2.1: **4.** Seja

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -2 & 1 & 4 \end{bmatrix}$$

Encontre

(a) M_{32} e C_{32}. (b) M_{44} e C_{44}.
(c) M_{41} e C_{41}. (d) M_{24} e C_{24}.

(a) $M_{32} = \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 4 \end{vmatrix} = 0 - 9 - 3 - 0 - 12 - 6 = -30$

$C_{32} = (-1)^{3+2} M_{32} = 30.$

(b) $M_{44} = \begin{vmatrix} 2 & 3 & -1 \\ -3 & 2 & 0 \\ 3 & -2 & 1 \end{vmatrix} = 4 - 6 + 6 + 9 = 13,$ $C_{44} = (-1)^{4+4} M_{44} = 13.$

2.1: ▶ Nos Exercícios **15–18**, encontre todos os valores de λ com os quais ~~(A) = 0~~. ◀ $\det(A) = 0$.

15. $A = \begin{bmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{bmatrix}$ **16.** $A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$

15. $\det \begin{pmatrix} \lambda-2 & 1 \\ -5 & \lambda+4 \end{pmatrix} = (\lambda-2)(\lambda+4) + 5 = \lambda^2 + 2\lambda - 8 + 5 = \lambda^2 + 2\lambda - 3$
$= (\lambda+3)(\lambda-1)$

$\lambda^2 + 2\lambda - 3 = 0$

$\iff \lambda = \frac{-2 \pm \sqrt{4+12}}{2} = \frac{-2 \pm 4}{2} = -1 \pm 2 \iff \lambda = 1$ ou $\lambda = -3$.

Portanto, $\det \begin{pmatrix} \lambda-2 & 1 \\ -5 & \lambda+4 \end{pmatrix} = 0$ se e só se $\lambda = -3$ ou $\lambda = 1$.

16. $\det \begin{pmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda-1 \end{pmatrix} = (\lambda-4)\lambda(\lambda-1) + 0 + 0 - 0 - 0 - (\lambda-4) 2 \cdot 3$
$= (\lambda-4)\lambda(\lambda-1) - 6(\lambda-4) =$
$= (\lambda-4)(\lambda(\lambda-1) - 6)$

$$= (\lambda-4)(\lambda^2-\lambda-6) = (\lambda-4)(\lambda-3)(\lambda+2).$$

$$\lambda^2-\lambda-6=0$$

$$\Leftrightarrow \lambda = \frac{1\pm\sqrt{1+24}}{2} = \frac{1\pm 5}{2} \Leftrightarrow \lambda=3 \text{ ou } \lambda=-2.$$

Portanto, $\det A = 0$ se e só se $\lambda=4$, $\lambda=3$ ou $\lambda=-2$.

2.1: Calcule o determinante:

25. $A = \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$

$$\begin{vmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{vmatrix} = -3\,C_{33} + 3\,C_{43} = -3\cdot 128 + 3\cdot 48 =$$
$$= -3\cdot 80 = -240$$

$$C_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{vmatrix} = \cancel{12} - \cancel{12} + 100 - 20 - 12 + 60$$
$$= 128$$

$$C_{43} = -1\cdot M_{43} = -\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{vmatrix} = -(-24 + 10 - 40 + 6) = +48$$

$L_4' = L_4 + L_3$

$$\begin{vmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 6 & 11 & 0 & 2 \end{vmatrix} = -3\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 6 & 11 & 2 \end{vmatrix} =$$
$$= -3(\cancel{12} - 36 + 110 - 60 - \cancel{12} + 66)$$
$$= -3\cdot 80 = -240$$

2.1: 23.

$$A = \begin{pmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{pmatrix}$$

$$\det A = k^2 C_{13} + k^2 C_{23} + k^2 C_{33}$$

$$= k^2 \left(M_{13} - M_{23} + M_{33} \right) = k^2 \left(\begin{vmatrix} 1 & k \\ 1 & k \end{vmatrix} - \begin{vmatrix} 1 & k \\ 1 & k \end{vmatrix} + \begin{vmatrix} 1 & k \\ 1 & k \end{vmatrix} \right)$$

$$= k^2 \cdot 0 = 0.$$

2.1: Calcule o determinante:

32. $\begin{bmatrix} -3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 40 & 10 & -1 & 0 \\ 100 & 200 & -23 & 3 \end{bmatrix} =: M,$

$$\det M = (-3)\,2 \cdot (-1)\,3 = 18.$$

2.1: (b) Duas matrizes quadradas A e B podem ter o mesmo determinante se, e só se, forem de mesmo tamanho.

