Polysymplectic and Multisymplectic Structures on Manifolds and Fiber Bundles

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Abstract

In this thesis, we introduce a new class of multilinear alternating forms and of differential forms called polylagrangian (in the case of vector-valued forms) or multilagrangian (in the csae of forms that are partially horizontal with respect to a given subspace or subbundle), characterized by the existence of a special type of maximal isotropic subspace or subbundle called polylagrangian or multilagrangian, respectively. As it turns out, these constitute the adequate framework for the formulation of an algebraic Darboux theorem. Combining this new algebraic structure with standard integrability conditions ($d\omega = 0$) allows us to derive a geometric Darboux theorem (existence of canonical local coordinates). Polysymplectic and multisymplectic structures, including all those that appear in the covariant hamiltonian formalism of classical field theory, are contained as special cases.

Contents

Introduction

Multisymplectic geometry is increasingly recognized as providing the appropriate mathematical framework for classical field theory from the hamiltonian point of view – just as symplectic geometry does for classical mechanics. Unfortunately, the development of this new area of differential geometry has so far been hampered by the non-existence of a fully satisfactory definition of the concept of a multisymplectic structure. Ideally, such a definition should be mathematically simple as well as in harmony with the needs of applications to physics.

The present thesis is devoted to presenting a new proposal for overcoming this problem.

As a first step, let us consider a simple analogy. The symplectic forms encountered in classical mechanics can locally all be written in the form

$$\omega \;=\; dq^i \wedge dp_i \;, \tag{1}$$

where $q^1, \ldots, q^N, p_1, \ldots, p_N$ are a particular kind of local coordinates on phase space known as canonical coordinates or Darboux coordinates. Introducing time t and energy E as additional variables (which is essential, e.g., for incorporating non-autonomous systems into the symplectic framework of hamiltonian mechanics), this equation is replaced by

$$\omega \;=\; dq^i \wedge dp_i \,+\, dE \wedge dt \;, \tag{2}$$

where $t, q^1, \ldots, q^N, p_1, \ldots, p_N, E$ can be viewed as canonical coordinates on an extended phase space. Similarly, the multisymplectic forms encountered in classical field theory over an n-dimensional space-time manifold M can locally all be written in the form

$$\omega \;=\; dq^i \wedge dp_i^\mu \wedge d^n x_\mu \,-\, dp \wedge d^n x \;, \tag{3}$$

where x^μ, q^i, p_i^μ, p $(1 \leqslant \mu \leqslant n, 1 \leqslant i \leqslant N)$ can again be viewed as canonical coordinates on some extended multiphase space. Here, the x^μ are (local) coordinates for M, while p is still a single energy variable (except for a sign), $d^n x$ is the (local) volume form induced by the x^μ and $d^n x_\mu$ is the (local) $(n-1)$-form obtained by contracting $d^n x$ with $\partial_\mu \equiv \partial/\partial x^\mu$:

$$d^n x_\mu \;=\; \mathrm{i}_{\partial_\mu} d^n x \;.$$

The idea of introducing "multimomentum variables" labeled by an additional space-time index μ (n multimomentum variables p_i^μ for each position variable q^i) goes back to the work of de Donder [5] and Weyl [20] in the 1930's (and perhaps even further) and has been recognized ever since as being an essential and unavoidable ingredient in any approach to a generally covariant hamiltonian formulation of classical field theory. Understanding the proper geometric setting for this kind of structure, however, has baffled both mathematicians and physicists for decades, as witnessed by the many attempts that can be found in the literature, almost all of which have ultimately failed, at some point or other, either because they did not properly take into account one or several of the various subtleties involved in the problem or else because they took "the easy way out" by artificially invoking additional structures. A typical example of such an additional structure is the choice of a background connection, which excludes the possibility to incorporate gauge theories where connections are part of the dynamical variables: obviously, a formalism that claims to be general but fails to accommodate gauge theories can hardly be of much use.

A notable exception is provided by the cojet bundle construction of multiphase space in first order classical field theories whose hamiltonian formulation is obtained from a lagrangian formulation via a Legendre transformation. Starting out from a fiber bundle E over space-time M whose sections are regarded as the basic fields of the model under consideration, one forms its first order jet bundle JE which is a fiber bundle over M (with respect to the source projection) whose sections represent the possible first order derivatives of the basic fields and which is the natural arena for the lagrangian formalism. Next, one makes use of the fact that JE is also an affine bundle over E (with respect to the target projection) to form its affine dual $P = J^{\circledast}E$ which turns out to be the natural arena for the hamiltonian formalism; in particular, it carries a globally defined multisymplectic form ω whose local representation in canonical coordinates has the form given in equation (3) above. Remarkably, in this case, the multisymplectic form ω is (up to a sign) the exterior derivative of a globally defined multicanonical form θ whose local representation in canonical coordinates reads

$$\theta \;=\; p_i^\mu \, dq^i \wedge d^n x_\mu + p \; d^n x \;. \tag{4}$$

We shall refrain from describing this construction in more detail, even though it provides the only known class of examples of multisymplectic structures known to date, since it is extensively documented in the literature [4, 6, 8, 9]. However, it should be noted that it is strictly analogous to the cotangent bundle construction of phase space in mechanics, which clearly indicates that it should not be the most general such construction. What is lacking is the multisymplectic analogue of other examples of symplectic manifolds which cannot be represented as vector bundles and where ω fails to be exact, such as Kähler manifolds or coadjoint orbits.

As a first attempt to approach the problem, consider the obvious option of extending the concept of a symplectic structure to that of a multisymplectic structure by defining a multisymplectic manifold to be a manifold P equipped with a differential form ω of degree $n+1$ which is closed ($d\omega = 0$) as well as everywhere non-degenerate, i.e., satisfies

$$\mathrm{i}_w \omega(p) = 0 \implies w = 0 \qquad \text{for } p \in P,\, w \in T_p P \ . \tag{5}$$

Obviously, this definition captures part of the relevant aspects but is much too broad to be of any practical use: in particular, it is insufficient to guarantee the validity of a Darboux type theorem (except when $n = 1$, of course).

The main problem with the definition just given is that it completely ignores the role of space-time. More precisely, it ignores the fact that the manifold P is not simply just a manifold but rather the total space of a fiber bundle over a base manifold M which not only determines the degree of the form in question ($\dim M = n \;\Rightarrow\; \deg \omega = n+1$) but also provides a notion of horizontal forms (the dx^μ and their exterior products) which, roughly speaking, allows us to split off an $(n-1)$-horizontal factor from ω and thus reduce it to a bunch of 2-forms, according to

$$\omega \;=\; \omega^{(\mu)} \wedge d^n x_\mu \ , \tag{6}$$

where

$$\omega^{(\mu)} \;=\; dq^i \wedge dp_i^\mu \;-\; dp \wedge dx^\mu \ , \tag{7}$$

since

$$dx^\mu \wedge d^n x_\nu \;=\; \delta_\nu^\mu \, d^n x \ . \tag{8}$$

Of course, such a splitting is coordinate dependent, but its existence is an important ingredient which must not be neglected but rather be reformulated in a coordinate independent manner.

In essence, this means that *there is no natural notion of a multisymplectic manifold but only that of a multisymplectic fiber bundle*, whose base space is to be interpreted as the space-time manifold on which classical fields are defined.[1] For our purposes here, the base space need not carry any additional structure, such as a Lorentz metric or, in the Euclidean setting, a Riemann metric, but it must be present as a carrier space. This may sound strange when compared with the situation in mechanics, where one is used to dealing with symplectic manifolds, rather than symplectic fiber bundles, but that is really a specific feature of non-relativistic mechanics for autonomous systems which gets lost when one passes to non-autonomous systems or to relativistic mechanics:

[1] Indeed, the most general mathematical concept of a field, in classical field theory, is that fields are sections of fiber bundles over space-time.

- Even in the realm of non-relativistic classical mechanics, the treatment of non-autonomous systems requires extending the conventional phase space of the theory, which is a symplectic manifold P_0, say, by incorporating a copy of the real line $\mathbb{R}$ as a "time axis" and, if one wants to avoid working with contact manifolds such as the "simply extended phase space" $\mathbb{R} \times P_0$, incorporating another copy of the real line $\mathbb{R}$ as an "energy axis": this allows us to remain within the realm of symplectic manifolds, specifically the "doubly extended phase space" $\mathbb{R} \times P_0 \times \mathbb{R}$, already considered by É. Cartan. Both of these extensions are, in a very obvious way, fiber bundles over the time axis $\mathbb{R}$, and it is this structure that will be reproduced naturally when our definition of a multisymplectic fiber bundle is specialized to the case of a one-dimensional base manifold.

- In relativistic classical mechanics, the notion of time suffers a radical change of interpretation when compared to the non-relativistic case. There, what prevails is Newton's notion of absolute time, which can be and is used as a universal parameter for the solutions of ordinary differential equations, for all possible dynamical systems at once. In the transition to the relativistic case, time loses its distinguished status and becomes a coordinate (almost) like any other one – after all, temporal and spatial coordinates mix under Lorentz transformations. Therefore, it is absolutely natural to combine it with the generalized positions and momenta of mechanics since it is no longer the independent variable but rather one of the dependent variables, whereas the new independent variable is something different: for particle trajectories, it is proper time as measured by a clock moving along with the particle along its trajectory, rather than coordinate time as measured in the reference frame of a distant observer.

In short, mechanics does provide a natural setting for the appearance of symplectic fiber bundles (over the real line).

Taking into account these considerations, we can make a second attempt at defining the concept of a multisymplectic structure, according to which a multisymplectic fiber bundle is a fiber bundle P over an n-dimensional manifold M equipped with a differential form ω of degree $n+1$ which is closed ($d\omega = 0$) as well as everywhere non-degenerate and $(n-1)$-horizontal, i.e., satisfies equation (5) as before and

$$\mathrm{i}_{v_1}\mathrm{i}_{v_2}\mathrm{i}_{v_3}\omega(p) = 0 \qquad \text{for } p \in P,\ v_1, v_2, v_3 \in V_pP \ , \tag{9}$$

where VP denotes the vertical bundle ($VP = \ker T\pi$ where $T\pi : TP \to TM$ is the tangent map to the bundle projection $\pi : P \to M$). As we shall see, these conditions allow us to

- Introduce a "joint vertical kernel" of ω, which is an involutive vector subbundle $K_\omega P$ of VP defined by

$$(K_\omega)_p P \ = \ \{v \in V_pP \mid \mathrm{i}_v \mathrm{i}_{v'}\omega(p) = 0 \ \text{ for all } \ v' \in V_pP\} \ , \tag{10}$$

and which turns out to be at most one-dimensional: it can either be trivial, as in the case of the "simply extended phase space" of mechanics, or else be one-dimensional, as in the "doubly extended phase space" of mechanics, where it represents the additional energy variable.

- Reduce the original "multisymplectic" form ω to a "polysymplectic" form $\hat{\omega}$ on the vertical bundle (i.e., a family of "polysymplectic" forms $\hat{\omega}_x$ on the fibers P_x parametrized by the points x in the base space M), which is formally a homomorphism

$$\hat{\omega} : \bigwedge^2 VP \longrightarrow \pi^*\big(\bigwedge^{n-1} T^*M\big) \tag{11}$$

of vector bundles over P, for which $K_\omega P$ is the "joint kernel".

Unfortunately, these conditions are still not sufficient to guarantee the (local) existence of canonical coordinates. In fact, simple counterexamples show that something important is missing, namely a further algebraic condition that allows us to find a Darboux basis in the tangent space at each point. A second problem is then to figure out whether (or under what assumptions) it is possible to even find a holonomic Darboux basis in a neighborhood of each point.

As will be shown in the present thesis, both questions have simple and elegant answers, thus giving rise to a new and, in our view, finally adequate definition of the notion of a multisymplectic structure. Hopefully, this conceptual clarification will open the way to new mathematical insights that promise to be highly relevant for physics.

The main body of this thesis is divided into two chapters: the first is purely algebraic while the second has a geometric flavor. At the algebraic level, the new mathematical structure that has emerged in the course of our investigation and has turned out to be remarkably general since it appears in a wide variety of different situations, can be resumed as the requirement of "existence of a special type of maximal isotropic subspace" which we shall call a "polylagrangian subspace" or "multilagrangian subspace", depending on the context. This concept allows to handle more general forms than the standard polysymplectic or multisymplectic forms and we propose to call them "polylagrangian" or "multilagrangian" forms, respectively, since they turn out to constitute the adequate framework for formulating an algebraic Darboux theorem which guarantees the existence of a canonical basis. Subsequently, when passing from the realm of (multi)linear algebra to that of differential geometry, this new structure reveals its full power since combining it with the standard integrability condition for differential forms ($d\omega = 0$), we obtain a geometric Darboux theorem which guarantees the existence of canonical local coordinates. In the third and final chapter, we specify the general results obtained before to polysymplectic and multisymplectic forms and also discuss the relation between our approach and previous attempts of other authors.

CHAPTER 1

Polylagrangian and multilagrangian forms: algebraic theory

In this chapter we investigate two specific classes of alternating multilinear forms on vector spaces that play an important role in classical field theory and which we shall call polylagrangian forms (containing polysymplectic forms as a special case) and multilagrangian forms (containing multisymplectic forms as a special case). The former are vector-valued while the latter are partially horizontal, i.e., restricted to satisfy a certain degree of horizontality relative to some fixed "vertical" subspace. We begin by defining a few general notions, such as that of the kernel, the support and the rank of an alternating multilinear form, and then introduce various types of orthogonal complements that will be used in the definition of certain classes of subspaces which generalize the isotropic, coisotropic, lagrangian and maximal isotropic subspaces of symplectic geometry. One of the central points of this thesis will be to show that there is a specific extension of the concept of a lagrangian subspace for which we shall introduce the term "polylagrangian subspace" (in the case of vector-valued forms) or "multilagrangian subspace" (in the case of partially horizontal forms) and whose existence turns out to be the key to developing a non-trivial theory of polysymplectic and multisymplectic forms covering all situations studied so far: the ones treated in the existing mathematical literature as well as the ones relevant for applications to physics, which have not yet been adequately treated.

The first section of this chapter will be devoted to some general properties of vector-valued forms. First, recall that given two finite-dimensional[1] vector spaces, V and $\hat{T}$, a **$\hat{T}$-valued k-form on V** is a linear mapping

$$\hat{\omega} : \bigwedge^k V \longrightarrow \hat{T} , \tag{1.1}$$

[1] The hypothesis that all vector spaces considered in this work should be finite-dimensional is imposed to simplify the presentation of the ideas but is by no means indispensable: it could be replaced by adequate conditions from functional analysis. This generalization will be left to a possible future investigation.

or in other words, an alternating k-multilinear mapping from $V \times \ldots \times V$ (k factors) to $\hat{T}$. The vector space of such forms will be denoted by

$$L_a^k(V;\hat{T}) \;\cong\; L\left(\bigwedge^k V;\hat{T}\right) \;\cong\; \left(\bigwedge^k V^*\right)\otimes\hat{T}\;. \tag{1.2}$$

Taking $\hat{T} = \mathbb{R}$ we recover the definition of an ordinary k-form on V.

The second section of this chapter will be devoted to ordinary forms which are partially horizontal. To understand what this means, fix a finite-dimensional [1] vector space W together with a subspace V, called the **vertical space**, and consider the quotient space $T := W/V$, called the **base space**. Denoting the canonical projection of W onto T by π, we have an **exact sequence** of vector spaces

$$0 \;\longrightarrow\; V \;\longrightarrow\; W \;\xrightarrow{\;\pi\;}\; T \;\longrightarrow\; 0\;. \tag{1.3}$$

In this situation, recall that a k-form $\alpha \in \bigwedge^k W^*$ on W is **horizontal** (relative to π) if its contraction $\mathrm{i}_v\,\alpha$ with any vertical vector $v \in V = \ker\pi$ vanishes [10, Vol. 2], [16, Vol. 1]. More generally, we say that a k-form $\alpha \in \bigwedge^k W^*$ on W is $(k-r)$**-horizontal** (relative to π), where $0 \leqslant r \leqslant k$, if its contraction with more than r vertical vectors vanishes, i.e., if

$$\mathrm{i}_{v_1}\ldots\,\mathrm{i}_{v_{r+1}}\,\alpha \;=\; 0 \qquad \text{for } v_1,\ldots,v_{r+1} \in V\;. \tag{1.4}$$

The vector space of $(k-r)$-horizontal k-forms on W will be denoted by $\bigwedge_r^k W^*$. Obviously, the condition of 0-horizontality ($r = k$) is void, so $\bigwedge_k^k W^* = \bigwedge^k W^*$, while the k-horizontal k-forms ($r = 0$) are precisely the horizontal forms as defined before, and we have a canonical isomorphism $\bigwedge_0^k W^* \cong \bigwedge^k T^*$. For posterior use, we note that in the direction $\leftarrow$, i.e., as a linear map

$$\begin{array}{ccc} \bigwedge^k T^* & \longrightarrow & \bigwedge_0^k W^* \\ \alpha_T & \longmapsto & \alpha_W \end{array} \tag{1.5}$$

it is given by pull-back with the projection π, i.e., by $\alpha_W = \pi^*(\alpha_T)$, or explicitly,

$$\alpha_W(w_1,\ldots,w_k) \;=\; \alpha_T(\pi(w_1),\ldots,\pi(w_k)) \qquad \text{for } w_1,\ldots,w_k \in W\;. \tag{1.6}$$

Furthermore, we have a sequence of inclusions

$$\bigwedge^k T^* \cong \bigwedge_0^k W^* \subset \ldots \subset \bigwedge_r^k W^* \subset \ldots \subset \bigwedge_k^k W^* = \bigwedge^k W^*\;, \tag{1.7}$$

where the first few terms can be trivial, since

$$\bigwedge_r^k W^* = \{0\} \qquad \text{if } k - r > \dim T\;. \tag{1.8}$$

This motivates the interpretation of such forms as "partially horizontal". More precisely, we we can justify this terminology by observing that if we introduce a basis $\{e_1^V, \ldots, e_m^V, e_1^T, \ldots, e_n^T\}$ of W such that the first m vectors span V while the last n vectors span a subspace complementary to V and hence isomorphic to T, then in terms of the dual basis $\{e_V^1, \ldots, e_V^m, e_T^1, \ldots, e_T^n\}$ of W^*, an arbitrary form $\alpha \in \bigwedge_r^k W^*$ can be represented as

$$\alpha \;=\; \sum_{s=0}^{r} \tfrac{1}{s!}\,\tfrac{1}{(k-s)!}\;\alpha_{i_1 \ldots\, i_s;\, \mu_1 \ldots\, \mu_{k-s}}\; e_V^{i_1} \wedge \ldots \wedge e_V^{i_s} \wedge e_T^{\mu_1} \wedge \ldots \wedge e_T^{\mu_{k-s}} \;, \tag{1.9}$$

showing that forms $\alpha \in \bigwedge_r^k W^*$ are $(k-r)$-horizontal in the sense of being represented as linear combinations of exterior products of 1-forms among which at least $k-r$ are horizontal, i.e., belong to the subspace $V^\perp$ (which is spanned by the 1-forms $e_T^1, \ldots, e_T^n$).[2] This also means that $\{\, e_V^{i_1} \wedge \ldots \wedge e_V^{i_s} \wedge e_T^{\mu_1} \wedge \ldots \wedge e_T^{\mu_{k-s}} \mid 0 \leqslant s \leqslant r\,,\, 1 \leqslant i_1 < \ldots < i_s \leqslant \dim V\,,$ $1 \leqslant \mu_1 < \ldots < \mu_{k-s} \leqslant \dim T \,\}$ is a basis of $\bigwedge_r^k W^*$ and so we have

$$\dim \textstyle\bigwedge_r^k W^* \;=\; \displaystyle\sum_{s=0}^{r} \binom{\dim V}{s} \binom{\dim T}{k-s} \;, \tag{1.10}$$

where it is to be understood that, by definition, $\binom{p}{q} = 0$ if $q > p$.

In the third section we shall discuss a construction that allows us to associate to each partially horizontal form a certain vector-valued form: the latter will be called the "symbol" of the former in order to stress the analogy with the concept of the (leading) symbol of a differential operator. This procedure provides the link between polysymplectic and multisymplectic structures. Briefly, in the notation of the previous paragraph, the **symbol** of a $(k-r)$-horizontal k-form α on W, $\alpha \in \bigwedge_r^k W^*$, is the $\bigwedge^{k-r} T^*$-valued r-form $\hat{\alpha}$ on V, $\hat{\alpha} \in \bigwedge^r V^* \otimes \bigwedge^{k-r} T^*$, given by

$$\hat{\alpha}(v_1, \ldots, v_r) \;=\; \mathrm{i}_{v_1} \ldots \mathrm{i}_{v_r} \alpha \;, \qquad \text{for } \; v_1, \ldots, v_r \in V \;. \tag{1.11}$$

where we use the identifications

$$L_a^r\big(V; \textstyle\bigwedge^{k-r} T^*\big) \;\cong\; L\big(\bigwedge^r V; \bigwedge^{k-r} T^*\big) \;\cong\; \bigwedge^r V^* \otimes \bigwedge^{k-r} T^* \;, \tag{1.12}$$

and

$$\textstyle\bigwedge^{k-r} T^* \;\cong\; \bigwedge_0^{k-r} W^* \;. \tag{1.13}$$

For a better understanding of this construction, introduce a basis $\{e_1^V, \ldots, e_m^V, e_1^T, \ldots, e_n^T\}$ of W with dual basis $\{e_V^1, \ldots, e_V^m, e_T^1, \ldots, e_T^n\}$ of W^*, as before; then if the form $\alpha \in \bigwedge_r^k W^*$

[2]The symbol "$.^\perp$" stands for the annihilator, that is, $V^\perp$ is the subspace consisting of all linear forms in W^* which vanish on V.

has the expansion

$$\alpha \;=\; \sum_{s=0}^{r} \tfrac{1}{s!}\,\tfrac{1}{(k-s)!}\;\alpha_{i_1\ldots i_s;\,\mu_1\ldots\mu_{k-s}}\,e_V^{i_1}\wedge\ldots\wedge e_V^{i_s}\wedge e_T^{\mu_1}\wedge\ldots\wedge e_T^{\mu_{k-s}}\;, \tag{1.14}$$

the form $\hat{\alpha} \in \bigwedge^r V^* \otimes \bigwedge^{k-r} T^*$ is given by the expansion

$$\hat{\alpha} \;=\; \tfrac{1}{r!}\,\tfrac{1}{(k-r)!}\;\alpha_{i_1\ldots i_r;\,\mu_1\ldots\mu_{k-r}}\,e_V^{i_1}\wedge\ldots\wedge e_V^{i_r}\otimes e_T^{\mu_1}\wedge\ldots\wedge e_T^{\mu_{k-r}}\;. \tag{1.15}$$

Hence passage to the symbol can be regarded as a projection

$$\begin{array}{ccc} \bigwedge_r^k W^* & \longrightarrow & \bigwedge^r V^* \otimes \bigwedge^{k-r} T^* \\ \alpha & \longmapsto & \hat{\alpha} \end{array} \tag{1.16}$$

whose kernel is the subspace $\bigwedge_{r-1}^k W^*$.

An auxiliary tool often employed in the context described in the last two paragraphs is the choice of a **splitting** of the exact sequence (1.3), i.e., a linear mapping $\mathfrak{s} : T \to W$ such that $\pi \circ \mathfrak{s} = \mathrm{id}_T$, or equivalently, the choice of a subspace H of W, called the **horizontal subspace**, which is complementary to V, i.e., which satisfies

$$W \;=\; V \oplus H\;, \tag{1.17}$$

with H the image of $\mathfrak{s}$ in W and $\mathfrak{s}$ the inverse of the restriction of the projection π to H. Then it is easy to see that

$$W^* \;=\; H^\perp \oplus V^\perp\;, \tag{1.18}$$

where $H^\perp \cong V^*$ and $V^\perp \cong H^*$, and more generally,

$$\bigwedge_r^k W^* \;\cong\; \bigoplus_{s=0}^{r} \bigwedge^s H^\perp \otimes \bigwedge^{k-s} V^\perp\;. \tag{1.19}$$

Returning to the canonical isomorphism $\bigwedge_0^k W^* \cong \bigwedge^k T^*$, we note that in the direction $\to$, i.e., as a linear map

$$\begin{array}{ccc} \bigwedge_0^k W^* & \longrightarrow & \bigwedge^k T^* \\ \alpha_W & \longmapsto & \alpha_T \end{array} \tag{1.20}$$

it is explicitly defined as the pull-back by $\mathfrak{s}$, i.e., $\alpha_T = \mathfrak{s}^*(\alpha_W)$ or

$$\alpha_T(t_1,\ldots,t_k) \;=\; \alpha_W(\mathfrak{s}(t_1),\ldots,\mathfrak{s}(t_k)) \qquad \text{for } t_1,\ldots,t_k \in T\;. \tag{1.21}$$

Thus we arrive at the following explicit formula for the symbol $\hat{\alpha} \in \bigwedge^r V^* \otimes \bigwedge^{k-r} T^*$ of a partially horizontal form $\alpha \in \bigwedge_r^k W^*$:

$$\begin{array}{c} \hat{\alpha}(v_1, \ldots, v_r)\big(t_1, \ldots, t_{k-r}\big) \;=\; \alpha(v_1, \ldots, v_r, \mathfrak{s}(t_1), \ldots, \mathfrak{s}(t_{k-r})) \\ \text{for } \; v_1, \ldots, v_r \in V, \; t_1, \ldots, t_{k-r} \in T \; . \end{array} \tag{1.22}$$

However, it is worthwhile to emphasize that the last two formulas, like all the constructions described in the last two paragraphs, do not depend on the choice of the splitting $\mathfrak{s}$.

