Autovalor e Autovetor (parte II)

Do último exemplo 7.

Se
$$M = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$
então $p_M(\lambda) = (1-\lambda)^2(2-\lambda)$,

autovalores : 1 e 2,

autovetores associados a 1 : $(x, y, 0)$, $x, y \in \mathbb{R}$, $\dim V_1 = 2$

autovetores associados a 2 : $(4z, z, z)$, $z \in \mathbb{R}$.

Exemplo 8. Calcule os autovalores e autovetores de
$$M = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

O pol. caract.
$$p_M(\lambda) = \det \begin{pmatrix} 1-\lambda & 1 & 4 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (1-\lambda)^2(2-\lambda)$$

Os autovalores são 1 e 2.

Autovetores associados a 1:
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+4z \\ y \\ 2z \end{pmatrix}$$

$$\Rightarrow \begin{cases} x + y + 4z = x \\ y = y \\ 2z = z \end{cases} \sim \begin{cases} y = 0 \\ z = 0 \end{cases}$$

$\therefore$ Os autovetores são da forma $(x, 0, 0)$, $x \in \mathbb{R}$. Daí

$V_1 = \{(x,0,0) \mid x \in \mathbb{R}\} = [(1,0,0)]$, $\dim V_1 = 1$.

Autovetores associados a 2:

$$\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = 2 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+4z \\ y \\ 2z \end{pmatrix}$$

$$\Rightarrow \begin{cases} x + y + 4z = 2x \\ y = 2y \\ 2z = 2z \end{cases} \sim \begin{cases} x = 4z \\ y = 0 \end{cases}$$

$\therefore$ Os autovetores são da forma $(4z, 0, z)$, $z \in \mathbb{R}$.
Daí $V_2 = \{(4z, 0, z) \mid z \in \mathbb{R}\}$, $\dim V_2 = 1$.

Exemplo 9. Encontre os autovalores e autovetores de

$$M = \begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix}.$$

Pol. carat.:

$$p_M(\lambda) = \det \begin{pmatrix} -1-\lambda & -1 \\ -3 & 1-\lambda \end{pmatrix} = (-1-\lambda)(1-\lambda) - (-3)(-1)$$
$$= -1-\lambda+\lambda+\lambda^2-3 = \lambda^2-4$$
$$= (\lambda+2)(\lambda-2)$$

Os autovalores são 2 e -2.

Autovet. associados 2:

$$2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Leftrightarrow$$

$$\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} -x-y \\ -3x+y \end{pmatrix}$$

Daí $\begin{cases} -x-y = 2x \\ -3x+y = 2y \end{cases} \sim \{ y = -3x$

$$\begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} = -2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Daí os autovet. são da forma $(x, -3x)$, $x \in \mathbb{R}$.

Autovet. associados a -2:

$$\begin{pmatrix} -2x \\ -2y \end{pmatrix} = \begin{pmatrix} -x-y \\ -3x+y \end{pmatrix}$$

Daí

$$\begin{cases} -x-y = -2x \\ -3x+y = -2y \end{cases} \sim \{x = y$$

Portanto, os autovet. são da forma (x, x), $x \in \mathbb{R}$.

$$\begin{pmatrix} -1 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -3 \end{pmatrix} \qquad \begin{pmatrix} -1 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = -2\begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

10. Calcule os autovalores e autvet. de $T: \mathbb{R}^3 \to \mathbb{R}^3$ tal que $T(1,0,0) = (2,0,0)$, $T(0,1,0) = (2,1,2)$, $T(0,0,1) = (3,2,1)$

$$[T] = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

Pol. caract.:

$$p_T(\lambda) = \det\begin{pmatrix} 2-\lambda & 2 & 3 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda)^2 - (2-\lambda)\cdot 4 =$$

$$= (2-\lambda)\left((1-\lambda)^2 - 4\right)$$

$$= (2-\lambda)(\lambda^2 - 2\lambda - 3) = (2-\lambda)(\lambda-3)(\lambda+1)$$

Os autovalores são 2, 3 e -1.

