Gradings and Polynomial Identities in Finite Matrix Algebras

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Motivation

In the section "Open Questions" of [16] the author makes the following question (that was posed to him by Zelmanov):

Question

Let D be a skew field $R = M_n(D)$ a matrix ring. Describe semigroups S and decompositions into direct sums of additive subrings $R = \oplus_{s \in S} R_s$ such that R is S-graded.

A complete answer to this question may be hard to obtain.

For algebras graded by $\mathbb{Z}_2$ this question is connected to Kemer's solution to the Specht problem.

Graded Wedderburn-Artin Theorem

Let D be an algebra graded by a group G. A G-grading on $M_t(\mathbb{K}) \otimes D$ is obtained from $\mathbf{g} = (g_1, \ldots, g_t) \in G^t$ where $e_{i,j} \otimes d_h$ is homogeneous of degree $g_i h g_j^{-1}$ for every $d_h \in D_h$.

Theorem

If R is graded simple of finite dimension then there exist a graded division algebra D and $\mathbf{g} \in G^t$ such that R is isomorphic to $M_t(\mathbb{K}) \otimes_{\mathbb{K}} D$ with the grading induced by $\mathbf{g}$.

To classify group gradings on matrix algebras it is sufficient to classify division gradings on these algebras.

Division gradings and projective representations

We say that $f : G \to GL(V)$ is a projective representation if such that $f(e) = Id$ and for all $g, h \in G$ we have $f(g)f(h) = \alpha(g, h)f(gh)$, where $\alpha(g, h) \in \mathbb{K}^\times$.

A projective representation is called irreducible if V has no non-trivial subspaces invariant under all $f(g)$, $g \in G$.

Over an algebraically closed field the dimension of an irreducible projective representation is bounded by $\sqrt{|G|}$.

In [4] it is proved that over an algebraically closed field the classification of all division gradings on matrix algebras equivalent to description of all finite groups with irreducible projective representations of maximal degree.

Pauli gradings

We assume that $\mathbb{K}$ contains a primitive n-th root of unity ϵ.

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} \epsilon^{n-1} & 0 & \cdots & 0 \\ 0 & \epsilon^{n-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then $A^n = B^n = I$ and $BA = \epsilon AB$, hence we obtain the following division $\mathbb{Z}_n \times \mathbb{Z}_n$-grading on $R = M_n(\mathbb{K})$:

$$R_{(\bar{k},\bar{l})} = \mathbb{K}A^k B^l.$$

We refer to this as the Pauli grading on $M_n(\mathbb{K})$.

Division gradings with abelian support

The complete classification is known for abelian groups and algebraically closed fields.

$M_n(\mathbb{K})$ admits a division grading with support a finite abelian group T if and only if $\operatorname{char}\mathbb{K} \nmid n$ and $T \cong \mathbb{Z}_{l_1}^2 \times \cdots \times \mathbb{Z}_{l_r}^2$, where $l_1 \cdots l_r = n$.

The division grading on the algebra $M_n(\mathbb{K})$ is equivalent to $M_{l_1}(\mathbb{K}) \otimes \cdots \otimes M_{l_r}(\mathbb{K})$ with the tensor product grading where each component has the Pauli grading.

Crossed products

Let $\mathbb{E}/\mathbb{F}$ be a finite Galois extension and let $G = \mathrm{Gal}(\mathbb{E}/\mathbb{F})$. We form a vector space A over $\mathbb{E}$ with basis $\{y_s \mid s \in G\}$, and so its elements have the form

$$\sum_{s\in G} \rho_s y_s, \ \rho_s \in \mathbb{E}.$$

Given a map

$$\begin{array}{rccc} k: & G\times G & \to & \mathbb{E}^\times \\ & (s,t) & \to & k_{s,t} \end{array} \tag{1}$$

define a product on A as follows:

$$\left(\sum_{s\in G}\rho_s y_s\right)\left(\sum_{t\in G}\delta_t y_t\right) = \sum_{s\in G}\sum_{t\in G} k_{s,t}\rho_s s(\delta_t) y_{st}, \ \rho_s, \delta_t \in \mathbb{E}.$$

Crossed products

The algebra $A = (\mathbb{E}, \text{Gal}(\mathbb{E}/\mathbb{F}), k)$ is associative if and only if

$$k_{s,t}k_{st,v} = s(k_{t,v})k_{s,tv} \tag{2}$$

for all $s, t, v \in \text{Gal}(\mathbb{E}/\mathbb{F})$, i. e., if k is a 2-cocycle (or a factor set) of $\text{Gal}(\mathbb{E}/\mathbb{F})$ in $\mathbb{E}^\times$.

Theorem

The crossed product $A = (\mathbb{E}, Gal(\mathbb{E}/\mathbb{F}), k)$ is a central simple algebra over $\mathbb{F}$. Moreover, $[A : \mathbb{F}] = m^2$ where $[\mathbb{E} : \mathbb{F}] = m$.

Proof.

See [12, Theorem 8.7]. □

Then $A \cong M_n(D)$ for some division algebra D. We use this to construct a division grading on $M_n(D)$.

