

Semidefinite Programming

Basics and SOS

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Available at: www.ime.usp.br/~fmario under talks

Conic programming

V is a real vector space

$\langle \cdot, \cdot \rangle$ is an inner product

Convex cone: set $C \subseteq V$ such that

$$\alpha x + \beta y \in C \quad \forall x, y \in C \text{ and } \alpha, \beta \geq 0$$

Given: $c, a_1, \dots, a_m \in V$ and $b_1, \dots, b_m \in \mathbb{R}$

$$\begin{aligned} & \max \quad \langle c, x \rangle \\ & \quad \langle a_i, x \rangle = b_i \quad \text{for } i = 1, \dots, m \\ & \quad x \in C \end{aligned}$$

Complexity: depends on the cone!

Examples

- $V = \mathbb{R}^n$ and $C = \mathbb{R}_+^n$: linear programming
- $V = \mathcal{S}_n$ and $C = \text{cone of PSD matrices}$: SDP

$$A, B \in \mathcal{S}_n$$

$$\langle A, B \rangle = \text{tr}(A^T B) = \sum_{i,j=1}^n A_{ij} B_{ij} = \text{trace inner product}$$

A is PSD (also $A \succeq 0$): $x^T A x \geq 0 \quad \forall x \in \mathbb{R}^n$

\iff all eigenvalues are nonnegative

\iff $A = Q^T Q$ for some $Q \in \mathbb{R}^{n \times n}$

\iff $A = \sum_{i=1}^n \lambda_i u_i u_i^T$, with $\lambda_i \geq 0$

Semidefinite programming

Given: $C, A_1, \dots, A_m \in \mathcal{S}_n$ and $b_1, \dots, b_m \in \mathbb{R}$

$$\begin{aligned} \max \quad & \langle C, X \rangle \\ & \langle A_i, X \rangle = b_i \quad \text{for } i = 1, \dots, m \\ & X \succeq 0 \end{aligned}$$

$$\begin{pmatrix} x & z \\ z & y \end{pmatrix} \succeq 0 \iff x, y \geq 0 \text{ and } xy - z^2 \geq 0$$

- Generalizes LP: make all matrices diagonal!
- Duality theory: comes from conic programming
- Solvable in poly-time: under some conditions (later)
- Numerical methods: efficient interior-point methods

Application: polynomial optimization

Is the polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$ nonnegative?

Sure, if it is a sum-of-squares: $p = q_1^2 + \dots + q_k^2$

Exercise: A univariate poly is nonnegative iff it is SOS

$P_{n,d}$ = nonnegative polys, n variables, degree d

$\Sigma_{n,d}$ = SOS polys, n variables, degree d

Theorem. $\Sigma_{n,d} \subseteq P_{n,d}$ with equality iff

- bivariate polys ($n = 2$)
- quadratics ($d = 2$)
- $n = 3, d = 4$



David Hilbert
1888

SOS is SDP

$p \in \mathbb{R}[x_1, \dots, x_n]$ degree d

$v(x)$ = vector with all monomials degree $\leq d/2$

Theorem. p is SOS iff $p(x) = \langle Q, v(x)v(x)^T \rangle$ for some $Q \succeq 0$.

$n = 2, d = 6$

$v(x) = (1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, y^3)$

$$\begin{pmatrix} 1 & x & x^2 & x^3 & y & xy & x^2y & y^2 & xy^2 & y^3 \\ x & x^2 & x^3 & x^4 & xy & x^2y & x^3y & xy^2 & x^2y^2 & xy^3 \\ x^2 & x^3 & x^4 & x^5 & x^2y & x^3y & x^4y & x^2y^2 & x^3y^2 & x^2y^3 \\ x^3 & x^4 & x^5 & x^6 & x^3y & x^4y & x^5y & x^3y^2 & x^4y^2 & x^3y^3 \\ y & xy & x^2y & x^3y & y^2 & xy^2 & x^2y^2 & y^3 & xy^3 & y^4 \\ xy & x^2y & x^3y & x^4y & xy^2 & x^2y^2 & x^3y^2 & xy^3 & x^2y^3 & xy^4 \\ x^2y & x^3y & x^4y & x^5y & x^2y^2 & x^3y^2 & x^4y^2 & x^2y^3 & x^3y^3 & x^2y^4 \\ y^2 & xy^2 & x^2y^2 & x^3y^2 & y^3 & xy^3 & x^2y^3 & y^4 & xy^4 & y^5 \\ xy^2 & x^2y^2 & x^3y^2 & x^4y^2 & xy^3 & x^2y^3 & x^3y^3 & xy^4 & x^2y^4 & xy^5 \\ y^3 & xy^3 & x^2y^3 & x^3y^3 & y^4 & xy^4 & x^2y^4 & y^5 & xy^5 & y^6 \end{pmatrix}$$

