Fraïssé and Ramsey properties of L_p-spaces

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Outline

1. Transitivities of isometry groups
2. Fraïssé spaces
3. Fraïssé properties of L_p spaces
4. KPT correspondence

Joint work with J. Lopez-Abad, M. Mbombo, S. Todorcevic
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Notations

S_X = unit sphere of X.

$\mathrm{Isom}(X)$= group of linear surjective isometries of X, with the Strong Operator Topology SOT.

$\mathrm{Emb}(F, X)$ = set of linear isometric embeddings of F into X, with SOT.

Usually F is finite dimensional, so SOT can be replaced by the distance induced by the norm on $\mathcal{L}(F, X)$.

p a real number in the separable Banach range: $1 \leq p < +\infty$

Classical isometry groups

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2. Fraïssé spaces
3. Fraïssé properties of L_p spaces
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Classical isometry groups

1. If H=Hilbert, then $\mathrm{Isom}(H)$ is the unitary group $\mathcal{U}(H)$. It acts transitively on S_H, meaning there is a single and full orbit for the action $\mathrm{Isom}(H) \curvearrowright S_H$.

2. For $1 \leq p < +\infty$, $p \neq 2$, every isometry on $L_p = L_p(0,1)$ is of the form
$$T(f)(.) = h(.)f(\phi(.)),$$
where ϕ is a measurable transformation of $[0,1]$ onto itself, and h such that $|h|^p = d(\lambda \circ \phi)/d\lambda$, λ the Lebesgue measure (Banach-Lamperti 1932-1958). So

3. $\mathrm{Isom}(L_p)$ acts almost transitively on S_{L_p}, meaning that the action $\mathrm{Isom}(L_p) \curvearrowright S_{L_p}$ admits dense orbits. $\forall x, y \in S_{L_p}, \forall \epsilon > 0, \exists T \in \mathrm{Isom}(L_p) : \|Tx - y\| \leq \epsilon$.

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A norm is *transitive (resp. almost transitive)* if the associated isometry group acts transitively (resp. almost transitively) on the associated unit sphere.

Mazur rotation problem

If $G = \mathrm{Isom(X)}$ acts transitively on S_X, must X be isometric? isomorphic? to a Hilbert space.

(a) if $\dim X < +\infty$: YES to both

(b) if $\dim X = +\infty$ and is separable: ???

(c) if $\dim X = +\infty$ and is non-separable: NO to both

Proof

(a) Average a given inner product by using the Haar measure on G and observe that this new inner product turns all $T \in G$ into unitaries and therefore, by transitivity, must induce the original norm.

$$[x, y] = \int_{T \in \mathrm{Isom}(X, \|.\|)} < Tx, Ty > d\mu(T),$$

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(c) Use ultrapowers.....

Ultrapowers

It is an easy observation that if X is almost transitive then for any non-principal ultrafilter $\mathcal{U}$, $X_{\mathcal{U}}$ is transitive. Actually the subgroup $\mathrm{Isom}(X)_{\mathcal{U}}$ of isometries T of the form

$$T((x_n)_{n\in\mathbb{N}}) = (T_n(x_n))_{n\in\mathbb{N}}$$

where $T_n \in \mathrm{Isom}(X)$, acts transitively on $X_{\mathcal{U}}$.

So we get (and using Henson 1976).

Proposition

The space $(L_p(0,1))_{\mathcal{U}} = L_p(\cup_{\mathfrak{c}}[0,1]^{\mathfrak{c}})$ *is transitive.*

Note that Cabello-Sanchez (1998) studies $\Pi_{n\in\mathbb{N}} L_{p_n}(0,1)$ for $p_n \to +\infty$ and obtains a transitive M-space.

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Note that for $p \neq 2$, L_p is not transitive, and ℓ_p not almost transitive. Furthermore

Theorem (Dilworth - Randrianantoanina, 2014)

Let $1 < p < +\infty, p \neq 2$. Then ℓ_p does not admit an equivalent almost transitive norm.

Question

Let $1 \leq p < +\infty, p \neq 2$. Show that the space $L_p([0,1])$ does not admit an equivalent transitive norm.

By the way, F. and Rosendal (2017) show that a transitive norm on a separable space must be strictly convex. So maybe the case $p = 1$ is not out of hand.

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Ultrahomogeneity

Definition

Let X be a Banach space.

