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Invariant polynomials and Molien functions

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The Molien function associated with a finite-dimensional representation of a compact Lie group is a useful tool in representation theory because it acts as a generating function for counting the number of invariant polynomials on the representation space. The main purpose of this paper is to introduce a more general (and apparently new) generating function which, in the same sense, counts not only the number of invariant real polynomials in a real representation or the number of invariant complex polynomials in a complex representation, as a function of their degree, but encodes the number of invariant real polynomials in a complex representation, as a function of their bidegree (the first and second component of this bidegree being the number of variables in which the polynomial is holomorphic and antiholomorphic, respectively). This is obviously an additional and non-trivial piece of information for representations which are truly complex (i.e., not self-conjugate) or are pseudo-real, but it provides additional insight even for real representations. In addition, we collect a number of general formulas for these functions and for their coefficients and calculate them in various irreducible representations of various classical groups, using the software package MAPLE.

I. INTRODUCTION

Determining the ring of invariant polynomials in arbitrary representations is perhaps one of the most important open problems in group theory. Trying to solve this problem in full generality—for example, in the framework of arbitrary finite-dimensional representations of compact Lie groups (including finite groups)—is presently considered to be a hopeless enterprise. What is, however, a tractable problem is to determine at least the number of (linearly independent) invariant polynomials, or more precisely, the dimension of the space of invariant homogeneous polynomials of any given degree, and this is often an extremely useful piece of information when it comes to calculating invariant polynomials for a concrete representation of a concrete group.

The present investigation originated from the recent work of J. E. M. Hornos and Y. M. M. Hornos[1] on the origin of the genetic code that has found great repercussion in the international scientific literature.[2,3] According to their proposal, the degeneracy of the universal genetic code for protein synthesis is not (as many molecular biologists used to and some continue to believe) purely accidental, but can be understood as resulting from an evolutionary process which involves symmetry breaking: evolution from a highly symmetric initial state to a final state in which this symmetry is strongly broken. This evolution must have occurred, in several consecutive steps, far back in earth's early history, and so is not accessible to direct observation. For the time being, the scheme proposed by Hornos and Hornos is purely group theoretical, its main virtue being that—within the limits of the originally proposed scheme—the initial symmetry and all intermediate steps in the sequence can be uniquely reconstructed from presently available data. (More recently, the scope of the scheme has been extended and, as a result, a second possibility has emerged.)[4]

The great challenge for the future is to identify a dynamical system modelling the underlying evolutionary process, so that the sequence of symmetry breakings found can be associated with a sequence of (generic) bifurcations. In fact, it is well known that in dynamical systems with symmetry and with external parameters, bifurcations that occur under appropriate variations of the parameters almost unavoidably lead to symmetry breaking.[5] In this general framework, of course, the variety of possibilities is enormous, so that for the time being, we have decided to perform our search in the more restricted class of Hamiltonian dynamical systems. But the most natural candidates for a Hamiltonian function capable of reproducing the desired symmetry breaking pattern are just the polynomial functions on the representation space which are invariant under the full symmetry group and which, if possible, should be of degree $\leqslant 4$. (This would correspond to some kind of anharmonic oscillator type model.) In the case at hand, the full symmetry group is the rank 3 symplectic group Sp(6) and the representation is the 64-dimensional irreducible representation of highest weight (1,1,0)—a complex representation which is self-conjugate but is pseudo-real, rather than real, and which has come to be called the codon representation of Sp(6).

Before attempting to explicitly construct all invariant polynomials up to a given degree, in any given representation of any group, it is obviously of great help to know precisely how many such polynomials there are. To see how useful this information can be and how it may help to avoid unpleasant surprises, consider as an example the problem of finding the invariant polynomials of degree 4 in, say, the spin 3 representation of the ordinary rotation group. (This example has been chosen because of certain similarities with the codon representation.) Given the fact that vectors in this representation space can be realized as totally symmetric tensors t of rank 3 over three-dimensional Euclidean space which are traceless in any pair of indices, it is easy to construct invariant polynomials of degree 4 by considering all possibilities of contracting indices in a product of four such tensors, using the invariant scalar product. Symmetry implies that the only relevant information is how many indices of any given tensor are contracted with how many indices of any other given tensor (not which with which), while tracelessness forbids contraction of two indices that appear within the same tensor. Therefore, there are three different possibilities to contract the 12 indices (each one ranging from 1 to 3) in the product

$$t_{i_1 j_1 k_1} t_{i_2 j_2 k_2} t_{i_3 j_3 k_3} t_{i_4 j_4 k_4}$$

which come to mind.

(1) Every tensor has only one partner for the contraction. Contract, for example, i_1 with i_2, j_1 with j_2, k_1 with k_2 and i_3 with i_4, j_3 with j_4, k_3 with k_4:

$$P_{4,1}(t) = P_2(t)^2, \quad P_2(t) = \sum_{i,j,k=1}^{3} t_{ijk}^2.$$

(2) Every tensor has two partners for the contraction. Contract, for example, i_1 with i_2, j_1 with j_2, k_1 with i_3, k_2 with i_4, j_3 with j_4, k_3 with k_4:

$$P_{4,2}(t) = \sum_{i,j,k,l,m,n=1}^{3} t_{ijk} t_{ijl} t_{kmn} t_{lmn}.$$

(3) Every tensor has three partners for the contraction. Contract, for example, i_1 with i_2, j_1 with i_3, k_1 with i_4, j_2 with j_3, k_2 with j_4, k_3 with k_4:

$$P_{4,3}(t) = \sum_{i,j,k,l,m,n=1}^{3} t_{ijk} t_{ilm} t_{jln} t_{kmn}.$$

The first possibility corresponds to the square of the quadratic polynomial stemming from the invariant scalar product in this representation, while the other two are genuinely quartic and apparently independent. Therefore, it comes as a surprise that the number of invariant polynomials of degree 4 in this representation, as computed by the techniques to be discussed (and further developed) in the present paper, turns out to be 2, and not 3. This means that the three polynomials obtained above must be linearly dependent! And indeed, writing out these polynomials as explicit functions of the seven variables $t_{112}, t_{122}, t_{113}, t_{133}, t_{223}, t_{233}, t_{123}$, we find, using MAPLE, the following linear relation:

$$P_{4,1} = 2(P_{4,2} + P_{4,3}).$$

This simple example shows that independent information on the correct number of invariant polynomials is crucial if one wants to avoid naive overcounting. It is equally crucial if one wants to avoid undercounting, which may occur as a result of overlooking non-obvious, ''hidden'' invariants. (One fairly well-known example that comes to mind is the Pfaffian—an invariant in the adjoint representation of the special orthogonal groups in even dimensions which is hard to detect by tensorial methods.)

The basic strategy for determining the number of invariant polynomials of any given degree is to encode all of them into a generating function, which is commonly called the Molien function and which can often be calculated in closed form, at least in sufficiently simple situations. But even when this is not possible in practice, the existing formulas for the generating function can be exploited to compute at least the first few coefficients.

Unfortunately, the standard Molien function M_π associated with a given finite-dimensional representation π of a given compact Lie group G is inadequate for handling the problem at hand in its full generality, due to a discrepancy between ground fields. On the one hand, the representation spaces encountered in group theory are always assumed to be complex (this guarantees that one can simultaneously diagonalize maximal commuting sets of linear transformations); the usual convention for handling real representations is then to view them as complex representations possessing an invariant antilinear involution. On the other hand, we are typically interested in finding all invariant real polynomials and not just the complex ones. The standard Molien function, however, does not allow one to identify the extent to which a real polynomial on the space of a complex representation is holomorphic or antiholomorphic in its variables. What is worse, it does not detect invariant polynomials of mixed type. The obvious prototype of such a polynomial is the invariant scalar product—a quadratic polynomial on the representation space, holomorphic in one variable and antiholomorphic in the other—which exists in any finite-dimensional representation of any compact Lie group and, in addition, is the only polynomial of its kind (up to a constant multiple) in case the representation is irreducible. The fact that the standard Molien function captures only purely holomorphic (or purely antiholomorphic) invariants, but fails to detect mixed invariants, including the invariant scalar product, can already be illustrated by looking at the simplest of all representations: the fundamental spin 1/2 representation of the ordinary rotation group (or rather its universal covering group SU(2)).

The natural way out of this dilemma, proposed and elaborated in the present paper, is to invent a new generating function F_π which generalizes the usual Molien function and is specifically designed to capture all real polynomials in complex representations, discriminating between holomorphic ones, purely antiholomorphic ones and mixed ones, according to their bidegree.

As far as the specific case of the codon representation of Sp(6) is concerned, the techniques developed in the present work allow one to conclude that: (a) there are no invariant quadratic polynomials of bidegree (2,0) (purely holomorphic) or of bidegree (0,2) (purely antiholomorphic), while there is one invariant quadratic polynomial of bidegree (1,1) (the scalar product), (b) there are no invariant cubic polynomials of any kind and (c) the numbers of invariant quartic polynomials are as follows: 3 of bidegree (4,0) (purely holomorphic), 3 of bidegree (0,3) (purely antiholomorphic), 6 of bidegree (3,1), 6 of bidegree (1,3) and finally 15 of bidegree (2,2). Since a Hamiltonian function must be real, we may therefore conclude that the general candidate for a Hamiltonian capable of describing the evolution of the genetic code through an anharmonic oscillator type model must be a linear combination of the invariant scalar product, its square and another 14 genuinely quartic invariant polynomials of bidegree (2,2). What remains to be determined are the explicit form of these polynomials and the conditions to be imposed on their coefficients in order to guarantee positivity of the energy. (The remaining final freedom of modifying the Hamiltonian by an additive constant may then be used to normalize its minimum value to 0.)

The paper is organized as follows. In Sec. II, we briefly review the definition of the Molien function, whereas in Sec. III we define our new generating function for counting invariant real polynomials in complex representations. Both sections contain comments on the relations between the analytic form of the generating functions and the structure of the (graded or bigraded) algebra of invariant polynomials, in terms of generators and relations. In Sec. IV, we derive integral formulas for both generating functions, with emphasis on their explicit form for unitary representations of compact connected Lie groups, in terms of roots and weights. In Sec. V, we present purely combinatorial formulas for the coefficients, involving the multiplicities of the weights and a set of integer coefficients called ''decomposition indices'' associated with the vectors in the root lattice. In Sec. VI, we discuss as an example the results we have obtained for the simplest among all compact connected simple Lie groups: SU(2). Finally, in Sec. VII, we present calculations for various irreducible representations of the rank 2 symplectic group Sp(4) and the rank 3 symplectic group Sp(6), including the fundamental representations and the other irreducible representations which appear in the symmetry breaking scheme of Hornos and Hornos.[1]

II. THE MOLIEN FUNCTION: DEFINITION AND ELEMENTARY PROPERTIES

Given an arbitrary finite-dimensional representation π of a compact Lie group G on some n-dimensional vector space V, and denoting by $c_k(\pi)$ the number of (linearly independent) G-invariant polynomials of degree k on V, one defines the corresponding Molien function M_π by the power series

$$M_\pi(z)=\sum_{k=0}^{\infty} c_k(\pi)z^k. \tag{1}$$

Note that identifying homogeneous polynomials of degree k on V with totally symmetric tensors of degree k over V^*, we easily obtain the estimate

$$0\leqslant c_k(\pi)\leqslant\binom{n+k-1}{k}=\binom{n+k-1}{n-1},$$

so $c_k(\pi)$ grows at most polynomially as $k\rightarrow\infty$ (the highest power being k^{n-1}); therefore, the above power series is absolutely convergent on the open unit disk in the complex z plane and hence M_π is a complex analytic function there—a function from which we may obviously recover all the numbers $c_k(\pi)$ as Taylor coefficients:

$$c_k(\pi)=\frac{1}{k!}\,\frac{d^k}{dz^k}\,M_\pi(z)\bigg|_{z=0}. \tag{2}$$

Note also that this definition can be used both in the real and in the complex setting, and more generally, for representations π of G by $\mathbb{F}$-linear transformations on finite-dimensional vector spaces V over $\mathbb{F}$, where $\mathbb{F}$ is an arbitrary field of characteristic 0.

Some initial information on the structure of the Molien function can be gained by describing the graded ring of G-invariant polynomials on V in terms of generators and relations. In fact, the Hilbert–Weyl theorem guarantees that this graded ring is finitely generated, i.e., that there exists a finite set $\{P_1,\ldots,P_N\}$ of homogeneous G-invariant polynomials on V such that every (homogeneous) G-invariant polynomial P on V can be written in the form

$$P(v)=p(P_1(v),\ldots,P_N(v)). \tag{3}$$

where p is some (homogeneous) polynomial on $\mathbb{F}^N$, provided we define homogeneity of polynomials on $\mathbb{F}^N$ as referring to a modified notion of degree, namely,

$$\deg\ p_{m_1,\ldots,m_N}=m_1\ \deg\ P_1+\ldots+m_N\ \deg\ P_N, \tag{4}$$

for the monomial $p_{m_1},\ldots,m_N$ given by

$$p_{m_1,\ldots,m_N}(u_1,\ldots,u_N)=u_1^{m_1}\ldots u_N^{m_N}. \tag{5}$$

(See, for example, Ref. 5, p. 46 for a statement and pp. 54–58 for a proof. However, I prefer to avoid the term ''Hilbert basis'' used in Ref. 5, which I consider to be potentially misleading and therefore unfortunate.) Note that the polynomial p is in general not uniquely fixed by the polynomial P because there may be relations, i.e., polynomials R on $\mathbb{F}^N$ satisfying

$$R(P_1(v),\ldots,P_N(v)) \equiv 0. \tag{6}$$

Note also that the relations R form a graded ring which is nothing but the kernel of the degree-preserving homomorphism, defined by (3), from the graded ring of polynomials p on $\mathbb{F}^N$ onto the graded ring of G-invariant polynomials P on V. Being a graded ideal in a polynomial ring, this kernel is finitely generated, i.e., there exists a finite set $\{R_1,\ldots,R_M\}$ of homogeneous polynomials on $\mathbb{F}^N$ such that every (homogeneous) polynomial R on $\mathbb{F}^N$ satisfying (6) can be written in the form

$$R(u_1,\ldots,u_N) = r(R_1(u_1,\ldots,u_N),\ldots,R_M(u_1,\ldots,u_n)), \tag{7}$$

where r is some (homogeneous) polynomial on $\mathbb{F}^M$, provided we define homogeneity of polynomials on $\mathbb{F}^M$ as referring to a modified notion of degree, namely,

$$\deg\ r_{n_1,\ldots,n_M} = n_1\ \deg\ R_1 + \ldots + n_M\ \deg\ R_M, \tag{8}$$

for the monomial $r_{n_1,\ldots,n_M}$ given by

$$r_{n_1,\ldots,n_M}(v_1,\ldots,v_M) = v_1^{n_1}\ldots v_M^{n_M}. \tag{9}$$

(See, for example, Ref. 5, Corollary 6.2, p. 54.) In general, neither the set $\{P_1,\ldots,P_N\}$ of generators nor the set $\{R_1,\ldots,R_M\}$ of relations is unique, but we can at least fix the number N of generators and the number M of relations by requiring both N and M to be minimal.