Falso.

Tome $A = (2)$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Então A e B possuem tamanhos distintos, porém

$$\det A = 2 = \det B.$$

2.3 ▶ Nos Exercícios **7–14**, use determinantes para decidir se a matriz é invertível. ◀

8. $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$

Relembre: M é invertível se e só se $\det M \neq 0$.

$$\begin{vmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{vmatrix} = -24 + 18 = -6 \neq 0.$$

Portanto, A é invertível.

2.3: **36.** Em cada parte, encontre o determinante, sabendo que A é uma matriz 4×4 com $\det(A) = -2$.

(a) $\det(-A)$ (b) $\det(A^{-1})$ (c) $\det(2A^T)$ (d) $\det(A^3)$

(a) $\det(-A) = \det((-1)\cdot A) = (-1)^4 \cdot \det A = -2.$

(b) Segue da teoria que

$$\det(A^{-1}) = \frac{1}{\det A} = -\frac{1}{2}.$$

Relembre: $\det A^T = \det A$

(c) $\det(2A^T) = 2^4 \cdot \det(A^T) = 16 \cdot \det(A) = 16(-2) = -32.$

Relembre: $\det(AB) = \det A \cdot \det B$

(d) $\det(A^3) = \det(A \cdot A^2) = \det(A)\det(A^2) = \det(A)\det(A \cdot A)$
$= \det(A) \cdot \det(A) \cdot \det(A) = (\det A)^3 = (-2)^3 = -8.$

2.3: **30.** Mostre que a matriz

$$A = \begin{bmatrix} \cos\theta & \operatorname{sen}\theta & 0 \\ -\operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

é invertível com qualquer valor de θ; em seguida, encontre A^{-1} usando o Teorema 2.3.6.

Basta mostrarmos que $\det A \neq 0$. Temos que

$$\det\begin{pmatrix} \cos\theta & \text{sen}\,\theta & 0 \\ -\text{sen}\,\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\cos\theta)^2 - (-\text{sen}\,\theta)\,\text{sen}\,\theta$$

$$= \cos^2\theta + \text{sen}^2\theta = 1 \neq 0.$$

Daí A é invertível, independente do valor de θ.

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \begin{pmatrix} \cos\theta & -\text{sen}\,\theta & 0 \\ \text{sen}\,\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$C_{11} = \cos\theta$, $C_{12} = (-1)(-\text{sen}\,\theta) = \text{sen}\,\theta$, $C_{13} = 0$
$C_{21} = (-1)^{2+1}\text{sen}\,\theta = -\text{sen}\,\theta$, $C_{22} = \cos\theta$, $C_{23} = 0$
$C_{31} = 0$, $C_{32} = 0$, $C_{33} = 1$

2.3: (c) Se A e B forem matrizes quadradas de mesmo tamanho e A for invertível, então

$$\det(A^{-1}BA) = \det(B)$$

Verdadeiro.
Pois

$$\det(A^{-1}BA) = \det(A^{-1})\det(BA) = \det(A^{-1})\det(B)\det(A)$$

$$= \frac{1}{\det(A)}\cdot\det(B)\det(A) = \frac{\cancel{\det(A)}\det(B)}{\cancel{\det(A)}} = \det(B)$$

Portanto,

$$\det(A^{-1}BA) = \det B.$$

2.1: Calcule o determinante:

26. $A = \begin{bmatrix} 4 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & -1 & 0 \\ 1 & 2 & 4 & 2 & 3 \\ 9 & 4 & 6 & 2 & 3 \\ 2 & 2 & 4 & 2 & 3 \end{bmatrix}$

