There is no largest proper operator ideal

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Outline

1. Background
2. A new class of proper ideals
3. Some details on the proof

Operator ideals in the sense of Pietsch

All notions from the 1979 Pietsch's book "Operator ideals"
An operator ideal U is a collection of subspaces $\mathrm{U(X, Y)}$ of $\mathrm{L(X, Y)}$ for all X, Y Banach spaces, such that

- $\mathrm{U(X, Y)}$ contains $\mathrm{F(X, Y)} = \{\text{finite rank operators}\}$
- (for all spaces Z, W and all appropriate operators T, V) $S \in \mathrm{U(X, Y)} \Rightarrow \mathrm{TSV} \in \mathrm{U(Z, W)}$

In particular the class $\mathrm{U(X)} := \mathrm{U(X, X)}$ is a two-sided ideal of $\mathrm{L(X)}$ in the usual sense.

Some classical operator ideals

- F=ideal of finite rank operators
- K=ideal of compact operators
- S=ideal of strictly singular

$$\mathrm{F} \subseteq \mathrm{F}^{\mathrm{closure}} \subseteq \mathrm{K} \subseteq \mathrm{S} \subseteq \mathrm{In}$$

(with strict inclusions)

- In=ideal of inessential operators

Note: recall that $S : X \to Y$ is strictly singular if $S_{|X'}$ is never an isomorphism into, for infinite dimensional $X' \subset X$.

Ideal of inessential operators $\mathrm{In}(X, Y)$

Defined by Kleinecke (63) when $X = Y$, Pietsch (78) in general.
Recall that an operator $T : X \to Y$ is Fredlholm
$\Leftrightarrow$ $\mathrm{Ker}\, T$ has finite dimension and $\mathrm{Im}\, T$ is closed of finite codimension.
$\Leftrightarrow$ T acts as an isomorphism between finite codimensional (closed) subspaces of X and Y.

Definition

$S : X \to Y$ is inessential iff $\forall T : Y \to X$, $Id_X - TS$ is Fredholm

When there exist Fredholm operators between X and Y, then more intuitively, S is inessential iff $\forall T : X \to Y$
T Fredholm $\Leftrightarrow$ $T + S$ Fredholm.

Facts about inessential operators

- that $S \subseteq In$ is a consequence of the Fredholm theory of strictly singular operators (Kato 58)
- $S \neq In$: one may use $i_{Y,X}$ where
  - X is a HI space and Y infinite codimensional subspace.
  - or X is Kalton-Peck space (79) and Y the canonical copy of ℓ_2 inside it.
  - ...

  In both cases, $\mathrm{L}(X, Y) = \mathrm{S}(X, Y)$, therefore $i_{Y,X}$ is inessential.

Proper operator ideals

Definition (Pietsch)

An ideal U is proper if for any Banach space X
Id_X belongs to U $\Leftrightarrow$ X is finite dimensional

More generally,

Definition

$\mathrm{Space(U)}$ *is the class of spaces X such that $Id_X \in \mathrm{U}$.*

Therefore U is proper iff
$\mathrm{Space(U)}=$ the class $\mathbb{F}$ of finite-dimensional spaces

This equivalently means that for any infinite-dimensional space X, $\mathrm{U(X)}$ is a proper ideal of $\mathrm{L(X)}$.

Proper ideals

Fact

The ideals $\mathrm{F}, \mathrm{K}, \mathrm{S}, \mathrm{In}$ *are proper.*

Proof (for In**).**

$T := Id_X$ is Fredholm on X; and if $\dim X = \infty$ then $T - Id_X = 0$ is not Fredholm. So $S := Id_X$ is inessential only if $\dim X < +\infty$. □

Note: more generally, Pietsch defines a procedure $\mathrm{U} \mapsto \mathrm{U}^{\mathrm{rad}} \supseteq \mathrm{U}$ for which

- $\mathrm{Space}(\mathrm{U}^{\mathrm{rad}}) = \mathrm{Space}(\mathrm{U})$ and
- $\mathrm{F}^{\mathrm{rad}} = \mathrm{In}$.

Two questions of Pietsch (79)

Question 1

Is In *the largest proper operador ideal?*

Question 2

Does there exist a largest proper operador ideal?

Theorem

The answer is no to both.

We shall try to answer the first, and shall find that we actually answer the second. We rely heavily on previous work by Aiena-González (00) and Gowers-Maurey (97) + notes by Maurey (96), and thank M. González for useful conversations. *Digression:* complex vs real...

Aiena-González's approach: improjective operators

Definition (Tarafdar 72)

An operator $T : X \to Y$ is projective if it induces an isomorphism between infinite dimensional complemented subspaces of X and Y respectively; and improjective otherwise.

