4.6 24. Se

$$P_{\beta\to\beta'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix},$$

$\beta' = \{(1,1,1), (1,1,0), (1,0,0)\}$, então $\beta = ?$

Denote $\beta = \{v_1, v_2, v_3\}$.

Então

$$[v_1]_\beta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_1 = 1\cdot v_1 + 0\cdot v_2 + 0\cdot v_3$$

$$[v_1]_{\beta'} = P_{\beta\to\beta'}[v_1]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow v_1 = 1\cdot(1,1,1) + 0\cdot(1,1,0) + 0\cdot(1,0,0) = (1,1,1)$$

Da mesma forma,

$$[v_2]_\beta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = 0\cdot v_1 + 1\cdot v_2 + 0\cdot v_3$$

$$[v_2]_{\beta'} = P_{\beta\to\beta'}[v_2]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow v_2 = 0\cdot(1,1,1) + 3(1,1,0) + 1\cdot(1,0,0) = (4,3,0)$$

Por fim,

$$[v_3]_{\beta'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$\Rightarrow v_3 = 0\cdot(1,1,1) + 2(1,1,0) + 1(1,0,0) = (3,2,0)$

$\therefore \beta = \{(1,1,1), (4,3,0), (3,2,0)\}$

<u>4.6</u> 23(b). Seja $P_{S\to\beta} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}$,

$S = \{e_1, e_2, e_3\}$ base canônica, $\beta = ?$

Uma estratégia para resolver:

→ calcula-se

$$P_{\beta\to S} = (P_{S\to\beta})^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}^{-1}$$

→ repetir o argumento do exercício anterior.

$$P_{\beta\to S} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}^{-1} = \frac{1}{-5}\begin{pmatrix} -4 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & -1 \end{pmatrix}$$

Denote $\beta = \{v_1, v_2, v_3\}$. Então

$$[v_1]_S = P_{\beta\to S}[v_1]_\beta = \frac{1}{-5}\begin{pmatrix} -4 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 1/5 \\ -2/5 \end{pmatrix}$$

$$[v_2]_S = P_{\beta\to S}[v_2]_\beta = \begin{pmatrix} 1/5 \\ -1/5 \\ 2/5 \end{pmatrix}, \quad [v_3]_S = \begin{pmatrix} -2/5 \\ 2/5 \\ 1/5 \end{pmatrix}.$$

$\therefore \beta = \{(4/5, 1/5, -2/5), (1/5, -1/5, 2/5), (-2/5, 2/5, 1/5)\}.$

4.9 7. Determine se $T: \mathbb{R}^3 \to \mathbb{R}^2$ é transf. linear:

(a) $T(x,y,z) = (0,0) \longrightarrow$ sim
(b) $T(x,y,z) = (1,1) \longrightarrow$ não
(c) $T(x,y,z) = (3x-4y, 2x-5z) \longrightarrow$ sim
(d) $T(x,y,z) = (y^2, z) \longrightarrow$ não
(e) $T(x,y,z) = (y-1, x) \longrightarrow$ não

(a) $T(x,y,z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

10. Encontre $[T]$:

(a) $T(x_1,x_2) = (2x_1 - x_2, x_1 + x_2)$
(b) $T(x_1,x_2) = (x_1, x_2)$
(c) $T(x_1,x_2,x_3) = (x_1 + 2x_2 + x_3, x_1 + 5x_2, x_3)$.

(a) $[T] = \begin{pmatrix} | & | \\ [T(1,0)] & [T(0,1)] \\ | & | \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$T(1,0) = (2.1 - 0, 1+0) = (2,1) \Rightarrow [T(1,0)] = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$T(0,1) = (-1,1) \Rightarrow [T(0,1)] = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(b) $[T] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $[T] = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4.10

4. Sejam $T_1(x_1, x_2, x_3) = (4x_1, -2x_1 + x_2, -x_1 - 3x_2)$ e $T_2(x_1, x_2, x_3) = (x_1 + 2x_2, -x_3, 4x_1 - x_3)$.

(a) Encontre as matrizes canônicas de T_1 e T_2.

(b) Encontre as matrizes canônicas de $T_2 \circ T_1$ e $T_1 \circ T_2$.

(c) Use as matrizes encontradas na parte (b) para encontrar fórmulas para $T_1(T_2(x_1, x_2, x_3))$ e $T_2(T_1(x_1, x_2, x_3))$.