Crossed products and division gradings

Let $\psi : T \to \mathrm{Gal}(\mathbb{E}/\mathbb{F})$ be an isomorphism of groups, denote $\psi(u) = \psi_u$. We construct a division grading on $M_n(D)$ as follows:

(a) Denote by $D_{T,\mathbb{E},\psi}$ the vector space over $\mathbb{E}$ with basis $\{Y_s \mid s \in T\}$, there exists an $\mathbb{E}$-linear isomorphism $D_{T,\mathbb{E},\psi} \to A = (\mathbb{E}, \mathrm{Gal}(\mathbb{E}/\mathbb{F}), k)$ such that $Y_s \to y_{\psi_s}$. This induces on $D_{T,\mathbb{E},\psi}$ a structure of algebra over $\mathbb{F}$.

(b) There exists an isomorphism $\varphi : D_{T,\mathbb{E},\psi} \to M_n(D)$ of algebras over $\mathbb{F}$, and so φ induces a G-grading on $M_n(D)$ with support T.

(c) A homogeneous element of degree $s \in T$ has the form ρY_s, $\rho \in \mathbb{E}$, and therefore is invertible. Hence we have a division grading on $M_n(D)$.

Crossed products and division gradings

Now we discuss the graded identities of $D_{T,\mathbb{E},\psi}$. Let $\mathbb{K}$ be an algebraically closed field that contains $\mathbb{E}$ as a subfield.

Then $D_{T,\mathbb{E},\psi} \otimes_{\mathbb{F}} \mathbb{K} \cong M_m(\mathbb{K})$. We consider on $M_m(\mathbb{K})$ a G-grading isomorphic to $D_{T,\mathbb{E},\psi} \otimes_{\mathbb{F}} \mathbb{K}$.

Theorem

Let $T = \{g_1, \ldots, g_m\}$. The grading above on $M_m(\mathbb{K})$ is isomorphic to the elementary determined by $(g_1, \ldots, g_m)$.

Proof.

See [2, Theorem 10]. □

Crossed product gradings

We recall the definition given in [2] of Crossed Product Gradings.

Definition

Let $A = M_m(\mathbb{K})$ be the algebra of $m \times m$-matrices over the field $\mathbb{K}$ and let G be a group of order m. We say that a G-grading on A is a crossed product grading if the grading is elementary and the corresponding m-tuple $(g_1 = 1, g_2, \ldots, g_m) \in G^m$ consists precisely of all elements of G.

The graded central polynomials for $M_m(\mathbb{K})$ with a crossed product grading were described in [2] under the hypothesis that $\operatorname{char} \mathbb{K} = 0$.

The same result for $M_n(\mathbb{K})$ with the Vasilovski grading was obtained previously in [6].

Graded identities of division gradings

What about the graded identities of the $\mathbb{F}$-algebra $D_{T,\mathbb{E},\psi}$?

Let $T = \{g_1, \ldots, g_m\}$. We consider $M_m(\mathbb{F})$ and $M_m(\mathbb{K})$ with the crossed product gradings determined by the tuple $(g_1, \ldots, g_m)$.

We have

$$D_{T,\mathbb{E},\psi} \otimes_{\mathbb{F}} \mathbb{K} \cong M_m(\mathbb{K}) \cong M_m(\mathbb{F}) \otimes_{\mathbb{F}} \mathbb{K}.$$

Therefore if the field $\mathbb{F}$ is infinite the algebras $D_{T,\mathbb{E},\psi}$ and $M_m(\mathbb{F})$ satisfy the same graded identities.

The graded identities for $M_m(\mathbb{F})$ with an elementary grading that has commutative neutral component were described in [3], [7].

Then it is interesting to study the graded identities for $D_{T,\mathbb{E},\psi}$ when $\mathbb{F}$ is a finite field.

In this case $T \cong \mathrm{Gal}(\mathbb{E}/\mathbb{F})$ is a cyclic group of order m and $A = (\mathbb{E}, \mathrm{Gal}(\mathbb{E}/\mathbb{F}), k)$ is isomorphic to $M_m(\mathbb{F})$, hence $D_{T,\mathbb{E},\psi}$ is isomorphic to a division grading on $M_m(\mathbb{F})$.

In [14] the authors classify the $\mathbb{Z}_2$-gradings on $M_2(\mathbb{F})$, where $\mathbb{F}$ is a finite field. There are, up to isomorphism, two non-trivial gradings Ω and Ω^α, here α is not a perfect square in $\mathbb{F}$, where:

$$\Omega_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, b \in \mathbb{F} \right\}, \quad \Omega_1 = \left\{ \begin{pmatrix} 0 & c \\ b & 0 \end{pmatrix} \mid c, d \in \mathbb{F} \right\}.$$

and

$$\Omega_0^\alpha = \left\{ \begin{pmatrix} a & d \\ \alpha d & a \end{pmatrix} \mid a, b \in \mathbb{F} \right\}, \quad \Omega_1^\alpha = \left\{ \begin{pmatrix} b & c \\ -\alpha c & -b \end{pmatrix} \mid c, d \in \mathbb{F} \right\}.$$

Gradings and identities on finite matrix algebras

Ω^α is isomorphic to $D_{\mathbb{Z}_2,\mathbb{E},\psi}$.

