There is no largest proper operator ideal

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Outline

Operator ideals in the sense of Pietsch

All notions from the 1979 Pietsch's book "Operator ideals"
An operator ideal U is a collection of subspaces $\mathrm{U(X,Y)}$ of $\mathrm{L(X,Y)}$ for all X, Y Banach spaces, such that

- $\mathrm{U(X,Y)}$ contains $\mathrm{F(X,Y)} = \{$finite rank operators$\}$
- (for all spaces Z, W and all appropriate operators T, V) $S \in \mathrm{U(X,Y)} \Rightarrow \mathrm{TSV} \in \mathrm{U(Z,W)}$

In particular the class $\mathrm{U(X)} := \mathrm{U(X,X)}$ is a two-sided ideal of $\mathrm{L(X)}$ in the usual sense.

Some classical operator ideals

- F=ideal of finite rank operators
- K=ideal of compact operators
- S=ideal of strictly singular
- In=ideal of inessential operators

$$\mathrm{F} \subsetneq \mathrm{F}^{\mathrm{closure}} \subsetneq \mathrm{K} \subsetneq \mathrm{S} \subsetneq \mathrm{In}$$

Inessential operators were defined by Kleinecke (63) when $X = Y$, by Pietsch (78) in general.

Definition

$S : X \to Y$ is inessential iff $\forall T : Y \to X$, $Id_X - TS$ is Fredholm

When there exist Fredholm operators between X and Y, then more intuitively, S is inessential iff $\forall T : X \to Y$
T Fredholm $\Leftrightarrow$ $T + S$ Fredholm.

Proper operator ideals

Definition (Pietsch)

A class U is proper if for any Banach space X, Id_X belongs to $\mathrm{U} \Leftrightarrow X$ is finite dimensional

Therefore an ideal U is proper iff for any infinite-dimensional space X, $\mathrm{U(X)}$ is a proper ideal of $\mathrm{L(X)}$.

Fact

The ideals $\mathrm{F}, \mathrm{K}, \mathrm{S}, \mathrm{In}$ are proper.

Two questions of Pietsch (79)

Question 1

Is In *the largest proper operador ideal?*

Question 2

Does there exist a largest proper operador ideal?

Theorem

The answer is no to both. Actually:

- In *is not even a maximal proper ideal, i.e. there exists* V *proper ideal such that* $\mathrm{In} \subsetneq V$
- *moreover, there exist two proper ideals* $\mathrm{In} \subsetneq V_1, V_{-1}$ *with* $V_1 + V_{-1}$ *not proper.*

We rely heavily on previous work by Aiena-González (00) and Gowers-Maurey (97) + notes by Maurey (96), and thank M. González and A. Martínez-Abejón for useful conversations.

Aiena-González's approach: improjective operators

Definition (Tarafdar 72)

An operator $T : X \to Y$ is projective if it induces an isomorphism between infinite dimensional complemented subspaces of X and Y respectively; and improjective otherwise.

Facts

- *Fredholm operators between ∞-dim spaces are projective.*
- Imp *is proper.*
- *We have the inclusion* $\mathrm{In} \subseteq \mathrm{Imp}$; *actually for an ideal* U, U *is proper* $\Leftrightarrow \mathrm{U} \subseteq \mathrm{Imp}$.

Aiena-González (00) investigated whether $\mathrm{In} = \mathrm{Imp}$ or at least whether Imp is an ideal. In which case Imp would be the largest proper ideal.

Aiena-González's results: $\mathrm{In} \subsetneq \mathrm{Imp}$

- If X is an HI space then

  $\mathrm{L(X)}$=Fredholm(X) $\cup$ $\mathrm{S(X)}$

  So $\mathrm{S(X)} = \mathrm{In(X)} = \mathrm{Imp(X)}$ is the largest proper ideal of $\mathrm{L(X)}$. This is useless for their purpose.

- Aiena-González note that if X is an indecomposable space, then

  $\mathrm{L(X)}$=Fredholm(X) $\cup$ $\mathrm{Imp}(X)$

So Aiena-González need an indecomposable, non HI space.

