I. Homotopy theory

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1. Overview [... meaning htpy theory of top. spaces + abstract homotopy thy]

[In alg. top. study alg. invariants of top. spaces, e.g. cohomology theories:]

Notations: Top := cat of top. sp., $\mathrm{Top}_* :=$ cat of pointed top sp., $I := [0,1] \subseteq \mathbb{R}$

Def.: • A reduced cohomology theory is a functor $\mathrm{Top}_* \xrightarrow{E} (\mathrm{Ab}^{\mathbb{Z}})^{op}$, $X \longmapsto E^*(X) = \bigoplus_{n\in\mathbb{Z}} E^n(X)$

s.t. 1. $X\times I \to X$ induces $E^*(X) \xrightarrow{\cong} E^*(X\times I)$.

2. Have nat. isos $E^n(X) \cong E^{n+1}(\Sigma X)$

(Remember: $\Sigma X = X\times I / X\times\{0\}\cup X\times\{1\}\cup\{pt\}\times I$

$= X\times S^1 / pt\times S^1 \cup X\times\{pt\} = S^1\wedge X$)

3. $E^*(\bigvee_{i\in I} X_i) \cong \prod_{i\in I} E^*(X_i)$

4. $A \hookrightarrow X \to X/A$ „good" inclusion 

induces exact sequence $E^*(X/A) \to E^*(X) \to E^*(A)$

• A cohomology theory is a functor of the form

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "(-)_+"] & \mathrm{Top}_* \arrow[r, "E"] & (\mathrm{Ab}^{\mathbb{Z}})^{op} \\ X \arrow[r, mapsto] & X \amalg \{pt\} & \end{tikzcd}$$

↖ reduced coh. theory.

Examples: Singular coh. H_{sing} / de Rham H_{dR}, top. complex/real K-theory KU/KO, complex cobordism MU, ...

(elliptic, Morava K/E, ...)

Cohomology theories factorize:

$$\begin{tikzcd} \mathrm{Top} \arrow[rr, "H^*_{sing}"] \arrow[d] & & (\mathrm{Ab}^{\mathbb{Z}})^{op} \\ \mathrm{Ho(Top)} \text{ "}\simeq\text{" } \mathrm{Top}[(X\times I\to X)^{-1}] =: \text{"Spaces"} \arrow[d, "(-)_+"] & & H\mathbb{Z}\text{-Mod} \simeq D(\mathbb{Z}\text{-Mod}) \arrow[u, "\text{coh. of chain complex}"'] \\ \mathrm{Spaces}_* \arrow[r, "\Sigma^\infty"] & \mathrm{Spaces}\langle\Sigma^{-1}\rangle =: \mathrm{Spectra} \arrow[ur] & \end{tikzcd}$$

e.g.

$MU, KU, H\mathbb{Z}$

„additive cat" - triangulated

have addition in $[\Sigma X, Y]$

$$H^n_{sing} = [\Sigma^n\Sigma^\infty X_+, H\mathbb{Z}]$$

$$\| $$

$$[H\mathbb{Z}\wedge(\Sigma^n\Sigma^\infty X_+), H\mathbb{Z}]_{H\mathbb{Z}\text{-Mod}}$$

$f: \Diamond \to Y$, $g: \Diamond \to Y$

2. Localisation of categories

a) Localisation $\mathcal{C}$ cat. $W \subseteq \mathcal{C}$ class of morphisms ("a relative category")

Def.: A (/the) localisation of $\mathcal{C}$ at W is a functor L:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "L"] \arrow[dr, "F"'] & \mathcal{C}[W^{-1}] \arrow[d, dashed, "\exists ! \bar{F}"] \\ & \mathcal{D} \end{tikzcd}$$

s.t. $\forall$ F s.t. $F(W) \subset \mathrm{Iso}(\mathcal{D})$

Examples: $\mathrm{Top} \longrightarrow \mathrm{Top}[\{X \times I \xrightarrow{pr} X \mid X \in \mathrm{Top}\}^{-1}] = \mathrm{Ho}(\mathrm{Top})$

$\mathcal{C} = \mathrm{Top}$, $W = \{f: X \to Y \text{ s.t. } \bigstar\}$

where $\bigstar := \begin{cases} \exists g: Y \to X \text{ s.t. } f \circ g \sim id_Y,\ g \circ f \sim id_X & \text{[hty equiv.]} \\ \pi_n(f): \pi_n(X) \to \pi_n(Y) \text{ iso } \forall n \\ E_n(f): E_n(X) \to E_n(Y) \text{ iso } \forall n, & E \text{ some cohomology theory} \end{cases}$

or $\mathcal{C} = \mathrm{Groupoids}$, $W = \{\text{eq.s of groupoids}\}$

Different $(\mathcal{C}, W), (\mathcal{D}, V)$ can give rise to the "same" localized cat: $\mathrm{Top} \underset{|\cdot|}{\overset{\mathrm{Sing}(\cdot)}{\rightleftarrows}} \mathrm{Set}^{\Delta^{op}}$ $\begin{matrix} \vdots \\ \downdownarrows \\ X_1 \\ \downdownarrows \\ X_0 \end{matrix}$

b) Derived functors

Given $(\mathcal{C}, W), (\mathcal{D}, V)$ rel. cats and $F: \mathcal{C} \to \mathcal{D}$, if $F(W) \subseteq V$ (or $F(W) \subseteq \mathrm{Iso}(\mathcal{D}[V^{-1}])$), then there is an induced functor:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "L"'] & \mathcal{D} \arrow[d, "L'"] \\ \mathcal{C}[W^{-1}] \arrow[r, dashed, "\bar{F}"] & \mathcal{D}[V^{-1}] \end{tikzcd}$$

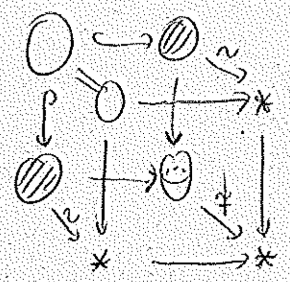
Often F does not satisfy this.
Then we take $\bar{F}$ to be the "best" approximation to an equivalence preserving functor:

$$\begin{tikzcd} \tilde{\mathcal{C}} \arrow[r, phantom, "\subseteq"] \arrow[d, "L"'] & \mathcal{C} \arrow[r, "F"] \arrow[d, "L"'] & \mathcal{D} \arrow[d, "L'"] \\ \tilde{\mathcal{C}}[W_{\tilde{\mathcal{C}}}^{-1}] \arrow[r, phantom, "\simeq"] & \mathcal{C}[W^{-1}] \arrow[r, dashed, "\bar{F}"'] \arrow[ur, phantom, "\Swarrow"] & \mathcal{D}[V^{-1}] \end{tikzcd}$$

(left/right Kan extension of $L' \circ F$ along L)

Construction: Find subcat $\tilde{\mathcal{C}} \subseteq \mathcal{C}$ s.t.
- $F|_{\tilde{\mathcal{C}}}$ preserves weq.s
- Every obj. in $\mathcal{C}$ is weq to one in $\tilde{\mathcal{C}}$

Example: $\mathrm{Top}^{\downarrow\rightarrow} \xrightarrow{\text{pushout}} \mathrm{Top}$



Fact: pushout is invariant under weak eqs for diagrams of the form

$$\begin{tikzcd} a \arrow[r, tail] \arrow[d, tail] & b \\ c & \end{tikzcd}$$

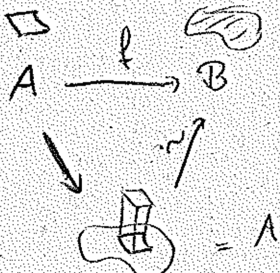
cofibrations

e.g. $\bigcirc \hookrightarrow$ 

=:

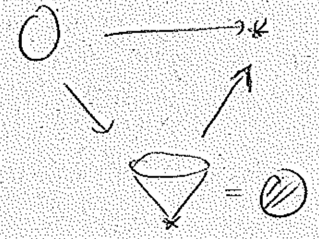
So: Construction of homotopy pushouts:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & \tilde{B} \arrow[r, "\sim"] \arrow[d] & B \\ \tilde{C} \arrow[r] \arrow[d] & P & \\ C & & \end{tikzcd}$$



$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ \bullet \arrow[ur, "\sim"'] & \end{tikzcd}$$

$= A \times I \sqcup B /_{(a,0) \sim f(a)}$

e.g. 

