6.1 Determine quais das seguintes funções são aplicações lineares:

(b) $g : \mathbb{R}^2 \to \mathbb{R}$ dada por $g(x,y) = xy$.

Não é transformação linear. De fato

$$g(1,0) + g(0,1) = 1\cdot 0 + 0\cdot 1 = 0,$$

$$g((1,0)+(0,1)) = g(1,1) = 1\cdot 1 = 1.$$

Portanto,

$$g((1,0)+(0,1)) \neq g(1,0) + g(0,1).$$

6.3 Resolva os items a seguir:

(a) Determine a transformação linear $T : \mathbb{R}^2 \to \mathbb{R}^3$ tal que $T(1,1) = (3,2,1)$ e $T(0,-2) = (0,1,0)$.

(b) Ache $T(1,0)$ e $T(0,1)$.

(c) Qual a transformação linear $S : \mathbb{R}^3 \to \mathbb{R}^2$ tal que $S(3,2,1) = (1,1)$, $S(0,1,0) = (0,-2)$ e $S(0,0,1) = (0,0)$?

(d) Calcule $P = S \circ T$.

(a) Primeiro, vamos escrever $(x,y) \in \mathbb{R}^2$ como comb. linear de $(1,1)$ e $(0,-2)$: vamos encontrar $a, b \in \mathbb{R}$ tais que

$$(x,y) = a(1,1) + b(0,-2) = (a, a-2b).$$

Daí

$$\begin{cases} a = x \\ a - 2b = y \end{cases} \sim \begin{cases} a = x \\ b = \frac{x-y}{2} \end{cases}.$$

Isso significa

$$(x,y) = x(1,1) + \left(\frac{x-y}{2}\right)(0,-2).$$

Assim,

$$T(x,y) = T\left(x(1,1) + \left(\frac{x-y}{2}\right)(0,-2)\right) =$$

$$= T(x(1,1)) + T\left(\left(\frac{x-y}{2}\right)(0,-2)\right) =$$

$$= x\,T(1,1) + \left(\frac{x-y}{2}\right) T(0,-2) = x(3,2,1) + \left(\frac{x-y}{2}\right)(0,1,0)$$

$$= (3x, 2x, x) + \left(0, \frac{x-y}{2}, 0\right) = \left(3x, \frac{5x-y}{2}, x\right).$$

(b) $T(1,0) = (3, 5/2, 1)$, $T(0,1) = (0, -1/2, 0)$.

(c) Vamos encontrar $a, b, c \in \mathbb{R}$ tais que

$$(x,y,z) = a(3,2,1) + b(0,1,0) + c(0,0,1) = (3a, 2a+b, a+c)$$

$$\therefore \begin{cases} 3a = x \\ 2a + b = y \\ a + c = z \end{cases} \sim \begin{cases} a = x/3 \\ b = y - 2x/3 \\ c = z - x/3 \end{cases}$$

Daí

$$(x,y,z) = \frac{x}{3}(3,2,1) + \left(y - \frac{2x}{3}\right)(0,1,0) + \left(z - \frac{x}{3}\right)(0,0,1).$$

Assim,

$$S(x,y,z) = \frac{x}{3} S(3,2,1) + \left(y - \frac{2x}{3}\right) S(0,1,0) + \left(z - \frac{x}{3}\right) S(0,0,1)$$

$$= \frac{x}{3}(1,1) + \left(y - \frac{2x}{3}\right)(0,-2) + \left(z - \frac{x}{3}\right)(0,0)$$

$$= \left(\frac{x}{3}, \frac{x}{3}\right) + \left(0, -2y + \frac{4x}{3}\right) = \left(\frac{x}{3}, -2y + \frac{5x}{3}\right).$$

Assim, $S(x,y,z) = \left(\frac{x}{3}, -2y + \frac{5x}{3}\right)$.

(d) $P = S \circ T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\qquad \mathbb{R}^2 \xrightarrow{T} \mathbb{R}^3 \xrightarrow{S} \mathbb{R}^2$

$$P(x,y) = S(T(x,y)) = S\left(3x, \frac{5x-y}{2}, x\right) =$$

$$= \left(\frac{3x}{3}, -2\left(\frac{5x-y}{2}\right) + \frac{5}{3}\cdot 3x\right) = (x, -5x + y + 5x) = (x,y)$$

Daí $P(x,y) = (x,y)$.