The simplest case is, of course, when $M=0$, i.e., there are no relations. Counting the number $c_k(\pi)$ of G-invariant polynomials of degree k on V gives

$$c_k(\pi) = \mathrm{card}\left\{ (m_1,\ldots,m_N) \in \mathbb{N}_0^N \bigg/ \sum_{j=1}^N m_j\ \deg\ P_j = k \right\}.$$

But this is just the coefficient of z^k in the power series expansion of the function

$$M_\pi(z) = \frac{1}{\prod_{j=1}^N (1 - z^{\deg\ P_j})}. \tag{10}$$

Conversely, it is clear that if the Molien function for the representation π of G on V has this form, then there can be no relations, because otherwise the number of G-invariant polynomial functions on V of degree k would have to be strictly less than the coefficient of z^k in the power series expansion of Eq. (10), at least for some k. Thus the Molien function detects the presence or absence of relations among the generators of the ring of invariant polynomials.

For later use, it is also of some interest to write down the corresponding result for the next-simplest case $M=1$, i.e., when there is a single relation $R = r(P_1,\ldots,P_N)$. Due to the fact that polynomials of the form $p(P_1,\ldots,P_N)\ r(P_1,\ldots,P_N)$ will vanish identically, counting the number $c_k(\pi)$ of G-invariant polynomials of degree k on V now gives

$$c_k(\pi) = \mathrm{card}\left\{ (m_1,\ldots,m_N) \in \mathbb{N}_0^N \bigg/ \sum_{j=1}^N m_j\ \deg\ P_j = k \right\}$$
$$- \mathrm{card}\left\{ (m_0;m_1,\ldots,m_N) \in \mathbb{N}_0^{N+1} \bigg/ \sum_{j=1}^N m_0 m_j\ \deg\ R\ \deg\ P_j = k \right\}.$$

But this is just the coefficient of z^k in the power series expansion of the function

$$M_\pi(z) = \frac{1 - z^{\deg R}}{\prod_{j=1}^{N}(1 - z^{\deg P_j})}. \tag{11}$$

Again, it is clear that if the Molien function for the representation π of G on V has this form, then there can be no other relations, because otherwise the number of G-invariant polynomial functions on V of degree k would have to be strictly less than the coefficient of z^k in the power series expansion of Eq. (11), at least for some k.

III. A NEW GENERATING FUNCTION

As remarked previously, the above definition of the Molien function applies equally well in the real and in the complex setting. Often, however, it is of interest to also determine the number of (linearly independent) G-invariant real polynomials in a complex representation, the typical example for a quadratic polynomial of this kind being the square of the norm in a unitary representation. Therefore, it is useful to introduce a generating function for counting the number of such invariants as well. The main new feature that must be taken into account is the fact that real polynomials over a complex vector space carry, over and above their usual degree, a bidegree that counts the number of variables in which they are holomorphic and antiholomorphic, respectively. This will lead to a generating function which depends on two variables, rather than one.

Indeed, given an arbitrary representation π of a compact Lie group G on some n-dimensional complex vector space V, let us first of all define $\bar{V}$ to be the n-dimensional complex vector space which is ''V with the opposite complex structure'' and $V_r = \bar{V}_r$ to be the $2n$-dimensional real vector space obtained from V or $\bar{V}$ by ''forgetting the complex structure''. In other words, V, $\bar{V}$ and V_r are identical as sets and as real vector spaces, while the complex structures on V and on $\bar{V}$ are in this picture encoded into real linear transformations $J: V_r \to V_r$ and $\bar{J}: V_r \to \bar{V}_r$, which are nothing but multiplication by i in V and in $\bar{V}$, respectively, so $J^2 = -1$ and $\bar{J}^2 = -1$; then ''opposite'' means that $\bar{J} = -J$. (The idea behind this construction is that it enables us to identify, for any complex vector space W, complex antilinear maps from V to W with complex linear maps from $\bar{V}$ to W.) Next, recall that homogeneous real polynomials of degree k on V can be identified with totally symmetric $\mathbb{R}$-multilinear mappings from $V_r \times \ldots \times V_r$ (k copies) to $\mathbb{R}$—or to $\mathbb{C}$ if we allow such polynomials to be complex-valued, as will be assumed throughout the following. We shall say that such a polynomial is homogeneous of bidegree (p,q), with $p+q=k$, if under this identification it corresponds to a totally symmetric $\mathbb{C}$-multilinear mapping from $V \times \ldots \times V \times \bar{V} \times \ldots \times \bar{V}$ to $\mathbb{C}$, with p copies of V and q copies of $\bar{V}$. For such a polynomial P, we call the number p its holomorphic degree, denoted by $\deg_h P$, and the number q its antiholomorphic degree, denoted by $\deg_a P$. It is easy to show that any homogeneous real polynomial of degree k may be uniquely decomposed into a sum of homogeneous polynomials of bidegree (p,q), as follows:

$$P_k = \sum_{\substack{p,q=0 \\ p+q=k}}^{k} P_{p,q}.$$

Namely, given P_k, we may set

$$\tilde{P}_{p,q}(v_1,\ldots,v_k) = \frac{1}{2^k} \sum_{l_1,\ldots l_k=0}^{1} (-i)^{l_1+\cdots+l_p}\, i^{l_{p+1}+\cdots+l_{p+q}}\, P_k(i^{l_1}v_1,\ldots,i^{l_k}v_k),$$

which defines a real multilinear function, complex linear and symmetric in the first p variables and complex antilinear and symmetric in the last q variables; then the above decomposition holds with $P_{p,q}$ obtained from $\tilde{P}_{p,q}$ by symmetrization in all k arguments,

$$P_{p,q}(v_1,\ldots,v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \tilde{P}_{p,q}(v_{\sigma(1)},\ldots,v_{\sigma(k)}).$$

This decomposition can be stated in more concrete terms by introducing an arbitrary basis $\{v_1,\ldots,v_n\}$ of V (over $\mathbb{C}$), together with the induced basis $\{v_1,iv_1,\ldots,v_n,iv_n\}$ of V_r (over $\mathbb{R}$), and expanding vectors in the representation space into components:

$$v=\sum_{j=1}^{n} \zeta_j v_j=\sum_{j=1}^{n} (\xi_j+i\eta_j)v_j .$$

Then any polynomial P on V_r can be written either as a linear combination of monomials which are products of powers of the real coordinates ξ_j and η_j or as a linear combination of monomials which are products of powers of the complex coordinates ζ_j and their complex conjugates $\overline{\zeta_j}$. Using the latter representation and employing multi-index notation, we have

$$P(v)=\sum_{\alpha,\beta} a_{\alpha,\beta}\zeta^{\alpha}\overline{\zeta^{\beta}},$$

so

$$P(v)=\sum_{k} P_k(v)=\sum_{p,q} P_{p,q}(v),$$

with

$$P_k(v)=\sum_{|\alpha|+|\beta|=k} a_{\alpha,\beta}\zeta^{\alpha}\overline{\zeta^{\beta}},\quad P_{p,q}(v)=\sum_{\substack{|\alpha|=p\\|\beta|=q}} a_{\alpha,\beta}\zeta^{\alpha}\overline{\zeta^{\beta}}.$$

(All sums are supposed to be finite.)

The crucial point is now that since G acts on V by complex linear transformations $\pi(g)$, these decompositions preserve G-invariance, that is, if P is G-invariant, so are not only the P_k but also the $P_{p,q}$. Therefore, denoting by $c_{p,q}(\pi)$ the number of (linearly independent) G-invariant polynomials of bidegree (p,q), we define the following generating function F_π of two variables, which for later convenience we shall assume to be mutually complex conjugate:

$$F_\pi(z,\overline{z})=\sum_{p,q=0}^{\infty} c_{p,q}(\pi)z^p\overline{z^q}. \tag{12}$$

As before, this power series is absolutely convergent on the open unit disk in the complex z plane and hence F_π is a real analytic function there—a function from which we may obviously recover all the numbers $c_{p,q}(\pi)$ as Taylor coefficients:

$$c_{p,q}(\pi)=\frac{1}{p!}\,\frac{1}{q!}\,\frac{\partial^p}{\partial z^p}\,\frac{\partial^q}{\partial\overline{z^q}}F_\pi(z,\overline{z})\Big|_{z=0,\overline{z}=0}. \tag{13}$$

As a first elementary property of this new generating function, note that it behaves naturally under complex conjugation. Namely, introducing an arbitrary conjugation σ on V, that is, an involutive antilinear transformation $\sigma:V\to V$ to define the complex conjugate representation $\overline{\pi}$ of π according to

$$\overline{\pi(g)}=\sigma\pi(g)\sigma\quad\text{for}\ g\in G, \tag{14}$$

we note that $c_{p,q}(\overline{\pi})=c_{q,p}(\pi)$ and hence

$$F_{\overline{\pi}}(z,\overline{z})=F_\pi(\overline{z},z). \tag{15}$$

Obviously, this relation does not depend on the choice of the conjugation σ because the representations $\overline{\pi_1}$ and $\overline{\pi_2}$ defined by means of two different conjugations σ_1 and σ_2 are equivalent (with $\sigma_2\sigma_1^{-1}$ as the intertwining operator). Note also that F_π contains the Molien functions M_π and $M_{\overline{\pi}}$ as special cases:

$$M_\pi(z)=F_\pi(z,0), \quad M_{\overline{\pi}}(z)=F_\pi(0,z). \tag{16}$$

As in the case of the ordinary Molien function M_π, the generating function F_π allows one to read off important information about the generators of the bigraded ring of G-invariant real polynomials on V and about the relations that exist between them. Indeed, observe first of all that the set $\{P_1,\dots,P_N\}$ of generators and the set $\{R_1,\dots,R_M\}$ of relations may without loss of generality (and at most at the expense of increasing the ''minimum'' number N of generators and the '' minimum'' number M of relations required) be assumed to consist of polynomials which are homogeneous in bidegree. Then it is not difficult to see that in the simplest case $M=0$ (no relations),

$$c_{p,q}(\pi)=\operatorname{card}\left\{(m_1,\dots,m_N)\in\mathbb{N}_0^N \Big/ \sum_{j=1}^N m_j \deg_h P_j=p,\ \sum_{j=1}^N m_j \deg_a P_j=q\right\}.$$

and hence

$$F_\pi(z,\overline{z})=\frac{1}{\prod_{j=1}^N(1-z^{\deg_h P_j}\overline{z}^{\deg_a P_j})}, \tag{17}$$

while in the next-simplest case $M=1$ (a single relation),

$$c_{p,q}(\pi)=\operatorname{card}\left\{(m_1,\dots,m_N)\in\mathbb{N}_0^N \Big/ \sum_{j=1}^N m_j \deg_h P_j=p,\ \sum_{j=1}^N m_j \deg_a P_j=q\right\}$$
$$-\operatorname{card}\left\{(m_0;m_1,\dots,m_N)\in\mathbb{N}_0^{N+1} \Big/ \begin{array}{l}\sum_{j=1}^N m_0 m_j \deg_h R \deg_h P_j=p\\ \sum_{j=1}^N m_0 m_j \deg_a R \deg_a P_j=q\end{array}\right\}.$$

and hence

$$F_\pi(z,\overline{z})=\frac{1-z^{\deg_h R}\,\overline{z}^{\deg_a R}}{\prod_{j=1}^N(1-z^{\deg_h P_j}\,\overline{z}^{\deg_a P_j})}. \tag{18}$$

IV. INTEGRAL FORMULAS

To begin with, we quote a well-known integral formula which allows one to compute the Molien function M_π in terms of an integral over the group. Namely, let μ_G be the biinvariant Haar measure on G, normalized so that the total volume of G with respect to μ_G is 1. Then

$$M_\pi(z)=\int_G d\mu_G(g)\,\frac{1}{\det(1-z\pi(g))}. \tag{19}$$

This formula is easily generalized to an integral formula for the generating function F_π; it reads

$$F_\pi(z,\overline{z})=\int_G d\mu_G(g)\,\frac{1}{\det(1-z\pi(g))}\,\frac{1}{\det(1-\overline{z\pi(g)})}. \tag{20}$$

The proof is similar to that for the usual Molien function (see, e.g., Ref. 6 p. 204) and is based on calculating the character χ_P of the representation π_P of G on the algebra of polynomial functions on V induced by the given representation π of G on V according to

$$(\pi_P(g)P)(v) = P(\pi(g)^{-1}v),$$

as follows: Since G is a compact Lie group, the given representation π of G on V may without loss of generality be assumed to be unitary (starting from some arbitrary scalar product on V, a G-invariant scalar product on V is obtained by integration over the group), so for fixed $g \in G$, $\pi(g)^{-1}$ can be diagonalized, i.e., there exists a basis $v_1,\ldots,v_n$ of V consisting of eigenvectors of $\pi(g)^{-1}$ with eigenvaleus $\lambda_1,\ldots,\lambda_n$. As a result, the monomials $\zeta^\alpha \overline{\zeta^\beta}$ (see above) form a basis of the space of polynomials on V of bidegree (p,q) consisting of eigenvectors of $\pi_P(g)$ with eigenvalues $\lambda^\alpha \overline{\lambda^\beta}$, so that the character $\chi_{p,q}$ of the representation $\pi_{p,q}$ on the space of homogeneous polynomials on V of bidegree (p,q) induced by the given representation π of G on V is given by

$$\chi_{p,q}(g) = \text{trace } \pi_{p,q}(g) = \sum_{\substack{|\alpha|=p\\|\beta|=q}} \lambda^\alpha \overline{\lambda^\beta}.$$

Multiplying by $z^p \overline{z^q}$ and summing over p and q gives, in the sense of formal power series,

$$\sum_{p,q=0}^{\infty} \chi_{p,q}(g) z^p \overline{z^q} = \sum_{p,q=0}^{\infty} \sum_{\substack{|\alpha|=p\\|\beta|=q}} (z\lambda)^\alpha (\overline{z\overline{\lambda}})^\beta$$
$$= \prod_{j=1}^{n} (1 - z\lambda_j)^{-1} \prod_{j=1}^{n} (1 - \overline{z\overline{\lambda}_j})^{-1} = \det(1 - z\pi(g))^{-1} \det(1 - \overline{z\pi(g)})^{-1}.$$

The result now follows due to a standard fact from the representation theory of compact groups, namely, that the dimension of the fixed subspace of a given representation—or to put it differently, the multiplicity with which the trivial representation occurs in a given representation—is equal to the integral of the character of that representation over the group.

An important aspect of Eqs. (19) and (20) which greatly facilitates the evaluation of the integrals is the fact that the determinants appearing under the integral signs are central functions on the group (i.e., are invariant under conjugation), so that the integral over the whole group can be reduced to an integral over the space of conjugacy classes.

Before performing this reduction, we note that the integral representations (19) and (20) are valid in two special cases which are at opposite extremes. One of these occurs when G is discrete, that is, a finite group, so that the integrals reduce to finite sums,

$$M_\pi(z) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - z\pi(g))}, \tag{21}$$

$$F_\pi(z,\overline{z}) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - z\pi(g))} \frac{1}{\det(1 - \overline{z\pi(g)})}, \tag{22}$$

which can be reduced to sums over conjugacy classes; their explicit evaluation, by means of various techniques, has been studied in the literature (see, e.g., Refs. 6, pp. 204–207, 7, and 8). The other and apparently much less studied case occurs when G is a compact connected Lie group, so that the integrals over the whole group G can be reduced to integrals over a maximal torus T: it is this situation that we shall now investigate in some detail.

Thus let G be a compact connected Lie group, let T be a maximal torus in G and let μ_T be the bi-invariant Haar measure on T, normalized so that the total volume of T with respect to μ_T is 1.