$\bigcirc \longrightarrow *$

$= \oslash$

So homotopy pushout of $\begin{tikzcd} S^n \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}$ is S^{n+1}

more generally: $$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner h"] & \Sigma X \end{tikzcd}$$

$\rightsquigarrow$ gives

$$\begin{tikzcd} \mathrm{Top}^{\downarrow\rightarrow} \arrow[r] \arrow[d] & \mathrm{Top} \arrow[d] \\ (\mathrm{Top}^{\downarrow\rightarrow})[W^{-1}] \arrow[r, dotted] \arrow[d, phantom, "\not\simeq"] & \mathrm{Top}[W^{-1}] \\ (\mathrm{Top}[W^{-1}])^{\downarrow\rightarrow} & \end{tikzcd}$$

Other example:

$$\begin{tikzcd} & R\text{-Mod} \arrow[r, "S\otimes_R -"] \arrow[d, phantom, "\cap"] & S\text{-Mod} \arrow[d, phantom, "\cap"] \\ \mathrm{Flat}_R \subseteq \arrow[d] & \mathrm{Chcomplex}_R \arrow[r, "S\otimes_R -"] \arrow[d] & \mathrm{Chcomplex}_S \arrow[d] \\ \mathrm{Flat}_R[H_*\text{-iso}^{-1}] \simeq & \mathrm{Ch}_R[H_*\text{-iso}^{-1}] \arrow[d, equal] & \mathrm{Ch}_S[H_*\text{-isos}^{-1}] \arrow[d, equal] \\ & D(R\text{-Mod}) \arrow[r, "S\otimes^{\mathbb{L}}_R -"] & D(S\text{-Mod}) \end{tikzcd}$$

3rd **Example**: Homotopy quotients

$$\begin{tikzcd} G\text{-Top} \arrow[r, "-/G"] \arrow[d] & \mathrm{Top} \arrow[d] \\ G\text{-Top}[G\text{-equiv. weq.s}^{-1}] \arrow[r, dashed, "-/_hG"] & \mathrm{Top}[\mathrm{weq}^{-1}] \end{tikzcd}$$

e.g. $G = \mathbb{Z}/2$

$$\begin{tikzcd} X := S^\infty := \mathrm{colim}(S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow \dots) \arrow[d] \\ * \end{tikzcd}$$

contractible / weq. to a point

$\pi_n(S^\infty) = [S^n, S^\infty]$

$0 \sim \tilde{f} \leftarrow f$

$\nearrow [S^n, S^{k+1}]$

$f \in [S^n, S^k]$

$S^\infty/_{\mathbb{Z}/2} \cong \mathbb{RP}^\infty = B\mathbb{Z}/2, \quad \pi_1(\mathbb{RP}^\infty) = \mathbb{Z}/2$

$*/_{\mathbb{Z}/2} \simeq * \qquad \pi_1(*) = 0.$

Fact: $-/G$ preserves equivalences on the subcat of free G-actions.

Can always replace a G-space X by an equivalent such G-space

$$\begin{tikzcd} G\times G\times G\times X \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \vdots \\ G\times G\times X \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & X \arrow[u, equal] \arrow[d, equal] \\ G\times X \arrow[r] & X \end{tikzcd}$$

in $\mathrm{Top}^{\Delta^{op}}$, then take geom. realization,

Example: $X = * \quad G\times X = \mathbb{Z}/2, \quad G\times G\times X = \mathbb{Z}/2\times\mathbb{Z}/2$

geom. real:

...

Classifying spaces

Fact: $*/_hG \simeq BG \simeq \left|\begin{matrix} \vdots \\ G\times G \\ \downarrow\downarrow\downarrow \\ G \\ \downarrow\downarrow \\ * \end{matrix}\right|$

$= EG/G =$

$EG = \left|\begin{matrix} G\times G\times G \\ \downarrow\downarrow\downarrow \\ G\times G \\ \downarrow\downarrow \\ G \end{matrix}\right|$

In $\mathrm{Top}^{\Delta^{op}}$: cocycle conditions

$$\begin{tikzcd} \vdots & \vdots \\ \amalg U_i\cap U_j\cap U_k \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \arrow[r] & G\times G \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\ \amalg U_i\cap U_j \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & G \arrow[d, shift left] \arrow[d, shift right] & \text{transition functions} \arrow[l] \\ X \simeq \amalg U_i \arrow[r] & * \simeq BG \end{tikzcd}$$

Remember: $\mathrm{Princ}_G(X) \cong [X, BG]$

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EG \arrow[d] \\ X \arrow[r] & BG \end{tikzcd} \longleftarrow\!\shortmid (X \to BG)$$

EG contractible with free G-action

Homotopy limits: $\mathrm{Top}^{\lrcorner} \xrightarrow{\text{pullback}} \mathrm{Top}$

Replace diagrams by fibrations, e.g.

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & * \arrow[d] & & \Omega X \arrow[r] \arrow[d] & PX \arrow[d, "ev_1"] \\ * \arrow[r] & X & \rightsquigarrow & * \arrow[r, "x"] & X \ni x \end{tikzcd}$$

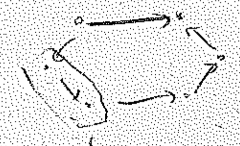
path space

$PX = \{I \xrightarrow{p} X \mid p(0) = x\}$

$\Omega X = \{I \xrightarrow{p} X \mid p(0) = p(1) = x\}$

Thus: $\mathrm{holim}(\cdot \to \overset{\overset{\cdot}{\downarrow}}{X}) = \Omega X$

3. Stabilization

Remember:  $H_{*+1}(\Sigma X) \cong H_*(X)$

Have functor $\text{Spaces} \longrightarrow \text{Spaces}_* = \text{Top}_*[W^{-1}]$

$X \longmapsto X_+ := X \amalg \{pt\}$

$W := \{$base-point preserving maps which are weqs$\}$

An adjunction on Spaces_*: $\Sigma \dashv \Omega$

$$\begin{tikzcd}
\text{Top}_*(X\times I, Y) \arrow[r, phantom, "\cong"] & \text{Top}_*(X, Y^I) \\
\text{Top}_*(\Sigma X, Y) \arrow[u, phantom, "\cup" rotate] \arrow[r, phantom, "\cong"] \arrow[d] & \text{Top}_*(X, \Omega X) \arrow[u, phantom, "\cup"] \arrow[d] \\
\text{Spaces}_*(\Sigma X, Y) \arrow[d, phantom, "\shortparallel"] & \text{Spaces}_*(X, \Omega Y) \arrow[d, phantom, "\shortparallel"] \\
{[\Sigma X, Y]} \arrow[r, phantom, "\cong"] & {[X, \Omega Y]}
\end{tikzcd}$$

$$\begin{tikzcd}
X\times I \arrow[r] \arrow[d] & Y \\
X\times I/\sim \arrow[ur] &
\end{tikzcd}$$

Want to make Σ invertible [Remember: $H^{*+1}(\Sigma X) \cong H^*(X)$].

~~Def.: $\mathcal{C}$ cat., $F: \mathcal{C} \to \mathcal{C}$ functor.~~

~~The stabilization of $\mathcal{C}$ wrt F is a functor~~

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "S"] \arrow[dr, "\forall T"'] & \mathcal{C}' \arrow[d, dashed, "\exists!"] \\
& \mathcal{D}, \mathcal{G}
\end{tikzcd}$$

~~$\mathcal{C}'$ cat w/ functor $F': \mathcal{C}' \to \mathcal{C}'$ s.t. $S \circ F \cong F' \circ S$~~ equivalence

~~s.t. $T \circ F = G \circ T$~~

Consider the following cat: **Obj:** $(\mathcal{C}, F)$ $F: \mathcal{C} \to \mathcal{C}$ endofunctor

Mor.: $(\mathcal{C}, F) \longrightarrow (\mathcal{D}, G)$

functors $\mathcal{C} \xrightarrow{T} \mathcal{D}$ s.t. $T \circ F \cong G \circ T$

~~[illegible]~~

Example: $(\text{Top}_*, \Sigma) \longrightarrow (Ab^{\mathbb{Z}}, \text{shift})$

(or $(\text{Spaces}_*, \Sigma)$)

More precisely: Will consider the $(\infty,1)$-category resp. $(2,1)$- Pres^L, where all cats $\mathcal{C}$ are presentable and functors T are left adjoints

Def.: $(\mathcal{C},F)$ cat. w/ endofunctor.

