Least degrees of identities

Plamen Koshlukov

Department of Mathematics
State University of Campinas, SP, Brazil

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Outline

Main Notions I

- F is a field, A an F-algebra, L a Lie algebra, J a Jordan algebra.
- $F\langle X\rangle$ the free associative algebra freely generated by $X = \{x_1, x_2, \ldots\}$ over F (we do not require $1 \in F\langle X\rangle$).
- The elements of $F\langle X\rangle$ are the (non-commutative) polynomials in the x_i.
- If $f(x_1, \ldots, x_n) \in F\{X\}$ and $f(a_1, \ldots, a_n) = 0$ for *every* $a_i \in A$ then f is a *polynomial identity* for A.
- The set of all PI's for A is a T-ideal $Id(A)$ in $F\langle X\rangle$.
- $L(X)$ is the free Lie algebra freely generated by X over F.
- $J(X)$ is the free Jordan algebra freely generated by X over F.

Polynomial identities I

- If A is associative, the product $[a, b] = ab - ba$ makes it a Lie algebra $A^{(-)}$. Every Lie algebra is a subalgebra of some $A^{(-)}$ (PBW Theorem).
- By a theorem of Witt, $L(X)$ is isomorphic to the Lie algebra of $F\langle X\rangle$ generated by X.
- If A is associative, and F of characteristic different from 2, the product $a \circ b = (ab + ba)/2$ makes it a Jordan algebra $A^{(+)}$. The Jordan algebras $A^{(+)}$ and their subalgebras are special, the remaining are exceptional.
- The Jordan algebra $SJ(X)$ inside $F\langle X\rangle^{(+)}$ generated by X is the free special Jordan algebra.
- Pay attention: $J(X) \cong SJ(X)$ if and only if $|X| \leq 2$. Moreover, $SJ(X)$ has homomorphic images that are exceptional.

Concrete identities I

- As in the associative case one defines identities for Lie and Jordan algebras.
- Standard polynomial $s_n = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ (associative).
  If $\dim A = n$ then A satisfies s_k, $k > n$. (The proof is easy: just recall how you prove a determinant of an $m \times m$ matrix is 0 whenever its rows, or columns, are LD.)

Concrete identities II

- (Amitsur, Levitzki) $M_n(F)$ satisfies s_{2n} and satisfies no identities of degree $< 2n$.
  This is much harder. Several proofs due to Amitsur and Levitzki, Kostant, Swan, Razmyslov, Procesi, and Rosset.
  For 2×2 matrices it is easy. Take a basis of $M_2(F)$: I, A, B, C. One writes s_4 as sums of products of two commutators easily:

$$2([x_1, x_2] \circ [x_3, x_4] - [x_1, x_3] \circ [x_2, x_4] + [x_1, x_4] \circ [x_2, x_3])$$

  Then one cannot substitute I in a commutator, hence 3 basic elements remain to substitute. But s_4 is alternating in 4 variables hence it vanishes.
  If $n > 2$ it requires rather delicate arguments.

Concrete identities III

- For $n = 2$ one knows the whole picture of the identities of $M_2(F)$. That is s_4 and $h = [[x_1, x_2]^2, x_3]$ generate all of them. In characteristic 0: Razmyslov, Drensky. If F is infinite and of characteristic $p > 3$ the same holds (PK). If $p = 3$, one more identity of degree 6 is required.
- A theorem of Kemer: in characteristic 0, the identities of $M_2(F)$ follow from s_4 "only": that is asymptotically, for n large enough, “all” identities of degree n for $M_2(F)$ follow from s_4. And the Hall polynomial $[[x_1, x_2]^2, x_3]$ is needed only to settle identities of low degree.
- For $n > 2$ it becomes worse: A big open problem!
  Not only computational but (my guess) lack of adequate methods.

Some reductions I

- If A is an algebra (associative, Lie, Jordan, or whatever), and $f(x_1, \ldots, x_n)$ an identity of A we can linearize it and obtain a multilinear identity for A. It is a consequence of f. How to linearize a polynomial: recall the method of producing a bilinear form starting from quadratic one (basic Linear Algebra). Sometimes this is called polarization. The reverse process: symmetrizing (restituting) is analogous.
- If F is an infinite field every identity for A is equivalent to a (finite) collection of multihomogeneous identities.
- If F is of characteristic 0, every identity for A is equivalent to a (finite) collection of multilinear identities.