Moreover, let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{t}\subset\mathfrak{g}$ be the Lie algebra of $T\subset G$. Introducing a G-invariant inner product (.,.) on $\mathfrak{g}$, we may decompose $\mathfrak{g}$ into the orthogonal direct sum

$$\mathfrak{g}=\mathfrak{t}\oplus\mathfrak{t}^{\perp}$$

of $\mathfrak{t}$ and its orthogonal complement $\mathfrak{t}^{\perp}$; this decomposition is $\mathrm{Ad}(T)$-invariant and does not depend on the choice of the inner product (.,.). Finally, let W_G be the Weyl group of G ($W_G = N_G(T)/T$ where $N_G(T)$ is the normalizer of T in G, defined by $N_G(T)=\{g\in G/gtg^{-1}\in T$ for all $t\in T\}$) and $|W_G|$ be its order. Then

$$\mathrm{Ad}(t)=1+\mathrm{Ad}^{\perp}(t) \quad \text{for } t\in T$$

and (see, e.g., Ref. 9, pp. 101–103)

$$M_{\pi}(z)=\frac{1}{|W_G|}\int_T d\mu_T(t)\,\frac{\det(1-\mathrm{Ad}^{\perp}(t))}{\det(1-z\pi(t))}, \tag{23}$$

$$F_{\pi}(z,\overline{z})=\frac{1}{|W_G|}\int_T d\mu_T(t)\,\frac{\det(1-\mathrm{Ad}^{\perp}(t))}{\det(1-z\pi(t))\det(1-\overline{z\pi(t)})}. \tag{24}$$

These integrals can be further evaluated in terms of the root system Δ of $\mathfrak{g}$ and the weight system Φ for the representation π. The procedure is standard when $\mathfrak{g}$ is semisimple, but to a certain degree it works just as well in the more general case when $\mathfrak{g}$ has a non-trivial center. The starting point is the fact that T being Abelian, the restriction from G to T of any unitary representation of G, such as the complexified adjoint representation Ad on the complexification $\mathfrak{g}^c$ of $\mathfrak{g}$ or the representation π on V, splits into the direct sum of irreducible one-dimensional representations. Grouping together all subspaces characterized by the same eigenvalues under all elements of T leads to the well-known root space decomposition

$$\mathfrak{g}^c=\mathfrak{t}^c\oplus\bigoplus_{\alpha\in\Delta}\mathfrak{g}_{\alpha}$$

of $\mathfrak{g}^c$ and to the weight space decomposition

$$V=\bigoplus_{\lambda\in\Phi}V_{\lambda}$$

of V; the dimension of the subspace V_λ is commonly known as the multiplicity $m(\lambda)$ of λ. The action of T on each of these subspaces is given by a character of T, i.e., a Lie group homomorphism from T to the unit circle, written as $t\to t^{\alpha}$ and $t\to t^{\lambda}$, respectively, according to

$$\mathrm{Ad}(t)(X)=t^{\alpha}X \quad \text{for } t\in T \text{ and } X\in\mathfrak{g}_{\alpha}, \tag{25}$$

and

$$\pi(t)(v)=t^{\lambda}v \quad \text{for } t\in T \text{ and } v\in V_{\lambda}, \tag{26}$$

respectively. At the Lie algebra level, Eqs. (25) and (26) imply the usual relations

$$\mathrm{ad}(H)(X)=\alpha(H)X \quad \text{for } H\in\mathfrak{t} \text{ and } X\in\mathfrak{g}_{\alpha}, \tag{27}$$

and

$$\dot{\pi}(H)(v)=\lambda(H)v \quad \text{for } H\in\mathfrak{t} \text{ and } v\in V_{\lambda}, \tag{28}$$

respectively. By complex conjugation of Eqs. (26) and (28), we obtain

$$\overline{\pi(t)}(v) = t^{-\lambda} v \quad \text{for } t \in T \text{ and } v \in \sigma V_\lambda, \tag{29}$$

$$\dot{\bar{\pi}}(H)(v) = -\lambda(H) v \quad \text{for } H \in \mathfrak{t} \text{ and } v \in \sigma V_\lambda. \tag{30}$$

We follow here the standard mathematical convention of considering roots α and weights λ as real linear forms on $\mathfrak{t}$, or as complex linear forms on the complexification $\mathfrak{t}^c$ of $\mathfrak{t}$, which—in accordance with the fact that $\mathfrak{g}$ is the compact real form of $\mathfrak{g}^c$—take purely imaginary values on $\mathfrak{t}$ (eigenvalues of antihermitean matrices are purely imaginary). Moreover, roots α and weights λ are transferred to generators H_α and H_λ using the isomorphism induced by the G-invariant nondegenerate complex bilinear form (.,.) on $\mathfrak{g}^c$ obtained from the G-invariant inner product (.,.) on $\mathfrak{g}$ by complex bilinear extension:

$$(H_\alpha, H) = \alpha(H), \quad (H_\lambda H) = \lambda(H) \quad \text{for all } H \in \mathfrak{t}^c.$$

This isomorphism is, by definition, an isometry:

$$(\alpha, \beta) = (H_\alpha, H_\beta), \quad (\lambda, \mu) = (H_\lambda, H_\mu).$$

The reality properties may then be summarized in the statement that roots α and weights λ belong to the real vector space $i\mathfrak{t}^*$, while the vectors H_α and H_λ belong to the real vector space $i\mathfrak{t}$. With this notation, we can rewrite the integrals in Eqs. (23) and (24) as follows:

$$M_\pi(z) = \frac{1}{|W_G|} \int_T d\mu_T(t) \, \frac{\Pi_{\alpha \in \Delta}(1 - t^\alpha)}{\Pi_{\lambda \in \Phi}(1 - z t^\lambda)^{m(\lambda)}}, \tag{31}$$

$$F_\pi(z, \overline{z}) = \frac{1}{|W_G|} \int_T d\mu_T(t) \, \frac{\Pi_{\alpha \in \Delta}(1 - t^\alpha)}{\Pi_{\lambda \in \Phi}(1 - z t^\lambda)^{m(\lambda)} (1 - \overline{z} \overline{t}^{-\lambda})^{m(\lambda)}}. \tag{32}$$

A further condition to be employed is that roots α and weights λ must be integral linear forms in the sense of taking values in $2\pi i \mathbb{Z}$ on the so-called unit lattice

$$L_1 = \{ H \in \mathfrak{t} / \exp(H) = 1 \} \tag{33}$$

of G: such integral linear forms are precisely the ones that arise as differentials of characters of T (Ref. 9, pp. 94–95). This lattice is the essential ingredient for understanding how to convert the integrals (31) and (32) into integrals over the product of r unit circles ($r = \text{rank } G = \dim T$), which can then be evaluated by an r-fold successive application of the residue theorem. Indeed, let us assume that $\{2\pi H_1, \ldots, 2\pi H_r\}$ is a basis of L_1 and define

$$\begin{aligned} h(\alpha) &= (h_1(\alpha), \ldots, h_r(\alpha)) = (-i\alpha(H_1), \ldots, -i\alpha(H_r)) \quad \text{for } \alpha \in \Delta, \\ h(\lambda) &= (h_1(\lambda), \ldots, h_r(\lambda)) = (-i\lambda(H_1), \ldots, -i\lambda(H_r)) \quad \text{for } \lambda \in \Phi. \end{aligned} \tag{34}$$

Then the map

$$\begin{aligned} S^1 \times \ldots \times S^1 &\to T, \\ w = (w_1, \ldots, w_r) = (e^{i\theta_1}, \ldots, e^{i\theta_r}) &\mapsto t = \exp\left(\sum_{j=1}^r \theta_j H_j \right) \end{aligned} \tag{35}$$

is an isomorphism of compact Abelian Lie groups such that, in multi-index notation (generalized to include negative integer powers),

$$t^{\alpha}=w^{h(\alpha)}=w_1^{h_1(\alpha)}\ldots w_r^{h_r(\alpha)} \quad \text{for } \alpha\in\Delta, \tag{36}$$

$$t^{\pm\lambda}=w^{\pm h(\lambda)}=w_1^{\pm h_1(\lambda)}\ldots w_r^{\pm h_r(\lambda)} \quad \text{for } \lambda\in\Phi.$$

Thus we obtain

Theorem 1: *Assume π is a finite-dimensional unitary representation of a compact connected Lie group G. Then in terms of the multi-index notation for roots and weights with respect to a basis of the unit lattice of G, as introduced in Eqs. (33–36), the generating functions M_π and F_π are given by*

$$M_\pi(z)=\frac{1}{|W_G|}\prod_{j=1}^{r}\oint\frac{dw_j}{2\pi i w_j}\frac{\Pi_{\alpha\in\Delta}(1-w^{h(\alpha)})}{\Pi_{\lambda\in\Phi}(1-zw^{h(\lambda)})^{m(\lambda)}}, \tag{37}$$

$$F_\pi(z,\overline{z})=\frac{1}{|W_G|}\prod_{j=1}^{r}\oint\frac{dw_j}{2\pi i w_j}\frac{\Pi_{\alpha\in\Delta}(1-w^{h(\alpha)})}{\Pi_{\lambda\in\Phi}(1-zw^{h(\lambda)})^{m(\lambda)}(1-\overline{zw}^{-h(\lambda)})^{m(\lambda)}}. \tag{38}$$

The expressions $w^{h(\alpha)}$ and $w^{\pm h(\lambda)}$ will often be abbreviated to w^{α} and $w^{\pm\lambda}$, respectively.

When G is semisimple, which is by far the most important case for applications, the exponents $h(\alpha)$ and $h(\lambda)$ are easily calculated from the root system Δ of $\mathfrak{g}$ and the weight system Φ for the representation π. To this end, it is convenient to introduce the following two lattices in $\mathfrak{t}$:

(a) the coroot lattice L_{cr}, which is dual to the standard weight lattice (Ref. 10, p. 67), in the sense that

$$L_{cr}=\{\check{\alpha}\in\mathfrak{t}/\lambda(\check{\alpha})\in 2\pi i\mathbb{Z} \quad \text{for all weights } \lambda\}, \tag{39}$$

and identical with the Δ-lattice which forms the translation part of the affine Weyl group (Ref. 11, p. 314) and generated by the vectors $4\pi i H_\alpha/(H_\alpha,H_\alpha)$ with $\alpha\in\Delta$ (Ref. 11, pp. 317–318),

(b) the coweight lattice L_{cw}, which is dual to the standard root lattice (Ref. 10, p. 67), in the sense that

$$L_{cw}=\{\check{\lambda}\in\mathfrak{t}/\alpha(\check{\lambda})\in 2\pi i\mathbb{Z} \quad \text{for all roots } \alpha\}, \tag{40}$$

and identical with the central lattice, defined as

$$L_c=\{H\in\mathfrak{t}/\exp(H)\in Z\}, \tag{41}$$

where Z is the center of G (Ref. 9, p. 95, Ref. 11, p. 311).

Obviously, the coroot lattice is contained in the coweight lattice, and the unit lattice lies in between:

$$L_{cr}\subset L_1\subset L_{cw}. \tag{42}$$

Note also that the unit lattice is sensitive to coverings, while the coroot lattice and the coweight lattice are not: they depend on the Lie group G only through its Lie algebra $\mathfrak{g}$. In fact, we have two extreme cases, between which the general case is intermediate:

(i) When G is simply connected, the unit lattice is minimal and coincides with the coroot lattice.
(ii) When G has trivial center, the unit lattice is maximal and coincides with the coweight lattice.

Therefore, when G is simply connected (which we may always assume to be the case, without loss of generality), one possible choice of the basis $\{2\pi H_1,\ldots,2\pi H_r\}$ is to set

$$H_j = \frac{2iH_{\alpha_j}}{(H_{\alpha_j},H_{\alpha_j})}, \tag{43}$$

where $\{\alpha_1,\ldots,\alpha_r\}$ is a basis of simple roots. In this case, the exponents $h(\alpha)$ and $h(\lambda)$ are precisely the coefficients in the expansion

$$\alpha = \sum_{j=1}^{r} h_j(\alpha)\lambda_j , \quad \lambda = \sum_{j=1}^{r} h_j(\lambda)\lambda_j \tag{44}$$

of a root $\alpha \in \Delta$ and of a weight $\lambda \in \Phi$ in terms of the basis $\{\lambda_1,\ldots,\lambda_r\}$ of fundamental weights, which is dual to the basis of simple roots in the usual sense:

$$\frac{2(\lambda_j , \alpha_k)}{(\alpha_k , \alpha_k)} = \delta_{jk} . \tag{45}$$

Observe that the relevant parts of this construction do not depend on the choice of the scalar product (.,.): a different choice would simply amount to a change of an overall normalization factor on each simple ideal which drops out of the definition of the generators appearing on the rhs of Eq. (43) or the definition (45) of the fundamental weights. When we want to be specific about normalization, we shall not use the Killing form, but rather the so-called standard form, which is normalized so that the long roots have length $\sqrt{2}$.

The additional assumption that G should be semisimple is less restrictive than it may seem. Indeed, when G is not semisimple, that is, has a non-discrete center Z, consider the orthogonal direct decomposition $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_s$ of $\mathfrak{g}$ into its center $\mathfrak{z}$ and its maximal semisimple ideal $\mathfrak{g}_s = [\mathfrak{g},\mathfrak{g}]$, together with the corresponding orthogonal direct decomposition $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}_s$ of the maximal Abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ into the center $\mathfrak{z}$ of $\mathfrak{g}$ and a maximal Abelian subalgebra $\mathfrak{t}_s$ of $\mathfrak{g}_s$. Then the roots α ($\alpha \in \Delta$) only generate the subspace $i\mathfrak{t}_s^*$ of $i\mathfrak{t}^*$ and the vectors H_α ($\alpha \in \Delta$) only generate the subspace $i\mathfrak{t}_s$ of $i\mathfrak{t}$, as real vector spaces. We can still define the unit lattice L_1 (cf. Eq. (33)) and introduce a basis $\{2\pi H_1,\ldots,2\pi H_r\}$ as before, as well as the exponents $h(\alpha)$ and $h(\lambda)$ (cf. Eq. (34)), but the unit lattice is now very flexible: any lattice in $\mathfrak{g}$ which contains the coroot lattice of $\mathfrak{g}_s$ and whose orthogonal projection to $\mathfrak{g}_s$ is contained in the coweight lattice of $\mathfrak{g}_s$ is admissible (Ref. 9, p. 97). Therefore, there is now no general way to proceed beyond Eqs. (37) and (38); their explicit evaluation must for each representation π be carried out separately.