The stabilization of $(\mathcal{C},F)$ is a map! $(\mathcal{C},F) \longrightarrow (\mathcal{C}',F')$

s.t. F' is an equivalence and s.t. $\forall$

$$\begin{tikzcd} (\mathcal{C},F) \arrow[r] \arrow[dr] & (\mathcal{C}',F') \arrow[d, dashed, "\exists!"] \\ & (\mathcal{D},G) \end{tikzcd}$$

G equivalence

How to construct the stabilization?

Localization of modules

R ring, $M \in R\text{-Mod}$, $r \in R$.

$M[r^{-1}] := \operatorname{colim}(M \xrightarrow{\cdot r} M \xrightarrow{\cdot r} M \xrightarrow{\cdot r} M \xrightarrow{\cdot r} \dots) = \bigoplus_{n \in \mathbb{N}} M \big/ (r \cdot m)_{n+1} \sim m_n$

Univ. prop.:

$$\begin{tikzcd} M \arrow[r, "\ell"] \arrow[dr, "f"'] & M[r^{-1}] = \operatorname{colim}(M \xrightarrow{\cdot r} M \xrightarrow{\cdot r} M \to \dots) \arrow[d, dashed, "\exists!"] \\ & N \end{tikzcd}$$

f, $r^{-1}f$, $r^{-2}f$

N s.t. $(-\cdot r)$ is invertible on N

here: $r^{-1} \cdot m_n = m_{n+1}$

Stabilization

Spaces_* is an "$(\mathbb{N},+)$-module" with $\underset{1}{n}$ acting as $\underset{\Sigma}{\Sigma^n}$.

$\mathrm{Spaces}_*\langle\Sigma^{-1}\rangle := \operatorname{colim}(\mathrm{Spaces}_* \xrightarrow{\Sigma} \mathrm{Spaces}_* \xrightarrow{\Sigma} \dots)$ colim taken in $\mathrm{Pr}^L \simeq (\mathrm{Pr}^R)^{op}$

$= \lim(\mathrm{Spaces}_* \xleftarrow{\Omega} \mathrm{Spaces}_* \xleftarrow{\Omega} \dots)$ in Pr^R

Remember: Limit of sets, $\lim(X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \leftarrow \dots) = \{(a_i) \in \prod_{i \in \mathbb{N}} X_i \mid f_i(a_i) = a_{i-1}\}$

(ho)Limit in Cat/Pr^R:

Objs: $\{(X_i)_{i \in \mathbb{N}} \mid X_i \in \mathrm{Spaces}_*,\ e_i: X_{i-1} \underset{weq}{\xrightarrow{\sim}} \Omega X_i\}$

Mor: $g: (X_i,e_i) \longrightarrow (Y_i,f_i)$

given by $g_i: X_i \to Y_i$

s.t. $\downarrow \circlearrowleft \downarrow$

$\Omega g_{i+1}: X_{i+1} \to Y_{i+1}$

Weak eq: (g_i) where all g_i are weqs in Spaces_*.

Def.: $\mathrm{Spectra} := \mathrm{Spaces}_*\langle\Sigma^{-1}\rangle$

Σ becomes shift: $(X_i,e_i) \longmapsto (X_{i+1},e_{i+1})$

Inverse is levelwise Ω: $(X_i,e_i) \longmapsto (\Omega X_i, \Omega e_i)$

Brown representability thm.:

If a functor $F: \mathrm{Ho}(\mathrm{Top}_*)^{op} \longrightarrow \mathrm{Set}$ satisfies

(1) $F(\bigvee_\alpha X_\alpha) \cong \prod_\alpha F(X_\alpha)$

and (2) $F\left(\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "s"] \\ C \arrow[r, "r"'] \arrow[ur, phantom, "\ulcorner h"] & D \end{tikzcd}\right) = \begin{tikzcd} F(D) \arrow[r, "s^*"] \arrow[d, "r^*"'] & F(B) \arrow[d, "i^*"] \\ F(C) \arrow[r, "j^*"'] & F(A) \end{tikzcd}$ weak pullback, i.e. $\forall \begin{smallmatrix} b \in F(B) \\ c \in F(C) \end{smallmatrix}$ s.t. $j^*(c) = i^*(b)$

$\exists\, d \in F(D)$ s.t. $r^*(d) = c$
$s^*(d) = b$

Then $\exists\, Z \in \mathrm{Top}_*$ s.t. $F(-) \cong [-, Z] \left(= \mathrm{Ho}(\mathrm{Top}_*)(-, Z)\right)$

[Lurie: HA 1.4.1.2 $\mathcal{C}$ presentable quasicat., $\{S_\alpha\}$ set of objects generating $\mathcal{C}$ under colims, each S_α compact, each S_α cogroup object in $\mathrm{Ho}(\mathcal{C})$.
Then $F: \mathrm{Ho}(\mathcal{C})^{op} \longrightarrow \mathrm{Set}$ is representable iff the above conditions hold]

In particular, if E^* is a cohomology theory, then each $E^n(-)$ satisfies (1), (2)

$$\Rightarrow \quad E^n(X) \underset{\substack{\uparrow \\ \text{Brown} \\ \text{rep.}}}{\cong} [X, E_n] \underset{\substack{\uparrow \\ \text{suspension} \\ \text{iso}}}{\cong} [\Sigma X, E_{n+1}] \underset{\substack{\uparrow \\ \Sigma \dashv \Omega}}{\cong} [X, \Omega E_{n+1}] \quad \text{nat isos.}$$

$\Rightarrow \exists$ weak eq. $e_i: E_n \xrightarrow{\sim} \Omega E_{n+1}$ [in particular all E_n are infinite loop spaces]

$\rightsquigarrow$ spectrum (E_i, e_i)

Thm: A cohomology theory determines, and is determined by, a spectrum

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More about the category of spectra:

(a) $\underset{[}{\mathrm{Ho}(\mathrm{Spectra})(X, Y)}$ is an abelian group: Use levelwise $\Omega E_n \times \Omega E_n \to \Omega E_n$ loop composition.

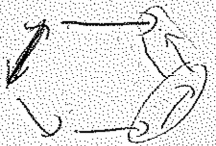
(b) have $\mathbb{Z}$-graded Hom groups: $[\![X, Y]\!] := \bigoplus_{n \in \mathbb{Z}} \mathrm{Ho}(\mathrm{Spectra})(\Sigma^n X, Y)$, ~~e.g. $[\![\Sigma^\infty X, E]\!] = E^*(X)$~~

(c) cohomology theories become representable: $[\![\Sigma^\infty X, E]\!] = E^*(X)$

e.g. $H^*_{sing}(X; \mathbb{Z})$ rep. by $(\mathbb{Z}, B\mathbb{Z}, B^2\mathbb{Z}, \dots)$ ($X \xrightarrow{\sim} \Omega BX$ equiv. for groups X)

$K^*_{top, \mathbb{C}}(X; \mathbb{Z})$ rep. by $(BU, \Omega BU, \overset{\simeq \Omega BU}{BU}, \Omega BU)$ $\boxed{\text{Bott periodicity}}$

(d) Spectra is monoidal and Σ^∞ is monoidal w.r.t. $\wedge$



Some cohomology theories have a $\overset{\text{graded}}{\text{ring}}$ structure, e.g. H_{sing}, $K_{\mathbb{C},top}$

$\cup$ = intersection (or $\wedge$ of diff. forms) ⊗ of v.bdls.

This corresponds to a ~~ring~~ monoid structure $E\wedge E\xrightarrow{\mu}E$ — a „ring spectrum" $\underset{\text{unit}}{\mathbb{S}\to E}$

$$\begin{tikzcd}
E^i(X)\times E^j(X) \arrow[r, phantom, "="] \arrow[dd, "\text{mult}"] & {[\Sigma^i\Sigma^\infty X_+, E]\times[\Sigma^j\Sigma^\infty X_+, E]} \arrow[dr, "\wedge"] & \\
 & & {[\Sigma^\infty\Sigma^i(X_+)\wedge\Sigma^\infty\Sigma^j(X_+), E\wedge E]} \arrow[d, equal] \\
E^{i+j}(X) \arrow[r, phantom, "="] & {[\Sigma^{i+j}\Sigma^\infty X_+, E]} & {[\Sigma^\infty\Sigma^{i+j}(X_+\wedge X_+), E\wedge E]} \arrow[d, equal] \\
 & & {[\Sigma^\infty\Sigma^{i+j}(X\times X)_+, E\wedge E]} \arrow[ul, "\mu\circ-\circ\Sigma^\infty\Sigma^{i+j}(X\xrightarrow{\Delta}X\times X)"']
\end{tikzcd}$$

Def: E ring spectrum.