In the fourth section we present the construction of canonical models of polylagrangian and multilagrangian forms. Finally, in the last two sections, we show that general polylagrangian and multilagrangian forms can always be brought into this canonical form, by means of an appropriate isomorphism. When expressed in terms of adapted bases, this statement can be considered an "algebraic precursor" of the Darboux theorem.

In what follows we shall adopt the general convention that objects related to vector-valued forms will be characterized by a hat, just as above. Moreover, since we are interested only in forms of degree $k > 1$, we shall relabel the parameter k and consider ordinary $(k+1)$-forms, $\omega \in \bigwedge^{k+1} W^*$, as well as vector-valued $(k+1)$-forms, $\hat{\omega} \in \bigwedge^{k+1} V^* \otimes \hat{T}$, with $k \geqslant 1$. Finally, the dimensions of the "auxiliary" spaces T and $\hat{T}$ will be denoted by n and by $\hat{n}$, respectively.

The results obtained in this chapter are all formulated for vector spaces over the real numbers, but they remain valid in the context of vector spaces over arbitrary fields of characteristic zero.

1.1 Vector-valued forms on linear spaces

In this section, we shall first present a few general notions from the theory of multilinear forms that can be easily extended from ordinary forms to the more general context of vector-valued forms. As a major new concept, we then introduce a special type of maximal isotropic subspace, called polylagrangian subspace, which plays a fundamental role in all that follows.

Let V and $\hat{T}$ be finite-dimensional vector spaces, with $\dim \hat{T} = \hat{n}$, and let $\hat{\omega}$ be a $\hat{T}$-valued $(k+1)$-form on V:

$$\hat{\omega} \in \bigwedge^{k+1} V^* \otimes \hat{T} \; . \tag{1.23}$$

The **kernel** of $\hat{\omega}$ is the subspace $\ker \hat{\omega}$ of V defined by

$$\ker \hat{\omega} \;=\; \{\, v \in V \mid \mathrm{i}_v \hat{\omega} = 0 \,\} \; , \tag{1.24}$$

that is,

$$\ker \hat{\omega} \;=\; \{\, v \in V \mid \hat{\omega}(v, v_1, \ldots, v_k) = 0 \;\text{ for all }\; v_1, \ldots, v_k \in V \,\} \; . \tag{1.25}$$

When $\ker\hat{\omega} = 0$ we say that $\hat{\omega}$ is **non-degenerate**. The **support** of $\hat{\omega}$ is the subspace $\operatorname{supp}\hat{\omega}$ of V^* defined as the annihilator of $\ker\hat{\omega}$, that is,[3]

$$\operatorname{supp}\hat{\omega} \;=\; (\ker\hat{\omega})^\perp \;=\; \{\, v^* \in V^* \mid \langle v^*, v\rangle = 0 \;\text{ for all } v \in \ker\hat{\omega} \,\} \;. \tag{1.26}$$

Conversely, we have

$$\ker\hat{\omega} \;=\; (\operatorname{supp}\hat{\omega})^\perp \;=\; \{\, v \in V \mid \langle v^*, v\rangle = 0 \;\text{ for all } v^* \in \operatorname{supp}\hat{\omega} \,\} \;. \tag{1.27}$$

According to a frequently employed convention (see, for instance, Ref. [18]), the **rank** $\operatorname{rk}(\hat{\omega})$ of $\hat{\omega}$ is defined to be the dimension of its support:

$$\operatorname{rk}(\hat{\omega}) \;=\; \dim\,\operatorname{supp}\hat{\omega} \;. \tag{1.28}$$

However, depending on the specific nature of $\hat{\omega}$, we shall often find it more convenient to adopt a numerically different definition. For example, the rank of a symplectic form on a $2N$-dimensional linear space is usually defined to be N, rather than $2N$. The same kind of phenomenon will appear for the various types of forms to be introduced below.

One useful way of thinking about vector-valued forms is to consider them as "multiplets" of ordinary forms, relative to some basis of the auxiliary space. More specifically, with each element $\hat{t}^* \in \hat{T}^*$ in the dual of the auxiliary space we can associate an ordinary $(k+1)$-form $\omega_{\hat{t}^*}$ on V, called the **projection** of $\hat{\omega}$ along $\hat{t}^*$, defined by

$$\omega_{\hat{t}^*} \;=\; \langle \hat{t}^*, \hat{\omega}\rangle \;. \tag{1.29}$$

Obviously, $\omega_{\hat{t}^*}$ depends linearly on $\hat{t}^*$, so if we choose a basis $\{\hat{e}_1, \ldots, \hat{e}_{\hat{n}}\}$ of $\hat{T}$, with dual basis $\{\hat{e}^1, \ldots, \hat{e}^{\hat{n}}\}$ of $\hat{T}^*$, and define the ordinary $(k+1)$-forms

$$\omega^a \;=\; \omega_{\hat{e}^a} \quad (1 \leqslant a \leqslant \hat{n}) \;, \tag{1.30}$$

we obtain

$$\hat{\omega} \;=\; \omega^a \otimes \hat{e}_a \;. \tag{1.31}$$

Then it becomes clear that

$$\ker\hat{\omega} \;=\; \bigcap_{\hat{t}^* \in \hat{T}^*} \ker\omega_{\hat{t}^*} \;, \tag{1.32}$$

and similarly

$$\ker\hat{\omega} \;=\; \bigcap_{a=1}^{\hat{n}} \ker\omega^a \;, \tag{1.33}$$

[3] The symbol $\langle\,.\,,.\,\rangle$ will be used to denote the natural bilinear pairing between a linear space and its dual.

and if we take the annihilator, that

$$\text{supp}\ \hat{\omega} \;=\; \sum_{\hat{t}^* \in \hat{T}^*} \text{supp}\ \omega_{\hat{t}^*} \ , \tag{1.34}$$

and similarly

$$\text{supp}\ \hat{\omega} \;=\; \sum_{a=1}^{\hat{n}} \text{supp}\ \omega^a \ . \tag{1.35}$$

The following characterization of the support is often used as a definition:

Lemma 1.1 *The support* $\text{supp}\ \hat{\omega}$ *of* $\hat{\omega}$ *is the smallest subspace of* V^* *which satisfies*

$$\hat{\omega} \in \left(\bigwedge^{k+1} \text{supp}\ \hat{\omega}\right) \otimes \hat{T} \ .$$

PROOF. Using a basis $\{\hat{e}_1, \ldots, \hat{e}_{\hat{n}}\}$ of $\hat{T}$ as before, together with a basis $\{e_1, \ldots, e_r\}$ of V composed of a basis $\{e_1, \ldots, e_s\}$ of $\ker \hat{\omega}$ followed by a basis $\{e_{s+1}, \ldots, e_r\}$ of a subspace of V complementary to $\ker \hat{\omega}$, and considering the corresponding dual basis $\{e^1, \ldots, e^r\}$ of V^*, we see that $\{e^{s+1}, \ldots, e^r\}$ is a basis of $\text{supp}\ \hat{\omega} = (\ker \hat{\omega})^\perp$, so expanding $\hat{\omega}$ in terms of the basis $\{\, e^{p_1} \wedge \ldots \wedge e^{p_{k+1}} \otimes \hat{e}_a \mid 1 \leqslant p_1 < \ldots < p_{k+1} \leqslant r \,,\, 1 \leqslant a \leqslant \hat{n} \,\}$ of $\bigwedge^{k+1} V^* \otimes \hat{T}$, we conclude that all the coefficients $\omega^a_{p_1 \ldots p_{k+1}}$ with $p_1 \leqslant s$ must vanish, which implies $\hat{\omega} \in \left(\bigwedge^{k+1} \text{supp}\ \hat{\omega}\right) \otimes \hat{T}$. On the other hand, if $S_{\hat{\omega}}$ is any subspace of V^* such that $\hat{\omega} \in \left(\bigwedge^{k+1} S_{\hat{\omega}}\right) \otimes \hat{T}$, then for every $v \in S_{\hat{\omega}}^\perp$ we have $\mathrm{i}_v \hat{\omega} = 0$ and hence $v \in \ker \hat{\omega}$, that is, $S_{\hat{\omega}}^\perp \subset \ker \hat{\omega}$, implying that $\text{supp}\ \hat{\omega} = (\ker \hat{\omega})^\perp \subset S_{\hat{\omega}}$. □

For forms of arbitrary degree $k+1$ ($k \geqslant 1$), the notion of orthogonal complement of a subspace depends on an additional parameter. Extending the definition given in Ref. [3] from ordinary to vector-valued forms, suppose that L is a subspace of V and ℓ is an integer satisfying $1 \leqslant \ell \leqslant k$; then the **$\ell$-orthogonal complement** of L (with respect to $\hat{\omega}$) is defined to be the subspace $L^{\hat{\omega},\ell}$ of V given by[4]

$$L^{\hat{\omega},\ell} \;=\; \{\, v \in V \mid \mathrm{i}_v \mathrm{i}_{v_1} \ldots \mathrm{i}_{v_\ell} \hat{\omega} = 0 \ \text{ for all } \ v_1, \ldots, v_\ell \in L \,\} \ . \tag{1.36}$$

Note that these orthogonal complements are related by the following sequence of inclusions:

$$L^{\hat{\omega},1} \subset \ldots \subset L^{\hat{\omega},\ell-1} \subset L^{\hat{\omega},\ell} \subset L^{\hat{\omega},\ell+1} \subset \ldots \subset L^{\hat{\omega},k} \ . \tag{1.37}$$

As in the case of 2-forms, the notion of orthogonal complement can be used to define various kinds of special subspaces:

[4] We discard the trivial case $\ell = 0$ since extrapolating the definition given here would lead to the conclusion that $L^{\hat{\omega},0}$ is simply the kernel of $\hat{\omega}$, independently of the subspace L of V.

- L is **ℓ-isotropic** (with respect to $\hat{\omega}$) if $L \subset L^{\hat{\omega},\ell}$;
- L is **ℓ-coisotropic** (with respect to $\hat{\omega}$) if $L \supset L^{\hat{\omega},\ell}$;
- L is **ℓ-lagrangian** (with respect to $\hat{\omega}$) if it is both ℓ-isotropic and ℓ-coisotropic, that is, if $L = L^{\hat{\omega},\ell}$.

Among the ℓ-isotropic subspaces, we distinguish the maximal ones, which are defined as usual: L is called **maximal ℓ-isotropic** (with respect to $\hat{\omega}$) if L is ℓ-isotropic (with respect to $\hat{\omega}$) and such that for any subspace L' of V which is also ℓ-isotropic (with respect to $\hat{\omega}$)

$$L \subset L' \implies L = L' \ .$$

Observe that both ℓ-lagrangian and maximal ℓ-isotropic subspaces contain the kernel of $\hat{\omega}$.

The following theorem assures that the concepts of ℓ-lagrangian and maximal ℓ-isotropic subspaces are equivalent and establishes relations between such subspaces for a vector-valued form and for its projections.

Theorem 1.1 *Let V and $\hat{T}$ be finite-dimensional vector spaces and let $\hat{\omega}$ be a $\hat{T}$-valued $(k+1)$-form on V. For a subspace L of V and $1 \leqslant \ell \leqslant k$, we have*

1. *L is maximal ℓ-isotropic with respect to $\hat{\omega}$ iff L is ℓ-lagrangian with respect to $\hat{\omega}$, and for all $\hat{t}^* \in \hat{T}^*$, L is maximal ℓ-isotropic with respect to $\hat{\omega}_{\hat{t}^*}$ iff L is ℓ-lagrangian with respect to $\hat{\omega}_{\hat{t}^*}$;*
2. *L is ℓ-isotropic with respect to $\hat{\omega}$ iff, for all $\hat{t}^* \in \hat{T}^*$, L is ℓ-isotropic with respect to $\hat{\omega}_{\hat{t}^*}$;*
3. *If, for all $\hat{t}^* \in \hat{T}^*$, L is maximal ℓ-isotropic with respect to $\hat{\omega}_{\hat{t}^*}$, then L is maximal ℓ-isotropic with respect to $\hat{\omega}$.*

It should be emphasized that the converse to the last statement is false: a subspace which is maximal isotropic with respect to a vector-valued form will be isotropic with respect to all of its projections but may fail to be maximal isotropic with respect to some of them.

PROOF.

1. If L is ℓ-isotropic, then for any vector $u \in L^{\hat{\omega},\ell} \setminus L$, the subspace $L' = L + \langle u \rangle$ of V is ℓ-isotropic as well, since for $u_i = v_i + \lambda_i u \in L'$ with $v_i \in L$ $(i = 0, \ldots, \ell)$, we have

$$\mathrm{i}_{u_0}\mathrm{i}_{u_1}\ldots\mathrm{i}_{u_\ell}\hat{\omega} \ = \ \sum_{i=0}^{\ell} \mathrm{i}_{v_0}\mathrm{i}_{v_1}\ldots\mathrm{i}_{v_\ell}\hat{\omega} + \sum_{i=0}^{\ell} (-1)^i \, \lambda_i \, \mathrm{i}_u \mathrm{i}_{v_0}\ldots\widehat{\mathrm{i}_{v_i}}\ldots\mathrm{i}_{v_\ell}\hat{\omega} \ = \ 0 \ .$$

Therefore, if L is maximal ℓ-isotropic, then $L' \subset L$ and hence $L^{\hat{\omega},\ell} \subset L$, which implies that L is ℓ-lagrangian. Conversely, if L is ℓ-lagrangian and L' is ℓ-isotropic with $L \subset L'$, then

$$L' \;\subset\; (L')^{\hat{\omega},\ell} \;\subset\; L^{\hat{\omega},\ell} \;=\; L~,$$

that is, L is maximal ℓ-isotropic. The same argument can be applied to each of the projected forms $\hat{\omega}_{\hat{t}^*}$ ($\hat{t}^* \in \hat{T}^*$), instead of $\hat{\omega}$.

2. For $v_i \in L$ ($i = 0, \ldots, \ell$), we have

$$\mathrm{i}_{u_0} \ldots \mathrm{i}_{u_\ell} \hat{\omega} \;=\; 0 \qquad \Longleftrightarrow \qquad \mathrm{i}_{u_0} \ldots \mathrm{i}_{u_\ell} \hat{\omega}_{\hat{t}^*} \;=\; 0 \quad \text{for all } \hat{t}^* \in \hat{T}^*~.$$

3. Suppose L is maximal ℓ-isotropic with respect to each of the projected forms $\hat{\omega}_{\hat{t}^*}$ ($\hat{t}^* \in \hat{T}^*$). According to the previous item, L is ℓ-isotropic with respect to $\hat{\omega}$. Moreover, if L' is a subspace of V which is ℓ-isotropic with respect to $\hat{\omega}$ and such that $L \subset L'$, then according to the previous item, L' is ℓ-isotropic with respect to each of the projected forms $\hat{\omega}_{\hat{t}^*}$ ($\hat{t}^* \in \hat{T}^*$) and hence $L' = L$, proving that L is in fact maximal ℓ-isotropic with respect to $\hat{\omega}$.

□

In what follows, we are mainly interested in the case $\ell = 1$, so in order to simplify the notation, we shall omit the prefix "1" whenever convenient.

Another way to approach the concept of (maximal) isotropic subspaces for vector-valued forms which will turn out to be particularly fruitful is based on the idea of contracting $\hat{\omega}$ with vectors. Recall that **contraction** with $\hat{\omega}$ is defined to be the linear map

$$\begin{array}{rccc} \hat{\omega}^\flat : & V & \longrightarrow & \big(\bigwedge^k V^*\big) \otimes \hat{T} \\ & v & \longmapsto & \mathrm{i}_v \hat{\omega} \end{array} . \tag{1.38}$$

Then it is clear that a subspace L of V will be isotropic if and only if

$$\hat{\omega}^\flat(L) \;\subset\; \big(\textstyle\bigwedge^k L^\perp\big) \otimes \hat{T}~, \tag{1.39}$$

which can be trivially rewritten in the form

$$\hat{\omega}^\flat(L) \;\subset\; \hat{\omega}^\flat(V) \,\cap\, \big(\textstyle\bigwedge^k L^\perp\big) \otimes \hat{T}~. \tag{1.40}$$

The merit of this – at first sight pointless – reformulation lies in its close analogy with the criterion that a subspace L of V containing $\ker \hat{\omega}^\flat = \ker \hat{\omega}$ will be maximal isotropic if and only if

$$\hat{\omega}^\flat(L) \;=\; \hat{\omega}^\flat(V) \,\cap\, \big(\textstyle\bigwedge^k L^\perp\big) \otimes \hat{T}~. \tag{1.41}$$

In fact, for any subspace L of V, we have the identity

$$\hat{\omega}^\flat\big(L^{\hat{\omega},1}\big) \;=\; \hat{\omega}^\flat\big(V\big) \cap \big(\textstyle\bigwedge^k L^\perp\big) \otimes \hat{T} \;, \tag{1.42}$$

since both sides are equal to the subspace of $\big(\bigwedge^k V^*\big) \otimes \hat{T}$ consisting of elements of the form $\mathrm{i}_v \hat{\omega}$ where $v \in V$ and $\mathrm{i}_u \mathrm{i}_v \hat{\omega} = 0$ for all $u \in L$.

Taking into account these considerations, we are led to introduce a concept that turns out to provide the key to the theory of polysymplectic forms: the existence of a special type of maximal isotropic subspace, which we shall call a polylagrangian subspace.

Definition 1.1 *Let V and $\hat{T}$ be finite-dimensional vector spaces and let $\hat{\omega}$ be a $\hat{T}$-valued $(k+1)$-form on V. We say that $\hat{\omega}$ is a **polylagrangian** form of **rank** N if V admits a subspace L of codimension N which is **polylagrangian**, i.e., such that*

$$\hat{\omega}^\flat\big(L\big) \;=\; \big(\textstyle\bigwedge^k L^\perp\big) \otimes \hat{T} \;. \tag{1.43}$$

*When $k = 1$, we call $\hat{\omega}$ a **polysymplectic** form.*

The next theorem shows that a polylagrangian subspace, when it exists, really is a special type of maximal isotropic subspace; in particular, it contains the kernel of $\hat{\omega}$.

Theorem 1.2 *Let V and $\hat{T}$ be finite-dimensional vector spaces, with $\hat{n} \equiv \dim \hat{T}$, and let $\hat{\omega}$ be a $\hat{T}$-valued polylagrangian $(k+1)$-form on V of rank N, with polylagrangian subspace L. Then $N \geqslant k$ (except if $\hat{\omega} \equiv 0$) and L contains the kernel of $\hat{\omega}$ as well as the kernel of each of the projected forms $\omega_{\hat{t}^*}$ $(\hat{t}^* \in \hat{T}^* \setminus \{0\})$:*

$$\ker\,\hat{\omega} \;\subset\; L \qquad , \qquad \ker\,\omega_{\hat{t}^*} \;\subset\; L \quad \textit{for all } \hat{t}^* \in \hat{T}^* \setminus \{0\} \;. \tag{1.44}$$

The dimension of L is given by

$$\dim\,L \;=\; \dim\,\ker\,\hat{\omega} \,+\, \hat{n} \binom{N}{k} \;. \tag{1.45}$$

PROOF. First we observe that if $N < k$, we have $\bigwedge^k L^\perp = \{0\}$ and thus both sides of the equation (1.43) vanish; therefore, L is contained in $\ker\,\hat{\omega}$ and so $\ker\,\hat{\omega}$ has codimension $< k$ in V, implying $\hat{\omega} \equiv 0$, since the $(k+1)$-form on the quotient space $V/\ker\,\hat{\omega}$ induced by $\hat{\omega}$ vanishes identically. (More generally, this argument shows that a nonvanishing vector-valued $(k+1)$-form does not permit isotropic subspaces of codimension $< k$.) Thus supposing that $\dim L^\perp = N \geqslant k$, we can for any vector $v \in V \setminus L$ find a linearly independent set of 1-forms $v_1^*, \ldots, v_k^* \in L^\perp$ such that $\langle v_1^*, v\rangle = 1$ and $\langle v_i^*, v\rangle = 0$ for $i > 1$. Given $\hat{t}^* \in \hat{T}^*$, take $\hat{t} \in \hat{T}$

such that $\langle \hat{t}^*, \hat{t} \rangle = 1$. According to the definition of a polylagrangian subspace, there is a vector $u \in L$ such that

$$\mathrm{i}_u \hat{\omega} \;=\; v_1^* \wedge \ldots \wedge v_k^* \otimes \hat{t} \quad \Rightarrow \quad \mathrm{i}_v \mathrm{i}_u \omega_{\hat{t}^*} \;=\; v_2^* \wedge \ldots \wedge v_k^* \;\neq\; 0$$

and so $v \notin \ker\, \omega_{\hat{t}^*}$. Hence it follows that $\ker\, \hat{\omega} \subset \ker\, \omega_{\hat{t}^*} \subset L$, and we obtain

$$\dim L \,-\, \dim \ker\, \hat{\omega} \;=\; \dim\, \hat{\omega}^\flat(L) \;=\; \dim\Big(\big(\textstyle\bigwedge^k L^\perp\big) \otimes \hat{T}\Big) \;=\; \hat{n} \binom{N}{k} \,.$$

□

The dimension formula stated in the previous theorem provides a simple criterion to decide whether a given isotropic subspace is polylagrangian.

Theorem 1.3 *Let V and $\hat{T}$ be finite-dimensional vector spaces, with $\hat{n} \equiv \dim \hat{T}$, and let $\hat{\omega}$ be a $\hat{T}$-valued $(k+1)$-form on V. If L is an isotropic subspace of V of codimension N containing $\ker\, \hat{\omega}$ and such that*

$$\dim\, L \;=\; \dim\, \ker\, \hat{\omega} \,+\, \hat{n} \binom{N}{k} \,, \tag{1.46}$$

then L is polylagrangian and $\hat{\omega}$ is a polylagrangian form.

PROOF. Since L is an isotropic subspace of V containing $\ker\, \hat{\omega}$, the restriction of the linear map $\hat{\omega}^\flat$ (see equation (1.38)) to L takes L to $\big(\bigwedge^k L^\perp\big) \otimes \hat{T}$ and induces an injective linear map from the quotient space $L/\ker\, \hat{\omega}$ to $\big(\bigwedge^k L^\perp\big) \otimes \hat{T}$ which, due to the condition (1.46), is a linear isomorphism, so the equality (1.43) follows. □

In the case of truly vector-valued forms, we can say much more.

Theorem 1.4 *Let V and $\hat{T}$ be finite-dimensional vector spaces, with $\hat{n} \equiv \dim \hat{T} \geqslant 2$, and let $\hat{\omega}$ be a $\hat{T}$-valued $(k+1)$-form on V which is polylagrangian of rank N, with polylagrangian subspace L. Then L is given by*

$$L \;=\; \sum_{t^* \in\, \hat{T}^* \backslash \{0\}} \ker\, \omega_{\hat{t}^*} \tag{1.47}$$

and, in particular, is unique. In terms of a basis $\{\hat{e}_1, \ldots, \hat{e}_{\hat{n}}\}$ of $\hat{T}$ with dual basis $\{\hat{e}^1, \ldots, \hat{e}^{\hat{n}}\}$ of $\hat{T}^*$,

$$L \;=\; \ker\, \hat{\omega} \oplus K_1 \oplus \ldots \oplus K_{\hat{n}} \,, \tag{1.48}$$

and for any $1 \leqslant a \leqslant \hat{n}$,

$$L \;=\; \ker\, \omega^a \oplus K_a \,, \tag{1.49}$$

where for any $1 \leqslant a \leqslant \hat{n}$, K_a is any subspace of V satisfying

$$\bigcap_{\substack{b=1 \\ b \neq a}}^{\hat{n}} \ker\, \omega^b \;=\; \ker\, \hat{\omega} \oplus K_a\;. \tag{1.50}$$

The dimensions of these various subspaces are given by

$$\dim\, \ker\, \omega^a \;=\; \dim\, \ker\, \hat{\omega} \,+\, (\hat{n}-1)\binom{N}{k} \quad , \quad \dim K_a \;=\; \binom{N}{k}\;. \tag{1.51}$$

PROOF. Fix a basis $\{\hat{e}_1, \ldots, \hat{e}_{\hat{n}}\}$ of $\hat{T}$ with dual basis $\{\hat{e}^1, \ldots, \hat{e}^{\hat{n}}\}$ of $\hat{T}^*$ and choose subspaces K_a of V $(a = 1, \ldots, \hat{n})$ as indicated above. Then the subspaces $\ker \hat{\omega}$ and $K_1, \ldots, K_{\hat{n}}$ of V have trivial intersection, so their sum is direct and defines a subspace of V which we shall, for the moment, denote by L'. According to the previous theorem, $L' \subset L$. To show that $L' = L$, it is therefore sufficient to prove that $\hat{\omega}^\flat(L) \subset \hat{\omega}^\flat(L')$, since both L and L' contain $\ker \hat{\omega}$. Using the definition of a polylagrangian subspace, we conclude that we must establish the inclusion

$$\big(\textstyle\bigwedge^k L^\perp\big) \otimes \hat{T} \;\subset\; \hat{\omega}^\flat\big(L'\big)\;.$$

But the equality (1.43) guarantees that for any $\alpha \in \bigwedge^k L^\perp$ and for any $1 \leqslant a \leqslant \hat{n}$, there is a vector $v_a \in L$ such that

$$\mathrm{i}_{v_a}\hat{\omega} \;=\; \alpha \otimes \hat{e}_a\;.$$

Since

$$\mathrm{i}_{v_a}\hat{\omega}^b \;=\; \alpha\,\langle \hat{e}^b, \hat{e}_a\rangle \;=\; \delta^b_a\,\alpha\;,$$

we see that

$$v_a \in \bigcap_{\substack{b=1 \\ b \neq a}}^{\hat{n}} \ker\, \hat{\omega}^b\;.$$

Decomposing v_a according to equation (1.50), we find a vector $u_a \in K_a$ such that

$$\mathrm{i}_{u_a}\hat{\omega} \;=\; \alpha \otimes \hat{e}_a\;,$$

so $\alpha \otimes \hat{e}_a \in \hat{\omega}^\flat(K_a) \subset \hat{\omega}^\flat(L')$. Finally, we observe that, due to the definition (1.50) of the spaces K_a, together with the relation (1.33), we have

$$\ker\, \omega^a \cap K_a \;=\; \{0\}$$

so that equation (1.49) follows from equation (1.48), whereas equation (1.51) is a direct consequence of the formulas (1.45) or (1.46), (1.48) and (1.49). □

But even for scalar forms, i.e., when $\dim \hat{T} = 1$, uniqueness of the polylagrangian subspace is assured, except for 2-forms $(k = 1)$.

