INTRODUÇÃO

Instanton bundles were first introduced in 1970

1977 Atiyah, Drinfeld, Hitchin and Manin as solutions of the Yang Mills equations. But I am sorry for those who are interested, I will not speak about physics but I will jump to

1986 Okonek Spindler define a

MATHEMATICAL INSTANTON BUNDLES on $\mathbb{P}^{2n+1}$ $(n\geq 1)$ as holomorphic rank $2n$ vector bundles $\mathcal{E}$ such that.

i) - $c_t(\mathcal{E}) = \left(\frac{1}{1-t^2}\right)^c$ chern polynomial

ii) - natural cohomology in the range $-2n-1 \leq \ell \leq 0$ → for each H^i there is at most one cohomology group among $H^i(\mathcal{E}(\ell))$ which is not zero

iii) - $\mathcal{E}$ has trivial splitting type

↳ Grothendieck's theorem $\mathcal{E}_{|\mathbb{P}^1} \simeq \oplus \mathcal{O}_{\mathbb{P}^1}(a_i)$

and in our case $\mathcal{E}_{|L} \simeq \mathcal{O}^{2n}$ for the generic line

iv) - $\mathcal{E}$ is simple $(\mathrm{Hom}(\mathcal{E},\mathcal{E})\simeq k)$ the only maps between $\mathcal{E}$ and himself are multiples of the identity

v) - $\mathcal{E}$ has a symplectic structure

In general we say $\mathcal{E}$ is SELF-DUAL (AUTO-DUAL) if there exists an isomorphism $\phi: \mathcal{E} \to \mathcal{E}^\vee$.

Moreover if $\phi^\vee = -\phi$ ⇝ SYMPLECTIC

$\phi^\vee = \phi$ ⇝ ORTHOGONAL

~~Ter a condições anteriores implica que podemos escrever~~

The previous conditions imply that we can write also an instanton bundle as the cohomology of the following MONAD

$$H^1(\mathcal{E}\otimes\Omega^2(1))\otimes\mathcal{O}(-1) \xrightarrow{\alpha} H^1(\mathcal{E}\otimes\Omega)\otimes\mathcal{O} \xrightarrow{\beta} H^1(\mathcal{E}(-1))\otimes\mathcal{O}(1)$$

$\dim H^1(\mathcal{E}(-1)) = c_2(\mathcal{E}) = c$ also called CHARGE

MONAD means

α injective

β surjective

$\beta\circ\alpha = 0$ ⇝ $\frac{\mathrm{Ker}\,\beta}{\mathrm{Im}\,\alpha} \simeq \mathcal{E}$ cohomology

Ancona, Ottaviani 1994 ~~show~~ show that (i) + (ii) = (iv) and ~~define~~ INSTANTON without v)

Obs. on $\mathbb{P}^3$ iii) and v) are superflous

becau rank 2 v.b on $\mathbb{P}^3$ simple $\Rightarrow$ stable $\rightarrow$ trivial splitting
Grauert-Mülich

v) is given considering

$$\mathrm{Hom}(\mathcal{E},\mathcal{E}) \simeq H^0(\mathcal{E}\otimes\mathcal{E}^\vee) = H^0(\mathcal{E}\otimes\mathcal{E}) = \underset{\mathbb{k}}{\underset{\shortparallel}{H^0(\Lambda^2\mathcal{E})}} \oplus \underset{=0}{\underset{\downarrow}{H^0(S^2\mathcal{E})}}$$

only remain the isomorphisms which give a symplectic structure

What happens for bigger projective spaces?

2009 Farnik, Frapporti, — prove that there is no orthogonal
2n rank instanton on $\mathbb{P}^{2n+1}$

$\rightsquigarrow$ stable $\rightsquigarrow$ unique symplectic structure

The rest of my time will be used to discuss a little b.t the following two ~~instantons~~ concepts

$\rightsquigarrow$ how can we describe a family of instantons bundles?

basically I would like to be able to solve a moduli problem, which means I would like a nice scheme structure of my family of objects.

$\rightsquigarrow$ give a look in the "quiver world" to see how we can deal with our problem in a different point of view.

There is an enormous literature on the moduli space of instanton bundles which took the work of many but I will mention, up to my knowledge, what are the latest results.