A final important observation concerns the form of the generating functions M_π and F_π when π is the adjoint representation Ad. In this case, we may appeal to Chevalley's theorem, which provides a complete description of the ring of invariant polynomials on a semisimple Lie algebra $\mathfrak{g}$: it is freely generated by $r = \mathrm{rank}(\mathfrak{g})$ elementary polynomials $P_1,\ldots,P_r$, whose degrees $p_1,\ldots,p_r$ are commonly known as the exponents of $\mathfrak{g}$. Therefore, the usual Molien function for the adjoint representation reads

$$M_{\mathrm{Ad}}(z) = \prod_{j=1}^{r} \frac{1}{1-z^{p_j}} . \tag{46}$$

Explicitly, for the classical groups, the polynomials $P_1,\ldots,P_r$ can (with one exception) be written as trace polynomials in the defining representation:

(i) $A_r:\mathfrak{g}=\mathfrak{sl}(r+1)$ or $\mathfrak{su}(r+1)$ (compact real form):

$$P_j(X)=\mathrm{tr}(X^{j+1})\quad \text{for } X\in\mathfrak{g},\ \ j=1,...,r.$$

(ii) $B_r:\mathfrak{g}=\mathfrak{so}(2r+1,\mathbb{C})$ or $\mathfrak{so}(2r+1)$ (compact real form):

$$P_j(X)=\mathrm{tr}(X^{2j})\quad \text{for } X\in\mathfrak{g},\ \ j=1,...,r.$$

(iii) $C_r:\mathfrak{g}=\mathfrak{sp}(2r,\mathbb{C})$ or $\mathfrak{sp}(2r)$ (compact real form):

$$P_j(X)=\mathrm{tr}(X^{2j})\quad \text{for } X\in\mathfrak{g},\ \ j=1,...,r.$$

(iv) $D_r:\mathfrak{g}=\mathfrak{so}(2r,\mathbb{C})$ or $\mathfrak{so}(2r)$ (compact real form):

$$P_j(X)=\mathrm{tr}(X^{2j})\quad \text{for } X\in\mathfrak{g},\ \ j=1,...,r-1,$$

$$P_r(X)=Pf(X)\quad \text{for } X\in\mathfrak{g},$$

where $Pf(X)$ denotes the Pfaffian of X. See, for example, Ref. 12, pp. 253–263.

It should be pointed out that Chevalley's theorem refers to invariant complex polynomials on complex semisimple Lie algebras, or equivalently, to invariant real polynomials on real semisimple Lie algebras (including compact real forms), but not to invariant real polynomials on complex semisimple Lie algebras. There is thus no reason to believe that this polynomial ring has an equally simple structure. In fact, it does not. To show this, we have calculated the generating function F_{Ad} for the simple Lie algebra $B_2=C_2$: the result (cf. Eq. (125) in Sec. VII below) exhibits a complicated structure, with lots of generators and relations.

V. COMBINATORIAL FORMULAS

In the following, we shall derive combinatorial formulas which allow one to determine the coefficients c_k and $c_{p,q}$ of the generating functions M_π and F_π solely in term of the root system Δ and the weight system Φ of the representation π.

Our starting point will be the integral representations (37) of M_π and (38) of F_π. First of all we need the following

Proposition: Let $f_1,\ldots,f_n$ be polynomials of degree 1 in the variable x (with coefficients that are rational functions of other variables $w_1,\ldots,w_r$). Then for non-negative integers $m_1,\ldots,m_n$ and k,

$$\frac{1}{k!}\,\frac{\partial^k}{\partial x^k}\left(\prod_{i=1}^{n} f_i^{-m_i}\right)=(-1)^k\sum_{|l|=k}\prod_{i=1}^{n}\binom{m_i-1+l_i}{l_i}(f_i^{-m_i-l_i})\left(\frac{\partial f_i}{\partial x}\right)^{l_i}, \tag{47}$$

where we use multi-index notation, i.e., $l=(l_1,\ldots,l_n)$, where the l_i are non-negative integers, and $|l|=l_1+\ldots+l_n$.

Proof: For $k=1$, the above formula reduces to the statement that

$$\frac{\partial}{\partial x}\left(\prod_{i=1}^{n} f_i^{-m_i}\right)=-\sum_{j=1}^{n}\prod_{i=1}^{n}(f_i^{-m_i})\left(\frac{m_j}{f_j}\,\frac{\partial f_j}{\partial x}\right),$$

which is obvious. The general case is proved by induction on k, using the hypothesis that

$$\frac{\partial^2 f_i}{\partial x}=0,$$

as follows:

$$\frac{1}{(k+1)!}\frac{\partial^{k+1}}{\partial x^{k+1}}\left(\prod_{i=1}^{n} f_i^{-m_i}\right)=\frac{(-1)^{k+1}}{k+1}\sum_{|p|=k}\sum_{j=1}^{n}\left[\prod_{\substack{i=1\\ i\neq j}}^{n}\binom{m_i-1+p_i}{p_i}(f_i^{-m_i-p_i})\left(\frac{\partial f_i}{\partial x}\right)^{p_i}\right]$$

$$\times\binom{m_j-1+p_j}{p_j}(m_j+p_j)(f_j^{-m_j-p_j-1})\left(\frac{\partial f_j}{\partial x}\right)^{p_j+1}$$

$$=(-1)^{k+1}\sum_{|p|=k}\sum_{j=1}^{n}\left[\prod_{\substack{i=1\\ i\neq j}}^{n}\binom{m_i-1+l_i}{l_i}(f_i^{-m_i-l_i})\left(\frac{\partial f_i}{\partial x}\right)^{l_i}\right]$$

$$\times\binom{m_j-1+l_j}{l_j}\frac{l_j}{k+1}(f_j^{-m_j-l_j})\left(\frac{\partial f_j}{\partial x}\right)^{l_j},$$

where l is defined in terms of p and j by putting $l_i=p_i$ for $i\neq j$ and $l_j=p_j+1$. Converting the double sum over p and j to a single sum over l yields the desired result.

As a result, we can explicitly differentiate the integrands of Eqs. (37) and (38):

$$\frac{1}{k!}\frac{\partial^k}{\partial z^k}\frac{1}{\Pi_{\lambda\in\Phi}(1-zw^{\lambda})^{m(\lambda)}}\bigg|_{z=0}=\sum_{|l|=k}N(l,m)w^{\Lambda(l)},$$

$$\frac{1}{p!}\frac{1}{q!}\frac{\partial^p}{\partial z^p}\frac{\partial^q}{\partial \bar{z}^q}\frac{1}{\Pi_{\lambda\in\Phi}(1-zw^{\lambda})^{m(\lambda)}(1-\bar{z}\bar{w}^{-\lambda})^{m(\lambda)}}\bigg|_{z=0,\bar{z}=0}=\sum_{|r|=p,|s|=q}N(r,m)N(s,m)w^{\Lambda(r-s)},$$

where

$$N(l,m)=\prod_{\lambda\in\Phi}\binom{m(\lambda)+l(\lambda)-1}{l(\lambda)},$$

and

$$\Lambda(l)=\sum_{\lambda\in\Phi}l(\lambda)\lambda.$$

In order to carry out the residue integrals in Eqs. (37) and (38), we must also expand the numerator in powers of w. The net result is most conveniently formulated in terms of the following concepts:

Definition 1: The extended root system $\tilde{\Delta}$ associated with a given root system Δ is the set of all linear combinations

$$\tilde{\alpha}=\sum_{\alpha\in\Delta}a(\alpha)\alpha \tag{48}$$

of roots with coefficients $a(\alpha)$ which are either 0 or 1. (Thus $\tilde{\Delta}$ is a (finite) subset of the root lattice generated by Δ.) To any such extended root $\tilde{\alpha}$, we associate its decomposition index $i(\tilde{\alpha})$, defined as the difference

$$i(\tilde{\alpha})=n_+(\tilde{\alpha})-n_-(\tilde{\alpha}), \tag{49}$$

between the number $n_+(\tilde{\alpha})$ of such decompositions of $\tilde{\alpha}$ into a sum of roots with an even number of nonzero coefficients and the number $n_-(\tilde{\alpha})$ of such decompositions of $\tilde{\alpha}$ into a sum of roots with an odd number of nonzero coefficients. In other words,

$$i(\tilde{\alpha}) = \sum_a (-1)^{|a|}, \tag{50}$$

where the sum is over all sequences $(a(\alpha))_{\alpha \in \Delta}$ of coefficients $a(\alpha) \in \{0,1\}$ satisfying Eq. (48), and

$$|a| \equiv \sum_{\alpha \in \Delta} a(\alpha). \tag{51}$$

We extend the definition of the decomposition index to the whole root lattice by setting

$$i(\tilde{\alpha}) = 0 \quad \text{if } \tilde{\alpha} \notin \tilde{\Delta}. \tag{52}$$

In these terms, we have

$$\prod_{\alpha \in \Delta} (1 - w^{\alpha}) = \sum_{\tilde{\alpha} \in \tilde{\Delta}} i(\tilde{\alpha}) w^{\tilde{\alpha}}. \tag{53}$$

Note that just like the usual root system, the extended root system is invariant under the action of the corresponding Weyl group W, and so is the decomposition index (it is constant along Weyl group orbits). Two particular values that can be computed immediately are

$$i(2\rho) = (-1)^{|\Delta|/2}, \quad i(0) = |W|, \tag{54}$$

where 2ρ is the vector obtained as the sum of all positive roots. (The first formula follows by observing that in this case there is only one possible sequence, namely (1,...,1), which has the parity stated above, while the second formula follows by combining the previous formula at $w=0$ with the fact that $M_{\pi}(0)=1$.) For the simplest rank 1 algebra A_1, for example, we have $\Delta = \{\alpha, -\alpha\}$ and $\tilde{\Delta} = \{\alpha, 0, -\alpha\}$ with $i(\alpha) = i(-\alpha) = -1$ and $i(0) = 2$. The result for the simple rank 2 algebras A_2, $B_2 = C_2$ and G_2 is shown in Figures 1–3.

Definition 2: For any positive integer k, the k-extended weight system $\tilde{\Phi}(k)$ associated with a given weight system Φ is the set of all linear combinations

$$\tilde{\lambda} = \sum_{\lambda \in \Phi} l(\lambda) \lambda \tag{55}$$

of weights with coefficients $l(\lambda)$ which are non-negative integers, such that

$$|l| \equiv \sum_{\lambda \in \Phi} l(\lambda) = k. \tag{56}$$

(Thus $\tilde{\Phi}(k)$ is a (finite) subset of the weight lattice.) To any such k-extended weight $\tilde{\lambda}$, we associate its k-extended multiplicity $m_k(\tilde{\lambda})$, defined as the sum of the combinatorial coefficients

$$N(l,m) = \prod_{\lambda \in \Phi} \binom{m(\lambda) - 1 + l(\lambda)}{l(\lambda)} = \prod_{\lambda \in \Phi} \binom{m(\lambda) - 1 + l(\lambda)}{m(\lambda) - 1} \tag{57}$$

over all such representations of $\tilde{\lambda}$. In other words,

$$m_k(\tilde{\lambda}) = \sum_l N(l,m), \tag{58}$$

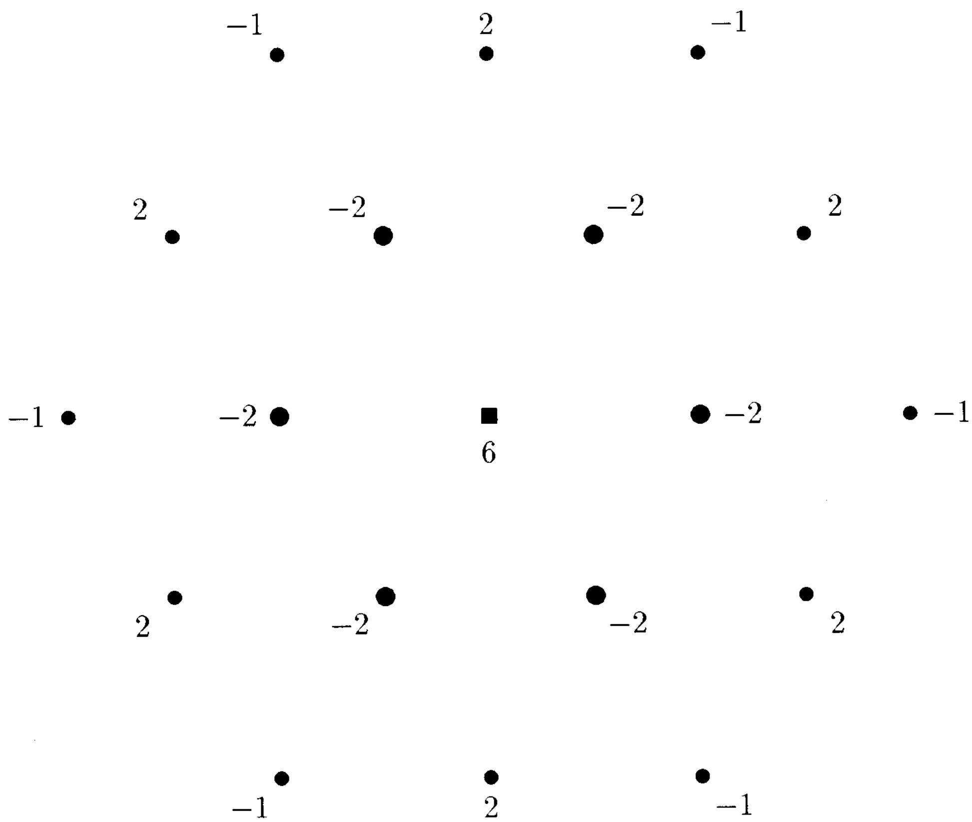

FIG. 1. Root system and extended root system, with decompostion indices: A_2.

where the sum is over all sequences $(l(\lambda))_{\lambda\in\Phi}$ of coefficients $l(\lambda)\in\mathbb{N}$ satisfying Eqs. (55) and (56).

Note that just like the usual weight system and the usual multiplicity, the k-extended weight system and the k-extended multiplicity are invariant under the action of the corresponding Weyl group W, and that $\tilde{\Phi}(1)=\Phi$, $m_1(\lambda)=m(\lambda)$.