An E-module is a spectrum M w/ „scalar multiplication map" $E\wedge M\xrightarrow{m}M$ satisfying associativity & unit axioms

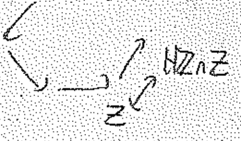
e.g.

$$\begin{tikzcd}
M\cong\mathbb{S}\wedge M \arrow[r] \arrow[dr, equal] & E\wedge M \arrow[d, "m"] \\
 & M
\end{tikzcd}$$

Category of E-modules $=: E\text{-Mod}$.

Free E-module over a spectrum Z: $E\wedge Z$

Obs.: The coh. theory $E^*(-)$ factors through E-Mod, in particular $H^*_{sing}(X;\mathbb{Z})\cong[\![\Sigma^\infty X_+, H\mathbb{Z}]\!]\cong[\![H\mathbb{Z}\wedge\Sigma^\infty X_+, H\mathbb{Z}]\!]_{H\mathbb{Z}\text{-Mod}}$



Facts: • $Ho(H\mathbb{R}\text{-Mod})\simeq D(\mathbb{R}\text{-Mod}) = \mathrm{Chcomplx}_{R\text{-Mod}}[H^*_*\text{-isos}]$

• $H\mathbb{Z}\wedge\Sigma^\infty X_+\simeq$ cellular complex / singular complex of X (in $D(\mathbb{Z}\text{-Mod})$))

II. $4\frac{1}{2}$

$$\begin{tikzcd} \text{CRing} \arrow[d] \\ \text{e.g. } \mathbb{Z}\text{-Alg} \arrow[d] \\ R\text{-Alg} \end{tikzcd}$$

Schemes$_{/R}$: Functors $R\text{-Alg} \longrightarrow \text{Set}$ satisfying axioms (Zariski sheaf + "good" covering by representables) II Hom(A ↓ F

Exs: (a) "circle": $R \longmapsto \{(r,s) \in R^2 \mid r^2 + s^2 = 1\}$

(b) affine line $\mathbb{A}^1$: $R \longmapsto R$ (as set) $\qquad \mathbb{A}^n : R \mapsto R^n$

(c) $\mathbb{G}_m$: $R \longmapsto R^* \cong \{(x,y) \in R^2 \mid x \cdot y = 1\}$

$x \longmapsto (x, x^{-1})$

(d) $\mathbb{A}^n \setminus 0$: $R \longmapsto \{(a_1, \dots, a_n) \in R^n \mid \exists i : a_i \text{ is a unit}\}_{Zar}$

Zariski sheafification: checking the defining condition of the presheaf on each of a collection of localizations at coprime ideals

e.g. $(\mathbb{A}^2 \setminus 0)(\mathbb{Z}) \ni (3,5)$

because $\mathbb{Z} \rightrightarrows \mathbb{Z}[\frac{1}{3}], \mathbb{Z}[\frac{1}{5}]$ is covering, as $\gcd(3,5) = 1$

$(\underset{\text{unit}}{3}, \overset{\text{unit in } \mathbb{Z}[\frac{1}{5}]}{5})$ in $\mathbb{Z}[\frac{1}{3}]$

(but $(\mathbb{A}^2 \setminus 0)(\mathbb{Z}) \not\ni (5,10)$)

Covering:

$$\begin{tikzcd} \mathbb{G}_m \times \mathbb{G}_m \arrow[r, hook] \arrow[d, hook] & \mathbb{G}_m \times \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \times \mathbb{G}_m \arrow[r] & \mathbb{A}^2 \setminus 0 \end{tikzcd}$$

intersection of the subpresheaves is given by a localization, i.e. by requiring more coordinates to be units.

"Diagonal" action $\mathbb{G}_m \times \mathbb{A}^n \setminus 0 \longrightarrow \mathbb{A}^n \setminus 0$

(e) $\mathbb{P}^{n-1} := \mathbb{A}^n \setminus 0 / \mathbb{G}_m$, points are $[a_0 : \dots : a_{n-1}]$

usual covering by $\mathbb{A}^{n-1}$'s ..

Field of definition: Coefficients of the defining polynomials

e.g. deformed circle : $\mathbb{R}\text{-Alg} \longrightarrow \text{Set}$

$R \longmapsto \{(r,s) \mid r^2 + \pi \cdot s^2 = 1\}$

"defined over $\mathbb{R}$"

but circle "defined over $\mathbb{Z}$".

Motivic homotopy theory

II.5 = III.1

(or CRing^{op} → Sm)

$$\begin{tikzcd}
 & \mathrm{Sm}_{/S} \arrow[rrr] \arrow[dl, "Y"'] & & & (\mathrm{Ab}^{\mathbb{Z}})^{op} \\
\mathrm{MotSpaces}_{/S} := \mathrm{sSet}^{\mathrm{Sm}_S^{op}}_{\mathrm{Nis}/\mathrm{Zar}}[X\times\mathbb{A}^1 \to X]^{-1} \arrow[dr, "(-)_+"'] & & & & \mathrm{M}\mathbb{Z}\text{-Mod} =: \mathrm{Motives} \arrow[u, dashed, shift left] \arrow[u, dashed, shift right] \\
 & \mathrm{MotSpaces}_* \arrow[rr, "\Sigma^\infty"] & & \mathrm{MotSpaces}\langle S^1\wedge -, \mathbb{G}_m\wedge -\rangle =: \mathrm{MotSpectra} \arrow[ur] &
\end{tikzcd}$$

$\cong$

$(\mathrm{MotSpaces}\langle \mathbb{P}^1\wedge -\rangle)$

$\$, K, \mathrm{M}\mathbb{Z} / \mathrm{M}\mathbb{Q}$

~~Schemes.~~

Plan: 1. Abstract motivic htpy theory (no individ. algebraic feats from - usual mot. htpy thy / - usual htpy thy)
2. Stable motivic htpy groups & quadratic forms.

Abstract mot. htpy theory: Replace $(\mathrm{MotSpaces}_{/S}, \mathbb{G}_m)$ with $(\mathcal{E}, \mathbb{G})$

where $\mathcal{E}$ is a presentable cart. closed ∞-cat.
$\mathbb{G} \in \mathcal{E}$ is a commutative group object.

[in terms of model cats: $\mathcal{C}$ a combinatorial model cat, monoidal for the product monoidal struct.
$\mathbb{G}$ a grouplike E_∞-algebra in $\mathcal{E}$]

Examples: 0. $(\mathrm{Top}[(X\times I \to X)^{-1}]$, S^1 w/ $\mathbb{C}$-multiplication or $\mathbb{C}^*$)

1. ($\mathrm{MotSpaces}_{/\mathbb{C}}$, $\mathbb{G}_m$ (i.e. $y(\mathbb{G}_m)$))

2. replace $\mathrm{Sm}_{/S}$ by
- analytic spaces (Berkovich sp. / $\mathbb{C}$-analytic spaces)
- C^∞-manifolds
- derived ~~aff.~~ schemes / spectral schemes
- log schemes
- $\mathbb{F}_1$-schemes (e.g. blue schemes)

pass to simplicial presheaves, sheafify wrt appropriate topology, contract affine line.