- X is called ultrahomogeneous when for every finite dimensional subspace E of X and every two isometric embeddings $i_1, i_2 : E \to X$ there is a linear isometry $g \in \mathrm{Isom}(X)$ such that $g \circ i_1 = i_2$;
- X is called approximately ultrahomogeneous (AuH) when for every finite dimensional subspace E of X, every two isometric embedding $i_1, i_2 : E \to X$ and every $\varepsilon > 0$ there is a linear isometry $g \in \mathrm{Isom}(X)$ such that $\|g \circ i_1 - i_2\| < \varepsilon$;

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Examples

Note that ultrahomogeneous $\Rightarrow$ transitive, and $(\mathrm{AuH})\Rightarrow$ almost transitive

Fact

Any Hilbert space is ultrahomogeneous.

Theorem

Are (AuH)*:*

- *The Gurarij space (Kubis-Solecki 2013)*
- $L_p[0,1]$ *for* $p \neq 4, 6, 8, \ldots$ *(Lusky 1978)*

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- the Gurarij is the unique separable, universal, (AuH) space (Lusky 1976 + Kubis-Solecki 2013).
- Lusky's result is based on the *equimeasurability theorem* by Rudin / Plotkin, 1976. His proof gives (AuH).
- L_p is not (AuH) for $p = 4, 6, 8, \ldots$:
  B. Randrianantoanina (1999) proved that for those p's there are two isometric subspaces of L_p (due to Rosenthal), with an unconditional basis, complemented/ uncomplemented.

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Fraïssé theory in one slide

- Given a (hereditary) class $\mathcal{F}$ of finite (or sometimes finitely generated) structures, Fraïssé theory (Fraïssé 1954) investigates the existence of a countable structure $\mathcal{A}$, universal for $\mathcal{F}$ and ultrahomogeneous (any t isomorphism between finite substructures extends to a global automorphism of $\mathcal{A}$)
- Fraïssé theory shows that this is equivalent to certain amalgamation properties of $\mathcal{F}$.
- Then $\mathcal{A}$ is unique up to isomorphism and called the Fraïssé limit of $\mathcal{F}$.

Well, maybe just one more slide for examples...

Example

if $\mathcal{F}$=$\{$*finite sets*$\}$, *then* $\mathcal{A} = \mathbb{N}$

In this case isomorphisms of the structure are just bijections.

Example

if $\mathcal{F}$=$\{$*finite ordered sets*$\}$, *then* $\mathcal{A} = (\mathbb{Q}, <)$.

Isomorphisms are order preserving bijections.

Many works exist about extension of this theory to the metric setting (i.e. with epsilons), but they are often at the same time too general and too restrictive for us. We focus on the Banach space setting.

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Fraïssé spaces

Given two Banach spaces E and X, and $\delta \geq 0$, let $\mathrm{Emb}_\delta(E, X)$ be the collection of all linear δ- embeddings $T : E \to X$, i.e. such that $\|T\|, \|T^{-1}\| \leq 1 + \delta$, equipped with the distance induced by the norm.

We consider the canonical action $\mathrm{Isom}(X) \curvearrowright \mathrm{Emb}_\delta(E, X)$

Definition (F., Lopez-Abad, Mbombo, Todorcevic)

X is weak Fraïssé if and only if for every $E \subset X$ of finite dimension, and every $\varepsilon > 0$ there is $\delta > 0$ such that whenever i_1, i_2 are δ-embeddings of E into X, there is a linear isometry g on X such that $\|g \circ i_1 - i_2\| \leq \varepsilon$.

X is Fraïssé if and only it is weak Fraïssé and δ depends only on ε and the dimension of E.

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X is Fraïssé if and only if for every $k \in \mathbb{N}$ and every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, k) > 0$ such that whenever i_1, i_2 are δ-embeddings of some E of dimension k into X, there is a linear isometry g on X such that $\|g \circ i_1 - i_2\| \leq \varepsilon$.
I.e. the action $\mathrm{Isom}(X) \curvearrowright \mathrm{Emb}_\delta(E, X)$ is "ε-transitive"

Note that Fraïssé $\Rightarrow$ weak Fraïssé $\Rightarrow$ (AuH)

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Examples of Fraïssé spaces

- Hilbert spaces are Fraïssé ($\varepsilon = \delta$, exercise);
- the Gurarij space is Fraïssé (actually $\varepsilon = 2\delta$) ;
- L_p is not Fraïssé for $p = 4, 6, 8, \ldots$ since not AUH.

On the other hand,

Theorem

(F.,Lopez-Abad, Mbombo, Todorcevic) The spaces $L_p[0,1]$ for $p \neq 4, 6, 8, \ldots$ are Fraïssé.

How can we get convinced that this is the relevant definition?

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Properties of Fraïssé spaces

Proposition

Assume that Y are Fraissé, and that X is separable. Then are equivalent:

1. (1) *X is finitely representable in Y*
2. (2) *every finite dimensional subspace of X embeds isometrically into Y*
3. (3) *X embeds isometrically in Y*

In particular (by Dvoretsky) ℓ_2 is the minimal separable Fraïssé space; and the Gurarij is the maximal one.