Proposition 1.1 *Let V be a finite-dimensional vector space and $\hat{\omega}$ a polylagrangian $(k{+}1)$-form of rank N on V, with polylagrangian subspace L. If $k > 1$, L is unique.*

A proof can be found in Ref. [19] (the additional hypothesis made there that $\hat{\omega}$ should be non-degenerate can easily be removed if we replace V and L by the respective quotient spaces $V/\ker\,\hat{\omega}$ and $L/\ker\,\hat{\omega}$).

In the case of symplectic forms ($\dim\hat{T} = 1$, $k = 1$, $\ker\,\hat{\omega} = \{0\}$), where the concept of a polylagrangian subspace reduces to the familiar concept of a lagrangian subspace, it is well known that such a uniqueness statement does not hold. The fundamental importance of polylagrangian subspaces in the process of generalizing lagrangian subspaces from the symplectic case to vector-valued forms stems from the fact that they are essential for obtaining a Darboux theorem.

Ordinary non-degenerate forms of degree > 2 ($\dim\hat{T} = 1$, $k > 1$, $\ker\,\hat{\omega} = \{0\}$) have been studied in the literature [3, 19] under the label "multisymplectic forms", but without emphasizing the central role played by the concept of a polylagrangian subspace, which appears only implicitly through the dimension criterion formulated in Theorem 1.3, employed as a definition. Anyway, the restriction imposed by the existence of such a subspace is a necessary condition to obtain a "Darboux basis" for the form $\hat{\omega}$.

1.2 Partially horizontal forms on linear spaces

In this section we present concepts analogous to those introduced in the previous section, but for ordinary forms subject to a condition of (partial) horizontality relative to a given vertical subspace, which requires some modifications. Once again, we introduce a special type of maximal isotropic subspace which will be called a multilagrangian subspace and which plays a fundamental role in the theory of multisymplectic forms.

Let W be a finite-dimensional vector space and V be a fixed subspace of W, the vertical space. Following the notation already employed at the beginning of this chapter, we denote the base space W/V by T, setting $\dim T = n$, and the canonical projection of W onto T by π, thus obtaining the exact sequence (1.3). Moreover, suppose that ω is a $(k+1)$-form on W which is $(k+1-r)$-horizontal, where $1 \leqslant r \leqslant k$ and $k+1-r \leqslant n$: [5]

$$\omega \in \bigwedge\nolimits_r^{k+1} W^* \, . \tag{1.52}$$

[5] The trivial cases $r = 0$ and $r = k+1$ will be excluded since they are already covered by the formalism developed in the previous section, with $\dim\hat{T} = 1$, whereas the additional condition $k+1-r \leqslant n$ (which need not be stated separately if $k \leqslant n$) can be imposed without loss of generality since otherwise, ω will have to vanish identically.

As we are dealing with a special case of the situation studied before, the definitions given at the beginning of the previous section can be applied without modification: this includes the concepts of kernel and of support, the notion of a non-degenerate form, the idea of rank and the definition of the ℓ-orthogonal complement of a subspace. The first important difference occurs in the study of isotropic subspaces, since in the case of partially horizontal forms, we shall consider only isotropic subspaces L of the total space W contained in the vertical subspace V, which leads to a concept of maximality different from the previous one: a subspace L of V will be called **maximal isotropic** in V (with respect to ω) if L is isotropic (with respect to ω) and such that for any subspace L' of V which is also isotropic (with respect to ω)

$$L \subset L' \quad \Longrightarrow \quad L = L' \; .$$

Using the restriction of the linear map

$$\begin{array}{rccc} \omega_W^\flat : & W & \longrightarrow & \bigwedge^k W^* \\ & w & \longmapsto & \mathrm{i}_w \omega \end{array}$$

defined by equation (1.38) to the vertical subspace V, which is a linear map

$$\begin{array}{rccc} \omega_V^\flat : & V & \longrightarrow & \bigwedge^k_{r-1} W^* \\ & v & \longmapsto & \mathrm{i}_v \omega \end{array} \quad , \tag{1.53}$$

and defining, for any subspace L of V,

$$\bigwedge^k_{r-1} L^\perp \;=\; \bigwedge^k L^\perp \cap \bigwedge^k_{r-1} W^* \; , \tag{1.54}$$

we conclude that a subspace L of V will be isotropic if and only if

$$\omega_V^\flat(L) \;\subset\; \bigwedge^k_{r-1} L^\perp \; , \tag{1.55}$$

which can be trivially rewritten in the form

$$\omega_V^\flat(L) \;\subset\; \omega_V^\flat(V) \,\cap\, \bigwedge^k_{r-1} L^\perp \; . \tag{1.56}$$

Similarly, we conclude that a subspace L of V containing $\ker \omega_V^\flat = V \cap \ker \omega$ will be maximal isotropic in V if and only if

$$\omega_V^\flat(L) \;=\; \omega_V^\flat(V) \,\cap\, \bigwedge^k_{r-1} L^\perp \; . \tag{1.57}$$

In fact, for any subspace L of V, we have the identity

$$\omega_V^\flat\big(L^{\omega,1} \cap V\big) \;=\; \omega_V^\flat(V) \,\cap\, \bigwedge^k_{r-1} L^\perp \; , \tag{1.58}$$

since both sides are equal to the subspace of $\bigwedge_{r-1}^{k} W^*$ consisting of elements of the form $\mathrm{i}_v \omega$ where $v \in V$ and $\mathrm{i}_u \mathrm{i}_v \omega = 0$ for all $u \in L$.

As in the case of polylagrangian forms, these considerations lead us to introduce a concept that turns out to provide the key to the theory of multisymplectic forms: the existence of a special type of maximal isotropic subspace, which we shall call a multilagrangian subspace.

Definition 1.2 *Let W be a finite-dimensional vector space and V be a fixed subspace of W, with $n = \dim(W/V)$, and let ω be a $(k+1-r)$-horizontal $(k+1)$-form on W, where $1 \leqslant r \leqslant k$ and $k+1-r \leqslant n$. We say that ω is a* ***multilagrangian*** *form of* ***rank*** *N and* ***degree of horizontality*** *$k+1-r$ if V admits a subspace L of codimension N which is* ***multilagrangian****, i.e., such that*

$$\omega_V^\flat(L) \;=\; \bigwedge\nolimits_{r-1}^{k} L^\perp \;. \tag{1.59}$$

When $k = n$, $r = 2$ and ω is non-degenerate, we call ω a ***multisymplectic*** *form.*

The following theorem shows that a multilagrangian subspace, when it exists, really is a special type of maximal isotropic subspace of V; in particular, it contains the kernel of ω.

Theorem 1.5 *Let W be a finite-dimensional vector space and V be a fixed subspace of W, with $n = \dim(W/V)$, and let ω be a multilagrangian $(k+1)$-form on W of rank N and degree of horizontality $k+1-r$, where $1 \leqslant r \leqslant k$ and $k+1-r \leqslant n$, and with multilagrangian subspace L. Then $N+n \geqslant k$ (except if $\omega \equiv 0$) and L contains the kernel of ω:*

$$\ker \omega \;\subset\; L \;. \tag{1.60}$$

The dimension of L is given by

$$\dim L \;=\; \dim \ker \omega + \sum_{s=0}^{r-1} \binom{N}{s} \binom{n}{k-s} \;, \tag{1.61}$$

where it is understood that, by definition, $\binom{p}{q} = 0$ if $q > p$.

Proof. First we observe that if $N+n < k$, we have $\bigwedge^k L^\perp = \{0\}$ and thus both sides of the equation (1.59) vanish; therefore, L is contained in $\ker \omega$ and so $\ker \omega$ has codimension $< k$ in W, implying $\omega \equiv 0$, since the $(k+1)$-form on the quotient space $W/\ker \omega$ induced by ω vanishes identically. (More generally, this argument shows that a nonvanishing $(k+1-r)$-horizontal $(k+1)$-form does not permit isotropic subspaces of codimension $< k-n$.) Thus supposing that $\dim L^\perp = N+n \geqslant k$ and using that $\dim V^\perp = n$ and $V^\perp \subset L^\perp$, we conclude that we can, for any vector $w \in W \setminus L$, find a linearly independent set of 1-forms

$w_1^*, \ldots, w_k^* \in L^\perp$ such that $w_r^*, \ldots, w_k^* \in V^\perp$ and $\langle w_1^*, w \rangle = 1$ and $\langle w_i^*, w \rangle = 0$ for $i > 1$. According to the definition of a multilagrangian subspace, there is a vector $u \in L$ such that

$$\mathrm{i}_u \omega \;=\; w_1^* \wedge \ldots \wedge w_k^* \quad \Rightarrow \quad \mathrm{i}_w \mathrm{i}_u \omega \;=\; w_2^* \wedge \ldots \wedge w_k^* \;\neq\; 0$$

and so $w \notin \ker \omega$. Hence it follows that $\ker \omega \subset L$, and we obtain

$$\dim L \,-\, \dim \ker \omega \;=\; \dim\, \omega_V^\flat\big(L\big) \;=\; \dim \textstyle\bigwedge_{r-1}^{k} L^\perp \;.$$

To calculate this dimension we proceed as at the beginning of this chapter, introducing a basis $\{e_1^L, \ldots, e_l^L, e_1^{L'}, \ldots, e_N^{L'}, e_1^T, \ldots, e_n^T\}$ of W such that the first l vectors form a basis of L, the following N vectors form a basis of a subspace L' complementary to L in V and the last n vectors form a basis of a subspace H complementary to V in W, isomorphic to T. Then in terms of the dual basis $\{e_L^1, \ldots, e_L^l, e_{L'}^1, \ldots, e_{L'}^N, e_T^1, \ldots, e_T^n\}$ of W^*, we conclude that $\{\, e_{L'}^{i_1} \wedge \ldots \wedge e_{L'}^{i_s} \wedge e_T^{\mu_1} \wedge \ldots \wedge e_T^{\mu_{k-s}} \mid 0 \leqslant s \leqslant r-1\,,\, 1 \leqslant i_1 < \ldots < i_s \leqslant N\,,$ $1 \leqslant \mu_1 < \ldots < \mu_{k-s} \leqslant n \,\}$ is a basis of $\bigwedge_{r-1}^{k} L^\perp$, which proves the formula (1.61). □

The dimension formula stated in the previous theorem provides a simple criterion to decide whether a given isotropic subspace is multilagrangian.

Theorem 1.6 *Let W be a finite-dimensional vector space and V be a fixed subspace of W, with $n = \dim(W/V)$, and let ω be a $(k+1-r)$-horizontal $(k+1)$-form on W, where $1 \leqslant r \leqslant k$ and $k+1-r \leqslant n$. If L is an isotropic subspace of V of codimension N containing $\ker \omega$ and such that*

$$\dim L \;=\; \dim \ker \omega \,+\, \sum_{s=0}^{r-1} \binom{N}{s} \binom{n}{k-s} \;, \tag{1.62}$$

where it is understood that, by definition, $\binom{p}{q} = 0$ if $q > p$, then L is a multilagrangian subspace and ω is a multilagrangian form.

PROOF. Since L is an isotropic subspace of V containing $\ker \omega$, the restriction of the linear map $\omega_V^\flat$ (see equation (1.53)) to L takes L to $\bigwedge_{r-1}^{k} L^\perp$ and induces an injective linear map from the quotient space $L/\ker \omega$ into $\bigwedge_{r-1}^{k} L^\perp$ which, due to the condition (1.62), is a linear isomorphism, so the equality (1.59) follows. □

A particularly simple case occurs when $r = 1$, since every k-horizontal multilagrangian $(k+1)$-form has rank 0 and multilagrangian subspace V. (In fact, the condition that ω shoud be k-horizontal is equivalent to the condition that V should be isotropic, and in this case, V does satisfy the remaining criteria of Theorem 1.6.)

1.3 The symbol

Our main goal in this section is to exhibit a simple and general relation between multilagrangian and polylagrangian forms: the symbol of a multilagrangian (multisymplectic) form is a polylagrangian (polysymplectic) form taking values in a space of forms of appropriate degree on the base space.

To explain this construction, we adopt the same notation as in the previous section, that is, W is a finite-dimensional vector space and V is a fixed subspace of W, the vertical space, whereas the quotient space W/V, the base space, will be denoted by T, its dimension by $n = \dim T$ and the canonical projection from W to T by π, giving rise to the exact sequence (1.3). We shall also suppose that ω is a $(k+1-r)$-horizontal $(k+1)$-form on W, where $1 \leqslant r \leqslant k$ and $k+1-r \leqslant n$:

$$\omega \in \textstyle\bigwedge_r^{k+1} W^* \, . \tag{1.63}$$

Then the symbol $\hat{\omega}$ of ω, defined as at the beginning of this chapter, will be a $\bigwedge^{k+1-r} T^*$-valued r-form on V:

$$\hat{\omega} \in \textstyle\bigwedge^r V^* \otimes \bigwedge^{k+1-r} T^* \, . \tag{1.64}$$

Using the canonical isomorphism $\bigwedge^{k+1-r} T^* \cong \bigwedge_0^{k+1-r} W^*$ as an identification, we have

$$\hat{\omega}(v_1, \ldots, v_r) \;=\; \mathrm{i}_{v_r} \ldots \mathrm{i}_{v_1} \omega \qquad \text{for } \; v_1, \ldots, v_r \in V \, . \tag{1.65}$$

More explicitly, if we use a splitting $\mathfrak{s}$ of π, we have

$$\begin{gathered} \hat{\omega}(v_1, \ldots, v_r)\big(t_1, \ldots, t_{k+1-r}\big) \;=\; \omega(v_1, \ldots, v_r, \mathfrak{s}(t_1), \ldots, \mathfrak{s}(t_{k+1-r})) \\ \text{for } \; v_1, \ldots, v_r \in V, \, t_1, \ldots, t_{k+1-r} \in T \, , \end{gathered} \tag{1.66}$$

as has already been mentioned at the beginning of this chapter.

A straightforward analysis of the formula (1.66) shows that a subspace L of the vertical space V which is isotropic with respect to ω will also be isotropic with respect to $\hat{\omega}$ (the converse statement is not true). Moreover, we have

Theorem 1.7 *Let W be a finite-dimensional vector space and V be a fixed subspace of W, with $n = \dim(W/V)$, let ω be a $(k+1-r)$-horizontal $(k+1)$-form on W, where $1 \leqslant r \leqslant k$ and $k+1-r \leqslant n$, and let $\hat{\omega}$ be the symbol of ω, which is a $\bigwedge^{k+1-r} T^*$-valued r-form on V. Suppose that ω is multilagrangian, with multilagrangian subspace L. Then $\hat{\omega}$ is polylagrangian, with polylagrangian subspace L. If ω is multisymplectic then $\hat{\omega}$ is polysymplectic.*

PROOF. Fixing an arbitrary horizontal subspace H of W and using the direct decompositions (1.17) and (1.18), with $H^\perp \cong V^*$, we note that in order to show that L is polylagrangian with respect to $\hat{\omega}$, we must establish the equality

$$\hat{\omega}^\flat(L) \;\cong\; \bigwedge^{r-1}(L^\perp \cap H^\perp) \otimes \bigwedge^{k+1-r} T^* \;.$$

To do so, we use the canonical isomorphism $\bigwedge^{k+1-r} T^* \cong \bigwedge_0^{k+1-r} W^* = \bigwedge^{k+1-r} V^\perp$ and the inclusion $V^\perp \subset L^\perp$, together with the fact that the space $\bigwedge^{r-1}(L^\perp \cap H^\perp) \otimes \bigwedge^{k+1-r} V^\perp$ is generated by elements which can be written in the form

$$\hat{\alpha} \;=\; \big(w_1^* \wedge \ldots \wedge w_{r-1}^*\big) \otimes \big(w_r^* \wedge \ldots \wedge w_k^*\big) \qquad \text{where} \qquad \alpha \;=\; w_1^* \wedge \ldots \wedge w_k^*$$

with $w_1^*, \ldots, w_{r-1}^* \in L^\perp \cap H^\perp$ and $w_r^*, \ldots, w_k^* \in V^\perp \subset L^\perp$. Since L is multilagrangian with respect to ω, there is a vector $u \in L$ such that $\alpha = \mathrm{i}_u \omega$ and so $\hat{\alpha} = \mathrm{i}_u \hat{\omega}$, showing that L is polylagrangian with respect to $\hat{\omega}$. □

We have shown that the multilagrangian subspace associated with a multilagrangian form ω is the same as the polylagrangian subspace associated with its symbol $\hat{\omega}$, but it is worthwhile to note that the kernels of the two forms need not coincide: The only relation that can be established in general is the inclusion

$$\ker\,\omega \;\subset\; \ker\,\hat{\omega}\;. \tag{1.67}$$

In particular, it may (and often does) happen that ω is non-degenerate while $\hat{\omega}$ is degenerate. In what follows, we shall frequently consider the projections $\hat{\omega}_{t_1,\ldots,t_{k+1-r}}$ of $\hat{\omega}$ along decomposable tensors $t_1 \wedge \ldots \wedge t_{k+1-r} \in \bigwedge^{k+1-r} T$: then using a splitting $\mathfrak{s}$ of π, we will have for the respective kernels

$$\ker\,\hat{\omega}_{t_1,\ldots,t_{k+1-r}} \;=\; \Big\{\, v \in V \;\Big|\; \begin{array}{c} \omega\big(v, v_1, \ldots, v_{r-1}, \mathfrak{s}(t_1), \ldots, \mathfrak{s}(t_{k+1-r})\big) \;=\; 0 \\ \text{para}\;\; v_1, \ldots, v_{r-1} \in V,\; t_1, \ldots, t_{k+1-r} \in T \end{array} \Big\}\;, \tag{1.68}$$

which implies that, as a consequence of the results obtained in the first section, $\ker\,\hat{\omega}$ is the intersection of all these subspaces and, for $0 < k+1-r < n$, L is their sum.

1.4 Canonical Examples

In this section we shall introduce the canonical examples of a polylagrangian and of a multilagrangian vector space. As we shall see later, every non-degenerate polylagrangian or multilagrangian form is equivalent to the canonical one presented here, that is, there is a linear isomorphism between the underlying vector spaces relating them by pull-back. Throughout the entire discussion, we shall fix the integer k that characterizes the degree of the form (which is $k+1$, with $k \geqslant 1$) and, in the multilagrangian case, the integer r that characterizes the horizontal degree of the form (which is $k+1-r$, with $1 \leqslant r \leqslant k$ and $k+1-r \leqslant n$, as before), in order to avoid overloading the notation with prefixes such as "k-" or "(k,r)-".

1.4.1 The canonical polylagrangian form

Let E and $\hat{T}$ be vector spaces of dimension N and $\hat{n}$, respectively. Set

$$V_0 \;=\; E \oplus \Big(\big(\textstyle\bigwedge^k E^* \big) \otimes \hat{T} \Big) \;. \tag{1.69}$$

Definition 1.3 *The* ***canonical polylagrangian form*** *of rank N is the non-degenerate $\hat{T}$-valued $(k+1)$-form $\hat{\omega}_0$ on V_0 defined by*

$$\hat{\omega}_0\big((u_0, \alpha_0 \otimes \hat{t}_0), \ldots, (u_k, \alpha_k \otimes \hat{t}_k)\big) \;=\; \sum_{i=0}^{k} (-1)^i \, \alpha_i(u_0, \ldots, \widehat{u_i}, \ldots, u_k) \, \hat{t}_i \;. \tag{1.70}$$

If $k=1$ we shall call $\hat{\omega}_0$ the ***canonical polysymplectic form****.*

For scalar forms ($\hat{T} = \mathbb{R}$) this construction can be found, e.g., in Refs [19] or [3].

To prove the statements contained in the above definition, note that it is a straightforward exercise to show that $\hat{\omega}_0$ is non-degenerate and that, considering E and

$$L \;=\; \big(\textstyle\bigwedge^k E^* \big) \otimes \hat{T} \tag{1.71}$$

as subspaces of V_0, we have the direct decomposition $V_0 = E \oplus L$ where

$$L \text{ is polylagrangian} \quad \text{and} \quad E \text{ is } k\text{-isotropic} \;, \tag{1.72}$$

since L is obviously isotropic and has the right dimension, as required by Theorem 1.3:

$$\dim L \;=\; \hat{n} \binom{N}{k} \quad \text{where} \quad \operatorname{codim}_{V_0} L \;=\; N \;. \tag{1.73}$$

In terms of bases, let $\{\, \hat{e}_a \mid 1 \leqslant a \leqslant \hat{n} \,\}$ be any basis of $\hat{T}$ with dual basis $\{\, \hat{e}^a \mid 1 \leqslant a \leqslant \hat{n} \,\}$ of $\hat{T}^*$ and let $\{\, e_i \mid 1 \leqslant i \leqslant N \,\}$ be any basis of E with dual basis $\{\, e^i \mid 1 \leqslant i \leqslant N \,\}$ of E^*. For $1 \leqslant a \leqslant \hat{n}$ and $1 \leqslant i_1 < \ldots < i_k \leqslant N$, define

$$e_a^{i_1 \ldots i_k} \;=\; e^{i_1} \wedge \ldots \wedge e^{i_k} \otimes \hat{e}_a \quad , \quad e^a_{i_1 \ldots i_k} \;=\; e_{i_1} \wedge \ldots \wedge e_{i_k} \otimes \hat{e}^a \;.$$

This provides a basis $\{ e_i, e_a^{i_1 \ldots i_k} \mid 1 \leqslant a \leqslant \hat{n}\,, 1 \leqslant i \leqslant N\,, 1 \leqslant i_1 < \ldots < i_k \leqslant N \}$ of V_0 with dual basis $\{ e^i, e^a_{i_1 \ldots i_k} \mid 1 \leqslant a \leqslant \hat{n}\,, 1 \leqslant i \leqslant N\,, 1 \leqslant i_1 < \ldots < i_k \leqslant N \}$ of V_0^*, both of which we shall refer to as a **Darboux basis**, such that

$$\hat{\omega}_0 \;=\; \tfrac{1}{k!} \left(e^a_{i_1 \ldots i_k} \wedge e^{i_1} \wedge \ldots \wedge e^{i_k} \right) \otimes \hat{e}_a \;. \tag{1.74}$$

In Section 1.5 we shall see that any polylagrangian vector space splits into the direct sum of a (its) polylagrangian subspace and a second component which is k-isotropic, as above. This property is precisely what is needed to show that every polylagrangian form is equivalent to the canonical one – a statement that can be viewed as a basis independent formulation of (the algebraic version of) the Darboux Theorem for polylagrangian vector spaces.