$\mathbb{G}$:= units of the structure sheaf (for ringed spaces)

3. $\mathbb{Z}/2$-Top, $\mathbb{G} = S^1$ w/ $\mathbb{C}^*$-multipl. & conjugation action

4. Universal example: Spaces free grouplike E_∞-Algs

Computing in MotSpaces

III.2

Should also derive $\mathrm{Hom}(-,Y) : \mathrm{MotSpaces}^{op} \longrightarrow \mathrm{Set} / \underset{\mathrm{sSet}}{\mathrm{Spaces}}$

$\rightsquigarrow$ replace Y by „good" equivalent object

$\overset{\text{def}}{\Leftrightarrow}$ Nis/Zar-sheaf & $\mathbb{A}^1$-rigid.   enough to check on X affine

$$\mathrm{Hom}(X,Y) \xrightarrow{\cong} \mathrm{Hom}(X\times\mathbb{A}^1, Y)$$

(Counter-)Examples:
- $\mathbb{G}_m$ is good; Zar-sheaf because morphisms glue

  $\mathbb{A}^1$-rigid: $\mathrm{Hom}(R, \mathbb{G}_m) \cong \mathbb{G}_m(R) \to \mathbb{G}_m(R[x])$

  (need smooth schemes: $(1+ax)(1-ax) = 1-\underset{=0}{a^2}x$) $\quad \overset{\shortparallel}{R^*} \qquad \overset{\shortparallel}{(R[x])^*}$

- proj. curves of genus >0 are good: $\mathbb{P}^1 \overset{\nexists}{\dashrightarrow} C$, $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$, $\mathbb{A}^1 \nearrow C$ $\quad g(\mathbb{P}^1)=0 < g(C)$ ↯

- $\mathbb{P}^1$ is not good; $\exists$ non-constant map $\mathbb{A}^1 \to \mathbb{P}^1$

- Have groupoid valued functor $\mathrm{Sm}/_S^{op} \longrightarrow \mathrm{Gpd}$

$$X \longmapsto \begin{pmatrix} \mathbb{G}_m(X) \\ \downarrow\downarrow \\ \{*\} \end{pmatrix} \simeq \text{cat of } \underline{\text{trivial}} \text{ line bdls \& automorphisms}$$

Take nerve $\rightsquigarrow$ $\mathrm{Sm}/_S^{op} \longrightarrow \mathrm{sSet}$

$$\begin{matrix} \mathbb{G}_m\times\mathbb{G}_m \\ \downarrow\downarrow\downarrow \\ \mathbb{G}_m \\ \downarrow\downarrow \\ * \end{matrix} = B\mathbb{G}_m$$

$\mathbb{A}^1$-rigid, because $\mathbb{G}_m$ ist,
but not a sheaf

sheafify: $\mathrm{Sm}/_S^{op} \longrightarrow \mathrm{sSet}$

$X \longmapsto \underline{[illegible]^{iso}}(X)$

$$\begin{tikzcd}[row sep=small]
\coprod_{ijk} U_i\cap U_j\cap U_k \arrow[d] \arrow[d, shift left] \arrow[d, shift right] \arrow[r] & X \arrow[d, equal] \arrow[rr, bend left] & & \mathbb{G}_m\times\mathbb{G}_m \arrow[d] \arrow[d, shift left] \arrow[d, shift right] \\
\coprod_{ij} U_i\cap U_j \arrow[d, shift left] \arrow[d, shift right] \arrow[r] & X \arrow[d, equal] \arrow[rr, bend left] \arrow[r] & {} & \mathbb{G}_m \arrow[d, shift left] \arrow[d, shift right] \\
\coprod_i U_i \arrow[r] & X \arrow[rr, bend left] & & *
\end{tikzcd}$$

eq.

commutativity = cocycle condition

transition maps

Def.: $\mathrm{Pic}(X) := \pi_0(\underline{\mathrm{LineBdl.}}^{iso}(X)) = \{\text{iso classes of line bdls}\}$

$\mathrm{Pic}(X)$ is not representable, otherwise it would be a sheaf.

Thm. (Morel-Voevodsky): $\mathrm{Pic}(X) = [X, \mathbb{P}^\infty]$

$\mathbb{P}^\infty = \mathrm{colim}(\mathbb{P}^1 \hookrightarrow \mathbb{P}^2 \hookrightarrow \dots)$

$[x_0:x_1] \mapsto [x_0:x_1:0]$

Abstract mot. htpy theory

$$\begin{tikzcd}
\mathrm{Sm}/S \arrow[rr] \arrow[d] & & (\mathcal{A}^{\mathbb{Z}})^{op} \\
\mathbb{G}_m \in \mathrm{MotSpaces} \arrow[d] & & H\mathbb{Z}\text{-Mod} \arrow[u] \\
\mathrm{MotSpaces}_* \arrow[r] & \mathrm{MotSpaces}_*\langle S^1\wedge -, \mathbb{G}_m\wedge -\rangle \arrow[ur] \arrow[d, phantom, "\shortparallel"] & \\
{} \arrow[uu] & \mathrm{MotSpectra} &
\end{tikzcd}$$

Starting point: Replace MotSpaces by a presentable, cartesian closed $(\infty,1)$-category $\mathcal{C}$ and $\mathbb{G}_m$ by a commutative group object $\mathbb{G}$ in there

Equivalently: $\mathcal{C}$ a cartesian combinatorial model category, $\mathbb{G}$ a grouplike E_∞-algebra in $\mathcal{C}$

Examples for the abstract setup:

0. $\mathcal{E} = \mathrm{Top}$, $\mathbb{G} = S^1$ w/ $\mathbb{C}$-multiplication.
1. $\mathcal{E} = \mathrm{MotSpaces}$, $\mathbb{G} = \mathbb{G}_m$
2. Motivic spaces from other geometries:

$$\left.\begin{array}{l}\text{Start with some geometric setting,}\\ \text{pass to simplicial presheaves,}\\ \text{sheafify, contract } \mathbb{A}^1\end{array}\right\} \mathcal{E}$$

e.g.
- analytic geometry (complex analytic / Berkovich / ...)
- derived / spectral alg geom.
- log geometry
- tropical geometry
- C^∞-manifolds
- $\mathbb{F}_1$-geometries (e.g. monoid schemes, Durov's / Haran's settings, Borger's Λ-rings, Connes-Consani, Lorscheid's blue schemes ...)

Most of these are ringed spaces. Then take $\mathbb{G}$ to be the sheaf of units of the ring

In the case of $\mathbb{F}_1$-geometry we always have an underlying monoid - take $\mathbb{G}$ to be the units of that monoid.

3. $\mathbb{Z}/2$-equivariant spaces: $\mathcal{E} = \mathbb{Z}/2\text{-Top}$
$\mathbb{G} = S^1$ with $\mathbb{C}$-multiplication and conjugation action.

4. Universal example: $\mathrm{Spaces}^{\text{fin. gen. free } E_\infty\text{-Algs}}$, $\mathbb{G}$ = underlying space of the E_∞-Alg

- every pair $(\mathcal{E}, \mathbb{G}')$ as above receives a product preserving left adjoint from this one, sending $\mathbb{G}$ to $\mathbb{G}'$. All subsequent constructions will be preserved by this functor.

Unstable results & constructions:

Punctured affine spaces: In alg. geom. have $(\mathbb{A}^n\setminus 0)(R) = \{(r_1,\dots,r_n) \mid \exists i : r_i \text{ invertible}$

Colimit description:
in Schemes:

$$\begin{tikzcd}
\mathbb{G}_m^3 \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{G}_m^2\times\mathbb{A}^1 \arrow[dr] \arrow[dd] & \\
& \mathbb{G}_m\times\mathbb{A}^1\times\mathbb{G} \arrow[rr, crossing over] & & \mathbb{G}_m\times(\mathbb{A}^1)^2 \arrow[dd] \\
\mathbb{A}^1\times\mathbb{G}^2 \arrow[rr] \arrow[dr] & & \mathbb{A}^1\times\mathbb{G}\times\mathbb{A}^1 \arrow[dr] & \\
& (\mathbb{A}^1)^2\times\mathbb{G} \arrow[rr] \arrow[from=uu, crossing over] & & \mathbb{A}^3\setminus 0
\end{tikzcd}$$

actually in schemes w/ $\mathbb{G}_m$-action
(all maps are $\mathbb{G}_m$-equivariant)

In MotSpaces: $\mathbb{A}^1 \simeq pt$; can delete the $\mathbb{A}^1$-factors

In an abstract setting define $\mathbb{A}^3\setminus 0$ as a colimit in $\mathbb{G}$-$\mathcal{E}$, i.e. in $\mathcal{E}$-objects with $\mathbb{G}$-action:

$$\begin{tikzcd}
\mathbb{G}^3 \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{G}^2 \arrow[dr] \arrow[dd] & \\
& \mathbb{G}^2 \arrow[rr, crossing over] & & \mathbb{G} \arrow[dd] \\
\mathbb{G}^2 \arrow[rr] \arrow[dr] & & \mathbb{G} \arrow[dr] & \\
& \mathbb{G} \arrow[rr] \arrow[from=uu, crossing over] & & \mathbb{A}^3\setminus 0
\end{tikzcd}$$

(Formally: $\mathbb{A}^n\setminus 0$ is the n-fold Day convolution product of $\mathbb{G}\to 1$ in $(\mathbb{G}$-$\mathcal{E})$