$\mathbb{P}^3$ $\rightsquigarrow$ smooth

Jardim, Verbitsky (2014)

$\rightsquigarrow$ irreducible

Tikhomirov (2012 / 2013)
odd / even

more in general in $\mathbb{P}^n$

~~Ancona - Ottaviani~~ denote

~~$M_{\mathbb{P}^{2n+1}}$~~ $\mathcal{M}_{\mathbb{P}^{2n+1}}(c)$ moduli space of STABLE instanton bundles on $\mathbb{P}^{2n+1}$ with charge c, then we get

$\mathcal{M}_{\mathbb{P}^{2n+1}}(2)$ smooth Ancona - Ottaviani 1995

$\mathcal{M}_{\mathbb{P}^{2n+1}}(c)$ singular $n \geq 2$, $c \geq 3$ Miró - Roig, Orus-Lacort 1997

$\mathcal{M}_{\mathbb{P}^{2n+1}}(c)$ affine Costa - Ottaviani 2003

rational Hoffmann ~~20~~ 2011

reducible (symplectic case) Costa, Hoffmann, Miró-Roig, Schmitt 2014

My goal is to convince you that we can study such family looking at their monadic definition, and we well look at the morphisms, which will be represented as matrices of linear forms. A good way to see this is for example in the paper of Costa and Ottaviani and I will now present the invariant they propose which comes from the monad and that helped them prove the affinity of the moduli space

Let us consider $\mathbb{P}^{2n+1} = \mathbb{P}(V)$ vector space dimension $2n+2$

and a SYMPLECTIC instanton bundle given by the monad

$$I^* \otimes \mathcal{O}(-1) \xrightarrow{A} W \otimes \mathcal{O} \xrightarrow{A} \cancel{A}\, I \otimes \mathcal{O}(1)$$

W, I vector spaces of dimension $2n+2c$ and c respectively

Remark

The "left and right" side of the monad look very much related one to the other. That is why, by a result available for example in the book of Okonek, Spindler, Schneider we have (for this monad)

morphism of $\mathcal{E}$ cohomology $\overset{1:1}{\longleftrightarrow}$ morphism of monads

In this case a symplectic structure $\phi: \mathcal{E} \to \mathcal{E}^\vee$ gives

$$\begin{array}{ccccc} \boxed{U \otimes \mathcal{O}(-1)} & \xrightarrow{B} & W \otimes \mathcal{O} & \xrightarrow{A} & I \otimes \mathcal{O}(1) \\ \downarrow \simeq & & J \downarrow \simeq & & \downarrow \simeq \\ I^* \otimes \mathcal{O}(-1) & \xrightarrow{A^t} & W^* \otimes \mathcal{O} & \xrightarrow{B^T} & U^* \otimes \mathcal{O}(1) \end{array}$$

here we denote the vector spaces as different

- the dual of an instanton bundle is the cohomology of the dual monad
- we have the wanted isomorphism and the pair (W, J) symplectic vector space.

We have that the matrix
~~Temos que a matriz~~ A defines ~~uma aplicação~~ a map

$$W \otimes \mathcal{O} \xrightarrow{A} I \otimes \mathcal{O}(1)$$ which is injective and this means that the evaluation at every point give a matrix of maximal rank.

This is equivalent to ask that (hold on on this)

$A^t \in \operatorname{Hom}(V^* \otimes I^*, W)$ is non degenerate ~~which means~~

Moreover we are looking at all elements

$$Q = \{ A^t \mid A^t \in \operatorname{Hom}(V^* \otimes I^*, W) \text{ and } \underbrace{A^t J A = 0}_{\text{describes the complex condition.}} \}$$

We can consider the action of

$(g, s) \in GL(I^*) \times Sp(W)$ ~ $(M^t J M = J)$ preserves symplectic forms.

$(g \cdot s) A^t = s \cdot A \cdot g$

PARENTESIS

$A^t \in \operatorname{Hom}(V^* \otimes I^*, W)$ is called degenerate if

$A(v \otimes i) = 0$ has a solution with $0 \neq v \in V^*$ e $i \neq 0 \in I^*$

numerous system ~ degenerate matrix are an irreducible variety of codimension $\mathbb{C}$.

Denote $Q^\circ \subset Q$ non degenerate matrix

From a result of Barth and Hulek of '78 we have ~~that~~ one to one correspondence between
- isomorphis classes of instantons (of normals)
- orbits ~~of~~ $GL(I) \times Sp(W)$ on Q_0
  (all orbits are closed)
  ↳ all points are stable

we can exclude trivial actions

$$G = \frac{GL(I) \times Sp(W)}{\pm (id, id)}$$

$Q^\circ \to Q^\circ / G$ GEOMETRIC QUOTIENT

Allows us to define

$$M_S I_{\text{pinn}}(c) \quad Q^\circ / GL(I) \times Sp(W)$$ a GIT - quotient ~~the~~

is the MODULI SPACE of symplectic c-instantons. It is a geometric quotient.