The graded identities for Ω and Ω^α are described in [14].

These results are modelled on the paper of Maltsev and Kuzmin [17] for ungraded identities.

The ungraded identities for $M_3(\mathbb{F})$ and $M_4(\mathbb{F})$ were described in [10] and [11], respectively.

Division gradings on finite matrix algebras

The division gradings by cyclic groups on finite matrix algebras are described in the following theorem:

Theorem

Let $D = M_m(\mathbb{F})$ be a G-graded division algebra, over a finite field $\mathbb{F}$, with support T . If T is a cyclic group and $\mathbb{E} = D_e$ then:

(a) *$\mathbb{E}$ is a finite field.*

(b) *$|T| = [\mathbb{E} : \mathbb{F}] = m$.*

(c) *There exists an isomorphism $\psi : T \to \mathrm{Gal}(\mathbb{E}/\mathbb{F})$ of groups such that $D \cong D_{T,\mathbb{E},\psi}$ as G-graded algebras over $\mathbb{F}$.*

Proof.

See [8, Theorem 4.1]. □

Division gradings on finite matrix algebras

Henceforth $\mathbb{F}$ is a finite field with $q = p^a$ elements, here $p = \operatorname{char} \mathbb{F}$, and T is a cyclic group of order m generated by $\mathbf{t}$.

We fix $\mathbb{E}_m$ an extension of $\mathbb{F}$ of degree m.

Let $\zeta : \mathbb{E}_m \to \mathbb{E}_m$ be the Frobenius map given by

$$\zeta(x) = x^p, \ x \in \mathbb{E}_m.$$

Then $\mathrm{Gal}(\mathbb{E}_m/\mathbb{F})$ is generated by ζ^a.

Let b be an integer with $\gcd(m, b) = 1$, $1 \leq b \leq m - 1$ and let

$$\psi^{(m,b)} : T \to \mathrm{Gal}(\mathbb{E}_m/\mathbb{F}) \ \text{such that} \ \psi^{(m,b)}(\mathbf{t}) = \zeta^{ab}. \qquad (3)$$

If $\psi : T \to \mathrm{Gal}(\mathbb{E}_m/\mathbb{F})$ is an isomorphism of groups, then $\psi = \psi^{(m,b)}$ for some b.

Division gradings on finite matrix algebras

Theorem

If $\mathbb{K}/\mathbb{F}$ is a field extension, $[\mathbb{K} : \mathbb{F}] = |T| = m$ and $\varphi : T \to \mathrm{Gal}(\mathbb{K}/\mathbb{F})$ is an isomorphism of groups, then there exists an integer b such that

$$\gcd(m, b) = 1, \ 1 \leq b \leq m - 1 \ \text{ and } \ D_{T,\mathbb{K},\varphi} \cong D_{T,\mathbb{E}_m,\psi^{(m,b)}}$$

as G-graded algebras over $\mathbb{F}$. Moreover, there are exactly $\phi(m)$ division G-gradings on $M_m(\mathbb{F})$ with support T not pairwise isomorphic.

Proof.

See [8, Theorem 4.4] □

Gradings on finite matrix algebras

Let G be a group such that every finite subgroup is cyclic, these results yield the classification of the G-gradings on finite matrix algebras with entries in $\mathbb{F}$.

If G is abelian then we have the following result on the graded identities of these algebras.

Theorem

Let $\mathbb{F}$ be a finite field with q elements, G an abelian group such that every finite subgroup of G is cyclic, $R = M_n(\mathbb{F})$ and $R' = M_{n'}(\mathbb{F})$ two G-graded matrix algebras over $\mathbb{F}$. Then R and R' are isomorphic G-graded algebras if and only if $Id_G(R) = Id_G(R')$.

Proof.

See [8, Theorem 7.8]. □

Future research

These results motivate the following problem:

Problem

Let $\mathbb{F}$ be a finite field. Classify the group gradings on $M_n(\mathbb{F})$ and describe its graded identities.

We hope that the problem above may be solved at least for $n \leq 4$. The following related question also seems interesting:

Question

Let G be a group. Let R and R' be finite graded simple algebras. Is it true that R and R' satisfy the same graded identities if and only if they are isomorphic as graded algebras?

Future research

The previous question might need a better understanding of the graded identities of finite algebras.

One may attempt partial questions, e. g., consider gradings on matrix algebras over more general groups or gradings on finite simple algebras.

The ideas in [1] and [15] for graded simple algebras over algebraically closed fields might be useful.

Group gradings on matrix algebras of order 2 and 3 over an arbitrary field were described in [13] and [5], respectively. The group gradings on matrix algebras of prime order are described in [9], here the field has to contain suitable primitive roots of the unity. These results might be useful to the previous problem.

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Thank you!