Problem(s) in LPP

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Introduction

- $R = (R, +, \cdot, 1)$ - unital ring (associative).

- **Definition**: Let $L : R \to R$ be an additive map. We say L leaves the set H invariant if $L(H) \subset H$. The map L strongly preserves H when $L(H) = H$.

- **Definition** Let $L : R \to R$ be an additive map. For a given relation $\sim$ in R, we say that L preserves $\sim$ when $L(A) \sim L(B)$ whenever $A \sim B$. When $L(A) \sim L(B)$ if and only if $A \sim B$ we say that L strongly preserves the relation $\sim$.

$R = M_n(\mathbb{F})$

- $M_n(\mathbb{F}) = M_n$ to denote the algebra of all $n \times n$ matrices with entries in $\mathbb{F}$ = algebraically closed field of zero characteristic.
- $R_k = \{A \in M_n \mid A \quad \text{has rank k}\}$ $(1 \leq k \leq n)$; $U = R_n$.
- $N = \{A \in M_n \mid A \quad \text{is nilpotent}\}$.
- $P = \{A \in M_n \mid A \quad \text{is potent}\}$.
- $\mathbb{E} = \{A \in M_n \mid A^2 = A\}$.
- $S = \{A \in M_n \mid A \quad \text{is algebraic of degree 2}\}$.

For now, let us suppose that the map $L : M_n \to M_n$ is linear and that also preserves one (and only one) of the above sets.

For instance, if the map L is either of the form (1) or (2) - the so called standard form - where

$$L(X) = M_1 X M_2, \tag{1}$$

and

$$L(X) = M_1 X^t M_2, \tag{2}$$

then L preserves any of the previous sets - here M_1, M_2 are suitable invertible matrices. The problem now is to decide - modulo the center of M_n and up to some nonzero constant (in some cases) - if every linear map that preserves any of the above mentioned sets has necessarily the standard form.

This question was answered positively:

- $H = R_k$ and $\mathbb{F} = \mathbb{C}$ by L. Beasley [1][1].
- $H = R_1$ by M. Marcus and B. Moyls [20][2]
- $H = N$ and $L(N) = N$ by P. Botta, S. Pierce, and W. Watkins [2][3] ($M_2 = M_1^{-1}$).
- $H = P$ by M. Brešar and P. Šemrl [5][4] ($\mathbb{F} = \mathbb{C}$ and $M_2 = M_1^{-1}$).

[1] L. B. Beasley. *Linear transformations on matrices: the invariance of rank k matrices*. Linear Algebra Appl., **3 (4)** (1970), 407-427. http://dx.doi.org/10.1016/0024-3795(70)90033-9.

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[4] M. Brešar, P. Šemrl. *Linear transformations preserving potent matrices*. Proc. Amer. Math. Soc., **119 (1)** (1993), 81–86. http://dx.doi.org/10.2307/2159827.

- $H = \mathbb{E}$ by M. Brešar and P. Šemrl [6][5] ($M_2 = M_1^{-1}$).
- $H = S$ and $L(S) = S$ by M. Alves and W. Franca [14][6] ($M_2 = M_1^{-1}$).
- $H = U$ by M. Marcus and R. Purves [21][7].

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The idea behind the proof for S and $n \geq 3$

For convenience, for each $\lambda \in \mathbb{F}$ and $k \in \{1, \ldots, n-1\}$, we set the following elements

$$A(\lambda, k) = \left[\begin{array}{c|c} \lambda \cdot I_k & 0 \\ \hline 0 & (\frac{-k\lambda}{n-k}) \cdot I_{n-k} \end{array}\right]. \tag{3}$$

$$\Gamma = \{A \in S \mid A^2 = 0\} = \{A \in M_n(\mathbb{F}) \setminus \{0\} \mid A^2 = 0\}.$$

- Lemma1: Let $T \in \Gamma$. Then, rank $T \leq [\frac{n}{2}]$.
- Lemma 2: Let $B \in S$ such that tr $(B) = 0$. Then, either $B \in \Gamma$, or there exist $\beta \in \mathbb{F}^*$ and $\tilde{k} \in \{1, \ldots, n-1\}$ such that B is similar to $A(\beta, \tilde{k})$ (as in (3)). In particular, either rank $B \leq [\frac{n}{2}]$ (when $B \in \Gamma$) or $B \in GL_n(\mathbb{F})$. In this last case we can assume that $\tilde{k} \in \{[\frac{n+1}{2}], \ldots, n-1\}$.

