On envelopes and L_p-spaces

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Outline

1. Envelopes
2. Fraïssé Banach spaces
3. New and old about L_p spaces

Work in progress with J. Lopez-Abad, and joint work with J. Lopez-Abad, B. Mbombo, and S. Todorcevic.

Envelopes

We call envelope map a map e which assigns to every subset A of a Banach space X its envelope, i.e. a closed subspace $e(A)$ of X satisfying the following properties:

a) $A \subseteq e(A)$.
b) If $B \subseteq A$, then $e(B) \subseteq e(A)$.
c) $e(\overline{\mathrm{span}}(A)) = e(A)$.
d) $e(e(A)) = e(A)$.

Examples of envelope maps:

- $A \mapsto \overline{\mathrm{span}}(A)$
- (X Banach lattice) $A \mapsto \mathrm{lat}(\mathrm{A})$
- (X reflexive strictly convex with strictly convex dual) $A \mapsto \mathrm{Env}_{min}(A)$, the smallest 1-complemented subspace of X containing A (see Calvert 76)

We shall be interested in what we call the Korovkin envelope associated to a bounded semigroup G of operators on X.

Definition

The (Korovkin) envelope $\mathrm{Env}_G(A)$ *of a subset* A *of* X *is the set of* $x \in X$ *such that whenever a net* $(T_\alpha)_\alpha \in G$ *converges pointwise on* A *to* Id_A*, then* $(T_\alpha(x))_\alpha$ *converges to* x*.*

Korovkin and others were mostly interested in $G =$ the semigroup of contractions, in which case the word "shadow" was used, and in the semigroup of positive contractions, when X is a lattice.

A *Korovkin set* A is a set such that $\mathrm{shadow}(A) = X$. The set $\{1, t, t^2\}$ is a Korovkin set for $C(0,1)$ (Wulbert 68, improving on Korovkin 60). The set $\{1, t\}$ is a Korovkin set for L_p (Bernau 74 for $1 < p < +\infty$, Berens-Lorentz 74 for $p = 1$).

In 75 Calvert proved that: if X is a reflexive space with LUR norm and dual norm, then $\mathrm{shadow}(A) = \mathrm{Env}_{min}(A)$.
(actually enough to assume the dual norm is strictly convex).

In particular, if A is a subset of $L_p, 1 < p < +\infty$ then

$$\mathrm{shadow}(A) = \mathrm{Env}_{min}(A) \ (= \mathrm{lat(A)\ if\ } 1 \in \mathrm{A.})$$

(also true if $p = 1$).

We wish to concentrate on the case $G = \mathrm{Isom}(X)$ (the isometry group of X), and w.l.o.g. we replace the subset A by a closed subspace Y. So we define $\mathrm{Env}(Y) := \mathrm{Env}_{\mathrm{Isom}(X)}(Y)$.

Equivalent definition

The envelope $\mathrm{Env}(Y)$ *of a subspace Y of X is the largest $Z \subseteq X$ such that whenever a net $(T_\alpha)_\alpha$ of isometries converges pointwise on Y, then $(T_\alpha)_\alpha$ converges pointwise on Z.*

Note that $\mathrm{shadow}(Y) \subseteq \mathrm{Env}(Y)$. Some easy examples:

- If X only admits trivial isometries (all isometries are multiple of the identity), then $\mathrm{Env}(Y) = X$ for all $Y \neq \{0\}$.
- If H is a Hilbert space then $\mathrm{Env}(Y) = Y$ for any $Y \subseteq X$.

The set of subspaces of X which are envelopes (i.e., $Y = \mathrm{Env}(Y)$) is some kind of "skeleton" of the isometric structure of X.

Proposition

The following hold:

(a) *all envelopes in a separable reflexive space X are* 1*-complemented,*

(b) *the Hilbert space is the only separable reflexive space for which all subspaces are envelopes,*

(c) *If $X = L_p, 1 \leq p < +\infty$ then for all Y, $\mathrm{Env}(Y) = \mathrm{Env}_{min}(Y)$*

(d) *the L_p's, $1 < p < +\infty$ are the only reflexive r.i. spaces on [0,1] for which all* 1*-complemented subspaces are envelopes.*

(a) uses LUR renormings (Lancien 93) and duality. One proof of (c)(d) uses Peller's description (1980) of the WOT closure of $\mathrm{Isom}(L_p)$ through the existence of "dilations" (as well as the same topic replacing L_p by a general r.i. space on $[0, 1]$).