Now we are ready to formulate the main result of this section:

Theorem 2: *Assume π is a finite-dimensional unitary representation of a compact connected Lie group G. Then in terms of the multi-index notation for roots and weights with respect to a basis of the unit lattice of G, as introduced in Eqs. (33–36), and with the notation introduced above, the number $c_k(\pi)$ of (linearly independent) G-invariant complex polynomials of degree k and the number $c_{p,q}(\pi)$ of (linearly independent) G-invariant real polynomials of bidegree (p,q) on the carrier space of π are given by the combinatorial formulas*

$$c_k(\pi)=\frac{1}{|W_G|}\sum_{\tilde{\lambda}\in\tilde{\Phi}(k)} i(\tilde{\lambda})m_k(\tilde{\lambda}), \tag{59}$$

and

$$c_{p,q}(\pi)=\frac{1}{|W_G|}\sum_{\tilde{\lambda}\in\tilde{\Phi}(p),\tilde{\mu}\in\tilde{\Phi}(q)} i(\tilde{\lambda}-\tilde{\mu})m_p(\tilde{\lambda})m_q(\tilde{\mu}), \tag{60}$$

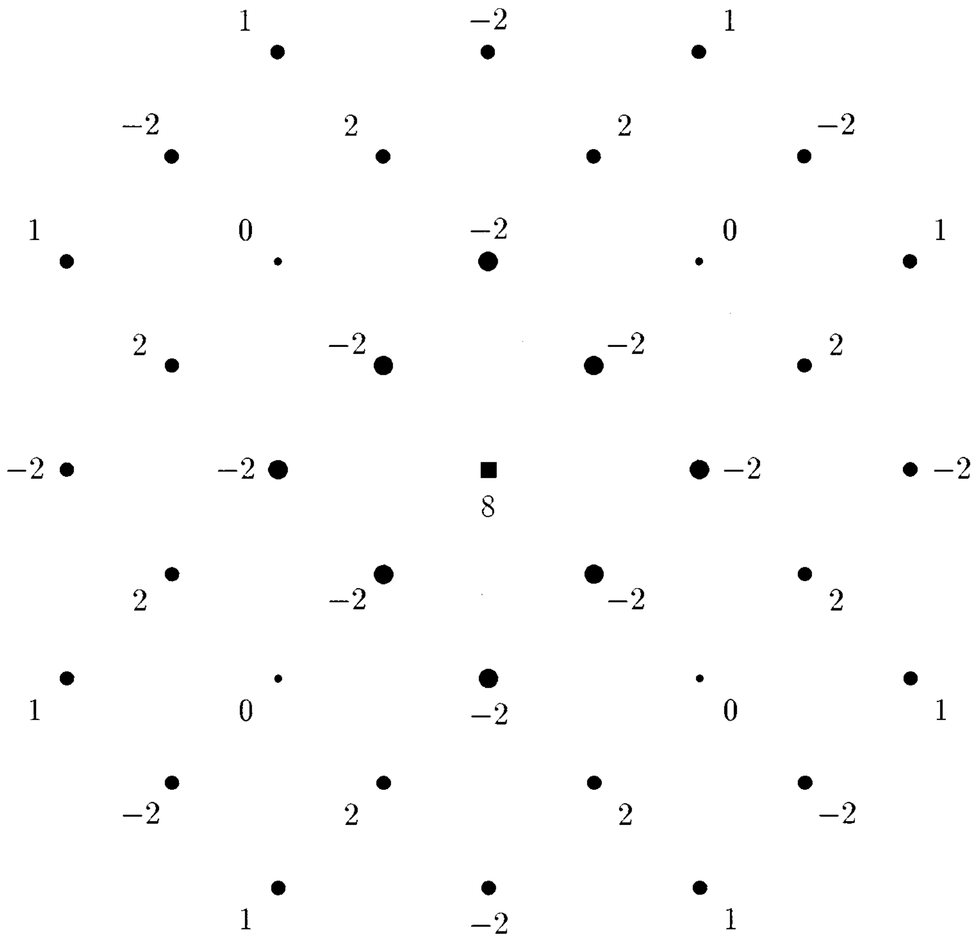

FIG. 2. Root system and extended root system, with decomposition indices: C_2.

respectively. (Note that the terms on the rhs of these equations yield non-vanishing contributions only when $\tilde{\lambda}$ and $\tilde{\lambda}-\tilde{\mu}$, respectively, belongs to the extended root system $\tilde{\Delta}$.)

We believe that on the basis of this theorem, it should be possible to develop a computer program for calculating the numbers c_k and $c_{p,q}$, up to reasonably high orders, for arbitrary groups and representations. The amount of computing time can be reduced by a factor of the order of $|W_G|$ by an appropriate implementation of the Weyl group symmetry.

VI. EXAMPLE: SU(2)

As a first example, let us apply Eqs. (37) and (38) to the case where $G=\mathrm{SU}(2)$, with maximal torus $T=\mathrm{U}(1)$ and $\pi=\pi_s$ the irreducible spin s representation. Then $\mathfrak{g}=\mathfrak{su}(2)$ is the Lie algebra of complex traceless antihermitean matrices and $\mathfrak{t}=\mathfrak{u}(1)$ the maximal Abelian subalgebra of imaginary traceless diagonal matrices, with invariant scalar product (.,.) given by

$$(X,Y)=\mathrm{tr}(XY) \quad \text{for } X,Y\in\mathfrak{su}(2). \tag{61}$$

Using the Pauli matrices

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{62}$$

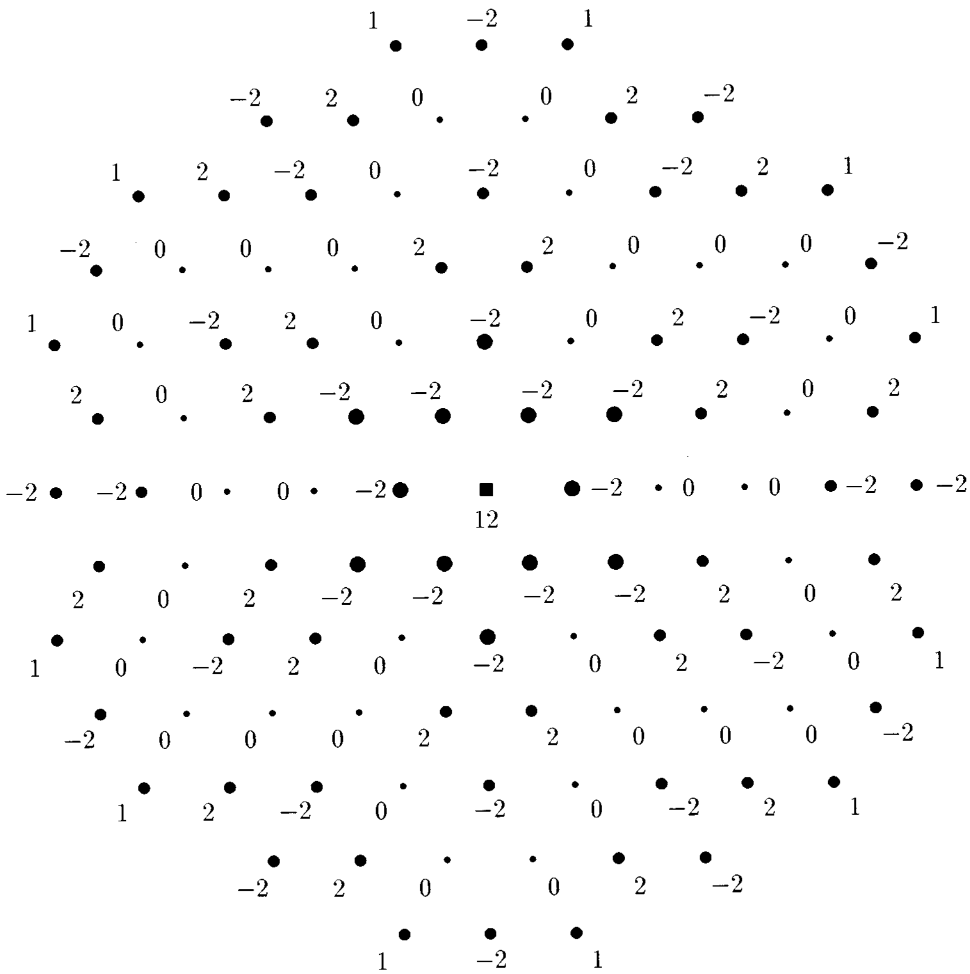

FIG. 3. Root system and extended root system, with decomposition indices: G_2.

which satisfy

$$\sigma_j \sigma_k = \delta_{jk} + i \epsilon_{jkl} \sigma_l , \tag{63}$$

we see that the nonzero roots are $\pm \alpha$, where

$$\alpha(H) = H_{11} - H_{22} = \mathrm{tr}(H \sigma_3) \quad \text{for } H \in \mathfrak{u}(1), \tag{64}$$

with root vectors

$$E_{\pm \alpha} = \sigma_1 \pm i \sigma_2 , \tag{65}$$

implying

$$H_\alpha = \sigma_3 , \tag{66}$$

which does have length $\sqrt{2}$, in accordance with our previous convention. The fundamental weight is given by

$$\lambda(H)=\tfrac{1}{2}(H_{11}-H_{22})=\tfrac{1}{2}\operatorname{tr}(H\sigma_3) \quad \text{for } H\in\mathfrak{u}(1), \tag{67}$$

so

$$H_\lambda=\tfrac{1}{2}\,\sigma_3. \tag{68}$$

Note that the root lattice is generated by α and the weight lattice is generated by λ, whereas the coroot lattice (which coincides with the unit lattice for SU(2) since SU(2) is simply connected) is generated by $2\pi H_\alpha$ and the coweight lattice (which coincides with the unit lattice for $\mathrm{SO}(3)=\mathrm{SU}(2)/\mathbb{Z}_2$ since $\mathbb{Z}_2$ is the center of SU(2)) is generated by $2\pi H_\lambda$. The highest weight of the irreducible spin s representation is precisely $2s\lambda$, and its complete weight system consists of the multiples $2m\lambda$ with m taking all integer (half-integer) values between $-s$ and s (inclusive) when s is integer (half-integer). Therefore, Eqs. (37) and (38) yield for this case

$$M_s(z)=\frac{1}{4\pi i}\oint\frac{dw}{w}\,\frac{(1-w^2)(1-w^{-2})}{\prod_{m=-s}^{s}(1-zw^{2m})}, \tag{69}$$

$$F_s(z,\overline{z})=\frac{1}{4\pi i}\oint\frac{dw}{w}\,\frac{(1-w^2)(1-w^{-2})}{\prod_{m=-s}^{s}(1-zw^{2m})(1-\overline{z}\overline{w}^{-2m})}. \tag{70}$$

(We have abbreviated M_{π_s} to M_s and F_{π_s} to F_s.) See Ref. 6, p. 94, noting that the formula given there is correct only for integer spin, in which case it can be deduced from Eq. (69) by the variable transformation $w=\exp(i\theta/2)$, together with the observation that the resulting integral from 0 to 4π may actually be reduced to an integral from 0 to π because the argument is a periodic function of θ with period 2π and, in addition, invariant under the reflection $\theta\to-\theta$.

For integer spin s, these equations are equivalent to

$$M_s(z)=-\frac{1}{2(1-z)}\,\frac{1}{2\pi i}\oint dw\,\frac{w^{s(s+1)-3}(1-w^2)^2}{\prod_{k=1}^{s}(1-zw^{2k})(w^{2k}-z)}, \tag{71}$$

$$F_s(z,\overline{z})=-\frac{1}{2(1-z)(1-\overline{z})}\,\frac{1}{2\pi i}\oint dw\,\frac{w^{2s(s+1)-3}(1-w^2)^2}{\prod_{k=1}^{s}(1-zw^{2k})(1-\overline{z}\overline{w}^{2k})(w^{2k}-z)(w^{2k}-\overline{z})}, \tag{72}$$

while for half-integer spin s, they are equivalent to

$$M_s(z)=-\frac{1}{2}\,\frac{1}{2\pi i}\oint dw\,\frac{w^{(s+1/2)^2-3}(1-w^2)^2}{\prod_{k=1}^{s+1/2}(1-zw^{2k-1})(w^{2k-1}-z)}, \tag{73}$$

$$F_s(z,\overline{z})=-\frac{1}{2}\,\frac{1}{2\pi i}\oint dw\,\frac{w^{2(s+1/2)^2-3}(1-w^2)^2}{\prod_{k=1}^{s+1/2}(1-zw^{2k-1})(1-\overline{z}\overline{w}^{2k-1})(w^{2k-1}-z)(w^{2k-1}-\overline{z})}. \tag{74}$$

For integer spin, however, it is more convenient to work with the complex variable $u=w^2$; then Eqs. (69), (70) and (71), (72) become

$$M_s(z)=\frac{1}{4\pi i}\oint\frac{du}{u}\,\frac{(1-u)(1-u^{-1})}{\prod_{m=-s}^{s}(1-zu^m)}, \tag{75}$$

$$F_s(z,\overline{z})=\frac{1}{4\pi i}\oint\frac{du}{u}\,\frac{(1-u)(1-u^{-1})}{\prod_{m=-s}^{s}(1-zu^m)(1-\overline{z}\overline{u}^{-m})}, \tag{76}$$

and

$$M_s(z) = -\frac{1}{2(1-z)}\frac{1}{2\pi i}\oint du\,\frac{u^{s(s+1)/2-2}(1-u)^2}{\Pi_{k=1}^{s}(1-zu^k)(u^k-z)}, \tag{77}$$

$$F_s(z,\overline{z}) = -\frac{1}{2(1-z)(1-\overline{z})}\frac{1}{2\pi i}\oint du\,\frac{u^{s(s+1)-2}(1-u)^2}{\Pi_{k=1}^{s}(1-zu^k)(1-\overline{zu}^k)(u^k-z)(u^k-\overline{z})}, \tag{78}$$

respectively. (Note that in the last four equations, a factor of 2 has disappeared because we must take into account that $u=w^2$ winds twice around the unit circle when w winds around once.)

We now proceed to calculate the generating functions M_s and F_s for a few irreducible representations of low spin, by applying the residue theorem (and remembering that $|z|<1$).

A. Spin 0

For the trivial representation, the integrands in Eqs. (77), (78) both have a double pole at $u=0$, and

$$M_0(z) = -\frac{1}{2(1-z)}\frac{1}{2\pi i}\oint du\,\frac{1-2u+u^2}{u^2},$$

$$F_0(z,\overline{z}) = -\frac{1}{2(1-z)(1-\overline{z})}\frac{1}{2\pi i}\oint du\,\frac{1-2u+u^2}{u^2},$$

that is,

$$M_0(z) = \frac{1}{1-z}, \tag{79}$$

$$F_0(z,\overline{z}) = \frac{1}{(1-z)(1-\overline{z})}. \tag{80}$$

This corresponds to the fact that for the one-dimensional trivial representation, every polynomial is invariant, and the ring of all (complex and real, respectively) polynomials in one variable is generated by the linear monomial(s) ζ and, ζ, $\overline{\zeta}$, respectively.

B. Spin 1/2

For spin 1/2, the integrand in Eq. (73) has a double pole at $w=0$, the coefficient of w in the Taylor expansion of the remaining factor around this pole being

$$\begin{aligned}\frac{d}{dw}\frac{(1-w^2)^2}{(1-zw)(w-z)}\bigg|_{w=0} &= \frac{(1-zw)(w-z)2(1-w^2)(-2w)-(1-w^2)^2(-z(w-z)+(1-zw))}{(1-zw)^2(w-z)^2}\bigg|_{w=0}\\ &= -\frac{1+z^2}{z^2},\end{aligned}$$

and a simple pole at $w=z$, whereas the integrand in Eq. (74) has simple poles at $w=0$, $w=z$ and $w=\overline{z}$, so

$$M_{1/2}(z) = -\frac{1}{2}\frac{1}{2\pi i}\oint dw\,\frac{(1-w^2)^2}{w^2(1-zw)(w-z)}$$
$$= -\frac{1}{2}\left\{-\frac{1+z^2}{z^2}+\frac{(1-z^2)^2}{z^2(1-z^2)}\right\} = \frac{1}{2}\left\{\frac{1+z^2}{z^2}-\frac{1-z^2}{z^2}\right\},$$

whereas

$$F_{1/2}(z,\bar{z}) = -\frac{1}{2}\frac{1}{2\pi i}\oint dw\,\frac{(1-w^2)^2}{w(1-zw)(1-\bar{z}w)(w-z)(w-\bar{z})}$$
$$= -\frac{1}{2}\left\{\frac{1}{z\bar{z}}+\frac{(1-z^2)^2}{z(1-z^2)(1-z\bar{z})(z-\bar{z})}+\frac{(1-\bar{z}^2)^2}{\bar{z}(1-z\bar{z})(1-\bar{z}^2)(\bar{z}-z)}\right\}$$
$$= -\frac{1}{2z\bar{z}(1-z\bar{z})(z-\bar{z})}\{(1-z\bar{z})(z-\bar{z})+\bar{z}(1-z^2)-z(1-\bar{z}^2)\},$$

that is

$$M_{1/2}(z)=1, \tag{81}$$

$$F_{1/2}(z,\bar{z}) = \frac{1}{1-z\bar{z}}. \tag{82}$$

This confirms the idea that for the two-dimensional spinor representation, there are no invariant complex polynomials except 1—in accordance with the fact that the jth symmetric tensor power of this fundamental representation is just the irreducible representation of spin $j/2$ and therefore cannot contain the trivial representation as a subrepresentation, except when $j=0$—whereas the ring of invariant real polynomials is freely generated by the quadratic form $\bar{\zeta}\cdot\zeta$, which is nothing but the invariant scalar product used in the definition of the group SU(2).

