4.2 Considere o produto interno usual de $\mathbb{R}^2$.

(a) Determine m de modo que os vetores $(1+m, 2)$ e $(3, m-4)$ sejam ortogonais.

(b) Determine todos os vetores de $\mathbb{R}^2$ que são ortogonais a $(2, 1)$.

(a) Queremos m tal que

$$0 = \langle (1+m, 2), (3, m-4) \rangle =$$
$$= (1+m)\cdot 3 + 2(m-4) = 3m + 3 + 2m - 8$$
$$= 5m - 5.$$

Daí $m = 1$.

(b) Queremos

$$\{(2,1)\}^{\perp} = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \perp (2,1)\}$$
$$= \{(x,y) \in \mathbb{R}^2 \mid \langle (x,y), (2,1) \rangle = 0\}$$
$$= \{(x,y) \in \mathbb{R}^2 \mid 2x + y = 0\}$$
$$= \{(x, -2x) \mid x \in \mathbb{R}\} = \{x(1,-2) \mid x \in \mathbb{R}\} = [(1,-2)].$$

4.5 Determine uma base ortonormal de cada um dos seguintes subespaços de $\mathbb{R}^4$, em relação ao produto interno usual, utilizando o processo de Gram-Schmidt:

(a) $W = [(1,1,0,0), (0,1,2,0), (0,0,3,4)]$.

(b) $W = [(2,0,0,0), (1,3,3,0), (3,-3,-3,0)]$.

(a) Defina

$$v_1 = (1,1,0,0),$$

$$v_2 = (0,1,2,0) - \frac{\langle (0,1,2,0), (1,1,0,0) \rangle}{\langle (1,1,0,0), (1,1,0,0) \rangle} (1,1,0,0) =$$
$$= (0,1,2,0) - \frac{1}{2}(1,1,0,0) = (-1/2, 1/2, 2, 0).$$

$$v_3 = (0,0,3,4) - \frac{\overbrace{\langle (0,0,3,4),(1,1,0,0)\rangle}^{0}}{\langle (1,1,0,0),(1,1,0,0)\rangle}(1,1,0,0) - \frac{\langle (0,0,3,4),(-1/2,1/2,2,0)\rangle}{\langle (-1/2,1/2,2,0),(-1/2,1/2,2,0)\rangle}(-1/2,1/2,2,0)$$

$$= (0,0,3,4) - \frac{6}{1/2+4}(-1/2,1/2,2,0)$$

$$= (0,0,3,4) - \frac{4}{3}(-1/2,1/2,2,0) = (0,0,3,4) - (-2/3, 2/3, 8/3, 0)$$

$$= (+2/3, -2/3, 1/3, 4)$$

Então

$$\{(1,1,0,0), (-1/2,1/2,2,0), (+2/3,-2/3,1/3,4)\}$$

é base ortogonal de W. Daí, uma base ortonormal de W é

$$\{(1/\sqrt{2}, 1/\sqrt{2}, 0, 0), \cancel{(-1/\sqrt{6}, 1/\sqrt{6}, 2/\sqrt{6}, 0)}, \cancel{(2/5, -2/5, 1/5, 4/5)}\}.$$

$$\left(-\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, 0\right),$$

$$\left(\frac{2}{3\sqrt{17}}, -\frac{2}{3\sqrt{17}}, \frac{1}{3\sqrt{17}}, \frac{4}{\sqrt{17}}\right)$$

(b) $W = [(2,0,0,0), (1,3,3,0), (3,-3,-3,0)]$

Defina

$$v_1 = (2,0,0,0)$$

$$v_2 = (1,3,3,0) - \frac{\langle (1,3,3,0),(2,0,0,0)\rangle}{\langle (2,0,0,0),(2,0,0,0)\rangle}(2,0,0,0)$$

$$= (1,3,3,0) - \frac{2}{4}(2,0,0,0) = (0,3,3,0).$$

$$v_3 = (3,-3,-3,0) - \frac{\langle (3,-3,-3,0),(2,0,0,0)\rangle}{\langle (2,0,0,0),(2,0,0,0)\rangle}(2,0,0,0)$$

$$- \frac{\langle (3,-3,-3,0),(0,3,3,0)\rangle}{\langle (0,3,3,0),(0,3,3,0)\rangle}(0,3,3,0)$$

$$= (3,-3,-3,0) - \frac{6}{4}(2,0,0,0) - \left(\frac{-18}{18}\right)(0,3,3,0)$$

$$= (3,-3,-3,0) - (3,0,0,0) + (0,3,3,0) = (0,0,0,0).$$

(o último vetor zerou porque $\{(2,0,0,0), (1,3,3,0), (3,-3,-3,0)\}$ é l.d.)