Theorem (Gowers-Maurey 97)

There exists a Banach space $\mathbb{X}$ such that for any ∞-dim subspace Y of $\mathbb{X}$, the following are equivalent

- *Y is finite codimensional*
- *Y is complemented in $\mathbb{X}$*
- *Y is isomorphic to $\mathbb{X}$*

The space $\mathbb{X}$ is indecomposable and the isomorphism between $\mathbb{X}$ and its hyperplanes is provided by the Right Shift operator R on the basis (e_n) of $\mathbb{X}$.

Theorem (Aiena-González, last line of p. 477)

We have $\mathrm{In}(\mathbb{X}) \subsetneq \mathrm{Imp}(\mathbb{X})$

Proof.

1 is in the essential spectrum of R, therefore $S_1 := Id_{\mathbb{X}} - R$ is not Fredholm. Therefore

(a) it is improjective

(b) but it is essential. Indeed $2Id_{\mathbb{X}} - S_1 = Id_{\mathbb{X}} + R$ is not Fredholm, since -1 is also in the essential spectrum of R

Note: for future use and wlog, replace S_1 by some compact perturbation T_1 taking value in some Y with $\dim \mathbb{X}/Y = \infty$. □

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If Imp were an ideal, then it would be the largest proper ideal. But:

Proposition (Aiena-González, second line of p. 478)

Imp *is not an ideal!*

Proposition (Aiena-González)

Imp *is not an ideal (actually* $\mathrm{Imp}(\mathbb{X})$ *is not a linear subspace of* $\mathrm{L}(\mathbb{X})$*).*

But we can use their work by shrinking Imp to an ideal:

Proposition

There exist a proper ideal $\mathrm{U}_1 \subseteq \mathrm{Imp}$ *such that* $T_1 \in \mathrm{U}_1$*.*

Since $T_1 \notin \mathrm{In}$, let us denote $V_1 := \mathrm{In} + U_1$:

Theorem

The ideal In *is not the largest proper ideal.*
More precisely, $\mathrm{In} \subsetneq V_1$ *and* V_1 *is a proper ideal.*

But one can actually deduce more...

Indeed Aiena-González observe the same as above holds for $Id_{\mathbb{X}} - \lambda R$, $|\lambda| = 1$ (and not only for $\lambda = 1$) and associated compact perturbation T_λ. I.e. T_λ is improjective $\forall \lambda$.
However $T_1 + T_{-1} = 2Id_{\mathbb{X}} + compact$ which is projective.
This is how they deduce that $\mathrm{Imp}(\mathbb{X})$ is not a subspace of $\mathrm{L}(\mathbb{X})$.
The case of T_{-1} is "the same as" T_1 so summing up:

Proposition

There exist two proper ideals U_1 and U_{-1} such that $T_i \in \mathrm{U_i}$, for $i = -1, 1$.

Therefore $Id_{\mathbb{X}} \in \mathrm{U}_1(\mathbb{X}) + \mathrm{U}_{-1}(\mathbb{X})$ and it follows

Theorem

There exist two proper ideals whose sum is not proper. In particular there is no largest proper ideal.

A new kind of proper ideal

Summing up everything boils down to proving:

Proposition

For $i = -1, 1$, there exist a proper ideal $\mathrm{U_i}$ such that $T_i \in \mathrm{U_i}$.

Definition

If X is an (∞-dim) Banach space, $\mathrm{Op}(X)$ is the class of operators factorizing through X.

This is an ideal as soon as, e.g., $X \simeq X^2$.

(if $T = AB$ and $T' = A'B'$ then $T + T' = \begin{pmatrix} A & A' \end{pmatrix} \begin{pmatrix} B \\ B' \end{pmatrix}$)

However this is never proper if X is infinite dimensional...