Have inclusions $\mathbb{A}^{n-1}\setminus 0 \xhookrightarrow{i_{n-1}} \mathbb{A}^n\setminus 0$ coming from taking the colimit along just one side of the cube.
Similarly have pushout square

$$\begin{tikzcd}
(\mathbb{A}^n\setminus 0)\times\mathbb{G} \arrow[r, "pr"] \arrow[d, "pr"'] & \mathbb{A}^n\setminus 0 \arrow[d, "i_n"] \\
\mathbb{G} \arrow[r] & \mathbb{A}^{n+1}\setminus 0
\end{tikzcd}$$

Can now define: $\mathbb{P}^n := \mathbb{A}^{n+1}\setminus 0/\mathbb{G}$

<u>Lemma</u>: $X \xrightarrow{f} Y$, $Z \xrightarrow{g} Y$, $1 \xrightarrow{p} X$ morphisms,

$$\begin{tikzcd}
Z\times X \arrow[r, "pr"] \arrow[d, "pr"'] & Z \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commutative diagram

Then g is constant, i.e. factors as
$$\begin{tikzcd}
Z \arrow[rr, "g"] \arrow[dr] & & Y \\
& 1 \arrow[ur] &
\end{tikzcd}$$

<u>Proof</u>:

$$\begin{tikzcd}
Z \arrow[r, "id_Z\times p"] \arrow[rrrr, bend left] & Z\times X \arrow[r, "pr"] \arrow[d, "pr"'] & Z \arrow[r, "id_Z\times p"] & Z\times X \arrow[r, "pr"] \arrow[d, "pr"'] & Z \arrow[d, "g"] \\
& X \arrow[r] & 1 \arrow[r, "p"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

The upper path gives g, the lower path is constant. □

<u>Cor.</u>: $\mathbb{A}^n\setminus 0 \xrightarrow{i_n} \mathbb{A}^{n+1}\setminus 0$ is constant

<u>Proof</u>:

$$\begin{tikzcd}
& (\mathbb{A}^n\setminus 0)\times G \arrow[r] \arrow[d] & \mathbb{A}^n\setminus 0 \arrow[d, "i_n"] \\
1 \arrow[r] & G \arrow[r] & \mathbb{A}^{n+1}
\end{tikzcd}$$

satisfies the hypotheses of the lemma.

unit map of the group G

□

<u>Cor.</u>: $\mathbb{A}^\infty\setminus 0 := \operatorname{colim}(\mathbb{A}^n\setminus 0 \overset{i_n}{\hookrightarrow} \mathbb{A}^{n+1}\setminus 0 \overset{i_{n+1}}{\hookrightarrow} \dots)$ is contractible

<u>Proof</u>:

$$\mathbb{A}^\infty\setminus 0 = \operatorname{colim}\left(\begin{tikzcd}
\mathbb{A}^n\setminus 0 \arrow[rr, "i_n"] \arrow[dr] & & \mathbb{A}^{n+1}\setminus 0 \arrow[rr, "i_{n+1}"] \arrow[dr] & & \mathbb{A}^{n+2}\setminus 0 \arrow[dr] & \\
& 1 \arrow[ur] \arrow[rr, equal] & & 1 \arrow[ur] \arrow[rr, equal] & & \dots
\end{tikzcd}\right)$$

cofinal subdiagram

□

<u>Thm</u>: $\mathbb{P}^\infty := \operatorname{colim}(\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n \hookrightarrow \dots) \simeq BG$

<u>Proof</u>:
$$\begin{aligned}
BG &= {*}/G \simeq \mathbb{A}^\infty\setminus 0/G \simeq \operatorname{colim}(\mathbb{A}^n\setminus 0 \hookrightarrow \mathbb{A}^{n+1}\setminus 0 \hookrightarrow \dots)/G \\
&\simeq \operatorname{colim}(\mathbb{A}^n\setminus 0/G \to \mathbb{A}^{n+1}/G \to \dots) \\
&\simeq \operatorname{colim}(\mathbb{P}^{n-1} \to \mathbb{P}^n \to \dots) = \mathbb{P}^\infty
\end{aligned}$$
□

<u>Cor</u>: $\mathbb{P}^\infty$ is a commutative group object again.

Stable results & constructions

Pass to $\mathcal{C} \xrightarrow{(-)_+} \mathcal{C}_* \xrightarrow{\Sigma^\infty_{\mathbb{P}^1}} \mathcal{C}_*\langle \mathbb{P}^1_\wedge - \rangle =: \mathbb{P}^1\text{-Spectra}(\mathcal{C})$

$X \longmapsto X \amalg 1$

↑ symm. monoidal stabilization

We also have $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$, hence $\mathbb{P}^1$-Spectra$(\mathcal{C})$ is a <u>stable</u> $(\infty,1)$-category.

$\Rightarrow$ It's homotopy category is triangulated, in particular additive.

In notation we suppress "$\Sigma^\infty_{\mathbb{P}^1}$", e.g. we simply write "$\mathbb{P}^\infty_+$" for $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$.

Bott multiplication map: $\mathbb{P}^1 \wedge \mathbb{P}^\infty_+ \longrightarrow \mathbb{P}^1_+ \wedge \mathbb{P}^\infty_+ \longrightarrow \mathbb{P}^\infty_+ \wedge \mathbb{P}^\infty_+ \xrightarrow{m} \mathbb{P}^\infty_+$

↑ multipl. map of the group structure of $\mathbb{P}^\infty_+$

Shifting down by one $\wedge$-factor of $\mathbb{P}^1$ we get the Bott mult. map:

$$\mathbb{P}^\infty_+ \xrightarrow{\cdot\beta} (\mathbb{P}^1)^{\wedge -1} \wedge \mathbb{P}^\infty_+$$

<u>Def.</u>: The Snaith spectrum is

$$\mathbb{P}^\infty[\beta^{-1}] := \operatorname{colim}\left(\mathbb{P}^\infty_+ \xrightarrow{\cdot\beta} \mathbb{P}^{1\wedge -1} \wedge \mathbb{P}^\infty_+ \xrightarrow{\cdot\beta} \mathbb{P}^{1\wedge -2} \wedge \mathbb{P}^\infty_+ \to \dots \right)$$

<u>Thm</u> (Snaith in topology, Gepner-Snaith / Spitzweck-Østvær)

- $\mathbb{P}^\infty[\beta^{-1}]$ is the spectrum for topological K-theory in case $\mathcal{C} = \text{Top}$
- $\mathbb{P}^\infty[\beta^{-1}]$ —— " —— alg. —— $\mathcal{C} = \text{MotSpaces}$

IV.1

Last time:

$$\begin{tikzcd}
G\in\mathcal{C} \arrow[dr, dashed] & \mathrm{Sm}/\mathbb{C} \arrow[r] \arrow[dr, "X\mapsto X(\mathbb{C})"] \arrow[d] & \mathcal{C}(\mathbb{G}) & & H\mathbb{Q}\text{-Mod} \arrow[d, dashed] \\
& \mathrm{MotSpaces}\ \mathcal{C}_* \arrow[d, dashed] \arrow[dr] & \mathrm{Top} \arrow[r] \arrow[dl] \arrow[d, dashed] & \mathcal{C}\langle(\mathbb{R}^1\wedge -)^{-1}\rangle \arrow[r] \arrow[d, dashed] & (\mathrm{Ab}) & H\mathbb{Z}\text{-Mod} \arrow[l] \arrow[d, dashed] \\
& \mathrm{Spaces} \arrow[dr] & \mathrm{MotSpaces}_* \arrow[r] \arrow[d, dashed] & \mathrm{MotSpaces}_*\langle(\mathbb{P}^1\wedge -)^{-1}\rangle \arrow[d, dashed] & & H\mathbb{Z}\text{-Mod} \\
& & \mathrm{Spaces}_* \arrow[r] & \mathrm{Spectra} \arrow[urr] & &
\end{tikzcd}$$

sends G to G

colim & product preserving

hence preserves $\mathbb{A}^n\setminus 0$, $\mathbb{P}^n$, $\mathbb{P}^\infty[\beta^{-1}]$

$\mathrm{colim}(\mathbb{P}^\infty \xrightarrow{\beta} (\mathbb{P}^1)^{\wedge -1}\wedge\mathbb{P}^\infty \to \ldots)$

Likewise with $\mathrm{Sm}/\mathbb{R}$, $X\mapsto X(\mathbb{R})$

Thm (Bachmann): $\mathbb{S}=\Sigma^\infty S^0 \xrightarrow{\rho} \Sigma\mathbb{G}_m$, $+\mapsto 1$, $*\mapsto -1$, ρ becomes eq. in Spectra because $\{\pm 1\}\simeq\mathbb{R}\setminus\{0\}$, S^0, $\mathbb{G}_m(\mathbb{R})$

$$\begin{tikzcd}
\mathrm{MotSpectra}[\rho^{-1}] \arrow[r, phantom, "\simeq"] & \mathrm{Spectra} \\
\mathrm{MotSpectra} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

HQ

$\boxed{\text{Next: Construct } H\mathbb{Q}.\quad \mathbb{P}^\infty[\beta^{-1}]_\mathbb{Q}\simeq\bigvee_i H\mathbb{Q}\wedge(\mathbb{P}^1)^{\wedge i}\ ;\quad \mathbb{S}_\mathbb{Q}\simeq\mathbb{S}_\mathbb{Q}^+\vee\mathbb{S}_\mathbb{Q}^-}$

A bit more about the Snaith K-theory spectrum $\mathbb{P}^\infty[\beta^{-1}]$:

Oriented cohomology theories: $E^*(-)$ with $x\in E^{2,1}(\mathbb{P}^\infty) = [\mathbb{P}^\infty_+, \mathbb{P}^1\wedge E]$

E ring spectrum ⇝ s.t.