(a) $[T_1] = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{pmatrix}$, $[T_2] = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{pmatrix}$

(b)

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 0 \\ -2 & -4 & -1 \\ -1 & -2 & 3 \end{pmatrix}$$

Da mesma forma, $[T_2 \circ T_1] = [T_2][T_1] = \cdots$

(c)

$$T_1 \circ T_2(x,y,z) = [T_1 \circ T_2]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 & 8 & 0 \\ -2 & -4 & -1 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x+8y \\ -2x-4y-z \\ -x-2y+3z \end{pmatrix}$$

$$T_1 \circ T_2(x,y,z) = (4x+8y, -2x-4y-z, -x-2y+3z).$$

5.1

12. Em cada parte, encontre os autovalores por inspeção.

(a) $\begin{bmatrix} -1 & 6 \\ 0 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

(a) -1 e 5 (b) $3, 7$ e 1 (c) $-1/3,\ 1,\ 1/2$

$$\left(p(\lambda) = \det\begin{pmatrix} x+1 & -6 \\ 0 & x-5 \end{pmatrix} = (x+1)(x-5) \right)$$

5.1 9-11 (b)

$$A = \begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$, autovalores, autovetores?

Calculemos o polinômio característico:

$$p_A(\lambda) = \det(\lambda \cdot I - A) = \det\begin{pmatrix} \lambda-10 & 9 & \multicolumn{2}{c}{0_{2\times 2}} \\ -4 & \lambda+2 & & \\ \multicolumn{2}{c}{0_{2\times 2}} & \lambda+2 & 7 \\ & & -1 & \lambda-2 \end{pmatrix} =$$

$$= \det\begin{pmatrix} \lambda-10 & 9 \\ -4 & \lambda+2 \end{pmatrix} \det\begin{pmatrix} \lambda+2 & 7 \\ -1 & \lambda-2 \end{pmatrix} =$$

$$= \Big((\lambda-10)(\lambda+2)+36\Big)\Big((\lambda+2)(\lambda-2)+7\Big)$$

$$= (\lambda^2-8\lambda+16)(\lambda^2+3) = (\lambda-4)^2(\lambda^2+3)$$

Daí 4 é o único autovalor real de A.

Calculando autovetores:

$$A\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = 4\cdot\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$\begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \qquad \begin{pmatrix} 4x \\ 4y \\ 4z \\ 4w \end{pmatrix}$$

$$\begin{pmatrix} 10x - 9y \\ 4x - 2y \\ -2z - 7w \\ z + 2w \end{pmatrix}$$

$$\Longleftrightarrow \begin{cases} 10x - 9y = 4x \\ 4x - 2y = 4y \\ -2z - 7w = 4z \\ z + 2w = 4w \end{cases} \sim \begin{cases} 6x - 9y = 0 \\ 4x - 6y = 0 \\ -6z - 7w = 0 \\ z - 2w = 0 \end{cases} \overset{\text{(verifique)}}{\sim} \begin{cases} 2x = 3y \\ z = 0 \\ w = 0 \end{cases}$$

Daí

$$V_4 = \{(x,y,z,w) \in \mathbb{R}^4 \mid 2x = 3y,\ z = w = 0\} =$$

$$= \{(\tfrac{3}{2}y, y, 0, 0) \mid y \in \mathbb{R}\} = \text{ger}\{(3/2, 1, 0, 0)\}.$$

$1 \leq \dim V_\lambda \leq$ mult. alg. λ

5.2. 5. Seja A uma matriz 6×6, com

$$p_A(\lambda) = \lambda^2(\lambda-1)^1(\lambda-2)^3.$$

Quais possíveis dimensões dos autoespaços?

Sabe-se que seus autovalores são $0, 1$ e 2.

Ainda,

(i) $\dim V_0$ é 1 ou 2

(ii) $\dim V_1 = 1$

(iii) $\dim V_2$ é $1, 2$ ou 3.

16–17. Encontre os autovalores, as mult. alg. e geométricas respectivas, e determine se A é diag. Se sim, encontre P tal que $P^{-1}AP$ é diagonal, e calcule $P^{-1}AP$.

(17) $A = \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}$

· Pol. caract.:

$$p_A(\lambda) = \det(\lambda I - A) = \det\begin{pmatrix} \lambda+1 & -4 & 2 \\ 3 & \lambda-4 & 0 \\ 3 & -1 & \lambda-3 \end{pmatrix} =$$

$$= (\lambda+1)(\lambda-4)(\lambda-3) - 6 - 6(\lambda-4) + 12(\lambda-3)$$

$$= (\lambda+1)(\lambda^2-7\lambda+12) - 6 - 6\lambda + 24 + 12\lambda - 36$$

$$= \lambda^3 - 7\lambda^2 + 12\lambda + \lambda^2 - 7\lambda + 12 - 6 - 6\lambda + 24 + 12\lambda - 36$$

$$= \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda-1)(\lambda^2-5\lambda+6) = (\lambda-1)(\lambda-2)(\lambda-3).$$

Seus autovalores são 1, 2, 3, suas mult. alg. são 1, e portanto, suas mult. geo. são 1 também.