Let

- $Age(X)$=the set of finite dimensional subspaces of X, and
- for $\mathcal{F}, \mathcal{G}$ classes of finite dimensional spaces, $\mathcal{F} \equiv \mathcal{G}$ mean that any element of $\mathcal{F}$ has an isometric copy in $\mathcal{G}$ and vice-versa.

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So separable Fraïssé spaces are uniquely determined, among Fraïssé spaces, by their age modulo $\equiv$.

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We also obtained internal characterizations of classes of finite dimensional spaces which are $\equiv$ to the age of some Fraïssé ("amalgamation properties"). For such a class $\mathcal{F}$ we write X=Fraïssé lim $\mathcal{F}$ to mean "X separable and $Age(X) \equiv \mathcal{F}$"

Fraïssé is an ultraproperty

Proposition

The following are equivalent.

1) X *is Fraïssé.*
2) $X_{\mathcal{U}}$ *is Fraïssé and* $(\mathrm{Isom}(X))_{\mathcal{U}}$ *is SOT-dense in* $\mathrm{Isom}(X_{\mathcal{U}})$,
3) $X_{\mathcal{U}}$ *is ultrahomogeneous under the action of* $(\mathrm{Isom}(X))_{\mathcal{U}}$
4) $X_{\mathcal{U}}$ *is ultrahomogeneous and* $(\mathrm{Isom}(X))_{\mathcal{U}}$ *is SOT-dense in* $\mathrm{Isom}(X_{\mathcal{U}})$.

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Fraïssé is an ultraproperty

In particular, it follows that if X is Fraïssé, then its ultrapowers are Fraïssé and ultrahomogeneous.

Corollary

The non-separable L_p-space $(L_p(0,1))_{\mathcal{U}}$ is ultrahomogeneous.

A similar fact was observed for the Gurarij, by Aviles, Cabello, Castillo, Gonzalez, Moreno, 2013.

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Is there a non-Hilbertian separable ultrahomogeneous space? an ultrahomogeneous renorming of $L_p(0,1)$?

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Note the result by G. Schechtman 1979 (as observed by D. Alspach 1983)

Theorem (Schechtman)

For any $1 \leq p < \infty$ any $\varepsilon > 0$, there exists $\delta = \delta_p(\epsilon) > 0$ such that

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for every $n \in \mathbb{N}$, and finite measure μ.

So the Fraïssé property in L_p is satisfied in a strong sense for subspaces isometric to an ℓ_p^n.

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We use:

Proposition

TFAE for X:

- *X is Fraïssé*
- *$\mathrm{Isom}(X) \curvearrowright \mathrm{Emb}_\delta(E, X)$ is ε-transitive for some δ depending on ε and E*
  each $\mathcal{B}_k(X)$ is compact in the Banach-Mazur distance, where $\mathcal{B}_k(X) =$ class of k-dim. spaces isom. embeddable in X.

- It is known that $\mathcal{B}_k(L_p)$ is closed.
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For $p \notin 2\mathbb{N}$, suppose that $(f_1, \ldots, f_n) \in L_p(\Omega_0, \Sigma_0, \mu_0)$ and $(g_1, \ldots, g_n) \in L_p(\Omega_1, \Sigma_1, \mu_1)$ and

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Equidistributed here means that for any Borel $B \in \mathbb{R}^n$,

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This was used by Lusky (1978) to prove

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it is possible to prove a "continuous" version of the above, in the sense that if

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Fraissé limits of non hereditary classes

It is also possible and useful to develop the theory with respect to certain classes of finite dimensional subspaces, which are not $\equiv$ to $\mathrm{Age}(X)$, because they are not hereditary.

For $L_p(0,1)$ we can use the family of ℓ_p^n's and the perturbation result of Schechtmann to give meaning to

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Extreme amenability

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2. Fraïssé spaces
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4. Kechris-Pestov-Todorcevic correspondence

Definition

A topological group G is called extremely amenable (EA) when every continuous action $G \curvearrowright K$ on a compact K has a fixed point; that is, there is $p \in K$ such that $g \cdot p = p$ for all $g \in G$.

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Examples of extremely amenable groups

1. The group $\mathrm{Aut}(\mathbb{Q}, <)$ of strictly increasing bijections of $\mathbb{Q}$ (with the pointwise convergence topology) (Pestov,1998);
2. The group of isometries of the Urysohn space with pointwise convergence topology. (Pestov, 2002);
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The KPT correspondence

For finite structures, when $\mathcal{A}$ is the Fraïssé limit of $\mathcal{F}$, then holds the Kechris-Pestov-Todorcevic correspondence.