Fraïssé Banach spaces

From now on X is always infinite dimensional. Let us recall Mazur rotation problem. Assume X is separable and the linear isometry group $\mathrm{Isom(X)}$ acts transitively on S_X. Must X be isomorphic, or even isometric, to a Hilbert space?

There are non-separable counterexamples: $(L_p)_{\mathcal{U}}$, for $1 \leq p < +\infty$, or $\mathbb{G}_{\mathcal{U}}$ (here $\mathbb{G}$ denotes the Gurarij space).

Problem

Find a property (P) of the Hilbert stronger than transitivity and for which there are non-separable non-Hilbertian examples. Investigate whether the Hilbert is the only separable space with (P).

One example of such a property (P) is "ultrahomogeneity", a multidimensional form of transitivity.

A space X is ultrahomogeneous if any partial isometry between finite dimensional subspaces of X extends to a (surjective) isometry of X.

Proposition

Are ultrahomogeneous

- $\mathbb{G}_{\mathcal{U}}$, *Aviles-Cabello-Castillo-González-Moreno (2013)*
- $(L_p)_{\mathcal{U}}$ *for* $1 \leq p < +\infty, p \neq 4, 6, 8, ...$, *F.-LopezAbad-Mbombo-Todorcevic (2019)*

Lusky (78) had proved that those L_p are "almost" ultrahomogeneous (AUH): "for any partial isometry t between finite dimensional subspaces E and F of X and $\varepsilon > 0$, there exists an isometry T on X such that $\|T_{|E} - t\| \leq \epsilon$". This used Plotkin-Rudin 76. This is not quite enough to deduce the Proposition.

The following assertions are equivalent and we called Fraïssé an infinite dimensional space satisfying them.

(1) $\forall \varepsilon > 0$ and $k \in \mathbb{N}$, $\exists \delta > 0$ such that for any $(1+\delta)$-isometric map t between subspaces E, F of X of dimension k, there exists $T \in \mathrm{Isom}(X)$ such that $\| T_{|E} - t \| < \varepsilon$,

(2) the subgroup $\mathrm{Isom(X)}_{\mathcal{U}} = \{(T_n)_n, T_n \in \mathrm{Isom}(X) \forall n\}$ of $\mathrm{Isom}(X_{\mathcal{U}})$ acts ultrahomogeneously on $X_{\mathcal{U}}$.

This is formally stronger than AUH. The Gurarij space $\mathbb{G}$ is Fraïssé from Kubis-Solecki (2013) but also:

Theorem

(F. - Lopez-Abad - Mbombo- Todorcevic 2019) The spaces L_p, $p \neq 4, 6, \ldots$ are Fraïssé.

Conjecture

The Gurarij space and the L_p spaces for appropriate p are the only separable Fraïssé or even AUH spaces.

Why the "Fraïssé" terminology?

A countable structure A is Fraïssé when it is ultrahomogeneous (with respect to the class $\mathrm{Age}(A)$ of its finite substructures). The KPT-correspondence (Kechris-Pestov-Todorcevic 05) states that the extreme amenability of $(\mathrm{Aut}(A), \mathit{ptwise})$ is equivalent to a Ramsey Property of embeddings between elements of $\mathrm{Age}(A)$ (a topological group is extremely amenable if any continuous action on a compact space admits a fixed point).

Our definition of Fraïssé Banach space was originally aimed at proving a KPT-correspondence to recover the extreme amenability of $\mathcal{U}(H)$ (Gromov-Milman 83) and $\mathrm{Isom}(L_p)$ (Giordano-Pestov 07) through Ramsey methods instead of concentration of measure.

The group $\mathrm{Isom}(\mathbb{G})$ is also extremely amenable (Bartosova - Lopez-Abad - Lupini - Mbombo 17).

Note that Fraïssé people usually "start" with $\mathrm{Age}(A)$ and construct A from it. In the L_p-situation this is somewhat reversed.