- Lemma 3: Let $\lambda \in \mathbb{F}^*$, $B \in S$ and $k \in \{1, \cdots, n-1\}$, where $k \neq \frac{n}{2}$. Let us consider $A = A(\lambda, k)$ as in (3). Let us suppose that

$$(A + B) \in \Gamma. \tag{4}$$

  Then $B \in S \setminus \Gamma$.
- Lemma 4: Let A and B be two elements of $S \setminus \Gamma$. If $\operatorname{tr}(A) = \operatorname{tr}(B) = 0$ and $(A + B) \in \Gamma$ then $(B - A) \in S$.
- Proposition 1: Let $X, Y \in \Gamma$. Let us suppose that $(X + Y)$ is similar to $A(\lambda, k)$ (as in (3)), for given $\lambda \in \mathbb{F}^*$ and $k \in \{1, \ldots, n-1\}$. Then, n is even and $k = \frac{n}{2}$. Moreover, $(X - Y) \in S$.
- Corollary: Let $X, Y \in \Gamma$. Let us suppose that $(X + Y) \in S$ and $(X - Y) \notin S$. Then, $(X + Y) \in \Gamma$.

- Proposition 2: Let $A \in M_n(\mathbb{F}) \setminus \{0\}$. Then $A = aI$ for some nonzero scalar a if and only if

$$A + B \in S \iff B \in S \tag{5}$$

for all $B \in M_n(\mathbb{F})$.

- Propostion 3: Let $f : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a linear map such that $f(S) = S$. Then, the map $g : M_n(\mathbb{F}) \to M_n(\mathbb{F})$; $X \longmapsto f(X) - \left(\frac{\operatorname{tr}(f(X)-X)}{n}\right) \cdot I$ satisfies the following conditions:
i) $g(S) = S$, that is, g strongly preserves S;
ii) g preserves trace, that is, $\operatorname{tr}(X) = \operatorname{tr}(g(X))$ for all $X \in M_n(\mathbb{F})$.

Question 1:

Can we obtain a similar result for S_j - the set of all matrices algebraic of degree j with $2 < j \leq m$.

Maps preserving multiplicative commutators

In 1961 I. N. Herstein, at the end of his AMS Hour talk, suggested the following problem:
Characterize all bijective additive mappings $\theta : R \to R'$, where R and R' are two unital simple rings, satisfying the condition $\theta(xyx^{-1}y^{1}) = \theta(x)\theta(y)\theta(x)^{-1}\theta(y)^{-1}$ for all invertible $x, y \in R$.

From the above we may derive directly the following:

- θ preserves invertibility, that is, $\theta(U) \subset U'$.
- θ preserves multiplicative commutators.

The condition $\theta(xyx^{-1}y^{-1}) = \theta(x)\theta(y)\theta(x)^{-1}\theta(y)^{-1}$ yields:

- $\theta(1) = 1$, that is, θ is unital.
- $\theta(x)\theta(y) = \theta(y)\theta(x)$ whenever $xy = yx$, where $x, y \in U$.

Hereafter, let us assume that:

- θ is bijective and linear (over its center), and $R = R'$.
- $\theta(U) = U$.

In this setting, we may define the map $G : R \times R \to R$, where $G(r_1, r_2) = \theta(\theta^{-1}(r_1)\theta^{-1}(r_2))$ for all $r_1, r_2 \in U$.

From the fact that each element $x \in U$ may be written as $x = \theta(y)$ with $y \in U$, we may infer that

$$[\theta(y^2), \theta(y)] = G(x,x)x - xG(x,x) = 0 \quad \text{for all} \quad x \in U. \tag{6}$$

An bilinear (biadditive) map G fulfilling the condition above possesses what is called a *commuting trace* on U. This is an example of a Functional Identity (F.I.) - in this case on a subset which is not closed under addition.

- This approach of constructing the map G as before was first introduced by M. Brešar when he was investigating linear bijective maps $L : S \to S'$ that locally preserves commutativity, that is, $[L(s^2), L(s)] = 0$ for all $s \in S$ [7, Theorem 5.5][8].
- In the case that S is a (unital) prime ring and $\text{char}(S) \neq 2$, it is known [7, Theorem 4.1] that a biadditive map $D : S \times S \to S$ whose trace is commuting on S has the form

$$D(s, s) = \lambda s^2 + \mu(s)s + \nu(s, s) \qquad (7)$$

where λ lies in $\mathcal{Z}(S)$ (=the center of S), $\mu : S \to \mathcal{Z}(S)$ is additive and $\nu : S \times S \to \mathcal{Z}(S)$ is biadditive.