C. Spin 1

For the vector representation, the integrand in Eq. (77) has simple poles at $u=0$ and at $u=z$, whereas the integrand in Eq. (78) has simple poles at $u=z$ and $u=\bar{z}$, so

$$M_1(z) = -\frac{1}{2(1-z)}\frac{1}{2\pi i}\oint du\,\frac{(1-u)^2}{u(1-zu)(u-z)} = \frac{1}{2(1-z)}\left\{\frac{1}{z}-\frac{(1-z)^2}{z(1-z^2)}\right\},$$

whereas

$$F_1(z,\bar{z}) = -\frac{1}{2(1-z)(1-\bar{z})}\frac{1}{2\pi i}\oint du\,\frac{(1-u)^2}{(1-zu)(1-\bar{z}u)(u-z)(u-\bar{z})}$$
$$= -\frac{1}{2(1-z)(1-\bar{z})}\left\{\frac{(1-z)^2}{(1-z^2)(1-z\bar{z})(z-\bar{z})}+\frac{(1-\bar{z})^2}{(1-z\bar{z})(1-\bar{z}^2)(\bar{z}-z)}\right\}$$
$$= -\frac{1}{2(1-z)(1-\bar{z})(1-z\bar{z})(z-\bar{z})}\left\{\frac{1-z}{1+z}-\frac{1-\bar{z}}{1+\bar{z}}\right\},$$

that is,

$$M_1(z) = \frac{1}{1-z^2}, \tag{83}$$

$$F_1(z,\bar{z}) = \frac{1}{(1-z^2)(1-z\bar{z})(1-\bar{z}^2)}. \tag{84}$$

This confirms the idea that for the three-dimensional vector representation, the ring of invariant (complex and real, respectively) polynomials is freely generated by the quadratic forms $\boldsymbol{\zeta}^2$ and $\boldsymbol{\zeta}^2$, $\boldsymbol{\zeta}\cdot\bar{\boldsymbol{\zeta}}$, $\bar{\boldsymbol{\zeta}}^2$, respectively. Alternatively, representing three-dimensional vectors as anti-symmetric (3×3)-matrices A, these generators may be written as $\mathrm{tr}(A^2)$ and $\mathrm{tr}(A^2)$, $\mathrm{tr}(A\bar{A})$, $\mathrm{tr}(\bar{A}^2)$, respectively.

For higher spin, these calculations become increasingly cumbersome because, according to Eqs. (77), (78) and (73), (74), the integrands have poles at $|z|^{1/p}$ times the pth roots of unity, for all integers p from 1 to s if s is integer and for all odd integers p from 1 to $2s$ if s is half-integer. They can, however, be simplified by combining a decomposition of the integrand into partial fractions with the fact that the residue integral with a single factor in the denominator can be easily evaluated, even when the numerator is a complicated polynomial in the integration variable, without having to sum over roots of unity. In fact, we may use the following elementary

Proposition: Let P be a polynomial in w with coefficients that are rational functions of z (and possibly of other variables $z_1,\ldots,z_r$):

$$P(z_1,\ldots,z_r,z,w) = \sum_{n=1}^{N} a_n(z_1,\ldots,z_r,z)\,w^{n-1}. \tag{85}$$

For any integer $k \geqslant 1$, let M be the largest integer such that $kM \leqslant N$, and define

$$Q_k(z_1,\ldots,z_r,z) = \sum_{m=1}^{M} a_{km}(z_1,\ldots,z_r,z)\,z^{m-1}. \tag{86}$$

Then

$$\oint \frac{dw}{2\pi i}\,\frac{P(z_1,\ldots,z_r,z,w)}{w^k - z} = Q_k(z_1,\ldots,z_r,z). \tag{87}$$

Proof: Let a be any kth root of z and ϵ be any primitive kth root of unity, e.g., $\epsilon = \exp(2\pi i/k)$. Then

$$\sum_{l=0}^{k-1} x^l = \frac{x^k - 1}{x-1} = \prod_{j=1}^{k-1} (x - \epsilon^j),$$

so taking the limit $x \to 1$, we obtain

$$\prod_{j=1}^{k-1} (1 - \epsilon^j) = k,$$

while putting $x = \epsilon$ gives

$$\sum_{l=0}^{k-1} \epsilon^l = 0.$$

More generally, we have

$$\sum_{l=0}^{k-1} \epsilon^{ln} = 0$$

if k and n are relatively prime, because in this case, ln mod k will assume every value between 0 and $k-1$ exactly once when l ranges from 0 to $k-1$. Still more generally, we have

$$\sum_{l=0}^{k-1} \epsilon^{ln} = \begin{cases} 0 & \text{if } n \text{ is not a multiple of } k \\ k & \text{if } n \text{ is a multiple of } k \end{cases},$$

because denoting by m the greatest common divisor of k and n, $p=k/m$ and $r=n/m$ are relatively prime while ϵ^m will be a primitive pth root of unity, so decomposing the summation variable l according to $l=pj+i$ and using the previous equation,

$$\sum_{l=0}^{k-1} \epsilon^{ln} = \sum_{l=0}^{k-1} (\epsilon^m)^{lr} = \sum_{j=0}^{m-1} \sum_{i=0}^{p-1} (\epsilon^m)^{ir} = m \sum_{i=0}^{p-1} (\epsilon^m)^{ir} = \begin{cases} 0 & \text{if } m \neq k \\ k & \text{if } m = k. \end{cases}$$

Now we are ready to prove the proposition (for simplicity, we suppress the variables $z_1,\ldots,z_r$):

$$\oint \frac{dw}{2\pi i} \frac{P(z,w)}{w^k - z} = \oint \frac{dw}{2\pi i} \frac{P(z,w)}{\prod_{m=0}^{k-1}(w-\epsilon^m a)} = \sum_{l=0}^{k-1} \left. \frac{P(z,w)}{\prod_{\substack{m=0 \\ m\neq l}}^{k-1}(w-\epsilon^m a)} \right|_{w=\epsilon^l a}$$

$$= \sum_{l=0}^{k-1} \frac{P(z,\epsilon^l a)}{\prod_{\substack{m=0 \\ m\neq l}}^{k-1}(\epsilon^l a - \epsilon^m a)} = \sum_{l=0}^{k-1} \frac{P(z,\epsilon^l a)}{(\epsilon^l a)^{k-1} \prod_{j=1}^{k-1}(1-\epsilon^j)} = \frac{1}{kz} \sum_{l=0}^{k-1} \epsilon^l a P(z,\epsilon^l a)$$

$$= \frac{1}{kz} \sum_{n=1}^{N} \sum_{l=0}^{k-1} a_n(z) \epsilon^{ln} a^n = \frac{1}{z} \sum_{\substack{n=1 \\ n \text{ multiple of } k}}^{N} a_n(z) z^{n/k} = Q_k(z).$$

This proposition is combined with partial fraction decompositions of the form

$$\prod_{k=1}^{s} \frac{1}{(w^k - z)(1 - zw^k)} = \sum_{k=1}^{s} \left(\frac{a_k(z,w)}{w^k - z} + \frac{b_k(z,w)}{1 - zw^k} \right),$$

$$\prod_{k=1}^{s} \frac{1}{(w^k - z)(w^k - \overline{z})(1 - zw^k)(1 - \overline{z}\overline{w}^k)}$$

$$= \sum_{k=1}^{s} \left(\frac{a_k(\overline{z},z,w)}{w^k - z} + \frac{a_k(z,\overline{z},\overline{w})}{w^k - \overline{z}} + \frac{b_k(\overline{z},z,w)}{1 - zw^k} + \frac{b_k(z,\overline{z},\overline{w})}{1 - \overline{z}\overline{w}^k} \right),$$

for integer spin s, where the a_k and b_k are polynomials in w of degree strictly less than k whose coefficients are rational functions of z and of z, $\overline{z}$, respectively, and of the form

$$\prod_{k=1}^{s+1/2} \frac{1}{(w^{2k-1} - z)(1 - zw^{2k-1})} = \sum_{k=1}^{s+1/2} \left(\frac{a_k(z,w)}{w^{2k-1} - z} + \frac{b_k(z,w)}{1 - zw^{2k-1}} \right),$$

$$\prod_{k=1}^{s+1/2} \frac{1}{(w^{2k-1} - z)(w^{2k-1} - \overline{z})(1 - zw^{2k-1})(1 - \overline{z}\overline{w}^{2k-1})}$$

$$= \sum_{k=1}^{s+1/2} \left(\frac{a_k(\overline{z},z,w)}{w^{2k-1} - z} + \frac{a_k(z,\overline{z},\overline{w})}{w^{2k-1} - \overline{z}} + \frac{b_k(\overline{z},z,w)}{1 - zw^{2k-1}} + \frac{b_k(z,\overline{z},\overline{w})}{1 - \overline{z}\overline{w}^{2k-1}} \right),$$

for half-integer spin s, where the a_k and b_k are polynomials in w of degree strictly less than $2k-1$ whose coefficients are rational functions of z and of z, $\overline{z}$, respectively; these functions and coefficients can be calculated recursively, by induction on the number of factors (that is, induction on s) and repeated application of the Euclid algorithm. Using these two techniques, we have developed, within MAPLE, a program which allows to calculate the function F up to spin 2 and the function M up to spin 4 within reasonable limits of a few minutes of computing time on a standard 386 PC, with the following results:

Spin 3/2:

$$M_{3/2}(z)=\frac{1}{1-z^4}, \tag{88}$$

$$F_{3/2}(z,\bar{z})=\frac{(1-z^4\bar{z}^4)(1-z^6\bar{z}^6)}{(1-z\bar{z})(1-z^4)(1-z^3\bar{z})(1-z\bar{z}^3)(1-\bar{z}^4)(1-z^3\bar{z}^3)}. \tag{89}$$

Spin 2:

$$M_2(z)=\frac{1}{(1-z^2)(1-z^3)}, \tag{90}$$

$$F_2(z,\bar{z})=\frac{1-z^6\bar{z}^6}{(1-z^2)(1-z\bar{z})(1-\bar{z}^2)(1-z^3)(1-z^2\bar{z})(1-z\bar{z}^2)(1-\bar{z}^3)(1-z^2\bar{z}^2)}. \tag{91}$$

Spin 5/2:

$$M_{5/2}(z)=\frac{1-z^{36}}{(1-z^4)(1-z^8)(1-z^{12})(1-z^{18})}. \tag{92}$$

Spin 3:

$$M_3(z)=\frac{1-z^{30}}{(1-z^2)(1-z^4)(1-z^6)(1-z^{10})(1-z^{15})}. \tag{93}$$

Spin 7/2:

$$\begin{aligned}\text{numer}(M_{7/2}(z))=1&+2z^8+4z^{12}+4z^{14}+5z^{16}+9z^{18}+6z^{20}+9z^{22}+8z^{24}+9z^{26}+6z^{28}+9z^{30}\\&+5z^{32}+4z^{34}+4z^{36}+2z^{40}+z^{48},\end{aligned} \tag{94}$$

$$\text{denom}(M_{7/2}(z))=(1-z^4)(1-z^8)(1-z^{12})^2(1-z^{20}).$$

Spin 4:

$$M_4(z)=\frac{1+z^8+z^9+z^{10}+z^{18}}{(1-z^2)(1-z^3)(1-z^4)(1-z^5)(1-z^6)(1-z^7)}. \tag{95}$$

These formulas lead to several interesting observations.

To begin with, let us comment on the result for the five-dimensional spin 2 representation, which can be interpreted most conveniently by realizing five-dimensional vectors as traceless symmetric (3×3)-matrices A. Equation (90) confirms the idea that the ring of invariant complex polynomials in this representation is freely generated by the quadratic form $\mathrm{tr}(A^2)$ and the cubic form $\mathrm{tr}(A^3)$, whereas Eq. (91) states that the ring of invariant real polynomials in this representation is generated by three quadratic forms of bidegree (2,0), (1,1) and (0,2), respectively, together with four cubic forms of bidegree (3,0), (2,1), (1,2) and (0,3), respectively, plus an extra quartic form of bidegree (2,2), and that these generators should satisfy a relation of bidegree (6,6).

Obviously, the quadratic forms are given by $\mathrm{tr}(A^2)$, $\mathrm{tr}(A\overline{A})$ and $\mathrm{tr}(\overline{A^2})$, respectively, and the cubic forms by $\mathrm{tr}(A^3)$, $\mathrm{tr}(A^2\overline{A})$, $\mathrm{tr}(A\overline{A^2})$ and $\mathrm{tr}(\overline{A^3})$, respectively, but the extra generator of bidegree (2,2) comes as a surprise, especially in view of the fact that the usual Molien function provides no hint toward its existence. To see why there should be such an extra generator, note first that the natural invariant quartic forms $\mathrm{tr}(A^4)$ of bidegree (4,0), $\mathrm{tr}(A^3\overline{A})$ of bidegree (3,1), $\mathrm{tr}(A\overline{A^3})$ of bidegree (1,3) and $\mathrm{tr}(\overline{A^4})$ of bidegree (0,4) are not independent, but can be expressed in terms of the natural invariant quadratic forms $\mathrm{tr}(A^2)$, $\mathrm{tr}(A\overline{A})$ and $\mathrm{tr}(\overline{A^2})$:

$$\mathrm{tr}(A^4) = \tfrac{1}{2}\,(\mathrm{tr}(A^2))^2,$$

$$\mathrm{tr}(A^3\overline{A}) = \tfrac{1}{2}\,\mathrm{tr}(A^2)\mathrm{tr}(A\overline{A}),$$

$$\mathrm{tr}(A\overline{A^3}) = \tfrac{1}{2}\,\mathrm{tr}(A\overline{A})\mathrm{tr}(\overline{A^2}),$$

$$\mathrm{tr}(\overline{A^4}) = \tfrac{1}{2}\,(\mathrm{tr}(\overline{A^2}))^2.$$

On the other hand, there are four natural invariant quartic forms of bidegree (2,2), namely

$$\mathrm{tr}(A^2\overline{A^2}),\quad \mathrm{tr}((A\overline{A})^2),\quad \mathrm{tr}(A^2)\mathrm{tr}(\overline{A^2}),\quad \mathrm{tr}(A\overline{A})^2,$$

between which there exists precisely one linear relation:

$$4\,\mathrm{tr}(A^2\overline{A^2}) + 2\,\mathrm{tr}((A\overline{A})^2) = \mathrm{tr}(A^2)\mathrm{tr}(\overline{A^2}) - 2\ \ \mathrm{tr}(A\overline{A})^2.$$

The extra generator of bidegree (2,2) can therefore be chosen to be any linear combination of $\mathrm{tr}(A^2\overline{A^2})$ and $\mathrm{tr}((A\overline{A})^2)$ which is not proportional to the lhs of this equation. As far as the relation of bidegree (6,6) is concerned, we have not been patient enough to determine its explicit form: this seems a formidable task in view of the fact that a power series expansion of Eq. (91) shows, using MAPLE, that the coefficient of $z^6\overline{z^6}$ in $F_2(z,\overline{z})$ is 36, so one has to find exactly one linear relation between 37 polynomials of bidegree (6,6) in 10 variables (5 holomorphic and 5 antiholomorphic)!