Lemma

The class $\mathrm{Op}(X) \cap \mathrm{Op}(X')$ is proper iff no ∞-dimensional complemented subspace of X is isomorphic to a complemented subspace of X'

(proof: $Id_Z \in \mathrm{Op}(X)$ if and only if $Id_Z = AB$ where $A \in \mathrm{L}(X, Z), B \in \mathrm{L}(Z, X)$; this is equivalent to saying that Z embeds (by B) as a subspace of X which is complemented (by the projection BA). So $Id_Z \in \mathrm{Op}(X) \cap \mathrm{Op}(X')$ iff Z embeds complementably into both X and X'.)

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So the properties of Gowers-Maurey's $\mathbb{X}$ mean that:

Proposition

The class $\mathrm{Op}(\mathbb{X}) \cap \mathrm{Op}(Y_i)$ *is proper.*

Furthermore recall that T_i is defined on $\mathbb{X}$ and takes values in ∞-codimensional Y_i; so $T_i \in \mathrm{Op}(\mathbb{X}) \cap \mathrm{Op}(Y_i)$.

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So we enlarge $\mathrm{Op}(X)$ to:

Definition

$\mathrm{Op}^{<\omega}(X) := \cup_{n\in\mathbb{N}}\mathrm{Op}(X^n)$ *is the ideal of operators factorizing through some power of* X,

and we enlarge $\mathrm{Op}(\mathbb{X}) \cap \mathrm{Op}(Y)$ to $\mathrm{Op}^{<\omega}(\mathbb{X}) \cap \mathrm{Op}^{<\omega}(Y)$ hoping it is still proper. And indeed:

Proposition

If Y *is infinite codimensional in* $\mathbb{X}$ *then the ideal* $\mathrm{Op}^{<\omega}(\mathbb{X}) \cap \mathrm{Op}^{<\omega}(Y)$ *is proper.*

This is enough: let $\mathrm{U_i} := \mathrm{Op}^{<\omega}(\mathbb{X}) \cap \mathrm{Op}^{<\omega}(\overline{\mathrm{ImT_i}})$... then $T_i \in \mathrm{U_i}$.

So our main result follows from the technical result:

Theorem

If Y is infinite codimensional in $\mathbb{X}$, and $m, n \in \mathbb{N}$, then no infinite dimensional complemented subspace Z of $\mathbb{X}^m$ embeds complementably into Y^n.

(which involves extending the techniques of Gowers-Maurey to the multidimensional setting, as is presented at the end of this talk)

A question

An abstract space ideal $\mathbb{A}$ (Pietsch) is a class of Banach spaces such that

- $\mathrm{F} \subseteq \mathbb{A}$
- $E_1, E_2 \in \mathbb{A} \Rightarrow E_1 \oplus E_2 \in \mathbb{A}$
- $F \in \mathbb{A}$ and E embeds complementably in $F \Rightarrow E \in \mathbb{A}$.

Examples:

- $\mathbb{F}$, $\mathrm{HILB} :=$ {hilbertian spaces}, REFL, SEP,
- $\mathrm{Space}(\mathrm{U}) := \{\mathrm{X} : \mathrm{Id}_\mathrm{X} \in \mathrm{U}\}$, for any ideal U

Pietsch proves that a space ideal $\mathbb{A}$ is always $\mathrm{Space}(\mathrm{U})$ for some ideal U , and asks (Problem 2.2.8) whether there is always a largest U such that $\mathbb{A} = \mathrm{Space}(\mathrm{U})$. We just proved that the answer is no for $\mathbb{A} = \mathbb{F}$ but we can also deal with other cases...

A question

Indeed by considering ideal of operators of the form $\mathbb{A} = \mathrm{Op}^{<\omega}(\mathbb{X} \oplus \ell_2) \cap \mathrm{Op}^{<\omega}(Y \oplus \ell_2)$, and proving that $\mathrm{Space}(\mathbb{A}) = \mathrm{HILB}$, we obtain:

Proposition

There is no largest ideal U *such that* $\mathrm{Space}(\mathrm{U}) = \mathrm{HILB}$.

This extends to space ideals whose elements are "different" enough from $\mathbb{X}$.