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 \,\underline{|\, \lambda - 1}$$
$$-5\lambda^2 \qquad \lambda^2 - 5\lambda + 6$$
$$6\lambda$$

Daí, A é diagonalizável, pois a mult. geo. de cada autovalor coincide com a resp. mult. alg.

Para determinar P: basta encontrar uma base de $\mathbb{R}^3$ formada por autovetores de A.

→ Calculo dos autoespaços:

$- V_1$:
$$\begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$\begin{pmatrix} -x+4y-2z \\ -3x+4y \\ -3x+y+3z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -x+4y-2z = x \\ -3x+4y = y \\ -3x+y+3z = z \end{cases} \sim \begin{cases} -2x+4y-2z = 0 \\ -3x+3y = 0 \\ -3x+y+2z = 0 \end{cases}$$

$$\underset{\text{(verifique)}}{\sim} \begin{cases} z = x \\ x = y \end{cases}$$

$$V_0 = \{(x,x,x) \mid x \in \mathbb{R}\} = \text{ger}\{(1,1,1)\}.$$

$-V_2$: $\begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} -x+4y-2z \\ -3x+4y \\ -3x+y+3z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -x+4y-2z = 2x \\ -3x+4y = 2y \\ -3x+y+3z = 2z \end{cases} \sim \begin{cases} -3x+4y-2z = 0 \\ -3x+2y = 0 \\ -3x+y+z = 0 \end{cases}$$

$$\underset{(\text{verifique})}{\sim} \begin{cases} 3x = 2y \\ y = z \end{cases}$$

$\therefore V_2 = \{(x,y,z) \in \mathbb{R}^3 \mid 3x = 2y,\ y = z\} =$

$= \{(\tfrac{2}{3}y, y, y) \mid y \in \mathbb{R}\} = \text{ger}\{(\tfrac{2}{3}, 1, 1)\}$

$-V_3$: $\begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} -x+4y-2z \\ -3x+4y \\ -3x+y+3z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -x+4y-2z = 3x \\ -3x+4y = 3y \\ -3x+y+3z = 3z \end{cases} \sim \begin{cases} -4x+4y-2z = 0 \\ -3x+y = 0 \\ -3x+y = 0 \end{cases}$$

(verifique) $\begin{cases} y = 3x \\ z = 4x \end{cases}$

$\therefore V_3 = \{(x, 3x, 4x) \mid x \in \mathbb{R}\} = ger\{(1,3,4)\}$.

$B = \{(1,1,1), (2/3, 1, 1), (1,3,4)\}$ é l.i. (pois estão associados a autovalores distintos), e portanto é base $\mathbb{R}^3$.

Daí

$$P = \begin{pmatrix} 1 & 2/3 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 4 \end{pmatrix}$$

é tal que $P^{-1}AP$ é diagonal (pela teoria desenvolvida).

Ainda,

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(nem precisa calcular P^{-1}!)

---

exemplo:

Se tomasse

$$M = \begin{pmatrix} 2/3 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 4 \end{pmatrix}, \quad \text{então}$$

$$M^{-1}AM = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

5.2 25. Dado

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

calcule A^n, em que $n \in \mathbb{N}$.

Passo 1: encontrar P tal que $P^{-1}AP$ é diagonal.

(verifique) se

$$P = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \text{ então } P^{-1} = \frac{1}{6}\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & -3 \\ -2 & 2 & -2 \end{pmatrix},$$

$$\text{e } P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Passo 2: Temos que

$$P^{-1}A^nP = (P^{-1}AP)^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}$$

$$\Rightarrow A^n = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}P^{-1} =$$

$$= \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}\begin{pmatrix} 1/6 & 1/3 & 1/6 \\ 1/2 & 0 & -1/2 \\ -1/3 & 1/3 & -1/3 \end{pmatrix} =$$

(verifique)

$$= \begin{pmatrix} 1+3^{n+1}+2\cdot 4^n & 2-2\cdot 4^n & 1-3^{n+1}+2\cdot 4^n \\ 2-2\cdot 4^n & 4+2\cdot 4^n & 2-2\cdot 4^n \\ 1-3^{n+1}+2\cdot 4^n & 2-2\cdot 4^n & 1+3^{n+1}+2\cdot 4^n \end{pmatrix}$$

4.3 Verifique se é l.i.

(a) $u_1 = (-1,2,4)$, $u_2 = (5,-10,-20)$ → não, pois $u_2 = -5u_1$

(b) $u_1 = (3,-1)$, $u_2 = (4,5)$, $u_3 = (-4,7)$ → não, pois são três vetores em $\mathbb{R}^2$