Some reductions II

- The PBW theorem produces a basis of $F\langle X\rangle$ starting from a basis of $L(X)$. We fix an order on the basis of $L(X)$ by the degree first and then for commutators of the same degree, arbitrarily.
- A standard argument shows that over an infinite field, if A is unital associative algebra, then its identities are determined by its proper identities.
- A polynomial in $F\langle X\rangle$ is proper if it is a linear combination of products of commutators.
- This simply means that if $f(x_1, \dots, x_m)$ is multihomogeneous of degree m_i in x_i, , substituting x_i by $x_i + 1$ and taking the component of degree $m_i - 1$ one gets 0 for every i.

Some reductions III

- If f is multilinear one can interpret the above as taking the partial derivative in x_i; that is why Specht called the proper polynomials "constants".

Identities in concrete algebras I

Let A be an associative algebra. Clearly the following problem arises.

Question

Find all polynomial identities for A.

But it is too general to expect a meaningful answer.

Question, slightly modified

Find generators for $T(A)$, the T-ideal of A.

Here we mean generators as a T-ideal (recall that these are ideals in the free algebra that are closed under endomorphisms).

Identities in concrete algebras II

Theorem, Kemer

If A is associative and if F is of characteristic 0, then $T(A)$ is finitely generated.

Analogous results hold for large classes of Lie algebras (Iltyakov), Jordan algebras (Vais and Zelmanov), alternative algebras (Iltyakov).
In characteristic $p > 0$ Kemer's theorem fails.
For Lie algebras the first examples were given by Vaughan-Lee (in characteristic 2), and later on by Drensky (in any characteristic $p > 0$).
For associative algebras, the examples were much harder. The first ones were given by Belov, Grishin, Shchigolev, almost simultaneously.

Identities in concrete algebras III

Question, further modified

Find the identities of least degree for A.
Analogously for Lie and for Jordan algebras.

Clearly we want to study "important" algebras A.

Associative algebras

Full matrix algebras

Amitsur and Levitzki!

- The least degree of an identity for $M_n(F)$ is $2n$.
- Up to a scalar multiple, s_{2n} is the only identity of degree $2n$.
- There is a little known exception. If $|F| = 2$ and $n = 2$ then there are two more identities of degree 4.
- If $n > 2$ and if F is of characteristic 0, then all identities of degree $2n + 1$ also follow from s_{2n}.
- The consequences of s_m were studied in detail by Benanti and Drensky.

So for matrix algebras we have a reasonably good answer to our problem.

Lie algebras

- There is no analogue of the AL Theorem for simple Lie algebras. Not even for $sl_n(F)$.
- The identities of $sl_2(F)$ are known: Razmyslov, Drensky in characteristic 0, and Vasilovsky, when F is infinite of characteristic $p \neq 2$.
- The identities of the remaining simple Lie algebras are not known. It is known that for the Witt algebra, the least degree identity is

$$\sum(-1)^{\sigma}[x_0, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}], \sigma \in S_4,$$

  of degree 5. This is the analog of the standard identity in the Lie case.
- The same polynomial is an identity for $sl_2(F)$. It is of least degree for $sl_2(F)$.

Jordan algebras

Even less is known for the identities in simple Jordan algebras.

- If V is a vector space with a symmetric bilinear form b then $B = F \oplus V$ becomes a Jordan algebra with the product

$$(\alpha + u) \circ (\beta + v) = (\alpha\beta + b(u, v)) + (\alpha v + \beta u).$$

  It is special. It is simple if b is nondegenerate and $\dim V > 1$.
- The identities of B are known (Iltyakov, Vasilovsky).
- The least degree of an identity for B equals 5.
- Thus the least degree of an identity for the 2×2 symmetric matrices is also 5. (This algebra is of the type B when $\dim V = 2$.)

Basics on standard polynomials I

- The standard polynomial s_m is not a Lie element (that is, it does not belong to $L(X)$) whenever $n > 2$.
- It is not a proper polynomial if m is odd. But it is proper if m is even.
- That is why we consider the polynomial

$$\ell_m = \sum (-1)^\sigma [x_0, x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)}], \sigma \in S_m,$$

  of degree $m+1$ and call it the Lie standard polynomial of degree m.
- Clearly if $\dim L = m$ then ℓ_k is an identity whenever $k > m$, thus ℓ_4 must be an identity for $sl_2(F)$.
- It follows from $\dim sl_m(F) = m^2 - 1$ that ℓ_{m^2} is an identity for $sl_m(F)$.
- But is it of least degree?

Concrete identities I

A negative answer

NO! ℓ_9 is of degree 10.