A small example

$$p(x, y) = x^4 + 2x^3y + x^2y^2 - 6x^2y - 6xy^2 + 10x^2 + 10xy + 9y^2 - 30y + 25$$

$$v(x) = (1, x, x^2, y, y^2, xy)$$

$$\left(\begin{array}{cccccc} & 1 & x & x^2 & y & y^2 & xy \\ - & q_{11} & q_{12} & q_{13} & q_{14} & q_{15} & q_{16} \\ 1 & q_{21} & q_{22} & q_{23} & q_{24} & q_{25} & q_{26} \\ x & q_{31} & q_{32} & q_{33} & q_{34} & q_{35} & q_{36} \\ x^2 & q_{41} & q_{42} & q_{43} & q_{44} & q_{45} & q_{46} \\ y & q_{51} & q_{52} & q_{53} & q_{54} & q_{55} & q_{56} \\ y^2 & q_{61} & q_{62} & q_{63} & q_{64} & q_{65} & q_{66} \end{array} \right)$$

$$\left(\begin{array}{cccccc} & 1 & x & x^2 & y & y^2 & xy \\ - & 1 & x & x^2 & y & y^2 & xy \\ 1 & 1 & x & x^2 & x^3 & xy & xy^2 \\ x & x^2 & x^3 & x^4 & x^2y & x^2y^2 & x^3y \\ x^2 & y & xy & x^2y & y^2 & y^3 & xy^2 \\ y & y^2 & xy^2 & x^2y^2 & y^3 & y^4 & xy^3 \\ y^2 & xy & x^2y & x^3y & xy^2 & xy^3 & x^2y^2 \\ xy & & & & & & \end{array} \right)$$

