1. Considere os seguintes subespaços vetoriais de $\mathbb{R}^3$:

$$U = \{(x,y,z) \in \mathbb{R}^3 \mid -2x - y = 0\},$$
$$V = \{(x,y,z) \in \mathbb{R}^3 \mid x + 2y - z = -x - y + z = 0\}.$$

Determine $U \cap V$ e $U + V$.

$$U \cap V = \{(x,y,z) \in \mathbb{R}^3 \mid -2x - y = x + 2y - z = -x - y + z = 0\}$$

$$(x,y,z) \in U \cap V \iff \begin{cases} -2x - y = 0 \\ x + 2y - z = 0 \\ -x - y + z = 0 \end{cases} \sim \begin{cases} -2x - y = 0 \\ x + 2y - z = 0 \\ y = 0 \end{cases} \sim \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

Daí $U \cap V = \{(0,0,0)\}$.

Para calcular a soma:

$$U = \{(x,y,z) \in \mathbb{R}^3 \mid -2x - y = 0\} = \{(x,-2x,z) \mid x, z \in \mathbb{R}\} =$$
$$= \{x(1,-2,0) + z(0,0,1) \mid x, z \in \mathbb{R}\} = [(1,-2,0), (0,0,1)]$$

Daí $\dim U = 2$.

$$V = \{(x,y,z) \in \mathbb{R}^3 \mid x + 2y - z = -x - y + z = 0\} =$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid y = 0,\ x = z\} = \{(x,0,x) \mid x \in \mathbb{R}\} = [(1,0,1)].$$

$$\begin{cases} x + 2y - z = 0 \\ -x - y + z = 0 \end{cases} \sim \begin{cases} x + 2y - z = 0 \\ y = 0 \end{cases} \sim \begin{cases} x = z \\ y = 0 \end{cases}$$

Daí $\dim V = 1$. Portanto

$$\dim(U+V) = \underbrace{\dim U}_{2} + \underbrace{\dim V}_{1} - \underbrace{\dim(U \cap V)}_{0} = 3.$$

Daí $U+V=\mathbb{R}^3$.

2. Sejam $\beta=\{(-2,-1),(-2,1)\}$ e $\beta'=\{(-2,0),(0,-1)\}$ bases de $\mathbb{R}^2$.
   a. Determine $[I]^{\beta'}_{\beta}$ e $[I]^{\beta}_{\beta'}$, as matrizes de mudança de base de β' para β, e de β para β', respectivamente.
   b. Dado $v=(2,4)$, determine $[v]_\beta$ e $[v]_{\beta'}$.

(a) Cálculo de $[I]^{\beta}_{\beta'}$:

$$(-2,-1) = a(-2,0)+b(0,-1) = (-2a,-b)$$

$$\therefore \begin{cases} -2a = -2 \\ -b = -1 \end{cases} \sim \begin{cases} a = 1 \\ b = 1 \end{cases}$$

$$\therefore [(-2,-1)]_{\beta'} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(-2,1) = a(-2,0)+b(0,-1) = (-2a,-b)$$

$$\therefore \begin{cases} -2a = -2 \\ -b = 1 \end{cases} \sim \begin{cases} a = 1 \\ b = -1 \end{cases}$$

$$\therefore [(-2,1)]_{\beta'} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Portanto

$$[I]^{\beta}_{\beta'} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Segue que

$$[I]^{\beta'}_{\beta} = \left([I]^{\beta}_{\beta'}\right)^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

(b) $(2,4) = a(-2,0)+b(0,-1) = (-2a,-b)$

$$\therefore \begin{cases} -2a = 2 \\ -b = 4 \end{cases} \sim \begin{cases} a = -1 \\ b = -4 \end{cases}$$

$$[(2,4)]_{\beta'} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}.$$

$$[(2,4)]_{\beta} = [I]^{\beta'}_{\beta}\,[(2,4)]_{\beta'} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} -1 \\ -4 \end{pmatrix} = \begin{pmatrix} -5/2 \\ 3/2 \end{pmatrix}.$$

3. A matriz mudança de base de β para $\beta' = \{w_1, w_2, w_3\}$ de $\mathbb{R}^3$ é

$$[I]^{\beta}_{\beta'} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 4 \\ 2 & 2 & -3 \end{pmatrix}.$$

Determine a base β.

$\beta = \{v_1, v_2, v_3\}$

Daí

$$[v_1]_{\beta'} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \Rightarrow v_1 = w_1 - w_2 + 2w_3$$

$$[v_2]_{\beta'} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \Rightarrow v_2 = 3w_2 + 2w_3$$

$$[v_3]_{\beta'} = \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix} \Rightarrow v_3 = 4w_2 - 3w_3$$

$$v_1 = 1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 \Rightarrow [v_1]_{\beta} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$[v_1]_{\beta'} = [I]^{\beta}_{\beta'}[v_1]_{\beta} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 4 \\ 2 & 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow v_1 = 1 \cdot w_1 + (-1) \cdot w_2 + 2w_3$$

5. Determine a dimensão do núcleo e a dimensão da imagem da transformação linear $T : \mathbb{R}^3 \to \mathbb{R}^2$, dada por $T(x, y, z) = (x + y + z, -2x + 2y - z)$.

$$\operatorname{Ker} T = \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,0)\} =$$

$$= \{(x,y,z) \in \mathbb{R}^3 \mid (x+y+z, -2x+2y-z) = (0,0)\}$$

$$= \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0,\ -2x+2y-z=0\} =$$

$$= \{(3y, y, -4y) \mid y \in \mathbb{R}\} = [(3,1,-4)].$$

$$(x,y,z) \in \operatorname{Ker} T \iff \begin{cases} x+y+z=0 \\ -2x+2y-z=0 \end{cases} \sim \begin{cases} x+y+z=0 \\ 4y+z=0 \end{cases} \sim \begin{cases} x = 3y \\ z = -4y \end{cases}$$

$$L_2' = L_2 + 2L_1$$

Daí $\dim \operatorname{Ker} T = 1$.