(Even if we are not familiar with GIT we can believe that we have our good moduli space and describe it in relation with the matrices)

Let us now take

$$\underbrace{W \otimes S^n I \xrightarrow{A \otimes id} V \otimes I \otimes \boxed{S^n I}}_{} \quad \rightsquigarrow \text{ denotes the symmetric power}$$

$$\downarrow id_V \otimes \underbrace{\pi}_{} \rightsquigarrow \text{ canonical projection}$$

$$\Delta_A : W \otimes S^n I \to V \otimes S^{n+1} I$$

the two vector spaces have the same dimension. $\rightsquigarrow \Delta_A$ is a squared matrix.

~~Remark~~ Define
$$D(A) = \det(\Delta_A)$$

Remark is $SL(V) \times SL(I) \times Sp(W)$ invariant. The super important part is

- If A^t is degenerate, then $D(A) = 0$

  $\rightsquigarrow$ because if we have $0 \neq v$ $0 \neq i$ $A(v \otimes i) = 0$

  then $\Delta_A^\vee (v \otimes S^{n+1} i) = 0$ hence the determinant vanishes

  the dual

- If A^t is non degenerate $\not\Rightarrow D(A) \neq 0$ does not imply that our invariant does not vanish (true in the case $k=1$)

  but we can surely state

thm If A defines an instanton $\Rightarrow D(A) \neq 0$

idea
$$0 \to K \to W \otimes \mathcal{O} \to I \otimes \mathcal{O}(1) \to 0 \quad \text{short exact sequence}$$

consider $(n+1)$-wedge power and we get

$$0 \to \Lambda^{n+1} K \to \Lambda^{n+1} W \otimes \mathcal{O} \to \dots \to \Lambda^2 W \otimes S^{n-1} I(n-1) \to$$

$$\to W \otimes S^{n} I \otimes \mathcal{O}(n) \to S^{n+1} I \otimes \mathcal{O}(n+1) \to 0$$

so I can tensor everything by $\mathcal{O}(-n)$ and get on the right

$$\dots \to W \otimes S^n I \otimes \mathcal{O} \to S^{n+1} I \otimes \mathcal{O}(1) \to 0$$

which in the induced long exact sequence in cohomology gives

$$W \otimes S^n I \to S^{n+1} I \otimes V \quad \text{that is exactly } \Delta_A$$

so every time we have an exact sequence we can break it in short exact ones and we get

$$\det \Delta_A \neq 0 \Leftrightarrow H^n(\Lambda^{n+1} K(-n)) = 0$$

Remember we had a monad so we also have

$$0 \to I^k \otimes \mathcal{O}(-1) \to K \to \mathcal{E} \to 0$$ take wedge as before (from the right side)

$$0 \to \overset{n}{\Lambda} K \otimes I^k(-n-1) \to \overset{n+1}{\Lambda} K(-n) \to \overset{n+1}{\Lambda} \mathcal{E}(-n) \to 0$$

we get that

$$\left.\begin{array}{l} H^{n+i}(\overset{n-1}{\Lambda} K(-n-i-1)) = 0 \quad i=0,\dots,n \\ \quad + \\ (*)\ H^n(\overset{n+1}{\Lambda} \mathcal{E}(-n)) = 0 \end{array}\right\} \Rightarrow H^n(\overset{n+1}{\Lambda} K(-n)) = 0$$

Finally, being $c_1(\mathcal{E}) = 0$ we get $\overset{n+1}{\Lambda} \mathcal{E} \simeq \overset{n-1}{\Lambda} \mathcal{E}$ and

$$H^{n+i}(\overset{n-1-i}{\Lambda}(K-n-i)) = 0 \Rightarrow (*) \quad i=0,\dots,n$$

and we check all this cohomological conditions using the first short exact sequence.

Thm. $MSI_{pen}(k)$ is affine ✓.

Remark If $\mathcal{E}$ is not symplectic we can define an analogue invariant using the two matrices of the monad and we arrive at the same conclusion.

Let us now give a look of a possible generalization

Our instanton bundles were defined also as the cohomology of a particular monad so we could wonder what happens if we take a monad of type

$$\mathcal{O}(-1)^a \to \mathcal{O}_{\mathbb{P}^n}^b \to \mathcal{O}_{\mathbb{P}^n}(1)^c$$

Flöystad in proves that (he proves more but I will explain what concerns us) its cohomology defines a vector bundle $\mathcal{E}$ whose ~~cohomology~~ rank is lower than the dimension of the base variety

$\Rightarrow a = c \quad n = 2k+1 \quad rk\,\mathcal{E} = 2k$

This means that our instanton bundles are the ones with the lowest possible rank coming from the monad

so it implies $H^0(\mathcal{E}) = 0$

What happens for higher rank?