$$q_{11} = 25$$

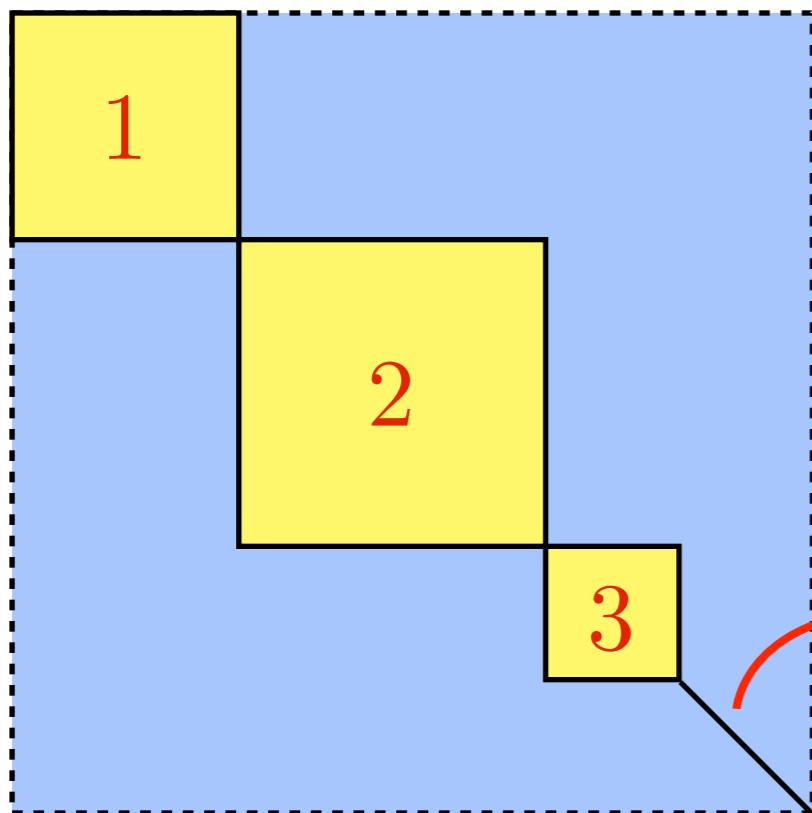
$$q_{23} + q_{32} = 0$$

$$q_{25} + q_{52} + q_{46} + q_{64} = -6$$

$$Q = \sum u_i u_i^\top \implies p = \sum (u_i^\top v(x))^2$$

Sparse SDPA format: Setup

$$\begin{aligned} & \text{maximize} && \langle F_0, Y \rangle \\ & && \langle F_i, Y \rangle = c_i \quad \text{for } i = 1, \dots, m \\ & && Y \succeq 0. \end{aligned}$$



Matrices have a block structure

Block 4, a diagonal block

Sparse SDPA format

$$\begin{aligned} \text{maximize} \quad & \langle F_0, Y \rangle \\ & \langle F_i, Y \rangle = c_i \quad \text{for } i = 1, \dots, m \\ & Y \succeq 0. \end{aligned}$$

Text file ‘problem.sdpa’:

Number of constraint matrices: m

Number of blocks: N

Block sizes: $s_1 \sqcup s_2 \sqcup \dots \sqcup s_N$

Right-hand side: $c_1 \sqcup c_2 \sqcup \dots \sqcup c_m$

Matrix entries: $\langle \text{matrix number} \rangle \sqcup \langle \text{block number} \rangle \sqcup \langle i \rangle \sqcup \langle j \rangle \sqcup \langle \text{entry} \rangle$

Remarks:

- Indices always start at 1
- Write the size of a block as a negative number to indicate a diagonal block
- Only upper-diagonal entries need to be given!

A feasibility problem

Is the polynomial

$$p(x) = x^4 + 6x^3 + 15x^2 - 10x + 17$$

a sum of squares?

$$v(x) = (1, x, x^2)$$

$$F_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad F_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad F_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$F_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad F_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad F_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$c_1 = 17, c_2 = -10, c_3 = 15, c_4 = 6, c_5 = 1$$

$$\begin{aligned} &\text{maximize} && \langle F_0, Y \rangle \\ & && \langle F_i, Y \rangle = c_i \quad \text{for } i = 1, \dots, 5 \\ & && Y \succeq 0. \end{aligned}$$

The SDPA file

$$F_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad F_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad F_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$F_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad F_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad F_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$c_1 = 17, c_2 = -10, c_3 = 15, c_4 = 6, c_5 = 1$$

File ‘sos.sdpa’:

```
5  
1  
3  
17 -10 15 6 1  
1 1 1 1 1.0  
2 1 1 2 1.0  
3 1 1 3 1.0  
3 1 2 2 1.0  
4 1 2 3 1.0  
5 1 3 3 1.0
```

- ▷ Number of constraint matrices
- ▷ We have only one block
- ▷ Which is 3x3
- ▷ The right-hand side
- ▷ Matrix entries

CSDP output

0.0	0.0	0.0	0.0	0.0	0.0
1	1	1	1	2.886751345948129001e-11	
1	1	2	2	2.886751345948129001e-11	
1	1	3	3	2.886751345948129001e-11	
2	1	1	1	1.7000000000000000e+01	
2	1	1	2	-5.0000000000000000e+00	
2	1	1	3	-2.965376342174380397e+00	
2	1	2	2	2.093075268434876079e+01	
2	1	2	3	3.0000000000000000e+00	
2	1	3	3	1.0000000000000000e+00	

- ▷ Dual solution, only numbers
- ▷ Dual solution, matrix
- ▷ Primal solution, matrix

$$Y = \begin{pmatrix} 17.000000000000 & -5.000000000000 & -2.96537634217438 \\ -5.000000000000 & 20.9307526843488 & 3.00000000000000 \\ -2.96537634217438 & 3.000000000000 & 1.00000000000000 \end{pmatrix}$$

$$p_1(x) = -0.719209404617218x^2 - 1.21267812518166x + 4.12310562561766$$

$$p_2(x) = 0.482351656352516x^2 + 4.41136763929902x$$

$$p_3(x) = 0.500074706343094x^2$$

$$p_1^2 + p_2^2 + p_3^2 = x^4 + 6x^3 + 15x^2 - 10x + 17$$

Other uses of SOS

Finding the minimum of a polynomial

$$\begin{aligned} \text{minimum of } p \text{ is } \lambda &\iff p - \lambda \text{ is nonnegative} \\ &\iff p \text{ is SOS} \end{aligned}$$