So the cases $\mathrm{REFL}, \mathrm{SEP}$ remain open.

Details on the proof of the main technical result

Theorem

If Y is infinite codimensional in $\mathbb{X}$, and $m, n \in \mathbb{N}$, then no infinite dimensional complemented subspace of $\mathbb{X}^m$ embeds into Y^n.

The proof (in the complex case) admits the following steps

(1) Any ∞-dimensional complemented subspace of $\mathbb{X}^m$ is isomorphic to $\mathbb{X}^p$ (for some $p \leq m$)

(2) "Any" embedding of $\mathbb{X}$ into $\mathbb{X}^n$ is complemented...

(3) Use (2) to show $\mathbb{X}$ does not embed into Y^n (*)

(*): if it did, then $(Id_X + s)\mathbb{X} \subseteq Y$

Sketch of (1)

- By Gowers-Maurey, $\exists$ projection and algebra homomorphism λ of $\mathrm{L}(\mathbb{X})$ onto some subalgebra $\mathcal{A}$ of operators (generated by R and the left shift L), such that
  - $\forall T$, $\lambda(T) - T$ is strictly singular
  - $\exists$ an isomorphism Ψ of $\mathcal{A}$ onto the Wiener algebra $A(\mathbb{T}) \subseteq C(\mathbb{T})$ of continuous functions with absolutely summable Fourier series.

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- "Wlog" let $P \in M_m(\mathcal{A})$ be a projection on $\mathbb{X}^m$. So $\Psi(P)$ is an idempotent of $M_m(A(\mathbb{T})) \subseteq M_m(C(\mathbb{T}))$.
  Therefore for each θ, $\Psi(P)(e^{i\theta})$ is an idempotent of $M_m(\mathbb{C})$, with rank $k(\theta)$, which is constant $= k$ by continuity.

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  Therefore for each θ, $\Psi(P)(e^{i\theta})$ is an idempotent of $M_m(\mathbb{C})$, with rank $k(\theta)$, which is constant $= k$ by continuity.
- If $m = 1$ we are done (GM), because then $\Psi(P) = 0$ or 1, and $P = 0$ or $Id_{\mathbb{X}}$.
  But if say $m = 2$ and $k = 1$, the rank 1 $(2,2)$-matrix $\Psi(P)(e^{i\theta})$ may and will vary with θ.

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- "Wlog" let $P \in M_m(\mathcal{A})$ be a projection on $\mathbb{X}^m$. So $\Psi(P)$ is an idempotent of $M_m(A(\mathbb{T})) \subseteq M_m(C(\mathbb{T}))$. Therefore for each θ, $\Psi(P)e^{i\theta}$ is an idempotent of $M_m(\mathbb{C})$, with rank $k(\theta)$, which is constant $= k$ by continuity.
- K-theory ($K_1(\mathbb{C}) = \{0\}$) tells us that this rank k is the unique similarity invariant ($a \sim b \Leftrightarrow \exists c : b = cac^{-1}$) for idempotents of $M_\infty(C(\mathbb{T}))$. So $\Psi(P) \sim$ the canonical "rank k" idempotent of $M_m(C(\mathbb{T}))$, i.e. $I_k = \begin{pmatrix} Id_k & 0 \\ 0 & 0 \end{pmatrix}$, acting on $C(\mathbb{T})^m$

- A multidimensional version of Wiener's Lemma tells us that this similarity occurs inside $M_m(A(\mathbb{T}))$. So we can lift this similarity back to $M_m(\mathcal{A}) \subseteq \mathrm{L}(\mathbb{X}^m)$. (*that working with $A(\mathbb{T})$ instead of $C(\mathbb{T})$ does not affect the use of K-theory took me some time to get convinced of, although this is probably obvious to K-theory specialists through the notion of "local Banach algebra"*)
- This means that P is similar to the natural projection P_k of $\mathbb{X}^m$ onto $\mathbb{X}^k$, which implies $P\mathbb{X}^m \simeq \mathbb{X}^k$.

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