A similar situation, though somewhat more complicated, occurs for the four-dimensional spin 3/2 representation. Here, we encounter one invariant quadratic form of bidegree (1,1) (the invariant scalar product, as usual) and four invariant quartic forms of bidegree (4,0), (3,1), (1,3) and (0,4), respectively, plus an extra invariant form of bidegree (3,3) (besides the cube of the invariant scalar product), as generators. There are two relations: one relation of bidegree (4,4), expressing a linear dependence between the four invariant polynomials $P(4,0)\cdot P(0,4)$, $P(3,1)\cdot P(1,3)$, $P(1,1)^4$ and $P(1,1)\cdot P(3,3)$, and one relation of bidegree (6,6), expressing a linear dependence between the seven invariant polynomials $P(4,0)\cdot P(0,4)\cdot P(1,1)^2, P(3,1)\cdot P(1,3)\cdot P(1,1)^2$, $P(1,1)^6$, $P(1,1)^3\cdot P(3,3)$, $P(4,0)\cdot P(1,3)^2$, $P(0,4)\cdot P(3,1)^2$ and $P(3,3)^2$, over and above the relation obtained by multiplying the previous one by $P(1,1)^2$. These two relations are, however, not independent, because the presence of an additional generator of bidegree (10,10) suggests that their product reduces to a trivial identity.

As far as the Molien functions for representations of spin >2 are concerned, the results indicate that for spin 5/2 and spin 3, the generators of the ring of invariant complex polynomials are subject to a single relation, while for spin 7/2 and spin 4, the structure of the relations themselves becomes complicated and no longer fits into the relatively simple scheme given by Eq. (11): there are lots of additional generators satisfying complicated relations, relations between the relations, etc. It is not even clear what is in general the most adequate way to present these functions, since numerator and denominator may have common factors. For example, the numerator and the denominator of the Molien function for spin 7/2 as given in Eq. (94) have a common factor $(1+z^6)(1+z^{10})$ which has been introduced to eliminate factors $1-z^6$ and $1-z^{10}$ from the denominator, so as to comply with the fact that there are no invariant polynomials of degree 6 and of degree 10 in this representation (as can be seen upon Taylor expansion). At any rate, the polynomials in the numerator of Eq. (94) and of Eq. (95) have roots of modulus $\neq 1$ and hence cannot possibly be reduced to an expression of the form $1-z^k$ or to a product of such expressions. (More precisely, a numerical calculation, using MAPLE, shows that the quotient of the numerator of Eq. (94) by $(1+z^6)(1+z^{10})$, considered as a polynomial of degree 16 in z^2, has 8 roots of modulus 1, 2 roots of modulus 1.46292, 2 roots of modulus 1.36453, 2 roots of modulus 0.73285 and 2 roots of modulus 0.68356, while the numerator of Eq. (95) has 14 roots of modulus 1, 2 roots of modulus 1.10697 and 2 roots of modulus 0.90377.)

In summary, everything indicates that with increasing spin, the situation becomes extremely complex. We shall therefore not pursue this matter any further and instead pass to other groups and representations.

VII. THE CODON REPRESENTATION AND ITS REDUCTIONS

Apart from the circle group, U(1), and the ordinary rotation group (or rather its universal covering group), SU(2), the compact simple Lie groups appearing in the symmetry breaking scheme of Hornos and Hornos[1] for describing the degeneracy of the genetic code involve the symplectic groups Sp(4) and Sp(6). With these applications in mind, we begin by collecting a few pertinent facts about the symplectic groups $\mathrm{Sp}(2r)$ and some of their irreducible representations, especially for the cases $r=2$ and $r=3$. (Note that we shall be dealing exclusively with the compact real form $\mathrm{Sp}(2r)$ of the complex symplectic group $\mathrm{Sp}(2r,\mathbb{C})$, which can be defined as a group of $(r\times r)$-matrices with quaternionic entries, not with the normal real form $\mathrm{Sp}(2r,\mathbb{R})$ that appears, e.g., in Hamiltonian mechanics.

The symplectic group $\mathrm{Sp}(2r)$ is a compact, connected, simply connected Lie group with center $\mathbb{Z}_2$, and its Lie algebra $\mathfrak{sp}(2r)$ is the compact real form of the complex simple Lie algebra $\mathfrak{sp}(2r,\mathbb{C})$; in the Cartan classification this is C_r, of rank r and dimension $r(2r+1)$. To construct its root system and the weight systems of various other irreducible representations besides the adjoint, we identify the spaces $\mathfrak{t}$ and $i\mathfrak{t}^*$ used before (cf. Sec. 4) with $\mathbb{R}^r$ by introducing bases $\{H_1,\ldots,H_r\}$ of $\mathfrak{t}$ and $\{e_1,\ldots,e_r\}$ of $i\mathfrak{t}^*$, dual to each other in the sense that

$$e_j(H_k)=i\,\delta_{jk}\,, \tag{96}$$

and orthonormal except for an overall normalization factor of $\sqrt{2}$; more precisely, we assume that

$$(e_j,e_k)=\tfrac{1}{2}\,\delta_{jk}\,,\quad (H_j,H_k)=2\,\delta_{jk}\,. \tag{97}$$

Then the root system Δ of $\mathfrak{sp}(2r)$, when written as the disjoint union

$$\Delta=\Delta_l\cup\Delta_s \tag{98}$$

of the set Δ_l of long roots (of length $\sqrt{2}$) and the set Δ_s of short roots (of length 1), is given by

$$\Delta_l=\{\pm 2e_j/1\leqslant j\leqslant r\}, \tag{99}$$

$$\Delta_s=\{\pm e_j\pm e_k/1\leqslant j<k\leqslant r\}. \tag{100}$$

(All signs are to be read independently.) We choose an ordering in this root system such that the set of positive roots becomes

$$\Delta^+=\Delta_l^+\cup\Delta_s^+\,, \tag{101}$$

where

$$\Delta_l^+=\{2e_j/1\leqslant j\leqslant r\}, \tag{102}$$

$$\Delta_s^+ = \{e_j \pm e_k / 1 \leqslant j < k \leqslant r\}, \tag{103}$$

leading to the following basis $\{\alpha_1,\ldots,\alpha_r\}$ of simple roots:

$$\alpha_1 = e_1 - e_2, \quad \ldots, \quad \alpha_{r-1} = e_{r-1} - e_r, \quad \alpha_r = 2e_r. \tag{104}$$

The highest root is

$$\delta = 2e_1. \tag{105}$$

Moreover, the vector ρ defined as half the sum of the positive roots, or equivalently, as the sum of the fundamental weights, which plays an important role in representation theory, is given by

$$\rho = re_1 + (r-1)e_2 + \ldots + 2e_{r-1} + e_r. \tag{106}$$

Passing to irreducible representations, we first compute the fundamental weights, defined by the condition (45), which in the present case leads to

$$\lambda_1 = e_1, \quad \lambda_2 = e_1 + e_2, \quad \ldots, \quad \lambda_{r-1} = e_1 + \ldots + e_{r-1}, \quad \lambda_r = e_1 + \ldots + e_r. \tag{107}$$

This implies that $\{e_1,\ldots,e_r\}$ is a basis of the weight lattice and $\{2\pi H_1,\ldots,2\pi H_r\}$ is a basis of the coroot lattice (which coincides with the unit lattice for $\mathrm{Sp}(2r)$ since $\mathrm{Sp}(2r)$ is simply connected); these are much more convenient than the basis $\{\lambda_1,\ldots,\lambda_r\}$ of fundamental weights and the basis $\{2\pi\check{\alpha}_1,\ldots,2\pi\check{\alpha}_r\}$ formed by the simple coroots, respectively, because they are orthonormal (except for the aforementioned overall normalization factor of $\sqrt{2}$).

With these generalities out of the way, we can proceed to write down the weight systems and, as a consequence, the generating functions M_π and F_π for the irreducible representations of Sp(4) and Sp(6) that appear in Ref. 1. For completeness, we also list their dimension and height, recalling that all irreducible representations of $\mathrm{Sp}(2r)$ are self-conjugate and that the height $\mathrm{ht}(\Lambda)$ of a self-conjugate representation of highest weight Λ allows one to decide whether the representation is real or pseudo-real: it is real iff $\mathrm{ht}(\Lambda)$ is even and pseudo-real iff $\mathrm{ht}(\Lambda)$ is odd (see Ref. 13, pp. 31–33). Finally, we list the coefficients $c_{p,q}$ in the Taylor expansion of F_π, up to fourth order, which have been calculated by differentiating under the integral sign and then computing the residues, using MAPLE. The results obtained for $c_{0,0}$, $c_{1,0}$, $c_{0,1}$ and $c_{1,1}$ are not listed because they come out to be what they must be for any irreducible representation:

$$c_{0,0} = 1, \quad c_{1,0} = 0 = c_{0,1}, \quad c_{1,1} = 1. \tag{108}$$

The result for $c_{0,0}$ reflects the correct normalization: there is (up to a constant multiple) always precisely one invariant polynomial of bidegree (0,0), namely the constant 1. The results for $c_{1,0}$ and for $c_{0,1}$ reflect the fact that an irreducible representation does not admit any invariant vectors, while the result for $c_{1,1}$ corresponds to the theorem that in an irreducible representation, the invariant scalar product is unique (up to a constant multiple), due to Schur's lemma.

A. Sp(4)

For an irreducible representation π_Λ of highest weight $\Lambda = a_1\lambda_1 + a_2\lambda_2$, we have

$$\dim(\pi_\Lambda) = (1+a_1)(1+a_2)\left(1 + \frac{a_1+a_2}{2}\right)\left(1 + \frac{a_2+2a_3}{3}\right), \tag{109}$$

$$\mathrm{ht}(\pi_\Lambda) = 3a_1 + 4a_2. \tag{110}$$

The Weyl group of Sp(4) is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_2$, generated by the reflections $e_1 \leftrightarrow -e_1$, $e_2 \leftrightarrow -e_2$ and the permutation of e_1 with e_2; it has order 8. Therefore,

$$M_\Lambda(z) = \frac{1}{8} \oint \frac{dw_1}{2\pi i} \oint \frac{dw_2}{2\pi i} \frac{w_1^{-5} w_2^{-5} f_N(w_1,w_2)^2}{f_{\Lambda,D}(z,w_1,w_2)}, \tag{111}$$

$$F_\Lambda(z,\overline{z}) = \frac{1}{8} \oint \frac{dw_1}{2\pi i} \oint \frac{dw_2}{2\pi i} \frac{w_1^{-5} w_2^{-5} f_N(w_1,w_2)^2}{f_{\Lambda,D}(z,w_1,w_2) f_{\Lambda,D}(\overline{z},w_1,w_2)}, \tag{112}$$

with

$$f_N(w_1,w_2) = (1-w_1^2)(1-w_2^2)(1-w_1w_2)(w_1-w_2). \tag{113}$$

(The definition of $f_{\Lambda,D}$ follows below.)

1. First fundamental representation (1,0)

The highest weight is $\Lambda = \lambda_1 = e_1$, the dimension is 4, the height is 3 (so the representation is pseudo-real), the complete weight system (cf. Figure 4) is

$$\Phi_{(1,0)} = \{\pm e_1, \pm e_2\} \tag{114}$$

(all weights have multiplicity 1; the signs are to be read independently), so

$$f_{(1,0),D}(x,w_1,w_2) = (1-xw_1)(1-xw_1^{-1})(1-xw_2)(1-xw_2^{-1}). \tag{115}$$

The non-trivial coefficients $c_{p,q}$ up to fourth order are

$$\begin{gathered} c_{2,0} = 0 = c_{0,2}, \\ c_{3,0} = 0 = c_{0,3}, \quad c_{2,1} = 0 = c_{1,2}, \\ c_{4,0} = 0 = c_{0,4}, \quad c_{3,1} = 0 = c_{1,3}, \quad c_{2,2} = 1. \end{gathered} \tag{116}$$

Without much difficulty, the generating function F can be computed in closed form; the result is

$$F_{(1,0)}(z,\overline{z}) = \frac{1}{1 - z\overline{z}}. \tag{117}$$

2. Second fundamental representation (0,1)

The highest weight is $\Lambda = \lambda_2 = e_1 + e_2$, the dimension is 5, the height is 4 (so the representation is real), the complete weight system (cf. Figure 4) is

$$\Phi_{(0,1)} = \{0, \pm e_1 \pm e_2\} \tag{118}$$

(all weights have multiplicity 1; the signs are to be read independently), so

$$f_{(0,1),D}(x,w_1,w_2) = (1-x)(1-xw_1w_2)(1-xw_1^{-1}w_2)(1-xw_1w_2^{-1})(1-xw_1^{-1}w_2^{-1}). \tag{119}$$

The non-trivial coefficients $c_{p,q}$ up to fourth order are

$$\begin{gathered} c_{2,0} = 1 = c_{0,2}, \\ c_{3,0} = 0 = c_{0,3}, \quad c_{2,1} = 0 = c_{1,2}, \end{gathered} \tag{120}$$

$$c_{4,0}=1=c_{0,4}, \quad c_{3,1}=1=c_{1,3}, \quad c_{2,2}=2.$$

Again, the generating function F can be computed in closed form; the result is

$$F_{(0,1)}(z,\overline{z})=\frac{1}{(1-z^2)(1-z\overline{z})(1-\overline{z}^2)}. \tag{121}$$

3. Adjoint representation (2,0)

The highest weight is $\Lambda=2\lambda_1=2e_1$, the dimension is 10, the height is 6 (so the representation is real), the complete weight system (cf. Figure 4) is the union of $\{0\}$ with the root system Δ,

$$\Phi_{(2,0)}=\{0\}\cup\{\pm 2e_1,\pm 2e_2,\pm e_1\pm e_2\} \tag{122}$$

(0 has multiplicity 2 and the roots have multiplicity 1; the signs are to be read independently), so

$$f_{(2,0),D}(x,w_1,w_2)=(1-x)^2(1-xw_1^2)(1-xw_1^{-2})(1-xw_2^2)(1-xw_2^{-2}) \\ \times(1-xw_1w_2)(1-xw_1^{-1}w_2)(1-xw_1w_2^{-1})(1-xw_1^{-1}w_2^{-1}). \tag{123}$$

The non-trivial coefficients $c_{p,q}$ up to fourth order are

$$c_{2,0}=1=c_{0,2},$$
$$c_{3,0}=0=c_{0,3}, \quad c_{2,1}=0=c_{1,2}, \tag{124}$$
$$c_{4,0}=2=c_{0,4}, \quad c_{3,1}=2=c_{1,3}, \quad c_{2,2}=4.$$