Então, uma base ortogonal de W é $\{(2,0,0,0), (0,3,3,0)\}$.
Uma base ortonormal é

$$\{(1,0,0,0), (0, 1/\sqrt{2}, 1/\sqrt{2}, 0)\}.$$

Solução alternativa

$$W = [(2,0,0,0), (1,3,3,0), (3,-3,-3,0)]$$

O conjunto $\{(2,0,0,0), (1,3,3,0), (3,-3,-3,0)\}$ é l.d., pois

$$(3,-3,-3,0) = +2(2,0,0,0) - (1,3,3,0).$$

Daí

$$W = [(2,0,0,0), (1,3,3,0)].$$

Aplicando Gram-Schmidt:

$$v_1 = (2,0,0,0)$$

$$v_2 = (1,3,3,0) - \frac{\langle (1,3,3,0),(2,0,0,0)\rangle}{\langle (2,0,0,0),(2,0,0,0)\rangle}(2,0,0,0) = (0,3,3,0)$$

Daí $\{(2,0,0,0),(0,3,3,0)\}$ é base ortogonal de W, e $\{(1,0,0,0),(0,1/\sqrt{2},1/\sqrt{2},0)\}$ é base ortonormal.

4.7 Seja $W = [(1,0,1),(1,1,0)]$. Considere $W^\perp$ em relação ao produto interno canônico de $\mathbb{R}^3$. Encontre uma base ortonormal para W e $W^\perp$.

Note que $\{(1,0,1),(1,1,0)\}$ é base de W. Queremos completar para uma base de $\mathbb{R}^3$.

O conjunto $\{(1,0,1),(1,1,0),(0,1,0)\}$ é base de $\mathbb{R}^3$, que contém uma base de W. Aplicando Gram-Schmidt:

$$v_1 = (1,0,1)$$

$$v_2 = (1,1,0) - \frac{\langle (1,1,0),(1,0,1)\rangle}{\langle (1,0,1),(1,0,1)\rangle}(1,0,1) = (1,1,0) - \frac{1}{2}(1,0,1)$$
$$= (1/2, 1, -1/2).$$

$$v_3 = (0,1,0) - \frac{\overbrace{\langle (0,1,0),(1,0,1)\rangle}^{0}}{\langle (1,0,1),(1,0,1)\rangle}(1,0,1) - \frac{\langle (0,1,0),(1/2,1,-1/2)\rangle}{\langle (1/2,1,-1/2),(1/2,1,-1/2)\rangle}(1/2,1,-1/2)$$

$$= (0,1,0) - \frac{1}{3/2}(1/2,1,-1/2) = (0,1,0) - (1/3, 2/3, -1/3)$$
$$= (-1/3, 1/3, 1/3).$$

Daí $\{(1,0,1),(1/2,1,-1/2)\}$ é base ortogonal de W, e $\{(-1/3,1/3,1/3)\}$ é base ortogonal de $W^\perp$.

Portanto,
$$\{(1/\sqrt{2},0,1/\sqrt{2}),(1/\sqrt{6},2/\sqrt{6},-1/\sqrt{6})\}$$
é base ortonormal de W, e $\{(-1/\sqrt{3},1/\sqrt{3},1/\sqrt{3})\}$ é base ortonormal de $W^\perp$.

M é simétrica se $M = M^T$

5.2 Seja A uma matriz quadrada. Mostre que $A + A^T$ é simétrica, em que A^T denota a transposta usual da matriz A.

$$(A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

Provamos que $(A+A^T)^T = A + A^T$. Portanto, $A + A^T$ é simétrica.

5.3 Seja $\mathbb{R}^3$ munido do produto interno canônico, e $T : \mathbb{R}^3 \to \mathbb{R}^3$ definida por $T(x,y,z) = (-x+y, x-2y+z, y-z)$.

(a) Mostre que T é um operador auto-adjunto, mas não ortogonal.

(b) Se $v = (2,-1,5)$ e $w = (3,0,1)$, verifique que $\langle T(v), w\rangle = \langle v, T(w)\rangle$.