Obviously we cannot ensure the stability of the bundle (hence we do not have a taxonomic moduli space) and the same invariant only works for the low rank case so we must use a different technique

~~Instanton~~ First of all, we will now focus on a higher rank bundle given by a monad $\mathcal{O}_{\mathbb{P}^n}(-1)^c \to \mathcal{O}_{\mathbb{P}^n}^{r+2c} \to \mathcal{O}_{\mathbb{P}^n}(1)^c$

If $\mathcal{E}_L \cong \mathcal{O}_{\mathbb{P}^n}^r$ for the generic line no Henni, Jardim, Martins ADHM construction to describe the morphism.

But I would like to focus on a different case, which was the thesis of my student Andrea. We do not require the splitting to be trivial on the generic line, but we do ask

$H^0(\mathcal{E}) = 0$ and $\varphi: \mathcal{E} \xrightarrow{\sim} \mathcal{E}^\vee$ $\varphi = \varphi^\vee$ orthogonal structure

Moreover $h^{n-1}(\mathcal{E}(-n)) = c$ and fix an isomorphism $f: H_c \to H^{n-1}(\mathcal{E}(-n))$

so we define two triples $(\mathcal{E}_1, \varphi_1, f_1)$ equivalent $(\mathcal{E}_2, \varphi_2, f_2)$ $g: \mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2$ isomorphism which behaves well with the other two structures

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{\varphi_1} & \mathcal{E}_1^\vee \\ g\downarrow & \circlearrowright & \uparrow g^\vee \\ \mathcal{E}_2 & \xrightarrow{\varphi_2} & \mathcal{E}_2^\vee \end{array} \qquad \begin{array}{ccc} H_c & \xrightarrow{f_1} & H^{n-1}(\mathcal{E}_1(-n)) \\ \lambda \mathrm{Id}\downarrow & & \downarrow g_* \\ H_c & \xrightarrow{f_2} & H^{n-1}(\mathcal{E}_2(-n)) \end{array}$$

We denote

$$\mathbb{E}(c,r) = \{ [\mathcal{E}, \varphi, f] \text{ equivalence classes} \}$$

without entering in details, from an equivalence class it is possible to define (starting from the Euler sequence tensored by the bundle) a morphism

$$A: H_c \otimes V \to H_c^* \otimes V^*$$

where V is the vector space which defines the projective space $\mathbb{P}^n = \mathbb{P}(V)$

with

$$A \in \bigwedge^2 H_c^* \otimes \bigwedge^2 V^*$$ so it is symmetric, moreover

① rank $A = 2c + r$

② A non degenerate

③ $\exists\, q: W \xrightarrow{\sim} W^*$ symmetric $\quad W = \dfrac{H_c \otimes V}{\ker A}$

we denote

$$\mathcal{A}[c,r] = \{ A \in \bigwedge^2 H_c^* \otimes \bigwedge^2 V^* \mid ①-③ \text{ hold} \}$$

Indeed we can prove that there is a one to one correspondence between such varieties and the equivalence classes of instanton.

theorem $\mathbb{E}[c,r] \overset{1:1}{\longleftrightarrow} \mathcal{M}[c,r]$

Moreover denote **KEY RESULT** $c = 1,2$ do not exist $c \geq 3 \Rightarrow$ rk $\mathcal{E} = (n-1)c$ irreducible

$\widetilde{\mathbb{E}}_c := \{[\mathcal{E}, e]\}$ we forget the isomorphism f.

Consider the action of the linear group

$$\alpha : GL(H_c) \times (\Lambda^2 H_c^* \otimes \Lambda^2 V^*) \longrightarrow \Lambda^2 H_c^* \otimes \Lambda^2 V^*$$

$$(h, A) \longmapsto (h \otimes Id) A (h^\vee \otimes Id)$$

and we prove the following steps.

- $\mathcal{M}_c$ is G-invariant
- $\widetilde{\mathbb{E}}_c \overset{1:1}{\longleftrightarrow}$ orbit space $\mathcal{M}_c / G$. Isotropy group $\{\pm Id_{H_c}\}$ $\quad G_0 = G/\{\pm Id_{H_c}\}$

so we are applying the action of a reductive group.

Thm. The geometric quotient

$$\mathcal{M}^o_{\mathbb{P}^n}(c) := \mathcal{M}_c /\!\!/ G_0$$

is reduced, irreducible, affine coarse moduli space of dimension $\binom{c}{2}\binom{n+1}{2} - c^2$

for orthogonal instanton bundle charge c rk $(n-1)c$ on $\mathbb{P}^n$ $H^0(\mathcal{E}) = 0$.
no global sections

As the last topic I will work is a relation between monads and quivers, let us recall basic definitions (Jardim, Prata 2015)

Def A quiver consists a pair (Q_0, Q_1) where Q_0 set of vertices, Q_1 set of arrows
and $t, h : Q_1 \to Q_0$
tail and head maps.