With considerable effort, the generating function F can be computed in closed form; the result is

$$\text{numer}(F_{(2,0)}(z,\overline{z}))=1+z^2\overline{z}^2+z^3\overline{z}^3+z^4\overline{z}^3+z^3\overline{z}^4+z^4\overline{z}^4+z^6\overline{z}^3+z^5\overline{z}^4 \\ +z^4\overline{z}^5+z^3\overline{z}^6+z^5\overline{z}^5+z^6\overline{z}^5+z^5\overline{z}^6+z^6\overline{z}^6+z^7\overline{z}^7+z^9\overline{z}^9,$$
$$\text{denom}(F_{(2,0)}(z,\overline{z}))=(1-z^2)(1-z\overline{z})(1-\overline{z}^2)(1-z^4)(1-z^3\overline{z})(1-z^2\overline{z}^2) \\ \times(1-z\overline{z}^3)(1-\overline{z}^4)(1-z^4\overline{z}^2)(1-z^2\overline{z}^4). \tag{125}$$

4. Reduced codon representation (1,1)

The highest weight is $\Lambda=\lambda_1+\lambda_2=2e_1+e_2$, the dimension is 16, the height is 7 (so the representation is pseudo-real), the complete weight system (cf. Figure 4) is the union

$$\Phi(1,1)=\Phi_{(1,1)}^{(1)}\cup\Phi_{(1,1)}^{(2)}, \tag{126}$$

where

$$\Phi_{(1,1)}^{(1)}=\{\pm 2e_1\pm e_2,\pm 2e_2\pm e_1\} \tag{127}$$

(these weights have multiplicity 1; the signs are to be read independently) and

$$\Phi_{(1,1)}^{(2)}=\{\pm e_1,\pm e_2\} \tag{128}$$

(these weights have multiplicity 2; the signs are to be read independently), so

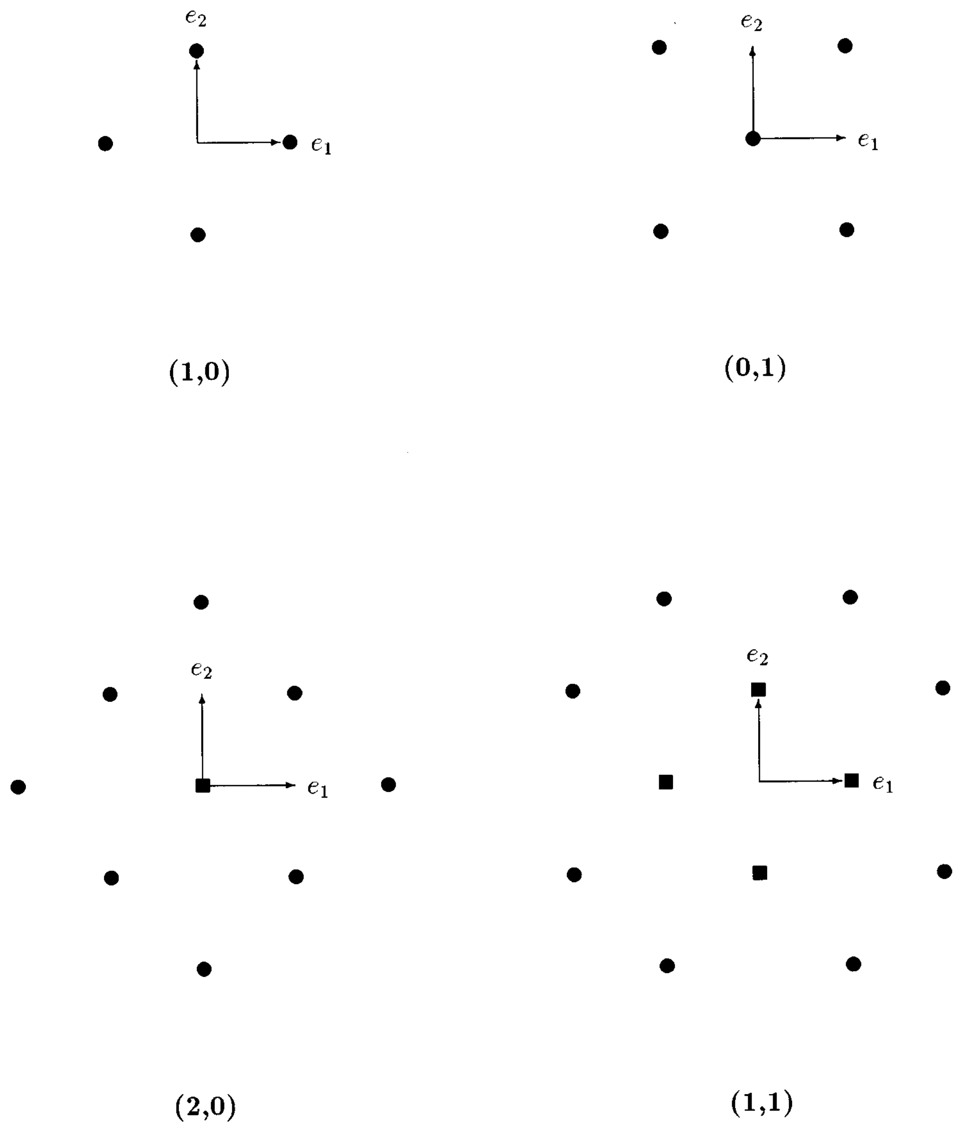

FIG. 4. Weight diagrams for some irreducible representations of Sp(4).

$$
\begin{aligned}
f_{(1,1),D}(x,w_1,w_2) &= (1-xw_1)^2(1-xw_1^{-1})^2(1-xw_2)^2(1-xw_2^{-1})^2 \\
&\quad \times (1-xw_1^2w_2)(1-xw_1^{-2}w_2)(1-xw_1^2w_2^{-1})(1-xw_1^{-2}w_2^{-1}) \\
&\quad \times (1-xw_1w_2^2)(1-xw_1^{-1}w_2^2)(1-xw_1w_2^{-2})(1-xw_1^{-1}w_2^{-2}). \qquad (129)
\end{aligned}
$$

The non-trivial coefficients $c_{p,q}$ up to fourth order are

$$
\begin{aligned}
& c_{2,0}=0=c_{0,2}, \\
& c_{3,0}=0=c_{0,3}, \quad c_{2,1}=0=c_{1,2}, \qquad (130) \\
& c_{4,0}=1=c_{0,4}, \quad c_{3,1}=2=c_{1,3}, \quad c_{2,2}=6.
\end{aligned}
$$

B. Sp(6)

For an irreducible representation π_Λ of highest weight $\Lambda = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3$, we have

$$\dim(\pi_\Lambda) = (1+a_1)(1+a_2)(1+a_3)\left(1+\frac{a_1+a_2}{2}\right)\left(1+\frac{a_2+a_3}{2}\right)\left(1+\frac{a_2+2a_3}{3}\right)$$
$$\times\left(1+\frac{a_1+a_2+a_3}{3}\right)\left(1+\frac{a_1+a_2+2a_3}{4}\right)\left(1+\frac{a_1+2a_2+2a_3}{5}\right), \tag{131}$$

$$\mathrm{ht}(\pi_\Lambda) = 5a_1 + 8a_2 + 9a_3. \tag{132}$$

The Weyl group of Sp(6) is $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times S_3$, generated by the reflections $e_i \leftrightarrow -e_i$ and the permutations of the e_i $(i=1,2,3)$; it has order 48. Therefore,

$$M_\Lambda(z) = \frac{1}{48}\oint\frac{dw_1}{2\pi i}\oint\frac{dw_2}{2\pi i}\oint\frac{dw_3}{2\pi i}\,\frac{w_1^{-7}w_2^{-7}w_3^{-7}f_N(w_1,w_2,w_3)^2}{f_{\Lambda,D}(z,w_1,w_2,w_3)}, \tag{133}$$

$$F_\Lambda(z,\overline{z}) = \frac{1}{48}\oint\frac{dw_1}{2\pi i}\oint\frac{dw_2}{2\pi i}\oint\frac{dw_3}{2\pi i}\,\frac{w_1^{-7}w_2^{-7}w_3^{-7}f_N(w_1,w_2,w_3)^2}{f_{\Lambda,D}(z,w_1,w_2,w_3)f_{\Lambda,D}(\overline{z},w_1,w_2,w_3)}, \tag{134}$$

with

$$f_N(w_1,w_2,w_3) = -(1-w_1^2)(1-w_2^2)(1-w_3^2)(1-w_1w_2)(1-w_1w_3)$$
$$\times(1-w_2w_3)(w_1-w_2)(w_1-w_3)(w_2-w_3). \tag{135}$$

(The definition of $f_{\Lambda,D}$ follows below.)

1. First fundamental representation (1,0,0)

The highest weight is $\Lambda = \lambda_1 = e_1$, the dimension is 6, the height is 5 (so the representation is pseudo-real), the complete weight system is

$$\Phi_{(1,0,0)} = \{\pm e_1, \pm e_2, \pm e_3\} \tag{136}$$

(all weights have multiplicity 1; the signs are to be read independently), so

$$f_{(1,0,0),D}(x,w_1,w_2,w_3) = (1-xw_1)(1-xw_1^{-1})(1-xw_2)(1-xw_2^{-1})(1-xw_3)(1-xw_3^{-1}). \tag{137}$$

The non-trivial coefficients $c_{p,q}$ up to fourth order are

$$c_{2,0} = 0 = c_{0,2},$$

$$c_{3,0} = 0 = c_{0,3}, \quad c_{2,1} = 0 = c_{1,2},$$

$$c_{4,0} = 0 = c_{0,4}, \quad c_{3,1} = 0 = c_{1,3}, \quad c_{2,2} = 1. \tag{138}$$

2. Second fundamental representation (0,1,0)

The highest weight is $\Lambda = \lambda_2 = e_1 + e_2$, the dimension is 14, the height is 8 (so the representation is real), the complete weight system is

$$\Phi_{(0,1,0)} = \{0\} \cup \{\pm e_1 \pm e_2, \pm e_1 \pm e_3, \pm e_2 \pm e_3\} \tag{139}$$

(0 has multiplicity 2 and all other weights have multiplicity 1; the signs are to be read independently), so

$$f_{(0,1,0),D}(x,w_1,w_2,w_3)=(1-x)^2(1-xw_1w_2)(1-xw_1^{-1}w_2)(1-xw_1w_2^{-1})(1-xw_1^{-1}w_2^{-1})\times(1-xw_1w_3)(1-xw_1^{-1}w_3)(1-xw_1w_3^{-1})(1-xw_1^{-1}w_3^{-1})\times(1-xw_2w_3)(1-xw_2^{-1}w_3)(1-xw_2w_3^{-1})(1-xw_2^{-1}w_3^{-1}). \quad (140)$$

The non-trivial coefficients $c_{p,q}$ up to fourth order are

$$c_{2,0}=1=c_{0,2},$$

$$c_{3,0}=1=c_{0,3}, \quad c_{2,1}=1=c_{1,2}, \quad (141)$$

$$c_{4,0}=1=c_{0,4}, \quad c_{3,1}=1=c_{1,3}, \quad c_{2,2}=3.$$

3. Third fundamental representation (0,0,1)

The highest weight is $\Lambda=\lambda_3=e_1+e_2+e_3$, the dimension is 14, the height is 9 (so the representation is pseudo-real), the complete weight system is

$$\Phi_{(0,0,1)}=\{\pm e_1,\pm e_2,\pm e_3,\pm e_1\pm e_2\pm e_3\} \quad (142)$$

(all weights have multiplicity 1; the signs are to be read independently), so

$$f_{(0,0,1),D}(x,w_1,w_2,w_3)=(1-xw_1)(1-xw_1^{-1})(1-xw_2)(1-xw_2^{-1})(1-xw_3)(1-xw_3^{-1})\times(1-xw_1w_2w_3)(1-xw_1^{-1}w_2w_3)(1-xw_1w_2^{-1}w_3)\times(1-xw_1^{-1}w_2^{-1}w_3)(1-xw_1w_2w_3^{-1})(1-xw_1^{-1}w_2^{-1}w_3^{-1}). \quad (143)$$

The non-trivial coefficients $c_{p,q}$ up to fourth order are

$$c_{2,0}=0=c_{0,2},$$

$$c_{3,0}=0=c_{0,3}, \quad c_{2,1}=0=c_{1,2}, \quad (144)$$

$$c_{4,0}=1=c_{0,4}, \quad c_{3,1}=1=c_{1,3}, \quad c_{2,2}=1.$$

4. Codon representation (1,1,0)

The highest weight is $\Lambda=\lambda_1+\lambda_2=2e_1+e_2$, the dimension is 64, the height is 13 (so the representation is pseudo-real), the complete weight system is the union

$$\Phi_{(1,1,0)}=\Phi_{(1,1,0)}^{(1)}\cup\Phi_{(1,1,0)}^{(2)}\cup\Phi_{(1,1,0)}^{(4)}, \quad (145)$$

where

$$\Phi^{(1)}_{(1,1,0)}=\{\pm 2e_1\pm e_2,\pm 2e_2\pm e_1,\pm 2e_1\pm e_3,\pm 2e_3\pm e_1,\pm 2e_2\pm e_3,\pm 2e_3\pm e_2\} \tag{146}$$

(these weights have multiplicity 1; the signs are to be read independently),

$$\Phi^{(2)}_{(1,1,0)}=\{\pm e_1\pm e_2\pm e_3\} \tag{147}$$

(these weights have multiplicity 2; the signs are to be read independently), and

$$\Phi^{(4)}_{(1,1,0)}=\{\pm e_1,\pm e_2,\pm e_3\} \tag{148}$$

(these weights have multiplicity 4; the signs are to be read independently), so

$$\begin{aligned} f_{(1,1,0),D}(x,w_1,w_2,w_3)=&(1-xw_1)^4(1-xw_1^{-1})^4(1-xw_2)^4(1-xw_2^{-1})^4(1-xw_3)^4(1-xw_3^{-1})^4\\ &\times(1-xw_1w_2w_3)^2(1-xw_1^{-1}w_2w_3)^2(1-xw_1w_2^{-1}w_3)^2\\ &\times(1-xw_1^{-1}w_2^{-1}w_3)^2(1-xw_1w_2w_3^{-1})^2(1-xw_1^{-1}w_2^{-1}w_3^{-1})^2\\ &\times(1-xw_1^2w_2)(1-xw_1^{-2}w_2)(1-xw_1^2w_2^{-1})(1-xw_1^{-2}w_2^{-1})\\ &\times(1-xw_1w_2^2)(1-xw_1^{-1}w_2^2)(1-xw_1w_2^{-2})(1-xw_1^{-1}w_2^{-2})\\ &\times(1-xw_1^2w_3)(1-xw_1^{-2}w_3)(1-xw_1^2w_3^{-1})(1-xw_1^{-2}w_3^{-1})\\ &\times(1-xw_1w_3^2)(1-xw_1^{-1}w_3^2)(1-xw_1w_3^{-2})(1-xw_1^{-1}w_3^{-2})\\ &\times(1-xw_2^2w_3)(1-xw_2^{-2}w_3)(1-xw_2^2w_3^{-1})(1-xw_2^{-2}w_3^{-1})\\ &\times(1-xw_2w_3^2)(1-xw_2^{-1}w_3^2)(1-xw_2w_3^{-2})(1-xw_2^{-1}w_3^{-2}). \end{aligned} \tag{149}$$

The non-trivial coefficients $c_{p,q}$ up to fourth order are

$$\begin{gathered} c_{2,0}=0=c_{0,2},\\ c_{3,0}=0=c_{0,3},\quad c_{2,1}=0=c_{1,2},\\ c_{4,0}=3=c_{0,4},\quad c_{3,1}=6=c_{1,3},\quad c_{2,2}=15. \end{gathered} \tag{150}$$

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