$x =$ „Chern class"

$$\begin{tikzcd}
x \arrow[d, mapsto] & {[\mathbb{P}^\infty_+, \mathbb{P}^1\wedge E]} \arrow[d] \\
x|_{\mathbb{P}^1} \arrow[d, mapsto] & {[\mathbb{P}^1, \mathbb{P}^1\wedge E]} \arrow[d, phantom, "\cong"] \\
1 \in & {[\mathbb{S}, E]}
\end{tikzcd}$$

unit map $\mathbb{S}\to E$

Thm: E oriented by $x\in E^{2,1}(\mathbb{P}^\infty)$.

Then $E^{*,*}(\mathbb{P}^n) = E^{*,*}(pt)[x]/(x^{n+1}) \xrightarrow{} E^{**}(pt)[x]/(x^n) \cong E^{**}(\mathbb{P}^{n-1})$ ($\mathbb{P}^{n-1}\hookrightarrow\mathbb{P}^n$ induces)

~~Example~~: $\mathbb{P}^\infty[\beta^{-1}]$ with $\mathbb{P}^\infty_+ \longrightarrow \mathbb{P}^\infty[\beta^{-1}] \xrightarrow{\cdot\beta^{-1}} \mathbb{P}^1\wedge\mathbb{P}^\infty[\beta^{-1}]$

Cor.: $\mathbb{P}^\infty\times\mathbb{P}^\infty\xrightarrow{\mu}\mathbb{P}^\infty$

$$\mu^*\left(\begin{array}{l} E^{*,*}(\mathbb{P}^\infty) = E^{**}(pt)[[x]] \quad x \\ E^{**}(\mathbb{P}^\infty\times\mathbb{P}^\infty) = E^{**}(pt)[[x,y]] \quad f(x,y) \end{array}\right.$$

$x\mapsto f(x,y)$, a 1-dimensional „formal group law"

Prop.: The formal group law of $\mathbb{P}^\infty[\beta^{-1}]$ is $x+y+x\cdot y\cdot\beta$ – „the" multiplicative fgl.

i.e. $f(f(x,y),z) = f(x,f(y,z))$

$f(x,y) = x+y+$ terms of higher degree

Think: Jet space of a Lie/alg. group at the neutral elt.

$\mathbb{G}_a$: $x+y = (0+x)+(0+y)$

$\mathbb{G}_m$: $(1+x)\cdot(1+y) = 1+x+y+xy$

IV.2

Compute endomorphisms of $\mathbb{P}^\infty[\beta^{-1}] =: K_{\mathbb{Q}}$

$$\underbrace{[\mathbb{P}^\infty[\beta^{-1}]}_{\operatorname{colim}(\mathbb{P}^\infty_+ \xrightarrow{\cdot\beta} (\mathbb{P}^1)^{\wedge -1} \wedge \mathbb{P}^\infty_+ \xrightarrow{\cdot\beta} \dots)}, \mathbb{P}^\infty[\beta^{-1}]] \overset{\text{over } \mathbb{Q}}{=} \lim [\mathbb{P}^\infty_+, \mathbb{P}^\infty[\beta^{-1}]] = \lim (K(\beta^{\pm})[[x]] \leftarrow K(\beta^{\pm})[[x]] \leftarrow \dots)$$

$$(x\cdot\beta+1)\cdot\frac{\partial f}{\partial x} \longleftarrow\!\shortmid f \qquad (*)$$

$$\| $$

$$\{(f_n) \mid f_n = (x\beta+1)\frac{\partial f_{n+1}}{\partial x}\}$$

Remark: $\mathbb{Z}[[x]] \leftarrow \mathbb{Z}[[x]] \leftarrow \dots$

$$(x\cdot\beta+1)\frac{\partial f}{\partial x} \longleftarrow\!\shortmid f$$

set of solutions only depends on the first element f_0 (Rioux) [question is whether ∃ lifts

— " — is uncountable (Adams Clarke)

Only known examples are linear combinations of $1+x$ and $\frac{1}{1+x}$

Over $\mathbb{Q}$: Know more...

(*) The computation: $\mathbb{P}^1 \wedge \mathbb{P}^\infty_+ \longrightarrow \mathbb{P}^\infty_+ \wedge \mathbb{P}^\infty_+ \longrightarrow \mathbb{P}^\infty_+ \wedge \mathbb{P}^\infty_+ \xrightarrow{\wedge} \mathbb{P}^\infty_+$

induces $\quad y\cdot K^*[[x]] \xleftarrow{pr} K^*[[x,y]]/(y^2) \longleftarrow K^*[[x,y]] \longleftarrow K^*[[x]]$

$$K^*[[x]] \cong y\cdot K^*[[x]]$$

$$K^*[[x,y]]/(y^2) = K^*[[x]] \oplus K^*[[x]]\cdot y$$

$$x+y+\beta xy \longleftarrow\!\shortmid x$$

$$(y(x\beta+1)+x)^n = (x+y+\beta xy)^n \longleftarrow\!\shortmid x^n = f$$

$$n\cdot(x\beta+1)x^{n-1} \leftarrow\!\shortmid n\cdot y(x\beta+1)x^{n-1} \longleftarrow\!\shortmid x^n + n\cdot y(x\beta+1)x^{n-1} \leftarrow\!\shortmid \sum_{i=0}^{n} \binom{n}{i} y^i (x\beta+1)^i \cdot x^{n-i}$$

$$\| $$

$$(x\beta+1)\frac{\partial f}{\partial x}$$

The map we look for is a K^*-module map

$(x\beta+1)\frac{\partial}{\partial x}$ is also a K^*-module map (because multiplying with $(x\beta+1)$ is and deriving is)

$\Rightarrow$ the map is the right one on all polynomials.

Polynomials are dense in $K^*[[x]]$ for the limit topology,

Bott mult. is continuous (because is determined by the restriction to the $\mathbb{P}^n \hookrightarrow \mathbb{P}^\infty$, ...)

$\Rightarrow$ the map is the right one on all power series.

Can write down explicit projectors $\overset{p \circ p = p}{}$ in $[K_{\mathbb{Q}}, K_{\mathbb{Q}}]$

$\rightsquigarrow$ get summands $K_{\mathbb{Q}} = \bigvee_i \underbrace{H\mathbb{Q}}_{\text{some spectrum}} \wedge (\mathbb{P}^1)^{\wedge i}$

$H\mathbb{Q}$ = Rational (motivic) Eilenberg-MacLane spectrum

$\rightsquigarrow$ $H\mathbb{Q}$-Mod =: Rational motives

2nd construction: $\tau \in [\mathbb{S}, \mathbb{S}] \cong [\mathbb{P}^1 \wedge \mathbb{P}^1, \mathbb{P}^1 \wedge \mathbb{P}^1]$ involution

Get idempotent endom. $\frac{1}{2}(id - \tau)$, $\frac{1}{2}(id + \tau)$

$$\left(\tfrac{1}{2}(id - \tau) \circ \tfrac{1}{2}(id - \tau) = \tfrac{1}{4} id^{\circ 2} - 2 \cdot \tfrac{1}{4} \tau + \tfrac{1}{4} \underbrace{\tau^{\circ 2}}_{= id} = \tfrac{1}{2}(id - \tau) \right)$$

$\rightsquigarrow$ $\mathbb{S}_{\mathbb{Q}} \simeq \mathbb{S}_{\mathbb{Q}+} \vee \mathbb{S}_{\mathbb{Q}-}$

corresp. to $\frac{1}{2}(id + \tau)$; here τ acts as id

<u>Thm</u>: $\mathbb{S}_{\mathbb{Q}+} \simeq H\mathbb{Q}$

<u>Big remark</u>: Reason why $K_{\mathbb{Q}}$ = sum of Eilenberg-MacLane spectra; in topology (& motivic htpy theory)

Complex cobordism: gen by  cobordism classes of maps of $\mathbb{C}$-mflds to X

$\rightsquigarrow$ $MU^*(X)$ coh. theory

<u>Thm</u> (Quillen): $MU^*(pt) \simeq$ Lazard ring $=: L \underset{\text{Lazard}}{\cong} \mathbb{Z}[x_1, x_2, x_3, \ldots]$

$\mathrm{Hom}_{Ring}(L, R) \simeq \{$ fgls in $R[[x]] \}$

Landweber exactness: $E^*(-)$ coh. theory, $\rightsquigarrow$ $E^*(X)$ an $E^*(pt)$-module (from $X \to pt$, $E^*(X) \leftarrow E^*(pt)$)

$M \in E^*(pt)$-Mod $\rightsquigarrow$ $E^*(-) \otimes_{E^*(pt)} M$ a new coh. theory?