6.4 Determinar as matrizes das seguintes transformações lineares em relação às bases canônicas dos respectivos espaços:

(a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ definida por $T(x, y, z) = (x + y, z)$,

$\beta = \{(1,0,0), (0,1,0), (0,0,1)\}, \quad \beta' = \{(1,0), (0,1)\},$

$T(1,0,0) = (1,0)$
$T(0,1,0) = (1,0)$, $[T]^{\beta}_{\beta'} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$T(0,0,1) = (0,1)$

6.5 Seja $F : \mathbb{R}^3 \to \mathbb{R}^2$ definida por $F(x, y, z) = (x + z, y - 2x)$. Determinar $[F]^{\beta}_{\beta'}$, em que $\beta = \{(1, 2, 1), (0, 1, 1), (0, 3, -1)\}$ e $\beta' = \{(1, 5), (2, -1)\}$.

$F(1,2,1) = (1+1, 2-2\cdot 1) = (2,0)$
$F(0,1,1) = (1,1)$
$F(0,3,-1) = (-1,3)$

Encontrando $[(2,0)]_{\beta'}$, $[(1,1)]_{\beta'}$ e $[(3,-1)]_{\beta'}$:

$(2,0) = a(1,5) + b(2,-1) = (a+2b, 5a-b)$

$\Rightarrow \begin{cases} a + 2b = 2 \\ 5a - b = 0 \end{cases} \sim \begin{cases} 11a = 2 \\ 5a - b = 0 \end{cases} \sim \begin{cases} a = 2/11 \\ b = 10/11 \end{cases}$

$(1,1) = a(1,5) + b(2,-1) = (a+2b, 5a-b)$

$\Rightarrow \begin{cases} a + 2b = 1 \\ 5a - b = 1 \end{cases} \sim \begin{cases} 11a = 3 \\ 5a - b = 1 \end{cases} \sim \begin{cases} a = 3/11 \\ b = 4/11 \end{cases}$

$(-1,3) = a(1,5) + b(2,-1) = (a+2b, 5a-b)$

$\Rightarrow \begin{cases} a + 2b = -1 \\ 5a - b = 3 \end{cases} \sim \begin{cases} 11a = 5 \\ 5a - b = 3 \end{cases} \sim \begin{cases} a = 5/11 \\ b = -8/11 \end{cases}$

$$[F]^{\beta}_{\beta'} = \begin{pmatrix} 2/11 & 3/11 & 5/11 \\ 10/11 & 4/11 & -8/11 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 2 & 3 & 5 \\ 10 & 4 & -8 \end{pmatrix}$$

6.6 Para cada uma das transformações lineares abaixo determinar uma base e a dimensão do núcleo e da imagem:

(a) $T : \mathbb{R}^3 \to \mathbb{R}$ dada por $T(x, y, z) = x + y - z$.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ dada por $T(x, y) = (2x, x + y)$.

(a) $\operatorname{Ker} T = \{(x,y,z) \in \mathbb{R}^3 \mid 0 = T(x,y,z) = x+y-z\}$

$= \{(x,y,z) \in \mathbb{R}^3 \mid z = x+y\} = \{(x,y,x+y) \mid x,y \in \mathbb{R}\}$

$= \{(x,0,x) + (0,y,y) \mid x,y \in \mathbb{R}\}$

$= \{x(1,0,1) + y(0,1,1) \mid x,y \in \mathbb{R}\} = [(1,0,1),(0,1,1)]$.

$\{(1,0,1),(0,1,1)\}$ é l.i, pois se $a,b \in \mathbb{R}$ são tais que

$(0,0,0) = a(1,0,1) + b(0,1,1) = (a,b,a+b)$

$\Rightarrow \begin{cases} a = 0 \\ b = 0 \\ a+b = 0 \end{cases} \Rightarrow a=b=0.$

Daí, $\{(1,0,1),(0,1,1)\}$ é base de $\operatorname{Ker} T$ e $\dim(\operatorname{Ker} T) = 2$.

$\boxed{\text{Teorema do Núcleo e Imagem: } \dim V = \dim(\operatorname{Im} T) + \dim \operatorname{Ker} T}$

Portanto,

$$\underbrace{\dim \mathbb{R}^3}_{3} = \dim(\operatorname{Im} T) + \underbrace{\dim \operatorname{Ker} T}_{2} \Rightarrow \dim(\operatorname{Im} T) = 1.$$

Isso implica que $\operatorname{Im} T = \mathbb{R}$ e uma base é $\{1\}$.