Do Teorema do Núcleo e Imagem:

$$\underbrace{\dim \mathbb{R}^3}_{3} = \underbrace{\dim(\operatorname{Ker} T)}_{1} + \dim(\operatorname{Im} T) \Rightarrow \dim(\operatorname{Im} T) = 2.$$

$$\operatorname{Im} T = \{T(x,y,z) \mid (x,y,z) \in \mathbb{R}^3\} = \{(x+y+z, -2x+2y-z) \mid x,y,z \in \mathbb{R}\}$$

$$= \{x(1,-2) + y(1,2) + z(1,-1) \mid x,y,z \in \mathbb{R}\} =$$

$$= [\underbrace{(1,-2), (1,2), (1,-1)}_{\text{não é um conjunto l.i.!}}] \Rightarrow \cancel{\dim(\operatorname{Im} T) = 3}$$

6. Ache os autovalores e autovetores da matriz seguinte:

$$M = \begin{pmatrix} -2 & -1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Polinômio característico:

$$P_M(\lambda) = \det\begin{pmatrix} -2-\lambda & -1 & -2 \\ 0 & -1-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (-2-\lambda)(-1-\lambda)^2.$$

Os autovalores são -2 e -1.

Autovetores associados a -2:

$$\begin{pmatrix} -2 & -1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix} \Rightarrow \begin{cases} -2x - y - 2z = -2x \\ -y + z = -2y \\ -z = -2z \end{cases}$$

$$\sim \begin{cases} y = 0 \\ z = 0 \end{cases}$$

$\therefore$ Os autovalores são da forma $(x,0,0)$, $x \in \mathbb{R}$.

Autovetores associados a -1:

$$\begin{pmatrix} -2 & -1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} \Rightarrow \begin{cases} -2x - y - 2z = -x \\ -y + z = -y \\ -z = -z \end{cases}$$

$$\sim \begin{cases} x = -y \\ z = 0 \end{cases}$$

$\therefore$ Os autovetores são da forma $(x,-x,0)$, $x \in \mathbb{R}$

(ou $(-y,y,0)$, $y \in \mathbb{R}$)

1. Considere os seguintes subespaços vetoriais de $\mathbb{R}^3$:

$$U = \{(x,y,z) \in \mathbb{R}^3 \mid -x - 2y + z = 0\},$$
$$V = \{(x,y,z) \in \mathbb{R}^3 \mid x + y - z = -x - 2y + z = 0\}.$$

Determine $U \cap V$ e $U + V$.

Seja $(x,y,z) \in V$. Então
$x + y - z = 0$ e $-x - 2y + z = 0$

Isso significa que $(x,y,z) \in U$. Daí $V \subseteq U$.

Portanto, $U \cap V = V$, e
$U \subseteq U + V \subseteq U \Rightarrow U + V = U$.

4. (a) Descreva a transformação linear $T : \mathbb{R}^2 \to \mathbb{R}^3$ tal que $T(1,1) = (4,-1,-2)$ e $T(1,-1) = (-2,-2,2)$. (b) Sendo β e β' são as bases canônicas de $\mathbb{R}^2$ e $\mathbb{R}^3$, respectivamente, determine $[T]^{\beta}_{\beta'}$, a matriz da transformação linear T nas bases β e β'.

(a) $(x,y) = a(1,1) + b(1,-1) = (a+b, a-b)$

$$\therefore \begin{cases} a+b = x \\ a-b = y \end{cases} \sim \begin{cases} a = \dfrac{x+y}{2} \\ b = \dfrac{x-y}{2} \end{cases}$$

Portanto $(x,y) = \left(\dfrac{x+y}{2}\right)(1,1) + \left(\dfrac{x-y}{2}\right)(1,-1)$.

Daí

$$T(x,y) = \left(\frac{x+y}{2}\right) T(1,1) + \left(\frac{x-y}{2}\right) T(1,-1)$$
$$= \left(\frac{x+y}{2}\right)(4,-1,-2) + \left(\frac{x-y}{2}\right)(-2,-2,2)$$
$$= \left(2x+2y - x + y, \frac{-x-y-2x+2y}{2}, -x-y+x-y\right) =$$
$$= \left(x+3y, \frac{-3x+y}{2}, -2y\right).$$

(b) $T(1,0) = (1, -3/2, 0)$, $[T(1,0)]_{\beta'} = \begin{pmatrix} 1 \\ -3/2 \\ 0 \end{pmatrix}$

$T(0,1) = (3, 1/2, -2)$, $[T(0,1)]_{\beta'} = \begin{pmatrix} 3 \\ 1/2 \\ -2 \end{pmatrix}$.

Dai

$$[T]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 3 \\ -3/2 & 1/2 \\ 0 & -2 \end{pmatrix}$$