Autovet. associados a 2:

$$\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 2y + 3z \\ y + 2z \\ 2y + z \end{pmatrix}$$

$$\therefore \begin{cases} 2x + 2y + 3z = 2x \\ y + 2z = 2y \\ 2y + z = 2z \end{cases} \sim \begin{cases} 2y + 3z = 0 \\ y = 2z \end{cases} \sim \begin{cases} y = 0 \\ z = 0 \end{cases}$$

Os autovet. são da forma $(x, 0, 0)$, $x \in \mathbb{R}$.

Checando: $\begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

Autovet. associados a 3:

$$\begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix} = \begin{pmatrix} 2x + 2y + 3z \\ y + 2z \\ 2y + z \end{pmatrix}$$

$$\therefore \begin{cases} 2x + 2y + 3z = 3x \\ y + 2z = 3y \\ 2y + z = 3z \end{cases} \sim \begin{cases} x = 5z \\ y = z \end{cases}$$

Portanto, os aut. vet. são da forma $(5z, z, z)$, $z \in \mathbb{R}$.

Checando: $\begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10+2+3 \\ 1+2 \\ 2+1 \end{pmatrix} = 3 \cdot \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 3 \\ 3 \end{pmatrix}$

Aut. vet. associados a -1:

$$\begin{pmatrix}-x\\-y\\-z\end{pmatrix} = \begin{pmatrix}2x+2y+3z\\ y+2z\\ 2y+z\end{pmatrix}$$

$$\therefore \begin{cases} 2x+2y+3z=-x\\ y+2z=-y\\ 2y+z=-z\end{cases} \sim \begin{cases}3x=-z\\ y=-z\end{cases} \sim \begin{cases} x=-\frac{1}{3}z\\ y=-z\end{cases}$$

Os aut. vet. são $\left(-\frac{1}{3}z, -z, z\right)$, $z\in\mathbb{R}$.

$$\begin{pmatrix}2&2&3\\0&1&2\\0&2&1\end{pmatrix}\begin{pmatrix}1\\3\\-3\end{pmatrix} = \begin{pmatrix}2+6-9\\3-6\\6-3\end{pmatrix} = \begin{pmatrix}-1\\-3\\3\end{pmatrix}$$

Ex. 7.2. (c) Autovalores e autovet. de $T:\mathbb{R}^3\to\mathbb{R}^3$,

$T(x,y,z)=(x+y,\ x-y+2z,\ 2x+y-z)$.

$$\begin{aligned} T(1,0,0)&=(1,1,2)\\ T(0,1,0)&=(1,-1,1)\\ T(0,0,1)&=(0,2,-1)\end{aligned} \Rightarrow [T]=\begin{pmatrix}1&1&0\\1&-1&2\\2&1&-1\end{pmatrix}$$

Pol. caract.:

$$p_T(\lambda)=\det\begin{pmatrix}1-\lambda&1&0\\1&-1-\lambda&2\\2&1&-1-\lambda\end{pmatrix} = (1-\lambda)\left((-1-\lambda)^2\right)+4+(1+\lambda)-2(1-\lambda)$$

$$=(1-\lambda)(\lambda^2+2\lambda+1)+4+1+\lambda-2+2\lambda$$

$$=\lambda^2+2\lambda+1-\lambda^3-2\lambda^2-\lambda+3+3\lambda = -\lambda^3-\lambda^2+4\lambda+4$$

(note que $\lambda=2$ é raiz)

$$-\lambda^3-\lambda^2+4\lambda+4 \;\underline{|\;\lambda-2}$$
$$-3\lambda^2 \qquad\qquad -\lambda^2-3\lambda-2$$
$$-2\lambda$$

$$\therefore\; p_T(\lambda) = -\lambda^3-\lambda^2+4\lambda+4 = (\lambda-2)(+\lambda^2+3\lambda+2)$$
$$= -(\lambda-2)(\lambda+1)(\lambda+2)$$

Então os autovalores são 2, -1 e -2.