Axioms 1, 2, 3 ok, for 4 need exactness of certain sequences.

Flatness too strong; „Landweber exactness" is enough.

Any fgl in $R[[x]]$ makes R into an $\underset{\substack{\shortparallel \\ MU^*(pt)}}{L}$-module

$\rightsquigarrow$ when Landweber exact, get new coh. theory

Ex.: $K^*(X) = MU^*(X) \otimes_{MU^*(pt)} \mathbb{Z}$ via mult. fgl $x+y+xy$

$H^*_{sing}(X) = MU(X) \otimes_{MU^*(pt)} \mathbb{Z}$ via add. fgl. $x+y$

(Fact: Every oriented coh. theory arises in this way & is [illegible] determined by its fgl (up to repetition of summands)

Fact Over $\mathbb{Q}$ (mult fgl) $\cong$ (add. fgl) via log series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \pm \dots$$

$$\Rightarrow \quad K^*_{\mathbb{Q}}(X) = \bigvee_i H^{*+i}_{sing\,\mathbb{Q}}(X)$$

[Other 1-dim. fgls: those of elliptic curves]

Applications of Motivic htpy theory

Mot. Spaces

Classifying Octonion algebras over schemes — Asok, Doran, Fasel, Hoyois, Wendt

Classifying vector bundles
e.g. [on $\mathbb{A}^1$-contractible schemes every vbdl. is trivial]

- see Asok Doran, vbdls on contractible smooth schemes p.3!!
and Asok Fasel, a cohomological classification of vbdls on 3-folds — cancellation thm.

Asok Doran: Classification of varieties up to $\mathbb{A}^1$-htpy type; $\mathbb{A}^1$ h-cobordism

Result: Rational smooth proper surfaces over an alg. closed field: ∃ arbitrary high-dim. moduli spaces of such, but only countably many $\mathbb{A}^1$-homotopy types; each uniquely determined by the iso class of its $\mathbb{A}^1$-π_1 (fundamental group sheaf)

Asok-Doran
Smooth varieties up to $\mathbb{A}^1$-htpy and h-cobordism

3.9 **Thm:** Any rational smooth proper surface over $k=\overline{k}$ is $\mathbb{A}^1$-weq. to either $\mathbb{P}^1\times\mathbb{P}^1$ or a blow-up ~~of~~ at fin. many pts. on $\mathbb{P}^2$

∃ covering space theory, fundamental group satisfying van Kampen thm.

MotSpectra

Bloch-Kato conjecture: computation of Steenrod alg.

Isaksen, ~~Röndigs, Spitzweck, Østvær~~: computation of ~~stable~~ htpy groups of spheres new results in
Wang, Xu 2016-17 (from ~~[illegible]~~ n= 62-93)

Computations in equivariant stable htpy theory using motivic computations:
$\mathrm{MotSpec}/\mathbb{R} \longrightarrow \mathbb{Z}/2\text{-eq-Spectra}$
$X \longmapsto X(\mathbb{C}) \circlearrowleft \mathbb{Z}/2$
[Guillou]

Alg. cobordism, Landweber exactness

Refined enumerative geometry

Unstable & stable operations on alg. K-theory
Riou

Motives

Milnor-conjecture — Voevodsky

Brown: Periods, relations between periods
Galois theory for transcendental nrs

Vishik: Invariants of quadrics

New calculations, e.g. of Chow groups, of decompositions of Weil cohomology theories

Interesting facts:

- DMT(k;R), mixed Tate motives, are coreflexive subcat; i.e. $\forall M \in DM \; \exists \; \overset{\text{DMT}}{\overset{\cup}{C(M)}} \to M$ inducing isos on motivic homology.
  ("cellular approximation")
- In topology: $\mathbb{S}\otimes\mathbb{Q} \simeq H\mathbb{Q}$ (because by Serre $\pi_n^{st}(\mathbb{S})$ is torsion $n>0$; hence only $\exists!$ htpy group $\neq 0$ of $\mathbb{S}\otimes\mathbb{Q} \Rightarrow$ Eilenberg-MacLane spectrum)

  In alg. geo.: (Riemann-Roch $\Rightarrow$ $K_0(X)\otimes\mathbb{Q} \simeq CH^*(X)\otimes\mathbb{Q}$

  In topology: $K^*(X)\otimes\mathbb{Q} \cong \bigoplus_i H_{sing}^{*+i}(X)\otimes\mathbb{Q}$ (?) because of Landweber exactness; all formal grp laws $\simeq$ additive FGL because $\exists$ exponential map $\boxed{\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \_}$

  $\|$

  $\Omega^*(X)\otimes\mathbb{Q} \leftarrow$ cobordism.

  All of these facts have a generalization to motivic htpy theory. – These have Landweber exactness
  $\pi_*^{st}(\mathbb{S})$ is not torsion, but still the first thing holds
- The unit map for alg. cobordism $\mathbb{1} \to MGL$ factors through $\mathbb{1} \to \mathbb{1}/\eta$. Hence $MGL \wedge \eta = 0$.
  Hence every MGL-module, and hence every module over an oriented ring spectrum is η-complete [Röndigs-Spitzweck-Østvær Lemma 2.1 – The motivic Hopf map solves the homotopy limit problem]
  i.e. $MGL \to \operatorname{holim}(\cdots MGL/\eta^3 \to MGL/\eta^2 \to MGL/\eta)$ is an equivalence

Isaksen:

$$\begin{tikzcd} X \arrow[r, mapsto] & X(\mathbb{C}) \\ \mathrm{Mot}/\mathbb{C} \arrow[r, "\text{idea: invert } \tau"'] & \mathrm{Top} \end{tikzcd}$$

$H^{*,*}(pt) = \mathbb{F}_2[\tau]$ $\qquad$ $H^*(pt) = \mathbb{F}_2$

Comes from $\tau \in \pi_{0,-1}(\mathbb{S})$ $\qquad$ $\mathbb{S}/\tau$ is a ring spectrum.

<u>Thm</u> (Gheorghe-Wang-Xu): $\mathrm{Ho}(\mathbb{S}/\tau\text{-Mod}) \simeq \mathrm{Ho}(BP_*BP\text{-comod})$

have Adams spec seq. for this $\mathbb{S} \to \mathbb{S}/\tau$ $\qquad$ ← gives Novikov spec seq.

$\mathbb{S}[\tau^{-1}]\text{-Mod} \overset{?}{\simeq} SH$

Bachmann: $S^0 \xrightarrow{\rho} \mathbb{G}_m$ in $\mathrm{MotSp}/\mathbb{R}$, <u>Thm</u>: $SH(\mathbb{R})[\rho^{-1}] \xrightarrow{\simeq} SH$

$\{-1,1\} \hookrightarrow \mathrm{Spec}\,\mathbb{R}[x,x^{-1}]$

becomes an equivalence
under real realization $X \mapsto X(\mathbb{R})$
because $\mathbb{R}\setminus\{0\} \simeq S^0$, hence factors

$$\begin{tikzcd} SH(\mathbb{R})[\rho^{-1}] \arrow[r, "\simeq"] & SH \\ SH(\mathbb{R}) \arrow[u] \arrow[ur, "X \mapsto X(\mathbb{R})"'] & \end{tikzcd}$$