- Drensky and Kasparian (1983) studied the identities in $M_3(F)$ in characteristic 0.
- They used the representation theory of the symmetric group S_n, direct and heavy computations, and the "equivalent" theory of polynomial representations of the general linear groups GL_m.
  Recall that passing to GL_m has the advantage of considering polynomials in "fewer" variables. But one loses the multilinearity.
- They proved that all identities of degree 8 for $M_3(F)$ follow from s_6.

Concrete identities II

- They also described the central polynomials of low degree for $M_3(F)$: they proved that the least degree of a central polynomial for this algebra equals 8.
- For $sl_3(F)$ their results read as follows.
- There is no identity for $sl_3(F)$ of degree 7 or less.
- The Lie algebra $sl_3(F)$ satisfies identities of degree 8.
- But it does not satisfy the identity ℓ_7.
- Hence one cannot expect a direct analogue of the AL Theorem in the case of $sl_m(F)$.

More general setting I

A paper by I. Benediktovich and A. Zalesskii (1979, in Russian only) studies the following.

- Let F be of characteristic 0. Define the almost standard polynomials as the multilinear polynomials

$$\sum_{\sigma\in S_m}\sum_{i=1}^{m+1}(-1)^{\sigma}\alpha_i x_{\sigma(1)}\cdots x_{\sigma(i-1)}x_{m+1}x_{\sigma(i)}\cdots x_{\sigma(m)}$$

  Denote the above by $a_m(k)$ whenever $\alpha_k=1$, and $\alpha_i=0$ if $i\neq k$.

- Denote by a_m^+ the sum of all $a_m(k)$ where k is even, and by a_m^- the one where k is odd.

Problems and results I

Problem

Describe the almost standard identities for $M_n(F)$.

- The polynomials a_m^+, a_m^-, and $a_m(k)$ for $k > n$ and $k < m - n + 2$ are consequences of s_n whenever $m \geq n$. (Direct computation, easy.)
- The polynomial $a_{4n-4}(2n-2)$ is not an identity for $M_n(F)$. This is trickier. As the polynomials are multilinear it suffices to evaluate them on the matrix units e_{ij}. Substitute the variables x_1 to x_{4n-4} as follows:
  The first $n-2$ of these by $e_{i,i+1}$
  The next n variables by $e_{i,i}$
  The next $n-1$ variables by $e_{i,n}$
  The next $n-1$ variables by $e_{n,i}$

Problems and results II

Put $e_{n,n}$ for x_{4n-3}.

- Then use the (obvious) interpretation in the language of directed graphs and paths: the vertices go from 1 to n, and the edges (arrows) are the $e_{i,j}$.
  The argument is not immediate though it goes along the lines of proof to the AL theorem given by R. Swan. In fact here it is easier (as one should expect!)

The theorem for matrices

Theorem

Let F be of characteristic 0. An almost standard polynomial f (with $m \geq 2n$) is an identity for $M_n(F)$ if and only if f is a linear combination of the polynomials a_m^-, a_m^+, and $a_m(k)$ where $k > 2n$ and $k < m - 2n + 2$.

Corollary

The Lie polynomial ℓ_m (of degree $m + 1$) is an identity for $sl_n(F)$ (or, to that effect, for $M_n(F)^{(-)}$) if and only if $m \geq 4n - 4$.

Remarks I

- The theorem and the corollary above give the least degree of a standard Lie identity for $sl_n(F)$. It equals $4n - 3$.
- When $n = 2$ this is the exact value. But recall that there is another identity of degree 5 for $sl_2(F)$ which does not follow from ℓ_4.
- When $n = 3$ the bound is not precise, it gives an identity of degree 9. And we saw there is one of degree 8 (although not standard Lie identity).
- What about the following conjecture:
  Does the least degree of an identity for $sl_n(F)$ equal $3n - 1$?
- What about the remaining simple f.d. Lie algebras? (Assuming F is of characteristic 0 and algebraically closed.)

Remarks II

- If L is simple but infinite dimensional the problem is wilder.
- For Jordan algebras it seems to be more difficult.
- If we take the upper triangular matrices UT_n as an associative or Lie algebra, the identities are easy to deduce.
- But what about the identities of the upper triangular matrices as a Jordan algebra? The least degree?
- If n is even, this least degree is $2n$ because $[x, y]^2$ is a Jordan element.
- But if n is odd? Does this least degree equal $2n$, or is it $2n + 1$?
- Clearly one may ask all of the above for graded algebras, for superalgebras.
- There are results due to many people concerning the involutive case.

THANK YOU!