Aut. vet. associados a 2:

$$\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x-y+2z \\ 2x+y-z \end{pmatrix}$$

$$\therefore \begin{cases} x+y = 2x \\ x-y+2z = 2y \\ 2x+y-z = 2z \end{cases} \sim \begin{cases} -x+y = 0 \\ x-3y+2z = 0 \\ 2x+y-3z = 0 \end{cases} \sim$$

$$\sim \begin{cases} -x+y = 0 \\ -2y+2z = 0 \\ 3y-3z = 0 \end{cases} \sim \begin{cases} -x+y = 0 \\ y-z = 0 \\ y-z = 0 \end{cases}$$

$$\sim \begin{cases} -x+y = 0 \\ y-z = 0 \end{cases} \sim \begin{cases} x = y \\ y = z \end{cases}$$

Daí os aut. vet. são (y,y,y), $y \in \mathbb{R}$.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

Autovet. associados a -1:

$$\begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} x+y \\ x-y+2z \\ 2x+y-z \end{pmatrix}$$

$$\therefore \begin{cases} x+y = -x \\ x-y+2z = -y \\ 2x+y-z = -z \end{cases} \sim \begin{cases} 2x+y = 0 \\ x + 2z = 0 \\ 2x+y = 0 \end{cases} \sim \begin{cases} y = -2x \\ z = -\frac{1}{2}x \end{cases}$$

Os aut. vet. são $(x, -2x, -\frac{1}{2}x)$, $x \in \mathbb{R}$.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$$

Autovet. associados a -2:

$$\begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix} = \begin{pmatrix} x+y \\ x-y+2z \\ 2x+y-z \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ -2 \end{pmatrix}$$

$$\therefore \begin{cases} x+y = -2x \\ x-y+2z = -2y \\ 2x+y-z = -2z \end{cases} \sim \begin{cases} 3x+y = 0 \\ x+y+2z = 0 \\ 2x+y+z = 0 \end{cases} \sim$$

$$\begin{cases} x+y+2z = 0 \\ -2y-6z = 0 \\ -y-3z = 0 \end{cases} \sim \begin{cases} x+y+2z = 0 \\ y+3z = 0 \end{cases} \sim \begin{cases} x-z=0 \\ y+3z=0 \end{cases}$$

$$\sim \begin{cases} x = z \\ y = -3z \end{cases}$$

Os aut. vet. são $(z, -3z, z)$, $z \in \mathbb{R}$.

Ex. 7.3 (f). Ache os autovalores e autovetores de

$$M = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Pol. caract.:

$$p_M(\lambda) = \det \begin{pmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{pmatrix} = \cdots = -(\lambda-1)(\lambda-4)(\lambda+1)$$

Os autovalores são: 1, 4 e -1.

Depois basta encontrar os autovet.

Autovet. associados a 1:

$$1 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+2z \\ x+2y+z \\ 2x+y+z \end{pmatrix}$$

$$\therefore \begin{cases} x+y+2z = x \\ x+2y+z = y \\ 2x+y+z = z \end{cases} \sim \begin{cases} y+2z = 0 \\ x+y+z = 0 \\ 2x+y = 0 \end{cases} \sim \begin{cases} y = -2z \\ x = z \end{cases}$$

Os autovet. são $(z, -2z, z)$, $z \in \mathbb{R}$.

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Autovet. associados a -1:

$$\begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} x+y+2z \\ x+2y+z \\ 2x+y+z \end{pmatrix}$$

$$\therefore \begin{cases} x+y+2z = -x \\ x+2y+z = -y \\ 2x+y+z = -z \end{cases} \sim \begin{cases} 2x+y+2z = 0 \\ x+3y+z = 0 \\ 2x+y+2z = 0 \end{cases} \sim$$

$$\sim \begin{cases} x+3y+z = 0 \\ -5y = 0 \end{cases} \sim \begin{cases} x = -z \\ y = 0 \end{cases}$$

Os autovet. são $(-z, 0, z)$, $z \in \mathbb{R}$.

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = -1 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \overset{(?)}{=} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Autovet. associados a 4:

$$\begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+2z \\ x+2y+z \\ 2x+y+z \end{pmatrix}$$

$$\therefore \begin{cases} x+y+2z = 4x \\ x+2y+z = 4y \\ 2x+y+z = 4z \end{cases} \sim \begin{cases} -3x+y+2z = 0 \\ x-2y+z = 0 \\ 2x+y-3z = 0 \end{cases}$$

$$\sim \begin{cases} x-2y+z = 0 \\ 5y-5z = 0 \\ -5y+5z = 0 \end{cases} \sim \begin{cases} x = z \\ y = z \end{cases}$$

Os autovet. são da forma (z, z, z), $z \in \mathbb{R}$.