1.4.2 The canonical multilagrangian form

Let F be a vector space of dimension $N+n$ and E be a fixed N-dimensional subspace of F. Denoting the n-dimensional quotient space F/E by T and the canonical projection of F onto T by ρ, we obtain the following exact sequence of vector spaces:

$$0 \;\longrightarrow\; E \;\longrightarrow\; F \;\xrightarrow{\;\rho\;}\; T \;\longrightarrow\; 0 \;. \tag{1.75}$$

Define

$$W_0 \;=\; F \oplus {\textstyle\bigwedge}^k_{r-1} F^* \quad , \quad V_0 \;=\; E \oplus {\textstyle\bigwedge}^k_{r-1} F^* \quad , \quad \pi_0 \;=\; \rho \circ \mathrm{pr}_1 \;, \tag{1.76}$$

where $\mathrm{pr}_1 : W_0 \to F$ is the canonical projection, which leads us to the following exact sequence of vector spaces:

$$0 \;\longrightarrow\; V_0 \;\longrightarrow\; W_0 \;\xrightarrow{\;\pi_0\;}\; T \;\longrightarrow\; 0 \;. \tag{1.77}$$

Definition 1.4 *The* ***canonical multilagrangian form*** *of rank N and horizontal degree $k+1-r$ is the $(k+1-r)$-horizontal $(k+1)$-form ω_0 on W_0 defined by*

$$\omega_0\big((u_0,\alpha_0), \ldots, (u_k,\alpha_k)\big) \;=\; \sum_{i=0}^{k} (-1)^i \, \alpha_i(u_0, \ldots, \widehat{u_i}, \ldots, u_k) \;. \tag{1.78}$$

If $k = n$ and $r = 2$ we shall call ω_0 the ***canonical multisymplectic form****.*

Note that when $r = 1$, the form ω_0 is degenerate, with

$$\ker \omega_0 \;=\; E \qquad \text{if } r = 1 \;, \tag{1.79}$$

whereas for $r > 1$, it is non-degenerate. In fact, suppose that $(u, \alpha) \in W_0 \setminus \{0\}$. If $\alpha \neq 0$, there exist vectors $u_1, \ldots, u_k \in F$ such that

$$\omega_0\big((u,\alpha), (u_1,0), \ldots, (u_k,0)\big) \;=\; \alpha(u_1, \ldots, u_k) \;\neq\; 0$$

and hence $(u,\alpha) \notin \ker \omega_0$. If $r = 1$ and $u \notin E$ or if $r > 1$ and $u \neq 0$, there exists a $(k+1-r)$-horizontal k-form $\beta \in \bigwedge_{r-1}^k F^*$ such that $\mathrm{i}_u \beta \neq 0$; hence we can find vectors $u_2, \ldots, u_k \in F$ such that

$$\omega_0((u,0),(0,\beta),(u_2,0),\ldots,(u_k,0)) \;=\; -\,\beta(u,u_2,\ldots,u_k) \;\neq\; 0 \;,$$

that is, $(u,0) \notin \ker \omega_0$.

In what follows, we shall assume that $N > 0$, since when $E = \{0\}$, we are back to the polylagrangian case on $T \oplus \bigwedge^k T^*$, with $\hat{T} = \mathbb{R}$, which has already been studied in Refs [19] and [3]. For the same reason, we shall also assume that $r > 1$, since for $r = 1$ we have $\bigwedge_0^k F^* \cong \bigwedge^k T^*$, so that after passing to the quotient by the kernel of ω_0, we are once again back to the polylagrangian case on $T \oplus \bigwedge^k T^*$, with $\hat{T} = \mathbb{R}$.

To prove the statements contained in the above definition (under the additional hypotheses that $N > 0$ and $r > 1$), note that it is a straightforward exercise to show that ω_0 is $(k+1-r)$-horizontal and that, considering F and

$$L \;=\; \bigwedge\nolimits_{r-1}^{k} F^* \tag{1.80}$$

as subspaces of W_0, we have the direct decompositions $W_0 = F \oplus L$ and $V_0 = E \oplus L$ where

$$L \text{ is multilagrangian} \quad , \quad F \text{ is } k\text{-isotropic} \quad , \quad E \text{ is } (r-1)\text{-isotropic} \;, \tag{1.81}$$

since L is obviously isotropic and has the right dimension, as required by Theorem 1.6:

$$\dim L \;=\; \sum_{s=0}^{r-1} \binom{N}{s} \binom{n}{k-s} \quad \text{where} \quad \mathrm{codim}_{V_0} L \;=\; N \;. \tag{1.82}$$

In terms of bases, let $\{\, e_i, e_\mu \,|\, 1 \leqslant i \leqslant N \,,\, 1 \leqslant \mu \leqslant n \,\}$ be a basis of F with dual basis $\{\, e^i, e^\mu \,|\, 1 \leqslant i \leqslant N \,,\, 1 \leqslant \mu \leqslant n \,\}$ of F^* such that $\{\, e_i \,|\, 1 \leqslant i \leqslant N \,\}$ is a basis of E and $\{\, e_\mu \,|\, 1 \leqslant \mu \leqslant n \,\}$ is a basis of a subspace H of F complementary to E, isomorphic to T. For $0 \leqslant s \leqslant r$, $1 \leqslant i_1 < \ldots < i_s \leqslant N$ and $1 \leqslant \mu_1 < \ldots < \mu_{k-s} \leqslant N$, define

$$\begin{array}{rcl} e_{i_1 \ldots\, i_s;\, \mu_1 \ldots\, \mu_{k-s}} & = & e_{i_1} \wedge \ldots \wedge e_{i_s} \wedge e_{\mu_1} \wedge \ldots \wedge e_{\mu_{k-s}} \\ e^{i_1 \ldots\, i_s;\, \mu_1 \ldots\, \mu_{k-s}} & = & e^{i_1} \wedge \ldots \wedge e^{i_s} \wedge e^{\mu_1} \wedge \ldots \wedge e^{\mu_{k-s}} \end{array} \;.$$

This provides a basis

$$\Big\{\, e_i, e_\mu, e^{i_1 \ldots\, i_s;\, \mu_1 \ldots\, \mu_{k-s}} \;\Big|\; 0 \leqslant s \leqslant r-1 \,,\, \begin{array}{c} 1 \leqslant i \leqslant N \,,\, 1 \leqslant i_1 < \ldots < i_s \leqslant N \\ 1 \leqslant \mu \leqslant n \,,\, 1 \leqslant \mu_1 < \ldots < \mu_{k-s} \leqslant n \end{array} \,\Big\}$$

of W_0 with dual basis

$$\{ e^i, e^\mu, e_{i_1 \ldots i_s; \mu_1 \ldots \mu_{k-s}} \mid 0 \leqslant s \leqslant r-1 \,, \begin{array}{c} 1 \leqslant i \leqslant N \,,\, 1 \leqslant i_1 < \ldots < i_s \leqslant N \\ 1 \leqslant \mu \leqslant n \,,\, 1 \leqslant \mu_1 < \ldots < \mu_{k-s} \leqslant n \end{array} \}$$

of W_0^*, both of which we shall refer to as a **Darboux basis**, such that

$$\omega_0 \;=\; \sum_{s=0}^{r-1} \tfrac{1}{s!} \tfrac{1}{(k-s)!} \, e_{i_1 \ldots i_s; \mu_1 \ldots \mu_{k-s}} \wedge e^{i_1} \wedge \ldots \wedge e^{i_s} \wedge e^{\mu_1} \wedge \ldots \wedge e^{\mu_{k-s}} \;. \tag{1.83}$$

and for the symbol

$$\hat{\omega}_0 \;=\; \tfrac{1}{(r-1)!} \tfrac{1}{(k+1-r)!} \left(e_{i_1 \ldots i_{r-1}; \mu_1 \ldots \mu_{k+1-r}} \wedge e^{i_1} \wedge \ldots \wedge e^{i_{r-1}} \right) \otimes \left(e^{\mu_1} \wedge \ldots \wedge e^{\mu_{k+1-r}} \right). \tag{1.84}$$

In Section 1.6 we shall see that every multilagrangian vector space splits into the direct sum of a (its) multilagrangian subspace and a second component which is k-isotropic and contains an $(r-1)$-isotropic subspace such that the quotient of these two components is isomorphic to the base space T, as above. This property is precisely what is needed to show that every multilagrangian form is equivalent to the canonical one – a statement that can be viewed as a basis independent formulation of (the algebraic version of) the Darboux Theorem for multilagrangian vector spaces.

1.5 Polylagrangian vector spaces

In this section we will show that every polylagrangian vector space admits a Darboux basis, or canonical basis, in terms of which the polylagrangian form is given by the standard expression of equation (1.74).

Definition 1.5 *Let V and $\hat{T}$ be finite-dimensional vector spaces, with $\hat{n} \equiv \dim \hat{T}$, and let $\hat{\omega}$ be a $\hat{T}$-valued polylagrangian $(k+1)$-form on V of rank N, with polylagrangian subspace L. A* ***(polylagrangian) Darboux basis*** *or* ***(polylagrangian) canonical basis*** *for $\hat{\omega}$ is a basis $\{ e_i, e_a^{i_1 \ldots i_k} \mid 1 \leqslant a \leqslant \hat{n} \,, 1 \leqslant i \leqslant N \,, 1 \leqslant i_1 < \ldots < i_k \leqslant N \}$ of any subspace of V complementary to $\ker \hat{\omega}$, with dual basis $\{ e^i, e^a_{i_1 \ldots i_k} \mid 1 \leqslant a \leqslant \hat{n} \,, 1 \leqslant i \leqslant N \,, 1 \leqslant i_1 < \ldots < i_k \leqslant N \}$ of the subspace $\mathrm{supp}\, \hat{\omega} = (\ker \hat{\omega})^\perp$ of V^*, together with a basis $\{ \hat{e}_a \mid 1 \leqslant a \leqslant \hat{n} \}$ of $\hat{T}$, with dual basis $\{ \hat{e}^a \mid 1 \leqslant a \leqslant \hat{n} \}$ of $\hat{T}^*$, such that*

$$\hat{\omega} \;=\; \tfrac{1}{k!} \left(e^a_{i_1 \ldots i_k} \wedge e^{i_1} \wedge \ldots \wedge e^{i_k} \right) \otimes \hat{e}_a \;. \tag{1.85}$$

If $k = 1$ we shall call such a basis a ***(polysymplectic) Darboux basis*** *or* ***(polysymplectic) canonical basis****.*

The existence of such a basis for any polylagrangian form follows from the existence of a direct decomposition $V = L \oplus E$ of V analogous to the one for V_0 in Section 1.4.1.

Theorem 1.8 *Let V and $\hat{T}$ be finite-dimensional vector spaces, with $\hat{n} \equiv \dim \hat{T}$, and let $\hat{\omega}$ be a $\hat{T}$-valued polylagrangian $(k+1)$-form on V of rank N, with polylagrangian subspace L. Then there exists a k-isotropic subspace E of V complementary to L, i.e., such that*

$$V \;=\; E \oplus L \;. \tag{1.86}$$

PROOF. Let E_0 be a k-isotropic subspace of V of dimension N' such that $E_0 \cap L = \{0\}$. (For instance, as long as $N' \leqslant k$, E_0 can be any subspace of V such that $E_0 \cap L = \{0\}$.) If $N' = N$, we are done. Otherwise, choose a basis $\{e_1, \ldots, e_N\}$ of a subspace of V complementary to L such that the first N' vectors constitute a basis of E_0, and denote the corresponding dual basis of $L^\perp$ by $\{e^1, \ldots, e^N\}$. We shall prove that there exists a vector $u \in V \setminus (E_0 \oplus L)$ such that the subspace E_1 of V spanned by u and E_0 is k-isotropic and satisfies $E_1 \cap L = \{0\}$; then since $\dim E_1 = N' + 1$, the statement of the theorem follows by induction. To this end, consider an arbitrary basis $\{\hat{e}_a \,|\, 1 \leqslant a \leqslant \hat{n}\}$ of $\hat{T}$ with dual basis $\{\hat{e}^a \,|\, 1 \leqslant a \leqslant \hat{n}\}$ of $\hat{T}^*$ and, choosing any subspace L' of L complementary to $\ker \hat{\omega}$, use the fact that L is polylagrangian to conclude that there exists a unique basis $\{\, e_a^{i_1 \ldots i_k} \,|\, 1 \leqslant a \leqslant \hat{n}\,,\, 1 \leqslant i_1 < \ldots < i_k \leqslant N \,\}$ of L' such that

$$\hat{\omega}^\flat(e_a^{i_1 \ldots i_k}) \;=\; e^{i_1} \wedge \ldots \wedge e^{i_k} \otimes \hat{e}_a \;.$$

Thus, for $1 \leqslant i_1 < \ldots < i_k \leqslant N$ and $1 \leqslant j_1 < \ldots < j_k \leqslant N$, we have

$$\omega^b(e_a^{i_1 \ldots i_k}, e_{j_1}, \ldots, e_{j_k}) \;=\; \delta_a^b \, \delta_{j_1}^{i_1} \ldots \, \delta_{j_k}^{i_k} \;.$$

Therefore, the vector

$$u \;=\; e_{N'+1} \;-\; \tfrac{1}{k!} \, \omega^a(e_{N'+1}, e_{i_1}, \ldots, e_{i_k}) \, e_a^{i_1 \ldots i_k}$$

does not belong to the subspace $E_0 \oplus L$ and, for $1 \leqslant j_1 < \ldots < j_k \leqslant N$, satisfies

$$\begin{aligned} &\omega^b(u, e_{j_1}, \ldots, e_{j_k}) \\ &\quad = \; \omega^b(e_{N'+1}, e_{j_1}, \ldots, e_{j_k}) \;-\; \tfrac{1}{k!} \, \omega^a(e_{N'+1}, e_{i_1}, \ldots, e_{i_k}) \, \omega^b(e_a^{i_1 \ldots i_k}, e_{j_1}, \ldots, e_{j_k}) \\ &\quad = \; 0 \;, \end{aligned}$$

which implies that since the subspace E_0 spanned by $e_1, \ldots, e_{N'}$ is k-isotropic, the subspace E_1 spanned by $e_1, \ldots, e_{N'}$ and u is so as well. □

Now it is easy to derive the algebraic Darboux theorem: let $\{\, \hat{e}_a \,|\, 1 \leqslant a \leqslant \hat{n} \,\}$ be an arbitrary basis of $\hat{T}$, with dual basis $\{\, \hat{e}^a \,|\, 1 \leqslant a \leqslant \hat{n} \,\}$ of $\hat{T}^*$, and let $\{\, e_i \,|\, 1 \leqslant i \leqslant N \,\}$ be an arbitrary basis of a k-isotropic subspace E complementary to L in V, with dual basis $\{ e^i \,|\, 1 \leqslant i \leqslant N \}$ of $L^\perp \cong E^*$. Choosing an arbitrary subspace L' of L complementary to $\ker \hat{\omega}$ and taking into account the identity (1.43), we define a basis $\{ e_a^{i_1 \ldots i_k} \,|\, 1 \leqslant a \leqslant \hat{n} \,,\, 1 \leqslant i_1 < \ldots < i_k \leqslant N \}$ of L' by

$$\hat{\omega}^\flat(e_a^{i_1 \ldots i_k}) \;=\; e^{i_1} \wedge \ldots \wedge e^{i_k} \otimes \hat{e}_a \;.$$

It is easy to see that the union of this basis with that of E gives a canonical basis of V (or more precisely, of $E \oplus L'$, which is a subspace of V complementary to $\ker \hat{\omega}$). Thus we have proved

Theorem 1.9 (Darboux theorem for polylagrangian vector spaces)
Every polylagrangian vector space admits a canonical basis.

Clearly, the inductive construction of a k-isotropic subspace E complementary to the polylagrangian subspace L, as presented in the proof of Theorem 1.8, provides an iterative and explicit method to construct polylagrangian bases in a way similar to the well known Gram-Schmidt orthogonalization process.

1.6 Multilagrangian vector spaces

In this section we will show that every multilagrangian vector space admits a Darboux basis, or canonical basis, in termos of which the multilagrangian form is given by the standard expression of equation (1.83).

Definition 1.6 *Let W be a finite-dimensional vector space and V be a fixed subspace of W, with $n = \dim(W/V)$, and let ω be a multilagrangian $(k{+}1)$-form on W of rank N and degree of horizontality $k+1-r$, where $1 \leqslant r \leqslant k$ and $k+1-r \leqslant n$, and with multilagrangian subspace L. A* ***(multilagrangian) Darboux basis*** *or* ***(multilagrangian) canonical basis*** *for ω is a basis*

$$\{ e_i, e_\mu, e^{i_1 \ldots i_s;\, \mu_1 \ldots \mu_{k-s}} \,|\, 0 \leqslant s \leqslant r-1 \,, \begin{array}{c} 1 \leqslant i \leqslant N \,,\, 1 \leqslant i_1 < \ldots < i_s \leqslant N \\ 1 \leqslant \mu \leqslant n \,,\, 1 \leqslant \mu_1 < \ldots < \mu_{k-s} \leqslant n \end{array} \}$$

of any subspace of W complementary to $\ker \omega$, with dual basis

$$\{ e^i, e^\mu, e_{i_1 \ldots i_s;\, \mu_1 \ldots \mu_{k-s}} \,|\, 0 \leqslant s \leqslant r-1 \,, \begin{array}{c} 1 \leqslant i \leqslant N \,,\, 1 \leqslant i_1 < \ldots < i_s \leqslant N \\ 1 \leqslant \mu \leqslant n \,,\, 1 \leqslant \mu_1 < \ldots < \mu_{k-s} \leqslant n \end{array} \}$$

of the subspace $\operatorname{supp} \omega = (\ker \omega)^\perp$ of W^*, such that

$$\omega \;=\; \sum_{s=0}^{r-1} \tfrac{1}{s!} \, \tfrac{1}{(k-s)!} \; e_{i_1 \ldots i_s;\, \mu_1 \ldots \mu_{k-s}} \wedge e^{i_1} \wedge \ldots \wedge e^{i_s} \wedge e^{\mu_1} \wedge \ldots \wedge e^{\mu_{k-s}} \;. \tag{1.87}$$

In this same basis we have for the symbol

$$\hat{\omega} \;=\; \tfrac{1}{(r-1)!}\,\tfrac{1}{(k+1-r)!}\,\big(e_{i_1\ldots\, i_{r-1}\,;\,\mu_1\ldots\,\mu_{k+1-r}} \wedge e^{i_1} \wedge \ldots \wedge e^{i_{r-1}}\big) \otimes \big(e^{\mu_1} \wedge \ldots \wedge e^{\mu_{k+1-r}}\big)\,. \tag{1.88}$$

If $k = n$ and $r = 2$ we shall call such a basis a ***(multisymplectic) Darboux basis*** *or* ***(multisymplectic) canonical basis****.*

The existence of such a basis for any multilagrangian form follows from the existence of direct decompositions $W = L \oplus F$ of W and $V = L \oplus E$ of V analogous to the ones for W_0 and V_0 in Section 1.4.2.

Theorem 1.10 *Let W be a finite-dimensional vector space and V be a fixed subspace of W, with $n = \dim(W/V)$, and let ω be a multilagrangian $(k{+}1)$-form on W of rank N and degree of horizontality $k+1-r$, where $1 \leqslant r \leqslant k$ and $k+1-r \leqslant n$, and with multilagrangian subspace L. Then there exists a k-isotropic subspace F of W such that the intersection $E = V \cap F$ is an $(r-1)$-isotropic subspace of V and*

$$W \;=\; F \oplus L \quad , \quad V \;=\; E \oplus L\,. \tag{1.89}$$

PROOF. First we construct an $(r-1)$-isotropic subspace E of V of dimension N which is complementary to L in V. If $r = 1$ there is nothing to prove since in this case the vertical subspace V is isotropic and so we have $L = V$, $E = \{0\}$ and $N = 0$. If $r > 1$ we apply Theorem 1.8 to the symbol $\hat{\omega}$ of ω to conclude that there is a subspace E of V of dimension N which is complementary to L in V and is $(r-1)$-isotropic with respect to $\hat{\omega}$. Now taking into account that the whole vertical subspace V is r-isotropic with respect to ω, it follows that E is $(r-1)$-isotropic with respect to ω as well. – Now let F_0 be a subspace of W of dimension $N + n'$ which is k-isotropic with respect to ω and such that $F_0 \cap V = E$. (For instance, if $n' = 0$, $F_0 = E$). If $n' = n$, we are done. Otherwise, choose a basis $\{e_1^E, \ldots, e_N^E, e_1, \ldots, e_n\}$ of a subspace of W complementary to L such that the first N vectors constitute a basis of E and the first $N + n'$ vectors constitute a basis of F_0, and denote the corresponding dual basis of $L^\perp$ by $\{e_E^1, \ldots, e_E^N, e^1, \ldots, e^n\}$. We shall prove that there exists a vector $u \in W \setminus (F_0 \oplus L)$ such that the subspace F_1 spanned by u and F_0 is k-isotropic and satisfies $F_1 \cap V = E$; then since $\dim F_1 = N + n' + 1$, the statement of the theorem follows by induction. To this end, choose any subspace L' of L complementary to $\ker \omega$ and use the fact that L is multilagrangian to conclude that there exists a unique basis

$$\Big\{\, e^{i_1\ldots\, i_s;\,\mu_1\ldots\,\mu_{k-s}} \;\Big|\; 0 \leqslant s \leqslant r-1\,,\; \begin{array}{c} 1 \leqslant i_1 < \ldots < i_s \leqslant N \\ 1 \leqslant \mu_1 < \ldots < \mu_{k-s} \leqslant n \end{array} \Big\}$$

of L' such that

$$\omega_V^\flat\big(e^{i_1\ldots\, i_s;\,\mu_1\ldots\,\mu_{k-s}}\big) \;=\; e_E^{i_1} \wedge \ldots \wedge e_E^{i_s} \wedge e^{\mu_1} \wedge \ldots \wedge e^{\mu_{k+1-s}}\,.$$

Thus, for $0 \leqslant s,t \leqslant r-1$, $1 \leqslant i_1 < \ldots < i_s \leqslant N$, $1 \leqslant j_1 < \ldots < j_t \leqslant N$, $1 \leqslant \mu_1 < \ldots < \mu_{k-s} \leqslant n$, $1 \leqslant \nu_1 < \ldots < \nu_{k-t} \leqslant n$, we have

$$\omega(e^{i_1 \ldots i_s ; \mu_1 \ldots \mu_{k-s}}, e_{j_1}^E, \ldots, e_{j_t}^E, e_{\nu_1}, \ldots, e_{\nu_{k-t}}) \;=\; \left\{ \begin{array}{cc} 0 & \text{if} \quad s \neq t \\ \delta_{j_1}^{i_1} \ldots \delta_{j_s}^{i_s} \, \delta_{\nu_1}^{\mu_1} \ldots \delta_{\nu_{k-s}}^{\mu_{k-s}} & \text{if} \quad s = t \end{array} \right\} .$$

Therefore, the vector

$$u \;=\; e_{n'+1} \;-\; \sum_{s=0}^{r-1} \tfrac{1}{s!} \, \tfrac{1}{(k-s)!} \, \omega(e_{n'+1}, e_{i_1}^E, \ldots, e_{i_s}^E, e_{\mu_1}, \ldots, e_{\mu_{k-s}}) \, e^{i_1 \ldots i_s ; \mu_1 \ldots \mu_{k-s}}$$

does not belong to the subspace $F_0 \oplus L$ and, for $0 \leqslant t \leqslant r-1$, $1 \leqslant j_1 < \ldots < j_t \leqslant N$ and $1 \leqslant \nu_1 < \ldots < \nu_{k-t} \leqslant n$, satisfies

$$\begin{aligned} & \omega(u, e_{j_1}^E, \ldots, e_{j_t}^E, e_{\nu_1}, \ldots, e_{\nu_{k-t}}) \\ & \;=\; \omega(e_{n'+1}, e_{j_1}^E, \ldots, e_{j_t}^E, e_{\nu_1}, \ldots, e_{\nu_{k-t}}) \\ & \qquad - \sum_{s=0}^{r-1} \tfrac{1}{s!} \, \tfrac{1}{(k-s)!} \, \omega(e_{n'+1}, e_{i_1}^E, \ldots, e_{i_s}^E, e_{\mu_1}, \ldots, e_{\mu_{k-s}}) \\ & \qquad\qquad\qquad \times \omega(e^{i_1 \ldots i_s ; \mu_1 \ldots \mu_{k-s}}, e_{j_1}^E, \ldots, e_{j_t}^E, e_{\nu_1}, \ldots, e_{\nu_{k-t}}) \\ & \;=\; 0 \, . \end{aligned}$$

which implies that since the subspace F_0 spanned by $e_1^E, \ldots, e_N^E, e_1, \ldots, e_{n'}$ is k-isotropic, the subspace F_1 spanned by $e_1^E, \ldots, e_N^E, e_1, \ldots, e_{n'}$ and u is so as well. □

Now it is easy to derive the algebraic Darboux theorem: let $\{ e_i, e_\mu \,|\, 1 \leqslant i \leqslant N \,, 1 \leq \mu \leq n \}$ be a basis of a k-isotropic subspace F complementary to L in W, with dual basis $\{ e^i, e^\mu \,|\, 1 \leqslant i \leqslant N \,, 1 \leq mu \leq n \}$ of $L^\perp \cong F^*$, such that $\{ e_i \,|\, 1 \leqslant i \leqslant N \}$ is a basis of the $(r-1)$-isotropic subspace $E = V \cap F$ which is complementary to L in V. Choosing any subspace L' of L complementary to $\ker \omega$ and taking into account the identity (1.59), we define a basis

$$\{ e^{i_1 \ldots i_s ; \mu_1 \ldots \mu_{k-s}} \,|\, 0 \leqslant s \leqslant r-1 \,, \begin{array}{c} 1 \leqslant i_1 < \ldots < i_s \leqslant N \\ 1 \leqslant \mu_1 < \ldots < \mu_{k-s} \leqslant n \end{array} \}$$

of L' by

$$\omega_V^\flat(e^{i_1 \ldots i_s ; \mu_1 \ldots \mu_{k-s}}) \;=\; e^{i_1} \wedge \ldots \wedge e^{i_s} \wedge e^{\mu_1} \wedge \ldots \wedge e^{\mu_{k+1-s}} \, .$$

It is easy to see that the union of this basis with that of F gives a canonical basis of W (or more precisely, of $F \oplus L'$, which is a subspace of W complementary to $\ker \omega$). Thus we have proved

Theorem 1.11 (Darboux theorem for multilagrangian vector spaces)
Every multilagrangian vector space admits a canonical basis.