A representation $R = (\{V_i\}, \{A_a\})$ of Q is a collection of finite dimensional $\mathbb{k}$-vector spaces

$\{V_i, i \in Q_0\}$ with a collection of linear maps $\{A_a : V_{t(a)} \to V_{h(a)} ; a \in Q_1\}$

morphism between representations is a collection of linear maps $\{f_i\}$ such that the following diagram is commutative

$$\begin{array}{ccc} V_{t(a)} & \xrightarrow{A_a} & V_{h(a)} \\ f_{t(a)} \downarrow & & \downarrow f_{h(a)} \\ W_{t(a)} & \xrightarrow{B_a} & W_{h(a)} \end{array}$$

$R_1 = (\{V_i\}, \{A_a\})$

$R_2 = (\{W_i\}, \{B_a\})$

The main result I would like to show in this topic is the following

Consider the monad ⊛ $\mathcal{A}^a \to \mathcal{B}^b \to \mathcal{C}^c$ on a projective smooth variety X with $\mathcal{A}, \mathcal{B}, \mathcal{C}$ locally free sheaves (we suppose also cohomology be locally free)

If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are simple vector bundles $(\mathrm{Hom}(\mathcal{A},\mathcal{A}) \cong \mathbb{K})$ then the category of monads of type ⊛ is equivalent to a full subcategory of the category of representations of ~~quivers~~ the following quiver.

$$\bullet \xrightarrow[m]{\vdots} \bullet \xrightarrow[n]{\vdots} \bullet$$

where $m = \dim \mathrm{Hom}(\mathcal{A}, \mathcal{B})$
$n = \dim \mathrm{Hom}(\mathcal{B}, \mathcal{C})$

Moreover, if $a^2+b^2+c^2-mab-nbc>1$ then cohomology sheaf is decomposable.

It is easy to see one part of the equivalence, let us just give a quick interpretation. The fact that the vector bundle is simple ~~tells us that~~ which means that all possible endomorphisms are the multiples of the identity, tells us that we can basically consider the bundle as ~~a~~ the field $\mathbb{K}$, so for the representation we consider a, b, c copies of the field as the vector spaces, and we take as many arrows to define a base of the homomorphism space between the two bundles

So this means that we must look better on what we mean by a subcategory because we have to check ~~what~~ how we can translate in terms of the quiver the properties that we have on the monad (injective, surjective, vanishing composition)

Let us present the idea of the proof.

We must construct a functor between

$$M_{\mathcal{A},\mathcal{B},\mathcal{C}} \longrightarrow R(K_{m,n})$$

monads given by the 3 chosen bundles

representation of the quiver presented above ~~once we have drawn~~

choose basis $\gamma = \{\gamma_1, \dots, \gamma_m\}$ of $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$

$\delta = \{\delta_1, \dots, \delta_n\}$ of $\mathrm{Hom}(\mathcal{B}, \mathcal{C})$

once we have chosen a basis we can write

$$\alpha = \sum_{i=1}^{m} A_i \otimes \gamma_i \quad \text{e} \quad \beta = \sum_{j=1}^{n} B_j \otimes \delta_j$$

where A_i $b \times a$ matrix
B_j $c \times b$ matrix
entries in $\mathbb{K}$.

so we have the correspondence

$$\underset{\substack{\text{with maps}\\ \alpha,\beta}}{M^\bullet} \longmapsto R = (\{\mathbb{K}^a, \mathbb{K}^b, \mathbb{K}^c\}, \{A_i\}, \{B_j\})$$

The first thing to prove is that our functor $G_{\gamma,\delta}$ behaves well with morphisms, indeed we get

$$\mathrm{Hom}(M_1^\bullet, M_2^\bullet) \xrightarrow{\sim} \mathrm{Hom}(G_{\gamma,\delta}(M_1^\bullet), G_{\gamma,\delta}(M_2^\bullet))$$ is an isomorphism

where a morphism of monads is given by three morphisms one for each pair, ~~that~~ ~~not~~ to have commutative diagrams

$\rightsquigarrow$ so we get a full subcategory

The conditions of having a monad are translated in the following way

α injective, β surjective gives a representation which is (γ,δ)-globally injective and surjective, which means

$$\alpha(p) = \sum A_i \otimes \gamma_i(p) \qquad \beta(p) = \sum B_j \otimes \delta_j(p)$$ injective and surjective for every $p \in X$

a priori this does not have to hold for every point of the variety, it is sufficient to have for the generic point in order to have injectivity as maps of sheaves, this means that we want the cohomology to be a vector bundle.