Theorem (Kechris-Pestov-Todorcevic)

The group $(\mathrm{Aut}(\mathcal{A}),$ ptwise cv topology$)$ is extremely amenable if and only if $\mathcal{F}$ satisfies the Ramsey property.

For example Pestov's result that $\mathrm{Aut}(\mathbb{Q}, <)$ is EA is a combination of "$(\mathbb{Q}, <) =$ Fraïssé limit of finite ordered sets" and of the classical finite Ramsey theorem on $\mathbb{N}$.

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The Approximate Ramsey Property

There is relatively well known form of the KPT correspondence, i.e. combinatorial characterization of the extreme amenability of an isometry group in terms of a Ramsey property of the Age, for metric structures.
This applies without difficulty to $(\mathrm{Isom}(X), SOT)$ for a Fraïssé Banach space X.

Definition

A collection $\mathcal{F}$ of finite dimensional normed spaces has the Approximate Ramsey Property (ARP) when for every $F, G \in \mathcal{F}$ and $r \in \mathbb{N}, \varepsilon > 0$ there exists $H \in \mathcal{F}$ such that every coloring c of $\mathrm{Emb}(F, H)$ into r colors admits an embedding $\varrho \in \mathrm{Emb}(G, H)$ which is ε-monochromatic for c.

Here ε-monochromatic means that for some color i, $\varrho \circ \mathrm{Emb}(F, G) \subset c^{-1}(i)_\varepsilon := \{\tau \in \mathrm{Emb}(F, H) : d(c^{-1}(i), \tau) < \varepsilon\}$.

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The Approximate Ramsey Property

Theorem (KPT correspondence for Banach spaces)

For X (AuH) the following are equivalent:

- $\mathrm{Isom}(X)$ *is extremely amenable.*
- $\mathrm{Age}(X)$ *has the approximate Ramsey property.*

The Approximate Ramsey Property for ℓ_p^n's

The KPT correspondence extends to the setting of ℓ_p^n-subspaces of L_p. This means we can recover the extreme amenability of $\mathrm{Isom}(L_p)$ through internal properties: i.e. through an approximate Ramsey property of isometric embeddings between ℓ_p^n's.

Theorem (Ramsey theorem for embeddings between ℓ_p^n's)

Given $1 \leq p < \infty$, integers d, m, r, and $\epsilon > 0$ there exists $n = n_p(d, m, r, \epsilon)$ such that whenever c is a coloring of $\mathrm{Emb}(\ell_p^d, \ell_p^n)$ into r colors, there is some isometric embedding $\gamma : \ell_p^m \to \ell_p^n$ which is ϵ-monochromatic.

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Comment and previous Ramsey results

- Odell-Rosenthal-Schlumprecht (1993) proved that that for every $1 \leq p \leq \infty$, every $m, r \in \mathbb{N}$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that for every coloring c of $S_{\ell_p^n}$ into r there is $Y \subset \ell_p^n$ isometric to ℓ_p^m so that S_Y is ϵ-monochromatic. Their proof uses tools from Banach space theory (like unconditionality) to find many symmetries;

- Note that Odell-Rosenthal-Schlumprecht is the case $d = 1$!

- Matoušek-Rödl (1995) proved the first result for $1 \leq p < \infty$ combinatorially (using spreads).

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Consequences

We recover the result of Giordano-Pestov through KPT correspondence, but also (through the Fraïssé Banach space notion) some non-separable versions of it.

Theorem

The topological group $(\mathrm{Isom}(L_p), SOT)$ *is extremely amenable (Giordano-Pestov).*
The topological group $(\mathrm{Isom}((L_p)_{\mathcal{U}}), SOT)$ *is also extremely amenable.*

What are the separable Fraïssé spaces?

Assume X reflexive and Fraïssé. Then X contains a copy of the Hilbert and furthermore every unitary on H extends uniquely and SOT-SOT continuously to an isometry on the envelope of H, some 1-complemented subspace of X containing H (which may be chosen to be the whole space if $X = L_p$).

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What are the separable Fraïssé spaces?

Question

Find a separable Fraïssé (or even AUH) space different from the Gurarij or some $L_p(0,1)$.

Question

Are the Hilbert and the Gurarij the only stable separable Fraïssé spaces (Fraïssé property independent of the dimension)?

Question

Are the $L_p(0,1)$ spaces stable Fraïssé for p non even?

Also:

Question

Show that $L_p(0,1)$ does not admit an ultrahomogeneous renorming if $p \neq 2$.

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