Once again, the inductive construction of a k-isotropic subspace F complementary to the multilagrangian subspace L, as presented in the proof of Theorem 1.10, provides an iterative and explicit method to construct multilagrangian bases in a way similar to the well known Gram-Schmidt orthogonalization process.

Chapter 2

Polylagrangian and multilagrangian forms: differential theory

The algebraic concepts introduced in the previous chapter constitute the basis for the development of a coherent differential theory that will be presented in this chapter. Our main result will be a Darboux theorem for polylagrangian and multilagrangian forms which guarantees that differential forms which are pointwise polylagrangian or multilagrangian admit holonomic canonical bases, or in other words, canonical local coordinates, if and only if they satisfy the standard integrability condition of being closed.

Regarding terminology, we shall freely use the standard notions of the theory of manifolds and of fiber bundles (in particular, vector bundles). A typical result that will be used repeatedly is the following.

Lemma 2.1 *Let E_1 and E_2 be vector subbundles of a vector bundle E. Then $E_1 \cap E_2$ has constant dimension if and only if $E_1 + E_2$ has constant dimension, and in this case, $E_1 \cap E_2$ and $E_1 + E_2$ are vector subbundles of E as well.*

A similar assertion holds for the intersection and sum of a finite number of vector subbundles of a given vector bundle.

As a particular case, we have the concept of a **distribution** on a manifold P, which is simply a vector subbundle D of the tangent bundle TP of P, thus formalizing the idea of a family of subspaces D_p of the tangent spaces T_pP parametrized by the points p of P in such a way that the subspaces D_p "depend smoothly on p" and "have constant dimension" (i.e., $\dim D_p$ does not depend on p). Such a distribution is **involutive** if for any two vector fields X and Y which are sections of D, their Lie bracket $[X, Y]$ is again a section of D, and it is **integrable** if it arises from a **foliation** of P, i.e., a decomposition of P into a disjoint union of submanifolds called **leaves** such that for any point of P the space D_p is the tangent space to the leaf passing through p. According to the Frobenius theorem, integrable distributions are involutive and, conversely, involutive distributions are integrable.

One well known example of an involutive distribution is the kernel of a closed differential form of constant rank. Briefly, the **kernel** $\ker\,\omega$ of a $(k+1)$-form ω on a manifold P, defined by

$$\ker_p \omega \;=\; \ker\,\omega_p \;=\; \{\, u_p \in T_p P \mid \mathrm{i}_{u_p}\omega = 0 \,\}\;, \tag{2.1}$$

gives rise to a distribution on P as soon as it has constant dimension, i.e., as soon as ω has constant rank.[1] Using the general formula for the exterior derivative of ω,

$$\begin{aligned} & d\omega\,(X_1,\ldots,X_{k+2}) \\ &\quad = \sum_{i=1}^{k+2} (-1)^{i-1}\, X_i \cdot \Big(\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{k+2})\Big) \\ &\qquad + \sum_{1\leqslant i<j\leqslant k+2} (-1)^{i+j}\, \omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{k+2}) \end{aligned}\;, \tag{2.2}$$

we conclude that if $d\omega = 0$ and if X_1 and X_2 take values in $\ker\,\omega$, then $[X_1,X_2]$ takes values in $\ker\,\omega$ as well, that is,

$$d\omega \;=\; 0 \quad\Longrightarrow\quad \ker\,\omega \;\; \text{involutive}\;. \tag{2.3}$$

The same procedure works for vector-valued differential forms.

2.1 Cartan calculus for vertical forms

In order to extend the structures studied in the previous chapter (specifically, the symbol) from a purely algebraic setting to the realm of differential geometry, we shall need a variant of Cartan's calculus, which in its standard formulation deals with differential forms on manifolds, to handle vertical differential forms on total spaces of fiber bundles.

Let E be a fiber bundle over a base manifold M, with projection $\pi : E \to M$, and let $\hat{T}$ a vector bundle over the same base manifold M, with projection $\hat{\tau} : \hat{T} \to M$. Consider the pull-back of $\hat{T}$ to E, which will be denoted by $\pi^*\hat{T}$, and the vertical bundle VE of E: both are vector bundles over E. Then the vector bundle

$$\textstyle\bigwedge^r V^*E \otimes \pi^*\hat{T}$$

[1] Although there exist different conventions regarding the concept of rank (the most common one being the codimension of the kernel), the notion of constant rank is unambiguously defined.

will be called the bundle of **vertical r-forms** on E, and its sections are called **differential r-forms** or simply **r-forms** on E, with **values** or **coefficents** in $\pi^*\hat{T}$: the space of such forms will be denoted by $\Omega_V^r(E\,;\pi^*\hat{T})$. On the other hand, the sections of the vertical bundle VE are called **vertical vector fields** or simply **vertical fields** on E: the space of such fields will be denoted by $\mathfrak{X}_V(E)$. Obviously, $\mathfrak{X}_V(E)$ and $\Omega_V^r(E\,;\pi^*\hat{T})$ are (locally finite) modules over the algebra $\mathfrak{F}(E)$ of functions on E.

It should be noted that there is a certain amount of "abuse of language" in this terminology, since the elements of $\Omega_V^r(E\,;\pi^*\hat{T})$ are really only equivalence classes of differential r-forms on E, since $\Omega_V^r(E\,;\pi^*\hat{T})$ is not a subspace of the space $\Omega^r(E\,;\pi^*\hat{T})$ of all differential r-forms on E but rather its quotient space

$$\Omega_V^r(E\,;\pi^*\hat{T}) \;=\; \Omega^r(E\,;\pi^*\hat{T})/\Omega^{\;r}_{r-1}(E\,;\pi^*\hat{T})$$

by the subspace $\Omega^{\;r}_{r-1}(E\,;\pi^*\hat{T})$ of all 1-horizontal differential r-forms on E.

An interesting aspect of this construction is that it is possible to develop a variant of the usual Cartan calculus for differential forms on a manifold M in which functions on M are replaced by functions f on E, vector fields on M are replaced by vertical fields X on E and differential forms on M are replaced by vertical differential forms α on E taking values in $\pi^*\hat{T}$, in such a way that all operations of this calculus are preserved and continue to satisfy the same rules. (See [10, Vol. 1, Probl. 8, p. 313] for the special case where $\hat{T}$ is a trivial fiber bundle $M \times \mathbb{R}$.) These operations are the **contraction**

$$\begin{array}{rccc} \mathrm{i}: & \mathfrak{X}_V(E) \times \Omega_V^r(E\,;\pi^*\hat{T}) & \longrightarrow & \Omega_V^{r-1}(E\,;\pi^*\hat{T}) \\ & (X,\alpha) & \longmapsto & \mathrm{i}_X\alpha \end{array},$$

the **vertical Lie derivative**

$$\begin{array}{rccc} L: & \mathfrak{X}_V(E) \times \Omega_V^r(E\,;\pi^*\hat{T}) & \longrightarrow & \Omega_V^r(E\,;\pi^*\hat{T}) \\ & (X,\alpha) & \longmapsto & L_X\alpha \end{array}, \tag{2.4}$$

and the **vertical exterior derivative**

$$\begin{array}{rccc} d_V: & \Omega_V^r(E\,;\pi^*\hat{T}) & \longrightarrow & \Omega_V^{r+1}(E\,;\pi^*\hat{T}) \\ & \alpha & \longmapsto & d_V\alpha \end{array}, \tag{2.5}$$

apart from various versions of the **exterior product**, for instance

$$\begin{array}{rccc} \wedge: & \Omega_V^r(E\,;\pi^*\hat{T}) \times \Omega_V^s(E\,;\pi^*\hat{T}') & \longrightarrow & \Omega_V^{r+s}(E\,;\pi^*(\hat{T}\otimes\hat{T}')) \\ & (\alpha,\alpha') & \longmapsto & \alpha\wedge\alpha' \end{array}.$$

The first and the last of these are purely algebraic operations, identical with the usual ones. To define the second and third, we use the same formulas as in the usual case, namely

$$L_X\alpha\,(X_1,\ldots,X_r) \;=\; X\cdot(\alpha(X_1,\ldots,X_r)) \,-\, \sum_{i=1}^{r} \alpha(X_1,\ldots,[X,X_i],\ldots,X_r)\;, \tag{2.6}$$

and

$$\begin{aligned} d_V\alpha\,(X_0,\ldots,X_r) \;=\; & \sum_{i=0}^{r} (-1)^i\, X_i\cdot\Big(\alpha(X_0,\ldots,\hat{X}_i,\ldots,X_r)\Big) \\ & +\sum_{0\leqslant i<j\leqslant r} (-1)^{i+j}\,\alpha([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_r) \end{aligned}\;, \tag{2.7}$$

where $X_0,X_1,\ldots,X_r \in \mathfrak{X}_V(E)$, which makes sense since VE is an involutive distribution on E (i.e., $\mathfrak{X}_V(E)$ is a subalgebra of $\mathfrak{X}(E)$ with respect to the Lie bracket), provided we correctly define the **vertical directional derivative**

$$\begin{array}{ccc} \mathfrak{X}_V(E)\times\Gamma(\pi^*\hat{T}) & \longrightarrow & \Gamma(\pi^*\hat{T}) \\ (X,\varphi) & \longmapsto & X\cdot\varphi \end{array}\;. \tag{2.8}$$

as an $\mathbb{R}$-bilinear operator which is $\mathfrak{F}(E)$-linear in the first entry and satisfies the Leibniz rule in the second entry,

$$X\cdot(f\varphi) \;=\; (X\cdot f)\,\varphi + f(X\cdot\varphi)\;. \tag{2.9}$$

Explicitly, for $X\in\mathfrak{X}_V(E)$ and $\varphi\in\Gamma(\pi^*\hat{T})$, $X\cdot\varphi\in\Gamma(\pi^*\hat{T})$ is defined as the standard directional derivative of vector valued functions along the fibers, that is, for any point m in M, $(X\cdot\varphi)\big|_{E_m}\in C^\infty(E_m,\hat{T}_m)$ is given in terms of $X\big|_{E_m}\in\mathfrak{X}(E_m)$ and $\varphi\big|_{E_m}\in C^\infty(E_m,\hat{T}_m)$ by

$$(X\cdot\varphi)\big|_{E_m} \;=\; X\big|_{E_m}\cdot\varphi\big|_{E_m}\;. \tag{2.10}$$

Since the Lie bracket is natural under restriction to submanifolds, we have

$$X\cdot\big(Y\cdot\varphi\big) \,-\, Y\cdot\big(X\cdot\varphi\big) \;=\; [X,Y]\cdot\varphi \qquad \text{for } X,Y\in\mathfrak{X}_V(E),\ \varphi\in\Gamma(\pi^*\hat{T})\;,$$

which implies that $d_V^2=0$. On the other hand, sections φ of $\pi^*\hat{T}$ obtained from sections $\hat{t}$ of $\hat{T}$ by composing with π are constant along the fibers and hence their vertical directional derivative vanishes:

$$X\cdot(\hat{t}\circ\pi) \;=\; 0 \qquad \text{for } X\in\mathfrak{X}_V(E),\ \hat{t}\in\Gamma(\hat{T})\;. \tag{2.11}$$

In the same way, substituting $\hat{T}$ by $\hat{T}^*$, we get

$$X\cdot(\hat{t}^*\circ\pi) \;=\; 0 \qquad \text{for } X\in\mathfrak{X}_V(E),\ \hat{t}^*\in\Gamma(\hat{T}^*)\;, \tag{2.12}$$

which means that

$$X \cdot \langle \hat{t}^* \circ \pi \,,\, \varphi \rangle ~=~ \langle \hat{t}^* \circ \pi \,,\, (X \cdot \varphi) \rangle \qquad \text{for}~~ X \in \mathfrak{X}_V(E),\, \varphi \in \Gamma(\pi^* \hat{T}) ~. \tag{2.13}$$

More generally, given a $\pi^* \hat{T}$-valued vertical r-form $\hat{\omega}$ on P and a section $\hat{t}^*$ of the dual vector bundle $\hat{T}^*$ of $\hat{T}$, we define the **projection** of $\hat{\omega}$ along $\hat{t}^*$ to be the ordinary vertical r-form $\hat{\omega}_{\hat{t}^*}$ on P given by [2]

$$\hat{\omega}_{\hat{t}^*}(p) ~=~ \langle \hat{t}^*(\pi(p)), \hat{\omega}(p) \rangle \qquad \text{for}~~ p \in P ~, \tag{2.14}$$

and obtain

$$d_V \hat{\omega}_{\hat{t}^*} ~=~ \big(d_V \hat{\omega} \big)_{\hat{t}^*} ~. \tag{2.15}$$

Hence, if $\hat{\omega}$ is closed, $\hat{\omega}_{\hat{t}^*}$ will be closed as well. Moreover, the **kernel** $\ker \hat{\omega}$ of $\hat{\omega}$ and the **kernel** $\ker \hat{\omega}_{\hat{t}^*}$ of $\hat{\omega}_{\hat{t}^*}$, defined by

$$\ker_p \hat{\omega} ~=~ \ker \hat{\omega}_p ~=~ \{\, u_p \in V_p P \mid \mathrm{i}_{u_p} \hat{\omega} = 0 \,\} \qquad \text{for}~~ p \in P ~, \tag{2.16}$$

and

$$\ker_p \hat{\omega}_{\hat{t}^*} ~=~ \ker(\hat{\omega}_{\hat{t}^*})_p ~=~ \{\, u_p \in V_p P \mid \mathrm{i}_{u_p} \hat{\omega}_{\hat{t}^*} = 0 \,\} \qquad \text{for}~~ p \in P ~, \tag{2.17}$$

satisfy $\ker \hat{\omega} \subset \ker \hat{\omega}_{\hat{t}^*} \subset VP$ and define distributions on P of P as long as they have constant dimension (in the case of $\ker \hat{\omega}$, this is equivalent to the condition that $\hat{\omega}$ has constant rank), and applying the same procedure as before, we conclude that

$$d_V \hat{\omega} ~=~ 0 \quad \Longrightarrow \quad \ker \hat{\omega} ~\text{and}~ \ker \hat{\omega}_{\hat{t}^*} ~\text{are involutive} ~. \tag{2.18}$$

2.2 Polylagrangian manifolds

In this section and the next, we shall transfer the poly- and multilagrangian structures introduced in the previous chapter from the algebraic to the differential context by adding the appropriate integrability condition.

Definition 2.1 *A **polylagrangian manifold** is a manifold P equipped with a $(k+1)$-form $\hat{\omega}$ taking values in a fixed $\hat{n}$-dimensional vector space $\hat{T}$, called the **polylagrangian form** and said to be of **rank** N, such that $\hat{\omega}$ is closed,*

$$d\hat{\omega} ~=~ 0 ~, \tag{2.19}$$

*and admits a distribution L on P, called the **polylagrangian distribution**, such that at every point p of P, $\hat{\omega}_p$ is a polylagrangian form of rank N on $T_p P$ with polylagrangian subspace L_p.*
*When $k = 1$, we say that P is a **polysymplectic manifold** and $\hat{\omega}$ is a **polysymplectic form**.*

[2]Here, it is important that $\hat{t}^*$ be a section of $\hat{T}^*$ and not of $\pi^* \hat{T}^*$.

Note that according to Theorem 1.2 and equation (1.47), L has constant dimension if and only if $\ker\,\hat{\omega}$ has constant dimension, that is, $\hat{\omega}$ has constant rank.

As in the purely algebraic context, we associate with each linear form $\hat{t}^* \in \hat{T}^*$ on the auxiliary space $\hat{T}$ an ordinary $(k+1)$-form $\omega_{\hat{t}^*}$ on P called the **projection** of $\hat{\omega}$ along $\hat{t}^*$ and defined by

$$\omega_{\hat{t}^*} \;=\; \langle \hat{t}^*, \hat{\omega} \rangle \;, \tag{2.20}$$

so taking a basis $\{\hat{e}_1, \ldots, \hat{e}_{\hat{n}}\}$ of $\hat{T}$ with dual basis $\{\hat{e}^1, \ldots, \hat{e}^{\hat{n}}\}$ of $\hat{T}^*$, and defining the ordinary $(k+1)$-forms

$$\omega^a \;=\; \omega_{\hat{e}^a} \quad (1 \leqslant a \leqslant \hat{n}) \;, \tag{2.21}$$

we obtain, as in the purely algebraic context,

$$\ker\,\hat{\omega} \;=\; \bigcap_{\hat{t}^* \in \hat{T}^*} \ker\,\omega_{\hat{t}^*} \;=\; \bigcap_{a=1}^{\hat{n}} \ker\,\omega^a \;. \tag{2.22}$$

When $\hat{n} \equiv \dim \hat{T} \geqslant 2$, we can use Theorem 1.4 to conclude that

$$L \;=\; \sum_{\hat{t}^* \in\, \hat{T}^* \setminus \{0\}} \ker\,\omega_{\hat{t}^*} \;. \tag{2.23}$$

In this case, it is sufficient to suppose that $\hat{\omega}$ is a $\hat{T}$-valued $(k+1)$-form of constant rank such that at every point p of P, L_p is the polylagrangian subspace of T_pP with respect to $\hat{\omega}_p$, since the dimension formulas (1.46) and (1.51) then guarantee that, for $1 \leqslant a \leqslant \hat{n}$, $\ker\,\omega^a$ has constant dimension and hence defines a distribution on P; therefore, L is a distribution on P as well. Moreover, for $1 \leqslant a \leqslant \hat{n}$, K_a has constant dimension and can be chosen to be a distribution on P. It should be emphasized, however, that this argument fails if $\hat{n} \equiv \dim \hat{T} = 1$, that is, for ordinary forms – notably in the symplectic case, where a lagrangian distribution is far from unique and a mere collection of lagrangian subspaces need not even be differentiable.

When $\hat{n} \equiv \dim \hat{T} \geqslant 3$, we can apply Theorem 1.4 to show that the polylagrangian distribution is necessarily involutive!

Theorem 2.1 *Let P be a polylagrangian manifold with polylagrangian $(k+1)$-form $\hat{\omega}$ taking values in a fixed vector space $\hat{T}$ of dimension $\hat{n} \geqslant 3$. Then the polylagrangian distribution L is involutive.*

PROOF. With the same notation as before, suppose that X and Y are vector fields which are sections of L. Using the decomposition

$$L \;=\; K_0 \oplus K_1 \oplus \ldots \oplus K_{\hat{n}} \;,$$

with $K_0 = \ker\,\hat{\omega}$ (see equation (1.48)), we can decompose X and Y according to

$$X \;=\; \sum_{a=0}^{\hat{n}} X_a \quad , \quad Y \;=\; \sum_{b=0}^{\hat{n}} Y_b \; ,$$

where X_a and Y_b are sections of K_a and K_b, respectively. Using that $\hat{n} \geqslant 3$, we can for each value of a and b find a value $c \neq 0$ such that $c \neq a$ and $c \neq b$; then $K_a \subset \ker\,\omega^c$ and $K_b \subset \ker\,\omega^c$. Since ω^c is closed and has constant rank, $\ker\,\omega^c$ is involutive. Therefore the vector field $[X_a, Y_b]$ is a section of $\ker\,\omega^c \subset L$. □

There are polylagrangian manifolds whose polylagrangian distribution is not involutive. According to Theorem 2.1, this can only occur when $\hat{n} = 1$ or $\hat{n} = 2$. An interesting example of this kind is the following.

Example 2.1 Let $P = S^3 = SU(2)$, $\hat{T} = \mathbb{R}^2$ and define α^1, α^2, α^3 to be the three components of the left invariant Maurer-Cartan form on P with respect to the standard basis of $\mathbb{R}^3$, which are dual to the three left invariant vector fields ξ_1, ξ_2, ξ_3 generated by this basis. Define

$$\omega^{(1)} \;=\; \alpha^3 \wedge \alpha^1 \quad , \quad \omega^{(2)} \;=\; \alpha^3 \wedge \alpha^2 \quad , \quad \omega^{(3)} \;=\; \alpha^1 \wedge \alpha^2 \; .$$

It follows immediately from the Maurer-Cartan structure equations that each of these 2-forms is closed. Thus omitting $\omega^{(3)}$, say, we see that

$$\hat{\omega} \;=\; \omega^{(1)} \hat{e}_1 + \omega^{(2)} \hat{e}_2$$

is a non-degenerate polysymplectic form of rank 1 on P whose polylagrangian distribution is spanned by ξ_1 (which generates the kernel of $\omega^{(2)}$) and ξ_2 (which generates the kernel of $\omega^{(1)}$), and this is certainly not involutive.

This implies that for $\hat{n} = 1$ or $\hat{n} = 2$, the condition of integrability of the polylagrangian distribution L needs to be imposed separately, whenever necessary or convenient.

2.3 Poly- and multilagrangian fiber bundles

Passing from manifolds to fiber bundles, suppose that P is a bundle over a base manifold M with projection $\pi : P \to M$ and consider the tangent mapping $T\pi : TP \to TM$, whose kernel is a vector subbundle VP of the tangent bundle TP of P, called the **vertical bundle**, giving rise to the following exact sequence of vector bundles over P:

$$0 \longrightarrow VP \longrightarrow TP \xrightarrow{T\pi} \pi^*TM \longrightarrow 0 \,. \tag{2.24}$$

This will play the same role as the exact sequence (1.3) of vector spaces in the previous chapter and is the basis for the notions of horizontality that will appear in this chapter.

To begin with, we introduce the notion of a polylagrangian fiber bundle, which formalizes the idea of a "family of polylagrangian manifolds smoothly parametrized by the points of a base manifold M". To do so, we shall need the concepts introduced in the first section of this chapter.

Definition 2.2 *A **polylagrangian fiber bundle** is a fiber bundle P over an n-dimensional manifold M equipped with a vertical $(k+1)$-form $\hat{\omega}$ on the total space P taking values in a fixed $\hat{n}$-dimensional vector bundle $\hat{T}$ over the same manifold M, called the **polylagrangian form along the fibers** of P, or simply **polylagrangian form**, and said to be of **rank** N, such that $\hat{\omega}$ is vertically closed,*

$$d_V\hat{\omega} \;=\; 0 \,, \tag{2.25}$$

*and admits a distribution L on P contained in the vertical bundle VP of P, called the **polylagrangian distribution**, such that at every point p of P, $\hat{\omega}_p$ is a polylagrangian form of rank N on V_pP with polylagrangian subspace L_p.*
*When $k = 1$, we say that P is a **polysymplectic fiber bundle** and $\hat{\omega}$ is a **polysymplectic form along the fibers** of P, or simply **polysymplectic form**.*

If P is a polylagrangian fiber bundle over a manifold M with polylagrangian form $\hat{\omega}$ taking values in a vector bundle $\hat{T}$ over M, then for any point m in M, the fiber P_m of P over m will be a polylagrangian manifold with polylagrangian form $\hat{\omega}_m$, the restriction of $\hat{\omega}$ to P_m, taking values in the vector space $\hat{T}_m$, the fiber of $\hat{T}$ over m, and whose polylagrangian distribution L_m is the restriction of the polylagrangian distribution L to P_m. The main point of the above definition is to guarantee regularity of this family of polylagrangian manifolds, parametrized by the points m in M, along M, that is, transversally to the fibers. Obviously, the involutivity theorem (Theorem 2.1) remains valid in this context.

In the same way, we define the notion of a multilagrangian fiber bundle:

Definition 2.3 *A **multilagrangian fiber bundle** is a fiber bundle P over an n-dimensional manifold M equipped with a $(k+1-r)$-horizontal $(k+1)$-form ω on the total space P, where $1 \leqslant r \leqslant k$ and $k+1-r \leqslant n$, called the **multilagrangian form** and said to be of **rank** N and **horizontal degree** $k+1-r$, such that ω is closed,*

$$d\omega \;=\; 0 \,, \tag{2.26}$$

and admits a distribution L on P, called the ***multilagrangian distribution****, such that at every point p of P, ω_p is a multilagrangian form of rank N and horizontal degree $k+1-r$ on T_pP with multilagrangian subspace L_p.*
When $k = n$, $r = 2$ and ω is non-degenerate, we say that P is a ***multisymplectic fiber bundle*** *and ω is a* ***multisymplectic form****.*

2.4 The symbol

In what follows we want to show how the close relation between multi- and polylagrangian structures found in the previous chapter can be extended from the purely algebraic context to that of differential geometry: the link is once again provided by the concept of symbol, which is now a prescription that to each multilagrangian form on a fiber bundle associates a polylagrangian form on that fiber bundle taking values in an appropriate bundle of differential forms on the base manifold. To this end, we shall once again need the concepts introduced in the first section of this chapter.