Moreover the condition $\beta \circ \alpha = 0$ is translated into the relations

$$\sum_{1 \le i \le j \le m} (B_i A_j + B_j A_i)(\delta_i \gamma_j) = 0$$

We denote $R^{gis}_{m,n}$ the subcategory defined by the conditions above
it is possible to prove $\rightsquigarrow$ it is closed under direct summands

$\rightsquigarrow$ $M^\bullet$ is decomposable if and only if the associated quiver is decomposable

$\hookrightarrow$ the monad splits as two monads

$\hookrightarrow$ splits as direct sum of two representations.

Extra possibility

~~Instanton bundles on other varieties~~

Possible applications of this equivalence

(–, Marques, Soares - 2018) we characterize the existence of monads on a projective variety (under certain hypothesis, ACM or not contained in a quadric) but we can reassume the main result in the following way

Thm X proj. variety dim n $L \to X$ line bundle

Suppose that there is a linear system $V \subseteq H^0(L)$ with no base points

$\leadsto$ define $X \dashrightarrow \mathbb{P}(V) \supset X'$ image is ACM $\left(\begin{array}{l} H^1_*(\mathcal{I}_{X'/\mathbb{P}^N}) = 0 \\ H^i_*(X', \mathcal{O}_{X'}) = 0 \quad 0 < i < n' = \dim X' \end{array} \right)$

Then there exists a monad of type

$$(L^\vee)^a \to \mathcal{O}_X^b \to L^c$$ if and only if

one of the two following conditions holds.

i) $b \geq a + c$ and $b \geq 2c + n - 1$

ii) $b \geq a + c + n$.

Moreover if L defines embedding and we want a vector bundle as cohomology of the monad of rank $< n$ $\Rightarrow$ $n = 2k+1$ $rk = 2k$, $a = c$.

We would like to study the family of bundle given to such monads, therefore we first trying to describe the family of monads and then we will try to look at their family of cohomologies.

Suppose we are in the case

$$X \xrightarrow{|L|} \mathbb{P}^N \quad L \text{ very ample}$$

we can rephrase our goal as follow: given a surjective map, when can we find another one that forms a monad? Let us give ~~some~~ more precise definitions

Consider set of all morphisms $g : \mathcal{O}_X^b \to L^c$

this is described as

$$\underbrace{B^* \otimes C}_{\text{vector spaces of dimension } b \text{ and } c} \otimes H^0(L)$$

So we have the projectivization

$$\mathbb{P} = \mathbb{P}(B^* \otimes C \otimes H^0(L)) \quad \text{and}$$

$$\mathcal{O}_{\mathbb{P}}(-1) \longrightarrow B^* \otimes C \otimes H^0(L) \otimes \mathcal{O}_{\mathbb{P}} \quad \text{and pay attention}$$

this is simply the tautological inclusion because the interpretation of $\mathcal{O}_{\mathbb{P}}(-1)$ is that the fiber on each point is the vector space associated to the projective point so this is a canonical inclusion. We therefore induce the morphisms

$$\begin{tikzcd} B \otimes H^0(L) \otimes \mathcal{O}_{\mathbb{P}}(-1) \arrow[r] \arrow[dr, "\varphi"'] & C \otimes H^0(L) \otimes H^0(L) \otimes \mathcal{O}_{\mathbb{P}} \\ & C \otimes H^0(L \otimes L) \otimes \mathcal{O}_{\mathbb{P}} \end{tikzcd}$$

and if we consider

$$Z = \{ g \in \mathbb{P} \mid rk_g(\varphi) \leq b(n+1) - a \}$$ we are considering all the morphism

$$0 \to K_g \to \mathcal{O}_X^b \xrightarrow{g} L^c \to 0$$ whose kernel K_g has

$$h^0(K_g \otimes L) \geq a.$$

so so the ones that give me a sound

If $c = 1, 2$ a-c holds for every g so irreducible

Very good, but how do we ensure the existence of the moduli space? coarse

We consider the previous equivalence and look at a result of King.

that states

th

Given a representation $R = \{ \overbrace{(c, 2k+2c, c)} \}$ → we look at vector bundles, ... we are this simpler notation in this case

it is called SEMISTABLE if there is a triple $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$ such that

$$\langle (\lambda_1, \lambda_2, \lambda_3), (c, 2k+2c, c) \rangle = 0$$

$$\langle (\lambda_1, \lambda_2, \lambda_3), (a', b', c') \rangle \geq 0 \quad \text{for every } (a', b', c') \text{ subrepresentation}$$

Moreover the existence of such λ guarantees the existence of a coarse moduli space

If $c = 1$ we always have it (and it is irreducible)

~~Other variede~~ Other varieties

The definition of instanton for other varieties has been dealt in two different way, either we generalize the description as a monad, either we consider the cohomological characterization. ~~twisted~~ Lovely with Malaspina and Pons-Llopis we define them for flag varieties. Why?