Formulate as SDP: $\max\{ \lambda : p - \lambda \text{ is SOS} \}$

Nonnegativity in a domain

$p \in \mathbb{R}[x]$ is nonnegative in $[a, b] \iff \exists q_1, q_2 \text{ SOS s.t.}$

$$p = (x - a)(b - x)q_1 + q_2$$

Duality in conic programming

$C \subseteq V$ a cone

Dual cone: $C^* = \{x \in V : \langle x, y \rangle \geq 0 \text{ for all } y \in C\}$

Primal standard form:

$$\begin{aligned} \max \quad & \langle \mathbf{c}, \mathbf{x} \rangle \\ & \langle \mathbf{a}_i, \mathbf{x} \rangle = b_i \quad \text{for } i = 1, \dots, m \\ & \mathbf{x} \in C \end{aligned}$$

strictly feasible: \mathbf{x} in interior of C

Dual standard form:

$$\begin{aligned} \min \quad & y_1 b_1 + \cdots + y_m b_m \\ & y_1 \mathbf{a}_1 + \cdots + y_m \mathbf{a}_m - \mathbf{c} \in C^* \end{aligned}$$

strictly feasible: $y_1 \mathbf{a}_1 + \cdots + y_m \mathbf{a}_m - \mathbf{c}$ in interior of C

Duality theorems

Weak duality:

$$x \text{ is primal feasible, } y \text{ is dual feasible} \implies \langle c, x \rangle \leq y^T b$$

Strong duality:

If: dual is bounded and strictly feasible

Then: primal attains optimal and values coincide

If: primal is bounded and strictly feasible

Then: dual attains optimal and values coincide

Applied to SDP...

Exercise: PSD cone is self-dual

Primal standard form:

$$\begin{aligned} \max \quad & \langle \textcolor{red}{C}, \textcolor{blue}{X} \rangle \\ & \langle \textcolor{red}{A}_i, \textcolor{blue}{X} \rangle = b_i \quad \text{for } i = 1, \dots, m \\ & \textcolor{blue}{X} \succeq 0 \end{aligned}$$

strictly feasible: $\textcolor{blue}{X}$ is positive definite

Dual standard form:

$$\begin{aligned} \min \quad & y_1 b_1 + \cdots + \textcolor{blue}{y}_m b_m \\ & \textcolor{red}{y}_1 A_1 + \cdots + \textcolor{blue}{y}_m A_m - \textcolor{red}{C} \succeq 0 \end{aligned}$$

strictly feasible: $y_1 A_1 + \cdots + \textcolor{blue}{y}_m A_m - \textcolor{red}{C}$ is positive definite

SDP and duality gone wrong

Optimal not attained:

$$\min \quad \beta \\ \begin{pmatrix} \alpha & 1 \\ 1 & \beta \end{pmatrix} \succeq 0$$

$$\max \quad -X_{12} - X_{21} \\ X_{11} = 0, X_{22} = 1$$

Positive duality gap:

$$\max \quad -X_{11} - X_{22} \\ X_{11} = 0 \\ 2X_{13} + X_{22} = 1 \\ X \succeq 0$$

Optimal: -1

$$\min \quad y_2 \\ \begin{pmatrix} y_1 + 1 & 0 & y_2 \\ 0 & y_2 + 1 & 0 \\ y_2 & 0 & 0 \end{pmatrix} \succeq 0$$

Optimal: 0

Summary and coming soon...

Semidefinite programming...

- Generalizes LP
- Fits framework of conic programming
- Solvable in poly time (under some “mild” assumptions)
- Has a duality theory (inherits from conic programming)
- One useful application: SOS polynomials

Tomorrow:

- Approximation algorithms
- Maximum-cut problem
- Grothendieck inequalities and applications