Facts

- *Fredholm operators between ∞-dim spaces are projective.*
- *We have the inclusion $\mathrm{In} \subseteq \mathrm{Imp}$*
- Imp *is proper.*

Aiena-González (00) investigated whether $\mathrm{In} = \mathrm{Imp}$

Aiena-González's results

- If X is an HI space then

  $L(X)$=Fredholm $\cup$ $S(X)$

  So $S(X) = In(X) = Imp(X)$ is the largest proper ideal of $L(X)$

- Aiena-González note that if X is an indecomposable space, then

  $L(X)$=Fredholm $\cup$ $Imp(X)$

So Aiena-González need an indecomposable, non HI space.

Theorem (Gowers-Maurey 97)

There exists a Banach space $\mathbb{X}$ such that for any ∞-dim subspace Y of $\mathbb{X}$, the following are equivalent

- *Y is finite codimensional*
- *Y is complemented in $\mathbb{X}$*
- *Y is isomorphic to $\mathbb{X}$*

The space $\mathbb{X}$ is indecomposable and the isomorphism between $\mathbb{X}$ and its hyperplanes is provided by the Right Shift operator R on the basis (e_n) of $\mathbb{X}$.

Note: this space of Gowers-Maurey, the "Shift Space", was a new prime space.

Theorem (Aiena-González, last line of p. 477)

We have $\mathrm{Imp}(\mathbb{X}) \neq \mathrm{In}(\mathbb{X})$

Proof.

1 is in the essential spectrum of R, therefore $S_1 := Id_{\mathbb{X}} - R$ is not Fredholm. Therefore

(a) it is improjective

(b) but it is essential. Indeed $2Id_{\mathbb{X}} - S_1 = Id_{\mathbb{X}} + R$ is not Fredholm, since -1 is also in the essential spectrum of R

Note: for future use and wlog, replace S_1 by some compact perturbation T_1 taking value in some Y with $\dim \mathbb{X}/Y = \infty$. □

All seems good, however....

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Proposition (Aiena-González, second line of p. 478)

Imp *is not an ideal!*

Proposition (Aiena-González)

Imp *is not an ideal.*

Indeed the same as above holds for $Id_{\mathbb{X}} - \lambda R$, $|\lambda| = 1$ and associated compact perturbation T_λ. I.e. T_λ is improjective $\forall \lambda$. However $T_1 + T_{-1} = 2Id_{\mathbb{X}} + \textit{compact}$ which is projective.

Inspired by this work we prove

Proposition

There exist two proper ideals U_1 *and* U_{-1} *such that* $T_i \in \mathrm{U_i}$, *for* $i = -1, 1$.

Therefore $Id_{\mathbb{X}} \in \mathrm{U}_1(\mathbb{X}) + \mathrm{U}_{-1}(\mathbb{X})$ and it follows

Theorem

There is no largest proper ideal.

A new kind of proper ideal

Definition

If X is a Banach space, $\mathrm{Op}(X)$ is the class of operators factorizing through X.

This is an ideal as soon as, e.g., $X \simeq X^2$.

(if $T = AB$ and $T' = A'B'$ then $T + T' = \begin{pmatrix} A & A' \end{pmatrix} \begin{pmatrix} B \\ B' \end{pmatrix}$)

$X \simeq X^2$ is unlikely to happen in Gowers-Maurey setting, so...

Definition

$\mathrm{Op}^{<\omega}(X) := \cup_{n \in \mathbb{N}} \mathrm{Op}(X^n)$ *is the ideal of operators factorizing through some power of X.*

We shall prove

Proposition

If Y is infinite codimensional in $\mathbb{X}$ then the ideal $\mathrm{Op}^{<\omega}(\mathbb{X}) \cap \mathrm{Op}^{<\omega}(Y)$ is proper.

This is enough: let $\mathrm{U_i} := \mathrm{Op}^{<\omega}(\mathbb{X}) \cap \mathrm{Op}^{<\omega}(\overline{\mathrm{ImT_i}})$... then $T_i \in \mathrm{U_i}$.

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Note that for any Z, saying that $Id_Z \in \mathrm{Op}(X)$ ($Id_Z = AB$) means that Z embeds (by B) as a subspace of X which is complemented (by the projection BA).
So the Proposition follows from

Theorem

If Y is infinite codimensional in $\mathbb{X}$, and $m, n \in \mathbb{N}$, then no infinite dimensional complemented subspace Z of $\mathbb{X}^m$ embeds into Y^n.

Details on the proof of the main theorem

Theorem

If Y is infinite codimensional in $\mathbb{X}$, and $m, n \in \mathbb{N}$, then no infinite dimensional complemented subspace of $\mathbb{X}^m$ embeds into Y^n.

The proof admits the following steps

(1) Any ∞-dimensional complemented subspace of $\mathbb{X}^m$ is isomorphic to $\mathbb{X}^p$ (for some $p \leq m$)

(2) "Any" embedding of $\mathbb{X}$ into $\mathbb{X}^n$ is complemented...

(3) Use (2) to show $\mathbb{X}$ does not embed into Y^n (*)

(4) (1)(2)(3) done in the complex case. For the real case, consider the real version of $\mathbb{X}$ and show that its complexification $(\mathbb{X})_{\mathbb{C}}$ satisfies the Theorem by the same methods, then go back to $\mathbb{X}$.