First Faenzi (2014) and Kuznetsov (2012) already consider Fano threefold of Picard one, and this is the first example with Picard two.

Moreover Hitchin 1981 proves $\mathbb{P}^3$ and flag variety are the only four dimensional real differential varieties which are Kähler (flag is the twistor space of $\mathbb{P}^2$)

Indeed for charge 1 (see in a while what we mean) they were already studied by Donaldson 1985 and Buchdal 2006.

Recall a little bit the geometry of the flag point-line

$F(0,1,2)$ general hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$

we get the Chow ring $\quad A(F) \cong \dfrac{\mathbb{Z}[h_1,h_2]}{(h_1^2 - h_1h_2 + h_2^2, h_1^3, h_2^3)}$ → the hyperplane classes of the two projective planes

We define

- Def: MATHEMATICAL INSTANTON BUNDLE (k-instanton) for short rank 2 vector bundle $\mathcal{E}$ μ-semistable $c_1(\mathcal{E}) = (0,0)$ $c_2(\mathcal{E}) = k h_1 h_2$ and $\underbrace{H^0(\mathcal{E})}$ $= H^1(\mathcal{E}(-1,-1)) = 0$

↳ this condition is not exactly necessary in the definition, but it is important ~~for the equivalence of stability.~~

To remark that

- Gieseker and μ (semi)-stability coincide.

We can classify directly the proper semistable case

$\mathcal{E}$ k-instanton not stable $\Rightarrow$ $k = \ell^2$ for $\ell \in \mathbb{Z}$ $\ell \neq 0$

and it is given as an extension

$$0 \to \mathcal{O}_F(\ell,-\ell) \to \mathcal{E} \to \mathcal{O}_F(-\ell,\ell) \to 0 \quad (\Lambda_\ell) \text{ or } (\Lambda_{-\ell})$$ switching the two line bundles

$\Lambda_\ell \cap \Lambda_{-\ell} = \mathcal{O}_F(-\ell,\ell) \oplus \mathcal{O}_F(\ell,-\ell)$ for each ℓ we have to families of semistable

As a consequence we have

If $\mathcal{E} \neq \mathcal{O}_F(-\ell,\ell) \oplus \mathcal{O}_F(\ell,-\ell)$ $\rightsquigarrow$ simple and we have unique symplectic structure

Describing the derived category of the flag variety, and in this case we have a good start because we can be defined also as the projectivization of the tangent bundle in $\mathbb{P}^2$, so knowing generators of derived category in $\mathbb{P}^2$ gives us generators for the flag. through this we have two equivalent monadic descriptions

- th $\mathcal{E}$ k-instanton is the cohomology of the monad

$$\mathcal{O}_F(-1,0)^k \oplus \mathcal{O}_F(0,-1)^k \to G_1(-1,0)^k \oplus G_2(0,-1)^k \to \mathcal{O}_F^{2k-2}$$

where $G_i \simeq p_i^* \Omega_{\mathbb{P}^2}(2)$

$$\begin{tikzcd} & F \arrow[dl, "p_1"'] \arrow[dr] & \\ \mathbb{P}^2 & & \mathbb{P}^{2\vee} \end{tikzcd}$$

considering the two canonical projections from the flag variety

and also the monad

$$\mathcal{O}_F(-1,0)^k \oplus \mathcal{O}_F(0,-1)^k \to \mathcal{O}_F^{4k+2} \to \mathcal{O}_F(1,0)^k \oplus \mathcal{O}_F(0,1)^k$$

↳ this is the monad useful to describe the symplectic structure we have. Indeed we see the monad is self dual and we can see the middle term as $W \otimes \mathcal{O}_F$ and (W, q) is symplectic vector space.

