mathcal{F}$	$T_\perp M$	$\chi(\mathcal{F}[1])$	$\Omega_{\mathcal{F}}(T_\perp M)$	LR_∞
$\mathbb{\Omega}(\mathbb{M})$	basic forms	$\wedge^\bullet T_\perp^* M$	$\Omega(\mathcal{F}[1])$	$\Omega_{\mathcal{F}}(\wedge^\bullet T_\perp^* M)$	∞-DGA
$\mathbb{X}^{\mathrm{sym}}_{\mathrm{poly}}(\mathbb{M})$	—	$S^\bullet T_\perp M$	$\chi^{\mathrm{alt}}_{\mathrm{sym}}(\mathcal{F}[1])$	$\Omega_{\mathcal{F}}(S^\bullet T_\perp M)$	P_∞
$\mathbb{X}^{\mathrm{alt}}_{\mathrm{poly}}(\mathbb{M})$	—	$\wedge^\bullet T_\perp M$	$\chi^{\mathrm{alt}}_{\mathrm{poly}}(\mathcal{F}[1])$	$\Omega_{\mathcal{F}}(\wedge^\bullet T_\perp M)$	G_∞
$\mathbb{d}_{\mathrm{dR}}(\mathbb{M})$	—	$D_\perp M = DM/DM \cdot \mathcal{F}$	$D(\mathcal{F}[1])$	$\Omega_{\mathcal{F}}(D_\perp M)$	A_∞
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	

For more details on the technical aspects the reader may have a look at the following three papers and references therein.

References

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