Among properties of Fraïssé spaces we have:

- if X, Y are separable Fraïssé and are finitely representable into each other, then they are isometric ($X \equiv Y$)
- if Y separable is finitely representable into a Fraïssé space X, then Y isometrically embeds into it

Therefore:

- every Fraïssé space contains an isometric copy of ℓ_2 (from Dvoretsky) - an unusual way of proving that ℓ_2 embeds isometrically into $L_p \ldots$
- separable Fraïssé spaces either have finite cotype or are isometric to $\mathbb{G}$

The envelope $\mathrm{Env}(Y)$ admits an equivalent definition in separable AUH spaces.

Proposition

Assume X is separable AUH and Y is a subspace of X. Then $\mathrm{Env}(Y)$ is the largest subspace Z containing Y such that

- *every isometric embedding $t : Y \to X$ extends uniquely to an isometric embedding $\tilde{t} : Z \to X$*
- *the map $t \mapsto \tilde{t}$ is SOT-SOT continuous*

Furthermore $\tilde{t}(\mathrm{Env}(Y)) = \mathrm{Env}(t(Y))$.

Corollary

If X is separable AUH, Y, Z subspaces of X then

$$Y \equiv Z \Rightarrow \mathrm{Env}(Y) \equiv \mathrm{Env}(Z).$$

Examples of envelopes in L_p

Proposition

The following subspaces Y of L_p have envelope isometric to L_p.

- $(1 \leq p < +\infty, p \neq 4, 6, \ldots)$
  $Y = \ell_2$ *and, unless* $p = 2$, $Y = \ell_2^n$, $n \geq 2$
- $(1 \leq p \leq q \leq 2)$ $Y = L_q$
- $(1 \leq p < q \leq 2)$ $Y = \ell_q$

Define a subspace Y of X to be full if $\mathrm{Env}(Y) = X$. As a consequence of the above:

- L_p $(p \neq 4, 6, \ldots)$ contains a full copy of ℓ_2 (we have no "explicit" description of such)
- this induces a topological embedding of $\mathcal{U}(\ell_2)$ as a subgroup of $\mathrm{Isom}(L_p)$

A curious consequence: the space L_p/ℓ_2, for p not even

There is a unique exact sequence (up to isometric equivalence) associated to full embeddings of ℓ_2 into L_p.
Indeed if

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ell_2 & \longrightarrow & L_p & \longrightarrow & L_p/\ell_2 & \longrightarrow & 0 \\
 & & {\scriptstyle t}\big\downarrow & & {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle \hat{T}} & & \\
0 & \longrightarrow & \ell_2 & \longrightarrow & L_p & \longrightarrow & L_p/\ell_2 & \longrightarrow & 0
\end{array}$$

are exact sequences associated to full copies of ℓ_2, and t an isometry between these two copies, then we get an extension T inducing an isometry $\hat{T}$ making the above diagram commute.

Therefore there is a isometrically unique quotient L_p/ℓ_2 of L_p by any choice of full copy of ℓ_2.

Question

Identify the space L_p/ℓ_2?

- if $p > 1$ then ℓ_2 is complemented so $L_p/\ell_2 \simeq L_p$: L_p/ℓ_2 is a certain renorming of L_p
- if $p = 1$ then ℓ_2 is uncomplemented and $L_1/\ell_2 \simeq ?$
- Same question for L_p/L_q or L_p/ℓ_q, $1 \le p < q < 2$.

What about possible other Fraïssé spaces?

If X is separable and Fraïssé then

- X contains an isometric copy of ℓ_2. Must it contain a full isometric copy of ℓ_2?
- if yes, then $\mathrm{Isom}(X)$ contains a subgroup isomorphic to $\mathcal{U}(\ell_2)$
- if $p(X) := sup\{p : X \text{ has type } p\}$, then X contains an isometric copy of $L_{p(X)}$ (through Maurey-Pisier); similarly for $q(X)$ if $< +\infty$
- (in case X is 1-complemented in its bidual) if Y is a K-complemented subspace of X then all isometric copies of Y inside X are K-complemented

Also

Proposition

If X separable Fraïssé admits a C_∞-bump function then $X \simeq \ell_2$.

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