[8] M. Brešar, *Commuting maps: a survey*, Taiwanese J. Math. **8** (2004), no. 3, 361-397.

The case $R = M_n$

- In [12][9] we showed that every biadditive map $G : M_n \times M_n \to M_n$ whose trace is commuting on U has the standard form (7).

  In the same work we have proved that if $\theta : M_n \to M_n$ is a bijective linear map satisfying the following conditions:
- $\theta(U) = U$.
- $[\theta(x^2), \theta(x)] = 0$ for all $x \in U$.

  Then, $\theta(y) = \lambda\phi(y)$ for all $y \in M_n$ where ϕ is a (anti)isomorphism.

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A more general case

- **Theorem** (2016): Let R be a unital simple ring and let $G : R \times R \to R$ be an biadditive map. Let us assume that the following conditions holds:
  i) The prime field of $\mathcal{Z}(R)$ contains at least 11 elements.
  ii) G is commuting on U.
  iii) $\mathbb{E} = \{e \in R \mid e^2 = e\}$ spans R as a vector space over its center $\mathcal{Z}(R)$.
  iv) $G(1, r) = G(r, 1) = r$ for all $r \in R$.

  Then, the trace of G has the form (7).

- **Corollary:** Let $\theta : R \to R$ be a bijective map which is linear over $\mathcal{Z}(R)$. In addition, let us suppose:

$$\theta(xyx^{-1}y^{-1}) = \theta(x)\theta(y)\theta(x)^{-1}\theta(y)^{-1} \quad \text{for all} \quad x, y \in U.$$

Furthermore, let us assume that the following conditions are fulfilled:
i) $\theta(U) = U$.
ii) The prime field of $\mathcal{Z}(R)$ contains at least 11 elements.
iii) R does not satisfy S_4.
iv) $\mathbb{E}$ spans R as a vector space over its center $\mathbb{Z}$. Besides, let us assume that there exist $e_1, e_2 \in \mathbb{E}$ such that $(e_1 + e_2) \in \mathbb{E}$.
Then, θ is an isomorphism.

Question2:

Can we obtain a similar result under the assumption that R is a (semi)simple Banach algebra?

THANK YOU!!!!

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Historical background

In 1961 Herstein posed several problems about Jordan, Lie, and group structure of rings. In particular, he was interested in the description of Jordan homomorphisms, Lie homomorphisms and maps preserving multiplicative commutators.

- **Definition**: Let R and S two rings. An additive map $f : R \to S$ is called a Lie homomorphism if $f(xy - yx) = f(x)f(y) - f(y)f(x)$ for all $x, y \in R$. In the same fashion, a map is called a Jordan homomorphism if $f(xy + yx) = f(x)f(y) + f(y)f(x)$ holds.

Examples of Lie maps: Homomorphisms and the negative of anti-homomorphisms.

- In 1951 Hua characterized Lie automorphisms of a simple Artinian ring $R = M_n(D)$ for $n \geq 3$, where D is a division ring whose characteristic is neither 2 nor 3. He showed that such maps are of the form $\phi + \tau$, where ϕ is an automorphism or the negative of an antiautomorphism of R and τ is an additive mapping of R into its center which send commutators to zero.

- In 1956 Herstein showed any Jordan authomorphism of a simple ring R, where its characteristic is neither 2 nor 3, is either an authomorphism or an anti-isomorphism.

- In 1963 Martindale characterized Lie isomorphisms $\theta : R \to S$ of simple rings under the assumption that characteristic of R is different from 2 and 3 and that R contains three nonzero orthogonal idempotents whose sum is the identity. He showed that such maps are of the form $\phi + \tau$, where ϕ is an isomomorphism or the negative of an anti-isomomorphism of R onto S and τ is an additive mapping of R into the center of S sending commutators to zero.

In 1993 Brešar used the description of commuting maps to characterize Lie homomorphisms of prime rings. This was the first result on Lie maps where the basic results of functional identities have been applied.

- **Theorem** : (Brešar, 1993) Let R and S be unital simple rings such that the characteristic of R and S is not 2. Let $\theta : R \to S$ be a Lie isomorphism. If neither R nor S satisfies S_4 then θ is of the form $\phi + \tau$, where ϕ is an isomomorphism or a negative of an antiisomomorphism of R into S, and τ is an additive mapping of R into the center of S sending commutators to zero.

The Theorem above was proved under the additional technical assumption that R and S do not satisfy S_4. This assumption was removed by Blau in 2002 who used the classical structure theory of PI rings together with some Martindale's results. Later, in 2003, Brešar and Šemrl found another more straightforward proof based only on commuting maps.