Thus let us suppose, as in the previous section, that P is a fiber bundle over an n-dimensional base manifold M and let us use the exact sequence (2.24). We also suppose, again as in the previous section, that ω is a $(k+1-r)$-horizontal $(k+1)$-form on P, where $1 \leqslant r \leqslant k$ and $k+1-r \leqslant n$:

$$\omega \,\in\, \Omega_r^{k+1}\big(P\big) \;=\; \Gamma\big(\textstyle\bigwedge_r^{k+1} T^*P\big) \;. \tag{2.27}$$

Explicitly, this means that contraction of ω with more than r vertical fields on P gives zero:

$$\mathrm{i}_{X_1} \ldots \mathrm{i}_{X_{r+1}} \omega \;=\; 0 \qquad \text{for } \; X_1, \ldots, X_{r+1} \in \mathfrak{X}_V(P) \;. \tag{2.28}$$

Then the symbol $\hat{\omega}$ of ω, defined pointwise (i.e., such that at every point p of P, $\hat{\omega}_p$ is the symbol of ω_p in the sense of the previous chapter), will be a vertical r-form on P taking values in the vector bundle $\pi^*\big(\bigwedge^{k+1-r} T^*M\big)$:

$$\hat{\omega} \,\in\, \Omega_V^r\big(P\,, \pi^*\big(\textstyle\bigwedge^{k+1-r} T^*M\big)\big) \;=\; \Gamma\big(\textstyle\bigwedge^r V^*P \otimes \pi^*\big(\bigwedge^{k+1-r} T^*M\big)\big) \;. \tag{2.29}$$

Using the canonical isomorphism $\pi^*\big(\bigwedge^{k+1-r} T^*M\big) \cong \bigwedge_0^{k+1-r} T^*P$ of vector bundles over P as an identification, we have

$$\hat{\omega}(X_1, \ldots, X_r) \;=\; \mathrm{i}_{X_1} \ldots \mathrm{i}_{X_r} \omega \qquad \text{for } \; X_1, \ldots, X_r \in \mathfrak{X}_V(P) \;. \tag{2.30}$$

More explicitly, using the horizontal lift

$$\begin{array}{ccc} \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(P) \\ \xi & \longmapsto & \xi^H \end{array}$$

of vector fields induced by some fixed connection on P, we have

$$\begin{array}{c} \hat{\omega}(X_1,\ldots,X_r)\cdot\big(\xi_1\circ\pi,\ldots,\xi_{k+1-r}\circ\pi\big) \;=\; \omega(X_1,\ldots,X_r,\xi_1^H,\ldots,\xi_{k+1-r}^H) \\ \text{for }\; X_1,\ldots,X_r\in\mathfrak{X}_V(P),\,\xi_1,\ldots,\xi_{k+1-r}\in\mathfrak{X}(M)\;, \end{array} \tag{2.31}$$

where it should be noted that although composition of vector fields on M with the projection π provides only a subspace of the vector space of all sections of the pull-back π^*TM of TM by π, this formula is sufficient to fix the value of $\hat{\omega}$ at each point p of P.

Theorem 2.2 *With the same notations as above, suppose that the form ω satisfies*

$$d\omega \;\in\; \Omega_r^{k+2}(P) \;=\; \Gamma\big(\textstyle\bigwedge_r^{k+2}T^*P\big)\;,$$

i.e., $d\omega$ is $(k+2-r)$-horizontal. Then the form $\hat{\omega}$ is vertically closed:

$$d_V\hat{\omega} \;=\; 0\;.$$

In particular, for $\hat{\omega}$ to be vertically closed, it is sufficient but not necessary that ω be closed.

PROOF. Let $X_0,\ldots,X_r$ be vertical fields on P and $\xi_1,\ldots,\xi_{k+1-r}$ be vector fields on M, and denote the horizontal lifts of the latter with respect to some fixed connection on P by $\xi_1^H,\ldots,\xi_{k+1-r}^H$, respectively. Then we compute

$$\begin{aligned}
&\Big(d_V\hat{\omega}\big(X_0,\ldots,X_r\big)\Big)\big(\xi_1\circ\pi,\ldots,\xi_{k+1-r}\circ\pi\big) \\
&= \left(\sum_{i=0}^{r}(-1)^i\,X_i\cdot\big(\hat{\omega}(X_0,\ldots,\hat{X}_i,\ldots,X_r)\big)\right. \\
&\qquad \left.+\sum_{0\leqslant i<j\leqslant r}(-1)^{i+j}\,\hat{\omega}([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_r)\right)\big(\xi_1\circ\pi,\ldots,\xi_{k+1-r}\circ\pi\big) \\
&= \sum_{i=0}^{r}(-1)^i\,X_i\cdot\Big(\hat{\omega}(X_0,\ldots,\hat{X}_i,\ldots,X_r)\big(\xi_1\circ\pi,\ldots,\xi_{k+1-r}\circ\pi\big)\Big) \\
&\qquad +\sum_{0\leqslant i<j\leqslant r}(-1)^{i+j}\,\hat{\omega}([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_r)\big(\xi_1\circ\pi,\ldots,\xi_{k+1-r}\circ\pi\big) \\
&= \sum_{i=0}^{r}(-1)^i\,X_i\cdot\omega(X_0,\ldots,\hat{X}_i,\ldots,X_r,\xi_1^H,\ldots,\xi_{k+1-r}^H) \\
&\qquad +\sum_{j=1}^{k+1-r}(-1)^{r+j}\,\xi_j^H\cdot\omega(X_0,\ldots,X_r,\xi_1^H,\ldots,\hat{\xi}_j^H,\ldots,\xi_{k+1-r}^H)
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{0 \leqslant i < j \leqslant r} (-1)^{i+j}\, \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_r, \xi_1^H, \ldots, \xi_{k+1-r}^H) \\
& + \sum_{i=0}^{r} \sum_{j=1}^{k+1-r} (-1)^i\, \omega([X_i, \xi_j^H], X_0, \ldots, \hat{X}_i, \ldots, X_r, \xi_1^H, \ldots, \hat{\xi}_j^H, \ldots, \xi_{k+1-r}^H) \\
& + \sum_{1 \leqslant i < j \leqslant k+1-r} (-1)^{i+j}\, \omega([\xi_i^H, \xi_j^H], X_0, \ldots, X_r, \xi_1^H, \ldots, \hat{\xi}_i^H, \ldots, \hat{\xi}_j^H, \ldots, \xi_{k+1-r}^H) \\
= \;\; & d\omega\big(X_0, \ldots, X_r, \xi_1^H, \ldots, \xi_{k+1-r}^H\big) \; ,
\end{aligned}
$$

where in the second equality we use equation (2.13) with $\varphi = \hat{\omega}(X_0, \ldots, \hat{X}_i, \ldots, X_r)$ and $\hat{t}^* = \xi_1 \wedge \ldots \wedge \xi_{k+1-r}$ and in the third equality we have added terms which vanish due to the condition that ω should be $(k+1-r)$-horizontal, since the Lie brackets $[X_i, \xi_j^H]$ are vertical. (More generally, if X is a projectable vector field on P and $\pi_* X$ its projection to M, then the pairs $(X, \pi_* X)$ and (ξ^H, ξ) are π-related, so the pair $([X, \xi^H], [\pi_* X, \xi])$ will also be π-related. But since X is vertical we have $\pi_* X = 0$.) □

From Theorems 1.7 and 2.2, it follows that

Theorem 2.3 *Let P be a fiber bundle over an n-dimensional manifold M, with projection $\pi : P \to M$, let ω be a $(k+1-r)$-horizontal $(k+1)$-form on P, where $1 \leqslant r \leqslant k$ and $k+1-r \leqslant n$, and let $\hat{\omega}$ be its symbol, which is a vertical r-form on P taking values in the bundle of $(k+1-r)$-forms on M (or more exactly, in $\pi^*\big(\bigwedge^{k+1-r} T^*M\big)$). Suppose that ω is multilagrangian, with multilagrangian distribution L. Then $\hat{\omega}$ will be polylagrangian, with polylagrangian distribution L. If ω is multisymplectic, then $\hat{\omega}$ will be polysymplectic.*

Regarding the question of involutivity of the poly- or multilagrangian distribution on fiber bundles, we can guarantee that

$$
L \text{ is involutive if } \hat{n} \geqslant 3, \text{ in the polylagrangian case}\; , \tag{2.32}
$$

and

$$
L \text{ is involutive if } \tbinom{n}{k+1-r} \geqslant 3, \text{ in the multilagrangian case}\; , \tag{2.33}
$$

the argument being exactly the same as that in the proof of Theorem 2.1, that is, we write L as a finite sum of involutive distributions which are kernels of projected forms $\hat{\omega}^a$. For multilagrangian fibrer bundles, it is interesting to spell out more explicitly under what circumstances L may fail to be involutive:

1. $n = k+1-r$: this includes the symplectic case ($r = 2$) with

$$\omega \in \Omega^{n+2}_{2}(P) \quad , \quad \hat{\omega} \in \Omega^{2}_{V}\big(P\,, \pi^{*}\big(\textstyle\bigwedge^{n} T^{*}M\big)\big) \,,$$

representing, under the additional hypothesis that $\ker \hat{\omega} = \{0\}$, a "family of symplectic forms smoothly parametrized by the points of a base manifold M" of dimension n. Here, it is not difficult to construct examples of lagrangian distributions which are not involutive.

2. $n > k+1-r$: the only possibility which is compatible with the condition $\binom{n}{k+1-r} = 2$ is $n = 2$ and $k = r$: this includes the multisymplectic case over a two-dimensional base manifold M ($n = k = r = 2$). An explicit example of this situation can be given by making use of the following construction, which is an adaptation of Example 2.1.

Example 2.2 Let M be a two-dimensional manifold admitting closed 1-forms θ_1 and θ_2 such that $\theta_1 \wedge \theta_2$ is a volume form on M (for instance, the two-dimensional torus or any of its coverings), and let P be the total space of a principal bundle over M with structure group $U(2)$. Given any connection form $A \in \Omega^1(P, \mathfrak{u}(2))$ on P with curvature form $F \in \Omega^2(P, \mathfrak{u}(2))$, and using the basis $\{\, i1\,, \sigma_1/2i\,, \sigma_2/2i\,, \sigma_3/2i \,\}$ of $\mathfrak{u}(2)$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;,\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;,$$

are the Pauli matrices, so that

$$A = A^0\, i1 + A^a\, \frac{\sigma_a}{2i} \quad , \quad F = F^0\, i1 + F^a\, \frac{\sigma_a}{2i} \;,$$

and

$$F^0 = dA^0 \quad , \quad F^a = dA^a + \tfrac{1}{2}\, \epsilon^{abc} A^b \wedge A^c \;,$$

we have that

$$\omega = A^3 \wedge A^1 \wedge \pi^*\theta_1 + A^3 \wedge A^2 \wedge \pi^*\theta_2 - A^0 \wedge \pi^*\theta_1 \wedge \pi^*\theta_2$$

is a multisymplectic form of rank 1 on P whose multilagrangian distribution L is spanned by the fundamental vector fields on P associated with the generators $\sigma_1/2i$, $\sigma_2/2i$ and $i1$, which is not involutive. Indeed, with L defined in this way and observing that the vertical bundle VP is spanned by the fundamental vector fields on P associated with the generators $\sigma_1/2i$, $\sigma_2/2i$, $\sigma_3/2i$ and $i1$, it becomes obvious that ω vanishes when contracted with at least three vector fields taking values in VP or two vector fields taking values in L, so all that remains to be shown is that ω is closed. But this is a straightforward computation, since

$$\begin{aligned} d\omega \;&=\; dA^3 \wedge A^1 \wedge \pi^*\theta_1 \,-\, A^3 \wedge dA^1 \wedge \pi^*\theta_1 \,+\, A^3 \wedge A^1 \wedge \pi^*(d\theta_1) \\ &\quad + dA^3 \wedge A^2 \wedge \pi^*\theta_2 \,-\, A^3 \wedge dA^2 \wedge \pi^*\theta_2 \,+\, A^3 \wedge A^2 \wedge \pi^*(d\theta_2) \\ &\quad - dA^0 \wedge \pi^*\theta_1 \wedge \pi^*\theta_2 \,+\, A^0 \wedge \pi^*(d\theta_1) \wedge \pi^*\theta_2 \,-\, A^0 \wedge \pi^*\theta_1 \wedge \pi^*(d\theta_2) \\ &=\; F^3 \wedge A^1 \wedge \pi^*\theta_1 \,-\, A^3 \wedge F^1 \wedge \pi^*\theta_1 \,+\, F^3 \wedge A^2 \wedge \pi^*\theta_2 \,-\, A^3 \wedge F^2 \wedge \pi^*\theta_2 \\ &\quad - F^0 \wedge \pi^*\theta_1 \wedge \pi^*\theta_2 \;, \end{aligned}$$

and since each of the terms in the last equation is at least 3-horizontal and hence must vanish because M is two-dimensional. Note that the symbol of this multisymplectic form is the polysymplectic form

$$\hat{\omega} \;=\; \big(A^3|_{VP} \wedge A^1|_{VP}\big) \otimes \theta_1 \,+\, \big(A^3|_{VP} \wedge A^2|_{VP}\big) \otimes \theta_2 \;,$$

which is essentially the polysymplectic form of Example 2.1 (trivially extended from $SU(2)$ to $U(2)$).

2.5 The Darboux Theorem

Now we are able to prove the Darboux theorem for poly- and multilagrangian forms. Here, the specific algebraic structure of poly- and multilagrangian subspaces identified in the previous chapter turns out to be crucial, in the sense that this central theorem can, in all cases, be proved by appropriately adapting the procedure used to prove the classical Darboux theorem for symplectic forms. (See, for instance, Ref. [1]).

Theorem 2.4 (Darboux theorem for polylagrangian manifolds) *Let P be a polylagrangian manifold with polylagrangian $(k+1)$-form $\hat{\omega}$ of rank N taking values in a fixed $\hat{n}$-dimensional vector space $\hat{T}$ and with polylagrangian distribution L, which is assumed to be involutive (recall this is automatic if $\hat{n} \geqslant 3$), and let $\{\, \hat{e}_a \,|\, 1 \leqslant a \leqslant \hat{n} \,\}$ be a basis of $\hat{T}$. Then around any point of P, there exists a system of local coordinates $(q^i, p^a_{i_1 \ldots i_k}, r^\kappa)$ $(1 \leqslant a \leqslant \hat{n}$, $1 \leqslant i \leqslant N$, $1 \leqslant i_1 < \ldots < i_k \leqslant N$, $1 \leqslant \kappa \leqslant \dim\ker\hat{\omega})$, called **Darboux coordinates** or **canonical coordinates**, such that*

$$\hat{\omega} \;=\; \tfrac{1}{k!}\, dp^a_{i_1 \ldots i_k} \wedge dq^{i_1} \wedge \ldots \wedge dq^{i_k} \otimes \hat{e}_a \;, \tag{2.34}$$

and such that (locally) L is spanned by the vector fields $\partial/\partial p^a_{i_1 \ldots i_k}$ and $\partial/\partial r^\kappa$ while $\ker\hat{\omega}$ is spanned by the vector fields $\partial/\partial r^\kappa$.

Theorem 2.5 (Darboux theorem for polylagrangian fiber bundles) *Let P be a polylagrangian fiber bundle over an n-dimensional manifold M with polylagrangian $(k+1)$-form $\hat{\omega}$ of rank N taking values in a fixed $\hat{n}$-dimensional vector bundle $\hat{T}$ over the same manifold M and with polylagrangian distribution L, which is assumed to be involutive (recall this is automatic if $\hat{n} \geqslant 3$), and let $\{\, \hat{e}_a \,|\, 1 \leqslant a \leqslant \hat{n} \,\}$ be a basis of local sections of $\hat{T}$. Then around any point of P (over the domain of the given basis of local sections), there exists a system of local coordinates $(x^\mu, q^i, p^a_{i_1 \ldots i_{r-1}}, r^\kappa)$ ($1 \leqslant \mu \leqslant n$, $1 \leqslant a \leqslant \hat{n}$, $1 \leqslant i \leqslant N$, $1 \leqslant i_1 < \ldots < i_{r-1} \leqslant N$, $1 \leqslant \kappa \leqslant \dim \ker \hat{\omega}$), called **Darboux coordinates** or **canonical coordinates**, such that*

$$\hat{\omega} \;=\; \tfrac{1}{k!}\, dp^a_{i_1 \ldots i_k} \wedge dq^{i_1} \wedge \ldots \wedge dq^{i_k} \otimes \hat{e}_a \;, \tag{2.35}$$

and such that (locally) L is spanned by the vector fields $\partial/\partial p^a_{i_1 \ldots i_k}$ and $\partial/\partial r^\kappa$ while $\ker \hat{\omega}$ is spanned by the vector fields $\partial/\partial r^\kappa$.

Theorem 2.6 (Multilagrangian Darboux theorem) *Let P be a multilagrangian fiber bundle over an n-dimensional manifold M with multilagrangian $(k+1)$-form ω of rank N and horizontal degree $k+1-r$, where $1 \leqslant r \leqslant k$ and $k+1-r \leqslant n$, and with multilagrangian distribution L, which is assumed to be involutive (recall this is automatic if $\binom{n}{k+1-r} \geqslant 3$). Then around any point of P, there exists a system of local coordinates $(x^\mu, q^i, p_{i_1 \ldots i_s;\, \mu_1 \ldots \mu_{k-s}}, r^\kappa)$ ($0 \leqslant s \leqslant r-1$, $1 \leqslant \mu \leqslant n$, $1 \leqslant i \leqslant N$, $1 \leqslant i_1 < \ldots < i_s \leqslant N$, $1 \leqslant \mu_1 < \ldots < \mu_{k-s} \leqslant N$, $1 \leqslant \kappa \leqslant \dim \ker \omega$), called **Darboux coordinates** or **canonical coordinates**, such that*

$$\omega \;=\; \sum_{s=0}^{r-1} \tfrac{1}{s!}\, \tfrac{1}{(k-s)!}\; dp_{i_1 \ldots i_s;\, \mu_1 \ldots \mu_{k-s}} \wedge dq^{i_1} \wedge \ldots \wedge dq^{i_s} \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{k-s}} \;, \tag{2.36}$$

and such that (locally) L is spanned by the vector fields $\partial/\partial p_{i_1 \ldots i_s;\, \mu_1 \ldots \mu_{k-s}}$ and $\partial/\partial r^\kappa$ while $\ker \omega$ is spanned by the vector fields $\partial/\partial r^\kappa$. In these coordinates, its symbol is given by

$$\hat{\omega} \;=\; \tfrac{1}{(r-1)!}\, \tfrac{1}{(k+1-r)!}\, \big(dp_{i_1 \ldots i_{r-1};\, \mu_1 \ldots \mu_{k+1-r}} \wedge dq^{i_1} \wedge \ldots \wedge dq^{i_{r-1}} \big) \otimes \big(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{k+1-r}} \big) \,. \tag{2.37}$$

Proof: For the sake of simplicity, we concentrate on the multilagrangian case: the proof for the other two cases is entirely analogous, requiring only small and rather obvious modifications.

Due to the local character of this theorem and since the kernel of ω, the multilagrangian subbundle L and the vertical bundle VP are all involutive, with $\ker \omega \subset L \subset VP$, we can without loss of generality work in a local chart of the manifold P around the chosen reference point in which the corresponding foliations are "straightened out", so we may assume that $P \cong \mathbb{R}^n \oplus \mathbb{R}^N \oplus L_0 \oplus K_0$ with $VP \cong \mathbb{R}^N \oplus L_0 \oplus K_0$, $L \cong L_0 \oplus K_0$ and $\ker \omega \cong K_0$ with fixed subspaces L_0 and K_0 and such that the aforementioned reference point corresponds to the origin. We also take ω_0 to be the constant multilagrangian form, with multilagrangian distribution L, obtained by spreading $\omega(0)$, the value of the multilagrangian form ω at the origin, all over the space P; then the existence of canonical coordinates for ω_0, in the form given by equation (2.36), is already guaranteed by the algebraic Darboux theorem of the previous chapter (Theorem 1.11).

Now consider the family of $(k+1)$-forms given by $\omega_t = \omega_0 + t(\omega - \omega_0)$, for every $t \in \mathbb{R}$. Obviously, $\omega_t(0) = \omega_0$ for every $t \in \mathbb{R}$, which is non-degenerate on $K_0' = \mathbb{R}^n \oplus \mathbb{R}^N \oplus L_0$ (a complement of K_0 in P). Since non-degeneracy is an open condition, and using a compactness argument with respect to the parameter t, we conclude that there is an open neighborhood of 0 such that, for all t satisfying $0 \leqslant t \leqslant 1$ and all points p in this neighborhood, $\omega_t(p)$ is non-degenerate on $K_0' = \mathbb{R}^n \oplus \mathbb{R}^N \oplus L_0$, that is, its kernel equals K_0. Moreover, for all t satisfying $0 \leqslant t \leqslant 1$ and all points p in this neighborhood, the subspace L_0, being isotropic for ω_0 as well as for $\omega(p)$, is also isotropic for $\omega_t(p)$ and, since it contains the kernel of $\omega_t(p)$ and has the right dimension as given by equation (1.62), is even multilagrangian for $\omega_t(p)$, according to Theorem 1.6. On the other hand, we have $d\omega_0 = 0$ (trivially) and $d\omega = 0$ (by hypothesis), so we can apply an appropriate version of the Poincaré lemma (see the Appendix) to prove, in some open neighborhood of the point 0 in P (contained in the previous one), existence of a k-form α satisfying $d\alpha = \omega_0 - \omega$ and $\alpha^\flat(L) = 0$. Now take X_t to be the unique time dependent vector field on P taking values in L_0[3] defined by

$$\mathrm{i}_{X_t}\omega_t \;=\; \alpha \;.$$

Let $F_t \equiv F_{(0,t)}$ be its flux beginning at 0, once again defined, for $0 \leqslant t \leqslant 1$, in some open neighborhood of the point 0 in P (contained in the previous one). Then it follows that

$$\begin{aligned}
\frac{d}{ds}\bigg|_{s=t} F_s^*\omega_s &= F_t^*\left(\frac{d}{ds}\bigg|_{s=t}\omega_s\right) + \frac{d}{ds}\bigg|_{s=t} F_s^*\omega_t \\
&= F_t^*\big(\omega - \omega_0 + L_{X_t}\omega_t\big) \\
&= F_t^*\big(\omega - \omega_0 + d(\mathrm{i}_{X_t}\omega_t)\big) \\
&= F_t^*\big(\omega - \omega_0 + d\alpha\big) \\
&= 0
\end{aligned}$$

Therefore, F_1 is the desired coordinate transformation, since $F_1^*\omega = F_1^*\omega_1 = F_0^*\omega_0 = \omega_0$. (For additional information, see [1].) □

[3]It is at thios point that we make essential use of the hypothesis that L_0 is multilagrangian and not just isotropic or even maxiaml isotropic (with respect to $\omega_t(p)$, in this case).

Chapter 3

Poly- and multisymplectic structures

In this chapter, we turn to a more specific study of poly- and multisymplectic forms. Our main goal is to provide, at least for these important classes of poly- and multilagrangian forms, respectively, a more substantial motivation for their definition. After all, our definition, based exclusively on the existence of a special type of maximal isotropic subspace (called a poly- or multilagrangian subspace, respectively), constitutes the central point of this work and should be compared with other proposals for defining the same concepts that can be found in the literature, which calls for a critical assessment.

A first attempt to define the notion of a multisymplectic manifold (P, ω) and prove a Darboux theorem is due to Martin [19]. The idea of postulating the existence of a "big" isotropic subspace already appeared in this paper, but the precise mathematical meaning of that condition remained unclear, partly because the criterion for being "big" was formulated merely as a numerical condition on the dimension. Soon it was noticed that the resulting formula for the total dimension (which follows from equation (1.45) or (1.46) by setting $\hat{n} = 1$ and $\ker \hat{\omega} = \{0\}$), namely

$$\dim P \;=\; N + \binom{N}{k} \;,$$

is inconsistent with the basic example from physics, because the dimension of the multiphase spaces (P, ω) that appear in the covariant hamiltonian formalism of classical field theory is given by a different formula (which follows from equation (1.61) or (1.62) by setting $k = n$, $r = 2$ and $\ker \omega = \{0\}$), namely

$$\dim P \;=\; (N+1)(n+1) \;,$$

which can also be obtained by a simple count of local coordinates: in the notation already employed in the introduction, we have n space-time coordinates x^μ, N position variables q^i, nN multimomentum variables p_i^μ and one energy variable p. Therefore, and taking into account that the term "multisymplectic" is already occupied in the literature at least since the beginning of the 1970s [13–15], we found it inadequate to use the same term for the structure studied by Martin in Ref. [19] and thus propose to replace it by the term "polylagrangian" – a convention that we have adopted in this work.

In later articles, such as [3] and [17], the necessity of modifying and extending the formalism by including horizontality conditions was already clearly realized, but so far, at least to our knowledge, there exists no comprehensive treatment of the subject, which would have to include complete proofs of the main statements.