(c) Exiba uma base de autovetores de T e verifique que é uma base ortogonal. A partir desta base, escreva uma base ortonormal de $\mathbb{R}^3$.

$$[T] = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

(a) $[T]$ é simétrica, e portanto, T é auto-adjunto.

$$[T] \circ [T]^T = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 \\ -3 & 6 & -3 \\ 1 & -3 & 2 \end{pmatrix} \neq I.$$

Portanto, T não é ortogonal.

(b) $T(v) = T(2,-1,5) = (-2-1, -2-2+5, -1-5) = (-3, 9, -6)$

$T(w) = T(3,0,1) = (-3, 4, -1)$

$$\langle T(v), w\rangle = \langle(-3,9,-6),(3,0,1)\rangle = -9 - 6 = -15$$

$$\langle v, T(w)\rangle = \langle(2,-1,5),(-3,4,-1)\rangle = -6 - 4 - 5 = -15$$

(c) $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

$$p_T(x) = \det \begin{pmatrix} -1-x & 1 & 0 \\ 1 & -2-x & 1 \\ 0 & 1 & -1-x \end{pmatrix} = (-1-x)^2(-2-x) - (-1-x) - (-1-x)$$

$$= (-1-x)\left((-1-x)(-2-x) - 2\right)$$

$$= -(x+1)(x^2+3x) = -x(x+1)(x+3)$$

Autovetores:

$\lambda = 0$:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x+y \\ x-2y+z \\ y-z \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -x+y = 0 \\ x-2y+z = 0 \\ y-z = 0 \end{cases} \sim \begin{cases} x = y \\ z = y \end{cases}$$

$V_0 = [(1,1,1)]$.

$\lambda = -1$:

$$\begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x+y \\ x-2y+z \\ y-z \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -x+y = -x \\ x-2y+z = -y \\ y-z = -z \end{cases} \sim \begin{cases} y = 0 \\ x-y+z = 0 \\ y = 0 \end{cases} \sim \begin{cases} x+z = 0 \\ y = 0 \end{cases}$$

$$V_{-1} = [(1,0,-1)].$$

$\lambda = -3$:

$$\begin{pmatrix} -3x \\ -3y \\ -3z \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x+y \\ x-2y+z \\ y-z \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -x+y = -3x \\ x-2y+z = -3y \\ y-z = -3z \end{cases} \sim \begin{cases} 2x+y = 0 \\ x+y+z = 0 \\ y+2z = 0 \end{cases}$$

$$\sim \begin{cases} x+y+z = 0 \\ -y-2z = 0 \\ y+2z = 0 \end{cases} \sim \begin{cases} x = -z-y = z \\ y = -2z \end{cases}$$

$$V_{-3} = [(1,-2,1)].$$

Verificando que uma base formada por autovetores de T realmente é ortogonal

$$\langle (1,1,1), (1,0,-1) \rangle = 1\cdot 1 + 1\cdot 0 + 1(-1) = 0$$
$$\langle (1,1,1), (1,-2,1) \rangle = 1\cdot 1 + 1(-2) + 1\cdot 1 = 0$$
$$\langle (1,0,-1), (1,-2,1) \rangle = 1\cdot 1 + 0(-2) + (-1)\cdot 1 = 0.$$

Assim, uma base ortogonal de $\mathbb{R}^3$ é

$$\{ (1,1,1), (1,0,-1), (1,-2,1) \}.$$

Normalizando:

$$u_1' = \frac{1}{\|(1,1,1)\|}(1,1,1) = \frac{1}{\sqrt{\langle (1,1,1),(1,1,1)\rangle}}(1,1,1) = \frac{1}{\sqrt{3}}(1,1,1) = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$$

$$u_2' = \frac{1}{\|(1,0,-1)\|}(1,0,-1) = \frac{1}{\sqrt{2}}(1,0,-1) = (1/\sqrt{2}, 0, -1/\sqrt{2})$$

$$u_3' = \frac{1}{\|(1,-2,1)\|}(1,-2,1) = \frac{1}{\sqrt{6}}(1,-2,1) = (1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6}).$$

Portanto, uma base ortonormal de $\mathbb{R}^3$ formada por autovetores de T é

$$\{(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}), (1/\sqrt{2}, 0, -1/\sqrt{2}), (1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6})\}.$$