(*): if it did, then $(Id_X + s)\mathbb{X} \subseteq Y$

Sketch of (1)

- By Gowers-Maurey, $\exists$ projection and algebra homomorphism λ of $\mathrm{L}(\mathbb{X})$ onto some subalgebra $\mathcal{A}$ of operators (generated by R and the left shift L), such that
  - $\forall T$, $\lambda(T) - T$ is strictly singular
  - $\exists$ an isomorphism Ψ of $\mathcal{A}$ onto the Wiener algebra $A(\mathbb{T}) \subseteq C(\mathbb{T})$ of continuous functions with absolutely summable Fourier series.

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- "Wlog" let $P \in M_m(\mathcal{A})$ be a projection on $\mathbb{X}^m$. So $\Psi(P)$ is an idempotent of $M_m(A(\mathbb{T})) \subseteq M_m(C(\mathbb{T}))$.
  Therefore for each θ, $\Psi(P)(e^{i\theta})$ is an idempotent of $M_m(\mathbb{C})$, with rank $k(\theta)$, which is constant $= k$ by continuity.

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  Therefore for each θ, $\Psi(P)(e^{i\theta})$ is an idempotent of $M_m(\mathbb{C})$, with rank $k(\theta)$, which is constant $= k$ by continuity.
- If $m = 1$ we are done (GM), because then $\Psi(P) = 0$ or 1, and $P = 0$ or $Id_{\mathbb{X}}$.
  But if say $m = 2$ and $k = 1$, the rank 1 $(2,2)$-matrix $\Psi(P)(e^{i\theta})$ may and will vary with θ.

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- "Wlog" let $P \in M_m(\mathcal{A})$ be a projection on $\mathbb{X}^m$. So $\Psi(P)$ is an idempotent of $M_m(A(\mathbb{T})) \subseteq M_m(C(\mathbb{T}))$. Therefore for each θ, $\Psi(P)e^{i\theta}$ is an idempotent of $M_m(\mathbb{C})$, with rank $k(\theta)$, which is constant $= k$ by continuity.
- K-theory ($K_1(\mathbb{C}) = \{0\}$) tells us that this rank k is the unique similarity invariant ($a \sim b \Leftrightarrow \exists c : b = cac^{-1}$) for idempotents of $M_\infty(C(\mathbb{T}))$. So $\Psi(P) \sim$ the canonical "rank k" idempotent of $M_m(C(\mathbb{T}))$, i.e. $I_k = \begin{pmatrix} Id_k & 0 \\ 0 & 0 \end{pmatrix}$, acting on $C(\mathbb{T})^m$

- A multidimensional version of Wiener's Lemma tells us that this similarity occurs inside $M_m(A(\mathbb{T}))$. So we can lift this similarity back to $M_m(\mathcal{A}) \subseteq \mathrm{L}(\mathbb{X}^m)$.
  This means that P is similar to the natural projection P_k of $\mathbb{X}^m$ onto $\mathbb{X}^k$, which implies $P\mathbb{X}^m \simeq \mathbb{X}^k$.

Questions

An abstract space ideal $\mathbb{A}$ (Pietsch) is a class of Banach spaces such that

- $\mathrm{F} \subseteq \mathbb{A}$
- $E_1, E_2 \in \mathbb{A} \Rightarrow E_1 \oplus E_2 \in \mathbb{A}$
- $F \in \mathbb{A}$ and E embeds complementably in $F \Rightarrow E \in \mathbb{A}$.

Examples:

- $\mathbb{F}$, {separable spaces}, {hilbertian spaces},
- $\mathrm{Space}(\mathrm{U})$ for any ideal U

Pietsch proves that a space ideal $\mathbb{A}$ is always $\mathrm{Space}(\mathrm{U})$ for some ideal U (*), and asks (Problem 2.2.8) whether there is always a largest U such that $\mathbb{A} = \mathrm{Space}(\mathrm{U})$. We just proved that the answer is no for $\mathbb{A} = \mathbb{F}$ but seems open for other cases...

(*) actually $\mathrm{U} = \mathrm{Op}(\mathbb{A})$

In addition: complex vs real ideals, the "forgetful functor"

Proposition

Let U *be a complex ideal, and let* u *be the real ideal defined by* $T \in \mathrm{u} \Leftrightarrow \mathrm{T}_{\mathbb{C}} \in \mathrm{U}$*. Then the following are equivalent:*

(a) *for any complex operator* T *between two complex spaces,* $T \in \mathrm{U}$ *if and only if* T *seen as real is in* u*,*

(b) U *is self-conjugate (i.e.* $\overline{T} \in \mathrm{U} \Leftrightarrow T \in \mathrm{U}$*)*

Definition

When this holds, we say that (u, U) *is a* regular *pair of ideals.*

Corollary

The pairs (s, S)*, and* (in, IN) *are regular.*
(Strictly singular, resp. Inessential operators, "respect the forgetful functor")

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