There are also approaches emphasizing other aspects that, in our opinion, turn out to be of little relevance. One example that we shall discuss in more detail are the hypotheses on the symmetric algebra built from exterior products between the members of a multiplet of 2-forms adopted in Ref. [11]: as we shall demonstrate, these conditions, apart from the apparent impossibility to extend them naturally from the polysymplectic to the multisymplectic case, are both insufficient and superfluous to achieve the main objective (namely the formulation and proof of an appropriate version of the algebraic Darboux theorem). But in spite of these drawbacks, Ref. [11] has played an important role in the development of our approach.

Regarding the relation between polysymplectic and multisymplectic forms, it is interesting to note that the latter are almost entirely determined by the former, by means of what we call the symbol, since this is a polysymplectic form whose kernel, as we shall see, is at most one-dimensional. What parametrizes this kernel is a variable which in classical field theory corresponds to energy.

3.1 Polysymplectic vector spaces

According to Definition 1.1, a 2-form $\hat{\omega}$ on a vector space V taking values in another vector space $\hat{T}$ of dimension $\hat{n}$ is said to be polysymplectic of rank N if there exists a polylagrangian subspace L of V of codimension N, characterized by the property that

$$\hat{\omega}^\flat\big(L\big) \;=\; L^\perp \otimes \hat{T} \; . \tag{3.1}$$

In this case, it is obvious that, according to equation (1.45) or (1.46),

$$\dim\, L \;=\; \dim\,\ker\,\hat{\omega} \,+\, \hat{n}N \; , \tag{3.2}$$

and hence

$$\dim\, V \;=\; \dim\,\ker\,\hat{\omega} \,+\, \big(\hat{n}+1\big)N \; . \tag{3.3}$$

In particular, the dimension of a vector space carrying a non-degenerate polysymplectic form taking values in an $\hat{n}$-dimensional vector space $\hat{T}$ has to be a multiple of $\hat{n}+1$:

this generalizes a well known property of symplectic forms, which are included as the special case $\hat{n} = 1$ (ordinary forms).[1] More generally, we have

$$\mathrm{rk}(\hat{\omega}) \;=\; N \;=\; \frac{1}{\hat{n}+1}\,\dim\,\mathrm{supp}\,\hat{\omega}\ , \tag{3.4}$$

which constitutes another example of the phenomenon already mentioned in the first chapter, according to which it may be convenient to use the term "rank" of a form for an expression that is numerically different from the dimension of its support but uniquely determined by it.

When $\hat{n} \geqslant 2$, we have the following criterion to decide whether a 2-form is polysymplectic:

Proposition 3.1 *Let V and $\hat{T}$ be finite-dimensional vector spaces, with $\hat{n} \equiv \dim \hat{T} \geqslant 2$, and let $\hat{\omega}$ be a $\hat{T}$-valued 2-form on V. Consider the subspace L of V spanned by the kernels of all the projections $\omega_{\hat{t}^*}$ ($\hat{t}^* \in \hat{T}^* \setminus \{0\}$) of $\hat{\omega}$:*

$$L = \sum_{\hat{t}^* \in \hat{T}^* \setminus \{0\}} \ker \omega_{\hat{t}^*}\ .$$

For the form $\hat{\omega}$ to be polysymplectic and L to be the pertinent polylagrangian subspace, the following conditions are necessary and sufficient:

- *L is isotropic;*
- $\dim L = (\dim \ker \hat{\omega} + \hat{n} \dim V)/(\hat{n}+1)$.

Proof. This is an immediate corollary of Theorems 1.3 and 1.4. □

To formulate the algebraic Darboux theorem in the specific context of polysymplectic forms, we begin by adapting Definition 1.5 to this context: explicitly, a **(polysymplectic) Darboux basis** or **(polysymplectic) canonical basis** is a basis $\{\, e_i, e_a^j \mid 1 \leqslant a \leqslant \hat{n}\,,\ 1 \leqslant i,j \leqslant N \,\}$ of a subspace of V complementary to $\ker \hat{\omega}$, with dual basis $\{\, e^i, e^a_j \mid 1 \leqslant a \leqslant \hat{n}\,,\ 1 \leqslant i,j \leqslant N \,\}$ of the subspace $\mathrm{supp}\ \hat{\omega} = (\ker \hat{\omega})^{\perp}$ of V^*, together with a basis $\{\, \hat{e}_a \mid 1 \leqslant a \leqslant \hat{n} \,\}$ of $\hat{T}$, with dual basis $\{\, \hat{e}^a \mid 1 \leqslant a \leqslant \hat{n} \,\}$ of $\hat{T}^*$, such that

$$\hat{\omega} \;=\; \left(e_i^a \wedge e^i\right) \otimes \hat{e}_a\ , \tag{3.5}$$

or in terms of the projected forms ω^a defined according to equations (1.29) and (1.30),

$$\omega^a \;=\; e_i^a \wedge e^i \quad (1 \leqslant a \leqslant \hat{n})\ . \tag{3.6}$$

[1] Note that for $\hat{n}=1$, the definition of a non-degenerate polysymplectic form reduces to that of a symplectic form, since in this case, the existence of a lagrangian subspace L can be derived as a theorem. The main difference between this case and those where $\hat{n}>1$ is that here, L is not unique.

Obviously, a 2-form $\hat{\omega} \in \bigwedge^2 V^* \otimes \hat{T}$ that admits such a canonical basis is a polysymplectic form, whose polylagrangian subspace L is the direct sum of the kernel of $\hat{\omega}$ with the $(\hat{n}N)$-dimensional subspace spanned by the vectors e_a^i $(1 \leqslant a \leqslant \hat{n},\ 1 \leqslant i \leqslant N)$. It is interesting to note that, on the other hand, the direct sum of the kernel of $\hat{\omega}$ with the N-dimensional subspace spanned by the vectors e_i $(1 \leqslant i \leqslant N)$ is a maximal isotropic subspace of V which is not polylagrangian.

Conversely, as a corollary of Theorem 1.9, we have

Theorem 3.1 (Darboux theorem for polysymplectic vector spaces)
Every polysymplectic vector space admits a canonical basis.

Although this theorem has already been proved in the first chapter, we want to present an alternative proof which is strictly parallel to the proof of the corresponding theorem for symplectic forms and for symmetric bilinear forms, based on an inductive "Gram-Schmidt" type process that can also be used to construct, inductively and explicitly, an isotropic subspace complementary to the polylagrangian subspace L (see, for instance, [1, Prop. 3.1.2, pp. 162-164]).

PROOF. Denoting the polylagrangian subspace of V by L, as always, and fixing a subspace L' of L complementary to $\ker \hat{\omega}$, we proceed by induction on $N = \dim V - \dim L$. To this end, we choose arbitrarily a 1-form $e^N \in L^\perp \setminus \{0\}$ and a vector $e_N \in V \setminus L$ such that

$$\langle e^N, e_N \rangle \;=\; 1 \;,$$

and using the property (3.1), define the vectors $e_a^N \in L'$ $(1 \leqslant a \leqslant \hat{n})$ by [2]

$$\hat{\omega}^\flat(e_a^N) \;=\; -\, e^N \otimes \hat{e}_a \;,$$

or in terms of the projected forms

$$(\omega^b)^\flat(e_a^N) \;=\; -\, \delta_a^b \, e^N \;,$$

and the 1-forms $e_N^a \in V^*$ $(1 \leqslant a \leqslant \hat{n})$ by

$$e_N^a \;=\; (\omega^a)^\flat(e_N) \;.$$

Thus we have

$$\langle e^N, e_1^N \rangle \;=\; -\,\omega^1(e_1^N, e_1^N) \;=\; 0 \quad \ldots \quad \langle e^N, e_{\hat{n}}^N \rangle \;=\; -\,\omega^{\hat{n}}(e_{\hat{n}}^N, e_{\hat{n}}^N) \;=\; 0$$

and

$$\omega^b(e_N, e_a^N) \;=\; \delta_a^b \;.$$

[2] Note that this determines the e_a^N uniquely in terms of e^N.

Using that L is isotropic with respect to $\hat{\omega}$ and hence also with respect to each of the projected forms, we get

$$\omega^c(e_a^N, e_b^N) \;=\; 0 \;.$$

This implies that we can decompose the space V into the direct sum

$$V \;=\; V_N \oplus V_N^{\hat{\omega}}$$

of the subspace V_N spanned by the vectors e_N and e_a^N $(1 \leqslant a \leqslant \hat{n})$, of dimension $\hat{n}+1$, and its simultaneous orthogonal complement $V_N^{\hat{\omega}}$, which is the intersection of the $\hat{n}$ different orthogonal complements of V_N with respect to each of the projected forms. What is more: the formula

$$P_N^{\hat{\omega}}(v) \;=\; v \,-\, \langle e^N, v\rangle\, e_N \,-\, \omega^a(e_N, v)\, e_a^N$$

provides an explicit definition of the projector $P_N^{\hat{\omega}}$ onto $V_N^{\hat{\omega}}$ along V_N. [To prove this assertion, note that

$$\begin{aligned} P_N^{\hat{\omega}}(e_N) \;&=\; e_N \,-\, \langle e^N, e_N\rangle\, e_N \,-\, \omega^a(e_N, e_N)\, e_a^N \;=\; 0 \;, \\ P_N^{\hat{\omega}}(e_a^N) \;&=\; e_a^N \,-\, \langle e^N, e_a^N\rangle\, e_N \,-\, \omega^b(e_N, e_a^N)\, e_b^N \;=\; 0 \;, \end{aligned}$$

and that, for every $v \in V$,

$$\begin{aligned} \omega^c(P_N^{\hat{\omega}}(v), e_N) \;&=\; \omega^c(v, e_N) \,-\, \langle e^N, v\rangle\, \omega^c(e_N, e_N) \,-\, \omega^b(e_N, v)\, \omega^c(e_b^N, e_N) \;=\; 0 \;, \\ \omega^c(P_N^{\hat{\omega}}(v), e_a^N) \;&=\; \omega^c(v, e_a^N) \,-\, \langle e^N, v\rangle\, \omega^c(e_N, e_a^N) \,-\, \omega^b(e_N, v)\, \omega^c(e_b^N, e_a^N) \;=\; 0 \;, \end{aligned}$$

showing that $P_N^{\hat{\omega}}$ vanishes on V_N and maps the entire space V to the subspace $V_N^{\hat{\omega}}$. Using again the definition of $P_N^{\hat{\omega}}$, we also conclude that $\ker P_N^{\hat{\omega}} = V_N$, since a linear independence argument shows that $P_N^{\hat{\omega}}$ cannot vanish on any vector that does not belong to the subspace spanned by the vectors e_N and e_a^N $(1 \leqslant a \leqslant \hat{n})$, and that $P_N^{\hat{\omega}}$ is the identity on $V_N^{\hat{\omega}}$, since using that $\langle e^N, v\rangle = -\,\omega^a(e_a^N, v)$ for any fixed value of a (no sum) shows that the 1-form e^N vanishes on $V_N^{\hat{\omega}}$. Therefore, $P_N^{\hat{\omega}}$ is a projector (i.e., satisfies $(P_N^{\hat{\omega}})^2 = P_N^{\hat{\omega}}$) with kernel V_N and image $V_N^{\hat{\omega}}$.] Passing to the dual, this direct decomposition of V induces a corresponding direct decomposition

$$V^* \;=\; V_N^* \oplus (V_N^{\hat{\omega}})^*$$

of V^*, where the dual of each subspace is naturally identified with the annihilator of the other:

$$V_N^* \;=\; (V_N^{\hat{\omega}})^\perp \quad , \quad (V_N^{\hat{\omega}})^* \;=\; V_N^\perp \;.$$

Note that under these decompositions L becomes the direct sum of the intersection $L \cap V_N^{\hat{\omega}}$ and the subspace spanned by the vectors e_a^N $(1 \leqslant a \leqslant \hat{n})$ while $L^\perp$ becomes the direct sum of the intersection $L^\perp \cap V_N^\perp$ and the one-dimensional subspace spanned by the 1-form e^N. Thus it becomes clear that we can repeat the same process for the restriction of the form $\hat{\omega}$ to the subspace $V_N^{\hat{\omega}}$, which is a polysymplectic form of rank $N-1$ (as can be seen, for instance, by analyzing the criteria of Proposition 3.1), so that after repeating it N times, we arrive at the conclusion of the theorem. $\square$

We conclude this section with a digression on general vector-valued 2-forms: this will help us to appreciate the simplicity and usefulness of the concept of a polysymplectic form adopted in this thesis, in comparison with other definitions that preceded ours. Certainly, finding the right path within the jungle of possible notions was the most difficult part of this work.

Suppose that $\hat{\omega} \in \bigwedge^2 V^* \otimes \hat{T}$ is an arbitrary $\hat{T}$-valued 2-form on V. To begin with, we recall that according to the convention adopted in this work (see, for instance, equation (3.4)), the rank of an ordinary 2-form ω is equal to half the dimension of its support. Thus we have, for every $\hat{t}^* \in \hat{T}^*$,

$$\mathrm{rk}\big(\omega_{\hat{t}^*}\big) \;=\; \tfrac{1}{2}\ \dim\ \mathrm{supp}\ \omega_{\hat{t}^*}\ . \tag{3.7}$$

Now note that the linear mapping

$$\begin{array}{ccc} \hat{T}^* & \longrightarrow & \bigwedge^2 V^* \\ \hat{t}^* & \longmapsto & \omega_{\hat{t}^*} \end{array} \tag{3.8}$$

induces, for every integer $k \geqslant 1$, a canonically defined linear mapping

$$\begin{array}{ccc} \bigvee^k \hat{T}^* & \longrightarrow & \bigwedge^{2k} V^* \\ P & \longmapsto & P(\hat{\omega}) \end{array} \tag{3.9}$$

where we have identified the space $\bigvee^k \hat{T}^*$ of covariant symmetric tensors of degree k over $\hat{T}$ with the space of homogeneous polynomials P of degree k on $\hat{T}$. Explicitly, in terms of a basis $\{\, \hat{e}_a \mid 1 \leqslant a \leqslant \hat{n} \,\}$ of $\hat{T}$, with dual basis $\{\, \hat{e}^a \mid 1 \leqslant a \leqslant \hat{n} \,\}$ of $\hat{T}^*$, we have

$$P \;=\; P_{a_1 \ldots a_k}\, \hat{e}^{a_1} \vee \ldots \vee \hat{e}^{a_k} \quad \Longrightarrow \quad P(\hat{\omega}) \;=\; P_{a_1 \ldots a_k}\, \omega^{a_1} \wedge \ldots \wedge \omega^{a_k}\ . \tag{3.10}$$

This allows us to introduce the following terminology:

Definition 3.1 *Let V and $\hat{T}$ be finite-dimensional vector spaces and let $\hat{\omega}$ be a $\hat{T}$-valued 2-form on V. We say that $\hat{\omega}$ has **constant rank** N if $\mathrm{rk}(\omega_{\hat{t}^*}) = N$ for every $\hat{t}^* \in \hat{T}^* \setminus \{0\}$ and that $\hat{\omega}$ has **uniform rank** N if the linear mapping (3.10) is injective for $k = N$ and identically zero for $k = N + 1$.*

Using multi-indices $\alpha = (\alpha_1, \ldots, \alpha_{\hat{n}}) \in \mathbb{N}^{\hat{n}}$, we set

$$\hat{e}^\alpha \;=\; (\hat{e}^1)^{\alpha_1} \vee \ldots \vee (\hat{e}^{\hat{n}})^{\alpha_{\hat{n}}} \qquad \text{where} \qquad (\hat{e}^a)^{\alpha_a} \;=\; \hat{e}^a \vee \ldots \vee \hat{e}^a \quad (\alpha_a \text{ times})$$

and

$$\omega^\alpha ~=~ (\omega^1)^{\alpha_1} \wedge \ldots \wedge (\omega^{\hat{n}})^{\alpha_{\hat{n}}} \qquad \text{where} \qquad (\omega^a)^{\alpha_a} ~=~ \omega^a \wedge \ldots \wedge \omega^a \quad (\alpha_a \text{ times})$$

to rewrite equation (3.10) in the form

$$P ~=~ \sum_{|\alpha|=k} P_\alpha \, \hat{e}^\alpha \quad \Longrightarrow \quad P(\hat{\omega}) ~=~ \sum_{|\alpha|=k} P_\alpha \, \omega^\alpha ~. \tag{3.11}$$

Since $\{\hat{e}^\alpha \,|\, |\alpha| = k\}$ is a basis of $\bigvee^k \hat{T}^*$, requiring $\hat{\omega}$ to have uniform rank N amounts to imposing the following conditions:

$$\begin{array}{c} \{\omega^\alpha \,|\, |\alpha| = N\} \text{ is linearly independent} \\ \omega^\alpha = 0 \text{ for } |\alpha| = N+1 \end{array} ~. \tag{3.12}$$

It is in this form that the requirement of uniform rank appears in the definition of a polysymplectic form adopted in Ref. [11].

To gain a better understanding for the conditions of constant rank and of uniform rank introduced above, we note first of all that they both generalize the standard notion of rank for ordinary forms. Indeed, when $\hat{n} = 1$, that is, given an ordinary 2-form ω of rank N on V, we can choose a canonical basis $\{e_1, \ldots, e_N, f^1, \ldots, f^N\}$ of a subspace of V complementary to the kernel of ω, with dual basis $\{e^1, \ldots, e^N, f_1, \ldots, f_N\}$ of the subspace $\operatorname{supp} \omega$ of V^*, to conclude that $\omega = e^i \wedge f_i$ and therefore

$$\omega^N ~=~ \pm~ e^1 \wedge \ldots \wedge e^N \wedge f_1 \wedge \ldots \wedge f_N ~\neq~ 0 \quad , \quad \omega^{N+1} ~=~ 0 ~.$$

In other words, the rank of ω can be characterized as that positive integer N for which $\omega^N \neq 0$ but $\omega^{N+1} = 0$. From this observation, it follows that, in the general case considered before, the requirement of uniform rank implies that of constant rank because it guarantees that for every $\hat{t}^* \in \hat{T}^* \setminus \{0\}$, we have $\omega_{\hat{t}^*}^N \neq 0$ and $\omega_{\hat{t}^*}^{N+1} = 0$, that is, $\operatorname{rk}(\omega_{\hat{t}^*}) = N$. However, the converse does not hold, as shown by the following

Example 3.1 $(\,\hat{n} = 2\,,~ N = 2\,,~ \dim V = 4\,,~ \ker \hat{\omega} = \{0\}\,)$:

Let $V = \mathbb{R}^4$, $\hat{T} = \mathbb{R}^2$ and consider the $\mathbb{R}^2$-valued 2-form $\hat{\omega}$ built from the following two ordinary 2-forms:

$$\omega^1 ~=~ dx \wedge dy \,+\, du \wedge dv \quad , \quad \omega^2 ~=~ dx \wedge du \,-\, dy \wedge dv ~.$$

Then for $\hat{t}^* = \hat{t}_a^* \, \hat{e}^a \in (\mathbb{R}^2)^*$, we have

$$\omega_{\hat{t}^*} ~=~ \hat{t}_a^* \, \omega^a ~=~ dx \wedge \big(\hat{t}_1^* \, dy \,+\, \hat{t}_2^* \, du\big) \,+\, dv \wedge \big(\hat{t}_2^* \, dy \,-\, \hat{t}_1^* \, du\big) ~.$$

Thus we obtain, for every $\hat{t}^* \neq 0$,

$$(\omega_{\hat{t}^*})^2 \ \equiv\ \omega_{\hat{t}^*} \wedge \omega_{\hat{t}^*} \ =\ \big((\hat{t}_1^*)^2 + (\hat{t}_2^*)^2\big)\ dx \wedge dy \wedge du \wedge dv \ \neq\ 0 \ ,$$

whereas, due to the fact that we are in a four-dimensional space,

$$(\omega_{\hat{t}^*})^3 \ \equiv\ \omega_{\hat{t}^*} \wedge \omega_{\hat{t}^*} \wedge \omega_{\hat{t}^*} \ =\ 0 \ ,$$

which guarantees that $\hat{\omega}$ has constant rank 2. However, $\hat{\omega}$ does not have uniform rank 2, since

$$\omega^1 \wedge \omega^2 \ =\ 0 \ .$$

On the other hand, polysymplectic forms do have uniform rank:

Proposition 3.2 *Let V and $\hat{T}$ be finite-dimensional vector spaces and let $\hat{\omega}$ be a $\hat{T}$-valued polysymplectic form of rank N on V. Then $\hat{\omega}$ has uniform rank N.*

PROOF. Introducing a (polysymplectic) canonical basis in which $\hat{\omega}$ assumes the form given by equations (3.5) and (3.6), suppose that $\alpha = (\alpha_1, \ldots, \alpha_{\hat{n}}) \in \mathbb{N}^{\hat{n}}$ is a multi-index of degree k (i.e., such that $|\alpha| = \alpha_1 + \ldots + \alpha_{\hat{n}} = k$) and consider the form

$$\begin{aligned} \omega^\alpha \ =\ \pm\ &\Big((e^{i_1^1} \wedge \ldots \wedge e^{i_{\alpha_1}^1}) \wedge \ldots \wedge (e^{i_1^{\hat{n}}} \wedge \ldots \wedge e^{i_{\alpha_{\hat{n}}}^{\hat{n}}})\Big) \\ &\wedge \Big(e^1_{i_1^1} \wedge \ldots \wedge e^1_{i_{\alpha_1}^1}\Big) \wedge \ldots \wedge \Big(e^{\hat{n}}_{i_1^{\hat{n}}} \wedge \ldots \wedge e^{\hat{n}}_{i_{\alpha_{\hat{n}}}^{\hat{n}}}\Big) \ . \end{aligned}$$

Obviously any such form vanishes when $k = N + 1$ since it then contains an exterior product of $(N+1)$ 1-forms e^i belonging to an N-dimensional subspace. On the other hand, all these forms are linearly independent when $k = N$ since ω^α then contains the exterior product $e^1 \wedge \ldots \wedge e^N$ multiplied by the exterior product of α_1 1-forms of type e_i^1 with ... with $\alpha_{\hat{n}}$ 1-forms of type $e_i^{\hat{n}}$; thus ω^α and ω^β belong to different subspaces of $\bigwedge^{2N} V^*$ whenever $\alpha \neq \beta$. $\Box$

The converse statement, as we shall see soon, is remote from being true. In fact, if it were true, then if $\hat{n} \geqslant 2$, it should be possible to construct the polylagrangian subspace as the sum of the kernels of the projected forms, as required by Proposition 3.1. Therefore, it should be possible to show that the subspace defined as the sum of these kernels is isotropic. And indeed, as a partial result in this direction, we have the following

Proposition 3.3 *Let V and $\hat{T}$ be finite-dimensional vector spaces and let $\hat{\omega}$ be a $\hat{T}$-valued 2-form of uniform rank N on V. Then for any $\hat{t}_1^*, \hat{t}_2^* \in \hat{T}^* \setminus \{0\}$, the kernel of $\omega_{\hat{t}_1^*}$ is isotropic with respect to $\omega_{\hat{t}_2^*}$.*

Proof. Given $u, v \in \ker \omega_{\hat{t}_1^*}$, we have

$$\mathrm{i}_u \omega_{\hat{t}_1^*}^N \;=\; N\, \mathrm{i}_u \omega_{\hat{t}_1^*} \wedge \omega_{\hat{t}_1^*}^{N-1} \;=\; 0 \quad , \quad \mathrm{i}_v \omega_{\hat{t}_1^*}^N \;=\; N\, \mathrm{i}_v \omega_{\hat{t}_1^*} \wedge \omega_{\hat{t}_1^*}^{N-1} \;=\; 0 \;,$$

and therefore

$$\omega_{\hat{t}_2^*}(u,v)\, \omega_{\hat{t}_1^*}^N \;=\; \mathrm{i}_u \mathrm{i}_v \big(\omega_{\hat{t}_2^*} \wedge \omega_{\hat{t}_1^*}^N \big) \;=\; 0 \;.$$

Using that $\omega_{\hat{t}_1^*}^N \neq 0$, it follows that $\omega_{\hat{t}_2^*}(u,v) = 0$. □

However, isotropy of the subspace defined as the sum of the kernels of all the projected forms, which is equivalent to the (stronger) condition that for any $\hat{t}_1^*, \hat{t}_2^*, \hat{t}_3^* \in \hat{T}^* \setminus \{0\}$, $\ker \omega_{\hat{t}_1^*}$ and $\ker \omega_{\hat{t}_2^*}$ are orthogonal under $\omega_{\hat{t}_3^*}$, i.e., that

$$\omega_{\hat{t}_3^*}(u_1, u_2) \;=\; 0 \qquad \text{for } u_1 \in \ker \omega_{\hat{t}_1^*} \text{ and } u_2 \in \ker \omega_{\hat{t}_2^*} \;,$$

cannot be derived from the condition of uniform rank. A nice counterexample is obtained by choosing V and $\hat{T}$ to be the same space, supposing it to be a Lie algebra $\mathfrak{g}$ and defining $\hat{\omega}$ to be the commutator in $\mathfrak{g}$. Then for $\hat{t}^* \in \mathfrak{g}^*$, the kernel $\ker \omega_{\hat{t}^*}$ and the support $\operatorname{supp} \omega_{\hat{t}^*}$ of the projected form $\omega_{\hat{t}^*}$ are the isotropy algebra of $\hat{t}^*$ and the tangent space to the coadjoint orbit passing through $\hat{t}^*$, respectively. There is one and only one semisimple Lie algebra for which $\hat{\omega}$ has constant rank, since this condition states that all coadjoint orbits except the trivial one, $\{0\}$, should have the same dimension: this is the algebra of type A_1, that is, $\mathbb{R}^3$ equipped with the vector product $\times$.