$C_1' = C_1 - 4C_4$

$$\det \begin{bmatrix} 4 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & -1 & 0 \\ 1 & 2 & 4 & 2 & 3 \\ 9 & 4 & 6 & 2 & 3 \\ 2 & 2 & 4 & 2 & 3 \end{bmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 & 0 \\ 7 & 3 & 3 & -1 & 0 \\ -7 & 2 & 4 & 2 & 3 \\ 1 & 4 & 6 & 2 & 3 \\ -6 & 2 & 4 & 2 & 3 \end{vmatrix} =$$

$$= (-1)^{1+4} \begin{vmatrix} 7 & 3 & 3 & 0 \\ -7 & 2 & 4 & 3 \\ 1 & 4 & 6 & 3 \\ -6 & 2 & 4 & 3 \end{vmatrix} = - \begin{vmatrix} 7 & 3 & 3 & 0 \\ -7 & 2 & 4 & 3 \\ 8 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -(-1)^{4+1} \begin{vmatrix} 3 & 3 & 0 \\ 2 & 4 & 3 \\ 2 & 2 & 0 \end{vmatrix}$$

$L_3' = L_3 - L_2$

$L_4' = L_4 - L_2$

$$= (-1)^{2+3} \cdot 3 \begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} = 0.$$

2.3: ▶ Nos Exercícios **15–18**, encontre os valores de k com os quais A é invertível. ◀

15. $A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$ **16.** $A = \begin{bmatrix} k & 2 \\ 2 & k \end{bmatrix}$

15. Basta encontrar k tais que $\det A \neq 0$.

$$\begin{vmatrix} k-3 & -2 \\ -2 & k-2 \end{vmatrix} = (k-3)(k-2) - 4 = k^2 - 5k + 6 - 4 = k^2 - 5k + 2$$

$$k^2 - 5k + 2 = 0 \iff k = \frac{5 \pm \sqrt{25-8}}{2} = \frac{5 \pm \sqrt{17}}{2}.$$

Então, p/ $k \neq \frac{5+\sqrt{17}}{2}$ e $k \neq \frac{5-\sqrt{17}}{2}$, A é invertível.

16.

$$\begin{vmatrix} k & 2 \\ 2 & k \end{vmatrix} = k^2 - 4 = (k+2)(k-2).$$

Então, para $k \neq -2$ e $k \neq 2$, A é invertível.

2.3:

▶ Nos Exercícios **24–29**, resolva usando a regra de Cramer, quando aplicável. ◀

24. $7x_1 - 2x_2 = 3$
$3x_1 + x_2 = 5$

25. $4x + 5y = 2$
$11x + y + 2z = 3$
$x + 5y + 2z = 1$

24. Seja

$$A = \begin{pmatrix} 7 & -2 \\ 3 & 1 \end{pmatrix}, \text{ então } \det A = 7 + 6 = 13 \neq 0.$$

Então podemos aplicar Regra de Cramer. A única solução é (x_1, x_2) em que

$$x_1 = \frac{1}{\det A} \begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = \frac{13}{13} = 1$$

$$x_2 = \frac{1}{\det A} \begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix} = \frac{35-9}{13} = 2.$$

25. Seja

$$A = \begin{pmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{pmatrix},$$

então $\det A = 8 + 10 - 110 - 40 = -132 \neq 0$.
Daí, a solução (x_1, x_2, x_3) é

$$x_1 = \frac{1}{\det A} \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = \frac{4 + 10 - 30 - 20}{-132} = \frac{-36}{-132} = \frac{3}{11}$$

$$x_2 = \frac{1}{\det A}\begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} = \frac{24+4-44-8}{-132} = \frac{-24}{-132} = \frac{2}{11}$$

$$x_3 = \frac{1}{\det A}\begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} = \frac{4+15+110-2-55-60}{-132} =$$

$$= \frac{129-117}{-132} = \frac{12}{-132} = -\frac{1}{11}$$