Alternativamente, pode-se descrever diretamente quem é $\operatorname{Im} T$:

$\operatorname{Im} T = \{T(x,y,z) \mid (x,y,z) \in \mathbb{R}^3\} = \{x+y-z \mid x,y,z \in \mathbb{R}\} = \mathbb{R}$.

(b) $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (2x, x+y)$.

$\text{Ker}\, T = \{(x,y) \in \mathbb{R}^2 \mid (0,0) = T(x,y) = (2x, x+y)\}$

$= \{(x,y) \in \mathbb{R}^2 \mid 2x = 0 \text{ e } x+y = 0\} = \{(0,0)\}$.

Assim, $\dim \text{Ker}\, T = 0$.

Do Teo. do Nucleo e Im:

$$\underbrace{\dim \mathbb{R}^2}_{2} = \underbrace{\dim \text{Ker}\, T}_{0} + \dim(\text{Im}\, T) \Rightarrow \dim \text{Im}\, T = 2.$$

Portanto, $\text{Im}\, T = \mathbb{R}^2$, e daí $\dim(\text{Im}\, T) = 2$ e $\{(1,0),(0,1)\}$ é base de $\text{Im}\, T$.

6.7 Sejam $\alpha = \{(1,-1),(0,2)\}$, $\beta = \{(1,0,-1),(0,1,2),(1,2,0)\}$ bases de $\mathbb{R}^2$ e $\mathbb{R}^3$ respectivamente e

$$[T]^\alpha_\beta = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}$$

(a) Determine T

(c) Ache uma base γ de $\mathbb{R}^3$ tal que

$$[T]^\alpha_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$(1,-1) = a(1,-1) + b(0,2)$

$a = 1$, $b = 0$

$\Rightarrow [(1,-1)]_\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$(0,2) = a(1,-1) + b(0,2) \Rightarrow a = 0$ e $b = 1$

$\Rightarrow [(0,2)]_\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(a) $[T(1,-1)]_\beta = [T]^\alpha_\beta [(1,-1)]_\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Portanto, $T(1,-1) = 1 \cdot (1,0,-1) + 1(0,1,2) + 0 \cdot (1,2,0) = (1,1,1)$

$[T(0,2)]_\beta = [T]^\alpha_\beta [(0,2)]_\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Daí $T(0,2) = 0 \cdot (1,0,-1) + 1 \cdot (0,1,2) + (-1) \cdot (1,2,0) = (-1,-1,2)$

Vamos achar $a, b \in \mathbb{R}$ tais que

$$(x,y) = a(1,-1) + b(0,2) = (a, 2b-a)$$

$$\Rightarrow \begin{cases} a = x \\ -a + 2b = y \end{cases} \sim \begin{cases} a = x \\ b = \frac{x+y}{2} \end{cases}$$

Então,

$$(x,y) = x(1,-1) + \left(\frac{x+y}{2}\right)(0,2)$$

Daí

$$T(x,y) = xT(1,-1) + \left(\frac{x+y}{2}\right)T(0,2) =$$

$$= x(1,1,1) + \left(\frac{x+y}{2}\right)(-1,-1,2)$$

$$= \left(x - \frac{x+y}{2}, x - \frac{x+y}{2}, x + x+y\right) = \left(\frac{x-y}{2}, \frac{x-y}{2}, 2x+y\right).$$

(c) Ache base γ de $\mathbb{R}^3$ tal que $[T]^{\alpha}_{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\gamma = \{v_1, v_2, v_3\}$.

$$[T(1,-1)]_{\gamma} = [T]^{\alpha}_{\gamma}[(1,-1)]_{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow T(1,-1) = 1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 = v_1$$

Daí $v_1 = (1,1,1)$

$$[T(0,2)]_{\gamma} = [T]^{\alpha}_{\gamma}[(0,2)]_{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow T(0,2) = v_3 \Rightarrow v_3 = (-1,-1,2)$$

Assim, podemos tomar $v_2 \in \mathbb{R}^3$ qualquer, de modo que $\gamma = \{(1,1,1), v_2, (-1,-1,2)\}$ seja base de $\mathbb{R}^3$. Por exemplo, $v_2 = (0,1,0)$ serve.