Example 3.2 $(\hat{n} = 3\,,\ N = 1\,,\ \dim V = 3\,,\ \ker \hat{\omega} = \{0\}\,)$:

Let $V = \hat{T} = \mathbb{R}^3$ and consider the $\mathbb{R}^3$-valued 2-form $\hat{\omega}$ built from the following three ordinary 2-forms:

$$\omega^1 \;=\; dy \wedge dz \quad , \quad \omega^2 \;=\; dz \wedge dx \quad , \quad \omega^3 \;=\; dx \wedge dy \;.$$

Then for $\hat{t}^* = \hat{t}_a^*\, \hat{e}^a \in (\mathbb{R}^3)^*$, we have

$$\omega_{\hat{t}^*} \;=\; \hat{t}_a^*\, \omega^a \;=\; \hat{t}_1^*\, dy \wedge dz \,+\, \hat{t}_2^*\, dz \wedge dx \,+\, \hat{t}_3^*\, dx \wedge dy \;.$$

Obviously, ω^1, ω^2 and ω^3 are linearly independent and hence $\hat{\omega}$ has uniform rank 1, since there exists no non-zero 4-form on a three-dimensional space. On the other hand, we have

$$\ker \omega_{\hat{t}^*} \;=\; \langle\, \hat{t}_1^* \frac{\partial}{\partial x} \,+\, \hat{t}_2^* \frac{\partial}{\partial y} \,+\, \hat{t}_3^* \frac{\partial}{\partial z} \,\rangle \;,$$

Therefore, the intersection of the three kernels $\ker \omega^1$, $\ker \omega^2$ and $\ker \omega^3$ is $\{0\}$ (i.e., $\hat{\omega}$ is non-degenerate). However, $\ker \omega^1$ and $\ker \omega^2$ are orthogonal under ω^1 and under ω^2 but not under ω^3. Now if there existed a polylagrangian subspace it would have to coincide with the sum of the kernels of all the projected forms, but that is the whole space $\mathbb{R}^3$, which is not isotropic. Thus $\hat{\omega}$ does not admit a polylagrangian subspace.

Finally, we observe that even if the sum of the kernels of all the projected forms is an isotropic subspace with respect to $\hat{\omega}$, it may still fail to be a polylagrangian subspace, as shown by the following

Example 3.3 $(\,\hat{n} = 2\,,\; N = 2\,,\; \dim V = 5\,,\; \ker\,\hat{\omega} = \{0\}\,)$:
Let $V = \mathbb{R}^5$, $\hat{T} = \mathbb{R}^2$ and consider the $\mathbb{R}^2$-valued 2-form $\hat{\omega}$ built from the following two ordinary 2-forms:

$$\omega^1 \;=\; dx^1 \wedge dx^4 \,+\, dx^2 \wedge dx^3 \quad , \quad \omega^2 \;=\; dx^1 \wedge dx^3 \,-\, dx^2 \wedge dx^5 \;.$$

Then for $\hat{t}^* = \hat{t}^*_a \, \hat{e}^a \in (\mathbb{R}^2)^*$, we have

$$\omega_{\hat{t}^*} \;=\; \hat{t}^*_a \, \omega^a \;=\; dx^1 \wedge \left(\hat{t}^*_1 \, dx^4 \,+\, \hat{t}^*_2 \, dx^3\right) \,+\, dx^2 \wedge \left(\hat{t}^*_1 \, dx^3 \,-\, \hat{t}^*_2 \, dx^5\right) \;.$$

Obviously, ω^1, ω^2 and the forms

$$\begin{aligned} (\omega^1)^2 \;&\equiv\; \omega^1 \wedge \omega^1 \;=\; 2\; dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \;, \\ \omega^1 \wedge \omega^2 \;&=\; dx^1 \wedge dx^2 \wedge dx^4 \wedge dx^5 \;, \\ (\omega^2)^2 \;&\equiv\; \omega^2 \wedge \omega^2 \;=\; 2\; dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^5 \;, \end{aligned}$$

are linearly independent and hence $\hat{\omega}$ has uniform rank 2, since there exists no non-zero 6-form on a five-dimensional space. On the other hand, we have

$$\ker\,\omega_{\hat{t}^*} \;=\; \langle\, \hat{t}^*_1 \hat{t}^*_2 \, \frac{\partial}{\partial x^3} \,-\, (\hat{t}^*_2)^2 \, \frac{\partial}{\partial x^4} \,+\, (\hat{t}^*_1)^2 \, \frac{\partial}{\partial x^5} \,\rangle \;.$$

The intersection of the two kernels $\ker\,\omega^1$ and $\ker\,\omega^2$ is $\{0\}$ (i.e., $\hat{\omega}$ is non-degenerate). Note that their (direct) sum is the two-dimensional subspace of V, say L', spanned by $\partial/\partial x^4$ and $\partial/\partial x^5$, whereas the subspace spanned by all the kernels $\ker\,\omega_{\hat{t}^*}$ $(\hat{t}^* \in \hat{T}^* \setminus \{0\})$ is the three-dimensional subspace of V, say L'', spanned by $\partial/\partial x^i$ with $i = 3, 4, 5$, and this is isotropic with respect to all the forms $\omega_{\hat{t}^*}$ $(\hat{t}^* \in \hat{T}^* \setminus \{0\})$. More than that: since its codimension is 2, it is maximal isotropic with respect to all the forms $\omega_{\hat{t}^*}$ $(\hat{t}^* \in \hat{T}^* \setminus \{0\})$. Now if there existed a polylagrangian subspace it would have to coincide with L' and also with L'', but these two are not equal and do not have the right dimension, which according to equation (3.2) would have to be 4: both of them are too small. Thus $\hat{\omega}$ does not admit a polylagrangian subspace.

To summarize, the examples given above show that the hypothesis of existence of a polylagrangian subspace is highly non-trivial and very restrictive: as it seems, it cannot be replaced by any other hypothesis that is not obviously equivalent. The examples also show the great variety of possibilities for the "relative positions" of the kernels of the various projected forms that prevails when such a subspace does not exist. In this sense, the definition adopted in Ref. [11] is quite inconvenient, since it makes no reference to this subspace, thus hiding the central aspect of the theory.

To conclude, we want to add some remarks about the relation between the polylagrangian subspace, when it exists, and the more general class of maximal isotropic subspaces. First, we emphasize that in contrast with a polylagrangian subspace, maximal isotropic subspaces always exist. To prove this, it suffices to start out from an arbitrary one-dimensional subspace L_1, which is automatically isotropic, and construct a chain $L_1 \subset L_2 \subset \ldots$ of subspaces where L_{p+1} is defined as the direct sum of L_p and the one-dimensional subspace spanned by some non-zero vector in its 1-orthogonal complement $L_p^{\hat{\omega},1}$. For dimensional reasons, this process must stop at some point, which means that at this point we have succeeded in constructing a maximal isotropic subspace. However, nothing guarantees that maximal isotropic subspaces resulting from different chains must have the same dimension, nor that there must exist some chain leading to a maximal isotropic subspace of sufficiently high dimension to be polylagrangian: this happens only in the special case of ordinary forms ($\hat{n} = 1$), where all maximal isotropic subspaces have the same dimension and where the notions of a polylagrangian subspace (or simply lagrangian subspace, in this case) and of a maximal isotropic subspace coincide.

Another important point concerns the relation between the notions of isotropic subspace and maximal isotropic subspace with respect to the form $\hat{\omega}$ and with respect to its projections. First, it is obvious that a subspace of V is isotropic with respect to $\hat{\omega}$ if and only if it is isotropic with respect to each of the projected forms $\omega_{\hat{t}^*}$ ($\hat{t}^* \in T^* \setminus \{0\}$) or ω^a ($1 \leqslant a \leqslant \hat{n}$). However, this no longer holds when we substitute the word "isotropic" by the expression "maximal isotropic": a subspace of V that is maximal isotropic with respect to each of the projections of $\hat{\omega}$ certainly will be maximal isotropic with respect to $\hat{\omega}$, but conversely, it can very well be maximal isotropic with respect to $\hat{\omega}$ (and hence isotropic with respect to each of the projections of $\hat{\omega}$) but even so fail to be maximal isotropic with respect to some of them. And finally, a polylagrangian subspace of V will certainly be maximal isotropic with respect to each of the projections of $\hat{\omega}$ (this follows from Proposition 3.1), but as we have seen in the last example above, the converse is false: a subspace can be maximal isotropic with respect to each of the projections of $\hat{\omega}$ without being polylagrangian. These statements illustrate the special nature of the polylagrangian subspace, already in the case of vector-valued 2-forms.

3.2 Multisymplectic vector spaces

According to Definition 1.2, an $(n+1)$-form ω on a vector space W which is $(n-1)$-horizontal with respect to a subspace V of W of codimension n in W is said to be multisymplectic of rank N if it is non-degenerate and if there exists a multilagrangian subspace L of V of codimension N in V (and hence of codimension $N + n$ in W), characterized by the property that

$$\omega_V^\flat(L) \;=\; {\textstyle\bigwedge_1^n} L^\perp \;\equiv\; {\textstyle\bigwedge^n} L^\perp \cap {\textstyle\bigwedge_1^n} W^* \;. \tag{3.13}$$

In this case, it is obvious that, according to equation (1.61) or (1.62),

$$\dim L \;=\; Nn + 1 \;, \tag{3.14}$$

and hence

$$\dim V \;=\; N(n+1) + 1 \;, \tag{3.15}$$

and

$$\dim W \;=\; (N+1)(n+1) \;. \tag{3.16}$$

In particular, we conclude that the dimension of a vector space carrying a multisymplectic form has to be a multiple of its degree $n+1$: this generalizes a well known property of symplectic forms, which are included as the special case $n = 1$ (2-forms).[3] More generally, we have

$$\mathrm{rk}(\omega) \;=\; N \;=\; \frac{1}{n+1}\,\dim\,\mathrm{supp}\,\omega \,-\, 1 \;, \tag{3.17}$$

which constitutes another example of the phenomenon already mentioned in the first chapter, according to which it may be convenient to use the term "rank" of a form for an expression that is numerically different from the dimension of its support but uniquely determined by it.

Given an $(n+1)$-form ω on a vector space W which is $(n-1)$-horizontal with respect to a subspace V of W of codimension n in W, we can define its **symbol**, which will be a 2-form on W taking values in the vector space $\hat{T} = \bigwedge^{n-1} T^*$, where $T \cong W/V$. Note that the auxiliary space $\hat{T}$ has the same dimension as the base space T itself: $\hat{n} = n$. Moreover, we know that if ω is multisymplectic, then $\hat{\omega}$ will be polysymplectic, and the polylagrangian subspace for $\hat{\omega}$ will be the same as the multilagrangian subspace for ω. Comparing equations (3.14) and (3.2), we conclude that in this case

$$\dim\,\ker\,\hat{\omega} \;=\; 1 \;. \tag{3.18}$$

In fact, since by hypothesis ω is non-degenerate, the linear mapping

$$\begin{array}{rccc} \omega_V^\flat : & V & \longrightarrow & \bigwedge_1^n W^* \\ & v & \longmapsto & \mathrm{i}_v \omega \end{array} \;, \tag{3.19}$$

(see equation (1.53)) is injective and $\ker\hat{\omega}$ is exactly the pre-image, under $\omega_V^\flat$, of the subspace $\bigwedge_0^n W^* \cong \bigwedge^n T^*$, which is one-dimensional.

When $n \geqslant 2$, these considerations, together with Proposition 3.1, lead to the following criterion to decide whether an $(n+1)$-form is multisymplectic:

[3] Note that for $n = 1$, the definition of a multisymplectic form reduces to that of a symplectic form, since in this case, the condition of 0-horizontality is empty and that of existence of a lagrangian subspace L can be derived as a theorem. The main difference between this case and those where $n > 1$ is that here, L is not unique.

Proposition 3.4 *Let W be a finite-dimensional vector space and V be a fixed subspace of W, with $n = \dim(W/V) \geqslant 2$, let ω be a non-degenerate $(n-1)$-horizontal $(n+1)$-form on W and let $\hat{\omega}$ be the symbol of ω, which is a $\hat{T}$-valued 2-form on V, where $T \cong W/V$ and $\hat{T} = \bigwedge^{n-1} T^*$. Consider the subspace L of V spanned by the kernels of all the projections $\omega_{\hat{t}}$ ($\hat{t} \in \hat{T}^* \setminus \{0\}$) of $\hat{\omega}$:*

$$L = \sum_{\hat{t} \in \hat{T}^* \setminus \{0\}} \ker\, \omega_{\hat{t}^*} \ .$$

For the form ω to be multisymplectic and L to be the pertinent multilagrangian subspace, the following conditions are necessary and sufficient:

- *L is isotropic;*
- $\dim L = (n \dim V + 1)/(n+1)$.

To formulate the algebraic Darboux theorem in the specific context of multisymplectic forms, we begin by adapting Definition 1.6 to this context: explicitly, a **(multisymplectic) Darboux basis** or **(multisymplectic) canonical basis** is a basis

$$\{\, e_\mu, e_i, e_\nu^j, e_0 \,|\, 1 \leqslant \mu, \nu \leqslant n \,,\, 1 \leqslant i, j \leqslant N \,\}$$

of W, with dual basis

$$\{\, e^\mu, e^i, e_j^\nu, e^0 \,|\, 1 \leqslant \mu, \nu \leqslant n \,,\, 1 \leqslant i, j \leqslant N \,\}$$

of W^*, such that

- $\{\, e_\mu \,|\, 1 \leqslant \mu \leqslant n \,\}$ is a basis of a subspace of W complementary to V, isomorphic to T, and $\{\, e^\mu \,|\, 1 \leqslant \mu \leqslant n \,\}$ is the corresponding dual basis of the subspace $V^\perp$ of W^*, so that if we define

  $$\hat{e}^0 \;=\; e^1 \wedge \ldots \wedge e^n \qquad \text{e} \qquad \hat{e}_\mu \;=\; \mathrm{i}_{e_\mu} \hat{e}^0 \quad (1 \leqslant \mu \leqslant n) \ ,$$

  then $\hat{e}^0$ spans the one-dimensional subspace $\bigwedge_0^n W^* \cong \bigwedge^n T^*$ and $\{\, \hat{e}_\mu \,|\, 1 \leqslant \mu \leqslant n \,\}$ is a basis of the n-dimensional subspace $\bigwedge_0^{n-1} W^* \cong \bigwedge^{n-1} T^*$;
- $\{\, e_i \,|\, 1 \leqslant i \leqslant N \,\}$ is a basis of a subspace of V complementary to L;
- $\{\, e_\mu^i, e_0 \,|\, 1 \leqslant \mu \leqslant n \,,\, 1 \leqslant i \leqslant N \,\}$ is a basis of L;
- e_0 spans the one-dimensional subspace $\ker \hat{\omega}$;

and such that

$$\omega \;=\; e^i \wedge e_i^\mu \wedge \hat{e}_\mu \,+\, e^0 \wedge \hat{e}^0 \;. \tag{3.20}$$

and for the symbol

$$\hat{\omega} \;=\; \left(e^i \wedge e_i^\mu\right) \otimes \hat{e}_\mu \;. \tag{3.21}$$

Obviously, an $(n+1)$-form $\omega \in \bigwedge_2^{n+1} W^*$ that admits such a canonical basis is a multisymplectic form. Conversely, as a corollary of Theorem 1.11, we have

Theorem 3.2 (Darboux theorem for multisymplectic vector spaces)
Every multisymplectic vector space admits a canonical basis.

Although this theorem has already been proved in the first chapter, we want to present an alternative proof based on the analogous theorem for polysymplectic forms (Theorem 3.1), using the symbol. This procedure uses strongly the fact that the kernel of the symbol of the multisymplectic form is only one-dimensional, which allows to reduce the problem to the polysymplectic case.

PROOF. We begin by choosing an arbitrary basis $\{\, e_\mu \mid 1 \leqslant \mu \leqslant n \,\}$ of a subspace of W complementary to V, isomorphic to T, with dual basis $\{\, e^\mu \mid 1 \leqslant \mu \leqslant n \,\}$ of the subspace $V^\perp$ of W^*, and define $\hat{e}^0 = e^1 \wedge \ldots \wedge e^n$ and $\hat{e}_\mu = \mathrm{i}_{e_\mu} \hat{e}^0$, as above. We also define a 1-form e^0 on W by $e^0 = (-1)^n \, \mathrm{i}_{e_1} \ldots \mathrm{i}_{e_n} \omega$, that is,

$$\langle e^0, w \rangle \;=\; \omega(w, e_1, \ldots, e_n) \quad \text{for } \; w \in W \;.$$

Since ω is non-degenerate, $\ker e^0 \cap \ker \hat{\omega} = \{0\}$. Now choose a vector $e_0 \in \ker \hat{\omega}$ such that $\langle e^0, e_0 \rangle = 1$ and pick a subspace W_0 of W complementary to $\ker \hat{\omega}$, setting $V_0 = V \cap W_0$ and $L_0 = L \cap W_0$. Then the restriction of the symbol of $\hat{\omega}$ of ω to V_0 is a non-degenerate polysymplectic form taking values in the auxiliary space $\hat{T} \;=\; \bigwedge_0^{n-1} W^* \;\cong\; \bigwedge^{n-1} T^*$, with polylagrangian subspace L_0, and so V_0 admits a (polysymplectic) canonical basis $\{\, e_i, e_\nu^j \mid 1 \leqslant \nu \leqslant n \,,\, 1 \leqslant i,j \leqslant N \,\}$ for $\hat{\omega}$ adapted to the basis $\{\, \hat{e}_\mu \mid 1 \leqslant \mu \leqslant n \,\}$ of $\hat{T}$. Now observing that the $(n-1)$-horizontal $(n+1)$-form

$$\omega - e^0 \wedge \hat{e}^0$$

is degenerate, with kernel equal to $\ker \hat{\omega}$, but has the same symbol $\hat{\omega}$ as the original $(n-1)$-horizontal $(n+1)$-form ω, we conclude that $\{\, e_\mu, e_i, e_\nu^j, e_0 \mid 1 \leqslant \mu,\nu \leqslant n \,,\, 1 \leqslant i,j \leqslant N \,\}$ is a (multisymplectic) canonical basis for ω. $\square$

3.3 Poly- and multisymplectic manifolds and fiber bundles

The Darboux thorems of the second chapter guarantee the existence of canonical local coordinates in which a polysymplectic form is expressed as

$$\hat{\omega} ~=~ \big(dq^i \wedge dp_i^a\big) \otimes \hat{e}_a ~,$$

and a multisymplectic form as

$$\omega ~=~ dq^i \wedge dp_i^\mu \wedge d^n x_\mu \,-\, dp \wedge d^n x ~,$$

where $d^n x_\mu = \mathrm{i}_{\partial_\mu} d^n x$, thus reproducing precisely the fundamental canonical structure of classical field theory.

Obviously, what has been achieved in this thesis is not the end but only the beginning. We hope that the theory of poly- and multisymplectic manifolds and fiber bundles, for which we believe to have laid the foundation, will in a not too distant future turn out to be as interesting and fruitful as symplectic geometry.

Appendix A

Poincaré Lemma

In this appendix, we state the version of the Poincaré lemma used in the proof of the Darboux theorem at the end of Chapter 2.

Theorem A.1 *Let $\omega \in \Omega^k(P,\hat{T})$ be a closed form on a manifold P taking values in a fixed vector space $\hat{T}$ and let L be an involutive distribution on P. Suppose that ω is $(k-r)$-horizontal (with respect to L), i.e., such that for any $p \in P$ and all $v_1, \ldots, v_{r+1} \in L_p$, we have*

$$\mathrm{i}_{v_1} \ldots \mathrm{i}_{v_{r+1}} \omega_p \;=\; 0 \;.$$

Then for any point of P there exist an open neighborhood U of that point and a $(k-1)$-form $\theta \in \Omega^{k-1}(U,\hat{T})$ on U which is also $(k-r)$-horizontal (with respect to L), i.e., such that for any $p \in U$ and all $v_1, \ldots, v_r \in L_p$, we have

$$\mathrm{i}_{v_1} \ldots \mathrm{i}_{v_r} \theta_p \;=\; 0 \;,$$

and such that $\omega = d\theta$ on U.

Proof. Due to the local character of this theorem and since the subbundle L of TP is involutive, we can without loss of generality work in a local chart of the manifold P around the chosen reference point in which the foliation defined by L is "straightened out", so we may assume that $P \cong K_0 \oplus L_0$ with $L \cong L_0$ with fixed subspaces K_0 and L_0 and such that the aforementioned reference point corresponds to the origin. (In what follows, we shall omit the index 0.) We also suppose that $\hat{T} = \mathbb{R}$, since we may prove the theorem separately for each component of ω and θ, with respect to some fixed basis of $\hat{T}$.

For $t \in \mathbb{R}$, define the "K-contraction" $F_t^K : P \to P$ and the "L-contraction" $F_t^L : P \to P$ by $F_t^K(x,y) = (tx,y)$ and $F_t^L(x,y) = (x,ty)$; obviously, F_t^K and F_t^L are diffeomorphisms if

$t \neq 0$ and are projections if $t = 0$. Associated with each of these families of mappings there is a time dependent vector field which generates it in the sense that, for $t \neq 0$,

$$X_t^K\big(F_t^K(x,y)\big) \;=\; \frac{d}{ds}\,F_s^K(x,y)\bigg|_{s=t} \qquad \text{and} \qquad X_t^L\big(F_t^L(x,y)\big) \;=\; \frac{d}{ds}\,F_s^L(x,y)\bigg|_{s=t} .$$

Explicitly, for $t \neq 0$,

$$X_t^K(x,y) \;=\; t^{-1}(x,0) \qquad \text{and} \qquad X_t^L(x,y) \;=\; t^{-1}(0,y) .$$

Define $\omega_0 = (F_0^L)^*\omega$ and, for $\epsilon > 0$,

$$\theta_\epsilon \;=\; \int_\epsilon^1 dt\, \Big((F_t^L)^*(\mathrm{i}_{X_t^L}\omega) \;+\; (F_t^K)^*(\mathrm{i}_{X_t^K}\omega_0)\Big) ,$$

as well as

$$\theta \;=\; \lim_{\epsilon\to 0}\, \theta_\epsilon \;=\; \int_0^1 dt\, \Big((F_t^L)^*(\mathrm{i}_{X_t^L}\omega) \;+\; (F_t^K)^*(\mathrm{i}_{X_t^K}\omega_0)\Big) .$$

To see that the $(k-1)$-forms θ and θ_ϵ are well defined, consider $k-1$ vectors $(u_i, v_i) \in K \oplus L$ $(1 \leqslant i \leqslant k-1)$ and observe that, for $t \neq 0$,

$$(F_t^L)^*(i_{X_t^L}\omega)_{(x,y)}\big((u_1,v_1),\ldots,(u_{k-1},v_{k-1})\big) \;=\; \omega_{(x,ty)}\big((0,y),(u_1,tv_1),\ldots,(u_{k-1},tv_{k-1})\big) ,$$

and

$$(F_t^K)^*(i_{X_t^K}\omega_0)_{(x,y)}\big((u_1,v_1),\ldots,(u_{k-1},v_{k-1})\big) \;=\; t^{k-1}\,\omega_{(tx,0)}\big((x,0),(u_1,0),\ldots,(u_{k-1},0)\big) .$$

Here we see easily that both expressions are differentiable in t and provide $(k-1)$-forms which are $(k-r)$-horizontal and $(k-1)$-horizontal with respect to L, respectively. Thus, θ_ϵ and θ are $(k-1)$-forms which are $(k-r)$-horizontal with respect to L. Moreover, since $d\omega = 0$ and $d\omega_0 = 0$,

$$\begin{aligned} d\theta_\epsilon \;&=\; \int_\epsilon^1 dt\, \Big((F_t^L)^*(\mathrm{i}_{X_t^L}\omega) \;+\; (F_t^K)^*(\mathrm{i}_{X_t^K}\omega_0)\Big) \;=\; \int_\epsilon^1 dt\, \Big((F_t^L)^*(L_{X_t^L}\omega) \;+\; (F_t^K)^*(L_{X_t^K}\omega_0)\Big) \\ &=\; \int_\epsilon^1 dt\, \Big(\frac{d}{dt}\big((F_t^L)^*\omega\big) \;+\; \frac{d}{dt}\big((F_t^K)^*\omega_0\big)\Big) \;=\; \omega \;-\; (F_\epsilon^L)^*\omega \;+\; \omega_0 \;-\; (F_\epsilon^K)^*\omega_0 . \end{aligned}$$

Taking the limit $\epsilon \to 0$, we get $(F_\epsilon^L)^*\omega \to \omega_0$ and $(F_\epsilon^K)^*\omega_0 \to 0$ and hence $d\theta = \omega$. □

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