Monadic description + symplectic structure allow us to define

$MI_F(k)$ geometric quotient and affine $k=1$

moduli space instanton on the flag. (conjecture is affine for each k)

What have I missed until now? I did not give you an example.

the idea is to construct a specific example for $k=1$ and then we construct inductively the other cases

CASE $k=1$ given by a monad

$$\mathcal{O}_F(-1,0) \oplus \mathcal{O}_F(0,-1) \xrightarrow{A} \mathcal{O}_F^6 \xrightarrow{B} \mathcal{O}_F(1,0) \oplus \mathcal{O}_F(1,0)$$

$\rightsquigarrow$ up to an action of the linear groups acting on the monad and because the cohomology bundle has no global sections, we can always describe the matrix B in the following way.

$$B = \begin{bmatrix} x_0 & x_1 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_0 & y_1 & y_2 \end{bmatrix}$$ using coordinates $(x_0 : x_1 : x_2) \times (y_0 : y_1 : y_2) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee}$

and the flag defined by $\sum x_i y_i = 0$

This forces the matrix A to be

$$A = \begin{bmatrix} f_1 & \gamma y_0 \\ f_2 & \gamma y_1 \\ f_3 & \gamma y_2 \\ \delta x_0 & g_1 \\ \delta x_1 & g_2 \\ \delta x_2 & g_3 \end{bmatrix}$$ where $(f_1, f_2, f_3)^T$, $(g_1, g_2, g_3)^T$ syzygies of (x_i), (y_i)

so we have a 7-dimensional family. $h^2(\mathcal{E} \otimes \mathcal{E}) = 0$ are Ulrich (ACM + $H^0_*(\mathcal{E})$ maximal number of generators)

and this case is completely described because

PROP $\mathcal{E}$ semistable $\Leftrightarrow$ one of the two syzygies is zero

sp. els $\Leftrightarrow$ both are zero

We could think to be on a super point because one could say, now we just diagonalize and we find examples for any charge but it is not so immediate the problems comes from the equation defining the flag.

So we construct example by INDUCTION starting from

$\mathcal{E}$ k-instanton stable $h^2(\mathcal{E} \otimes \mathcal{E}) = 0$ $\quad \mathcal{E}_{|C} \cong \mathcal{O}_C^2$ trivial restriction on the generic conic, but this is not necessary to ask it is indeed implicit in the definition of instanton

and we consider what is usually called an elementary transformation

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{I}_C(1) \to 0$$

where C is a generic conic.

We then prove that $\mathcal{F}$ can be deformed into a stable vector bundle

$c_1(\mathcal{F}) = 0 \quad c_2(\mathcal{F}) = (k+1) h_1 h_2 \quad h^1(\mathcal{F}) = h^1(\mathcal{F}(-1,-1)) = 0 \Rightarrow k+1$-instanton

and $h^2(\mathcal{F} \otimes \mathcal{F}) = 0$ and we iterate.

We show

$M_F I_F(k)$ has a generically smooth irreducible component of dimension $8k-3$

We conclude saying couple of word on the locus of jumping conics (which are the corresponding object of a line in $\mathbb{P}^3$)

Recall that we have a $\mathbb{P}^2 \times \mathbb{P}^2$ of conics in the flag

$$\begin{array}{c}\mathbb{P}^2 \times \mathbb{P}^2 \\ \cup \\ \mathbb{F}\end{array} \quad \text{smooth conics are a flag variety}$$

We know that for the generic one the splitting is trivial but

- **Def.** $C \in F$ is called JUMPING CONIC of order (a,b) if

$$h^1(E_{|C}(-1,0)) = a \qquad h^1(E_{|C}(0,-1)) = b$$

↳ we manage to prove

Thm $\mathcal{E}$ is k-instanton $\Rightarrow$ $D_{\mathcal{E}}$ locus of jumping conics is a divisor of type (k,k)

and we can be a little more specific if we consider

- **Prop** $\mathcal{E}$ k instanton prop. semistable $(k = \ell^2)$

$$C \in D_{\mathcal{E}} \iff \underset{\text{reducible}}{C = L_1 \cup L_2}$$

Take one possible family (given by the extension)

$$0 \to \mathcal{O}_F(\ell,-\ell) \to \mathcal{E} \to \mathcal{O}_F(-\ell,\ell) \to 0$$

the possible splittings are $\quad \mathcal{O}_C(\ell,-a) \oplus \mathcal{O}_C(-\ell,a) \quad 0 \le a \le \ell$

Moreover we can find $\mathcal{E}$ s.t.c. semist. and $\ell+1$ conics $C_j \subset F$

s.t. $\quad \mathcal{E}_{|C_j} \simeq \mathcal{O}_{C_j}(\ell,-j) \oplus \mathcal{O}_{C_j}(-\ell,j)$ which means we can find all possible splittings

and for higher jump

$$\mathcal{E} \text{ } k\text{-instanton generic} \Rightarrow \underset{\text{at least } (-2,2)}{S_{\mathcal{E}} \overset{}{=}} \text{ is a curve}$$

this is interesting because at some point we thought we would have only even splitting but this statement disproves it.