Group actions on exact sequences of Banach spaces

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FIRST PART: exact sequences of Banach spaces

Some technical details will be overlooked during this talk! Usually these technicalities disappear if the spaces are assumed uniformly convex.

Objects are Banach spaces, arrows are bounded linear maps.

$$0 \longrightarrow X \xrightarrow{i} Z \xrightarrow{q} Y \longrightarrow 0$$

Note that since $\mathrm{Im}\ i = \mathrm{Ker}\ q$, i has closed image, so by the *open mapping theorem* it is an isomorphic embedding, and Y identifies with the *quotient* space Z/X; Z is said to be a twisted sum of X and Y.

The sequence splits (or is trivial) when q admits a bounded linear lifting $T : Y \to Z$ ($qT = Id_Z$), which means that iX is complemented in Z (by $p = Id_Z - Tq$).

Exact sequences of (quasi)-Banach spaces

According to Kalton-Peck (1979), let $\Omega : Y \to X$ be quasi-linear, i.e. homogeneous and $\|\Omega(y + y') - \Omega(y) - \Omega(y')\| \leq K(\|y\| + \|y'\|)$. Then

$$\|(x, y)\|_\Omega := \|x - \Omega y\| + \|y\|.$$

defines a quasi-norm (i.e. the triangular inequality is replaced by $\|x + x'\| \leq C(\|x\| + \|x'\|)$ and if $X \oplus_\Omega Y$ denotes the quasi-Banach space $\{(x, y) \in X \times Y : \|(x, y)\|_\Omega < +\infty\}$, then

$$0 \to X \to^i X \oplus_\Omega Y \to^q Y \to 0,$$

with the obvious maps $i(x) = (x, 0)$ and $q(x, y) = y$. Under some conditions (e.g. uniform convexity), this is equivalent to a norm, and we have an exact sequence of Banach spaces.

Note: if $\Omega = 0$ but also if $\Omega = L$ linear (unbounded) then the sequence splits: use $Ty = (\Omega y, y)$.

Exact sequences of Banach spaces

Kalton-Peck: the sequence splits if and only if Ω is trivial, i.e. *linear* (unbounded) plus *bounded* (as a non-linear homogeneous map). More generally, $\mathrm{Ext}(Y, X)$ denotes the set of exact sequences $0 \to X \to Z \to Y \to 0$ modulo the equivalence relation: "$\exists T$ isomorphism such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Z_1 & \longrightarrow & Y & \longrightarrow & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & Z_2 & \longrightarrow & Y & \longrightarrow & 0." \end{array}$$

Kalton-Peck prove that

- every element of $\mathrm{Ext}(Y, X)$ is equivalent to some $0 \longrightarrow X \xrightarrow{i} X \oplus_\Omega Y \xrightarrow{q} Y \longrightarrow 0$
- The sequences associated to Ω_1, Ω_2 are equivalent in $\mathrm{Ext}(Y, X)$ if and only if $\Omega_1 - \Omega_2$ is trivial.

Classical solutions to the "Palais problem"

Theorem (Enflo-Lindenstrauss-Pisier 75)

There exists a non-trivial twisted Hilbert space, i.e., there exists a non-trivial exact sequence

$$0 \to \ell_2 \to ELP \to \ell_2 \to 0.$$

Theorem (Kalton-Peck 79)

This can be achieved with Ω_{KP} *defined on* ℓ_2 *by* $\Omega_{KP}(x) = x \log(x/\|x\|_2)$. *The associated exact sequence*

$$0 \longrightarrow \ell_2 \xrightarrow{i} Z_2 \xrightarrow{q} \ell_2 \longrightarrow 0$$

is even "singular" (i.e. q is strictly singular).

Entangling quasi-linear maps

Aim: use Kalton-Peck's description through quasi-linear maps in richer structure, instead of the abstract homological methods.

Important technical note:
Consider

$$0 \to X \to Z \to Y \to 0$$

Through completion and (algebraically) linear extensions, actually enough to define Ω *from* Y_0 *dense subspace of* Y *into* $X_\infty \supset X$ *(instead of from* Y *to* X*), as soon as* $\Omega(y) + \Omega(y') - \Omega(y + y')$ *stays in* X *for* $y \in Y_0$*. We write* $\Omega : Y \curvearrowright X$*.*

So for example $\Omega_{KP} : \ell_2 \curvearrowright \ell_2$ *actually goes from* ℓ_2 *to* $\mathbb{C}^{\mathbb{N}}$ *or from* c_{00} *to* ℓ_2

Objects are operator spaces and arrows are completely bounded operators.

One important difference is that there is no open mapping theorem for operator spaces. An extension sequence (Wood 99)

$$0 \longrightarrow Y \xrightarrow{i} Z \xrightarrow{q} X \longrightarrow 0$$

is such that q is a *complete surjection*, meaning that a complete isomorphism is induced between Z/Y and X.

[Corrêa 2018]: a general study of extension sequences.

Definition

The two extension sequences below are completely equivalent if there exists a complete isomorphism T:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Z_1 & \longrightarrow & Y & \longrightarrow & 0 \\ & & \| & & \downarrow T & & \| & & . \\ 0 & \longrightarrow & X & \longrightarrow & Z_2 & \longrightarrow & Y & \longrightarrow & 0. \end{array}$$

An extension sequence is completely trivial, or splits in the operator space category, if it is completely equivalent to the trivial one (i.e. $0 \to X \to X \oplus Y \to Y \to 0$).

[Paulsen 98, Wood 99]: a study of extension sequences which split (called "$\mathbb{C}$-$\mathbb{C}$-split" or "admissible" respectively).

A solution of "Palais' problem for OH"

Theorem (Corrêa 2018)

There exists an extension sequence

$$0 \to OH \to Z \to OH \to 0$$

which does not split, although it splits at the Banach space level.

In particular there exists an operator space structure Z on the Hilbert space, not completely isomorphic to OH, containing a completely isometric copy of OH such that Z/OH is completely isometric to OH.

- This extension sequence is actually "completely singular".
- To understand Corrêa's construction we need first to go back to Banach spaces.

Interpolation scales and exact sequences

[Rochberg-Weiss 1983] Calderón-Zygmund complex method $(X_\theta)_{0\leq\theta\leq 1}$ of interpolation induces a derived space, i.e., for $0 < \theta < 1$, an exact sequence

$$0 \longrightarrow X_\theta \xrightarrow{i} X_\theta \oplus_{\Omega_\theta} X_\theta \xrightarrow{q} X_\theta \longrightarrow 0$$

in such a way that

- the trivial scale $(X_0 \equiv X_1)$ induces $\Omega_\theta = 0$
- a weighted scale $(X_1 \equiv X_0(w))$ of function spaces induces $\Omega_\theta(f) = (\log\ w)f$ (on $X_0(w^\theta)$) and therefore a trivial exact sequence
- the scale of L_p-spaces (i.e. $L_2(\mu) = (L_\infty(\mu), L_1(\mu))_{1/2}$) induces the Kalton-Peck map $f \mapsto f \log(|f|/\|f\|_2)$ and therefore the associated non-trivial twisted Hilbert.

Digressions

Digression 1: some profound and beautiful work of Kalton on this. In particular, all "reasonable" twisted sums of function spaces arise from the Rochberg-Weiss theory (Kalton 1992)

Digression 2: For those who know a bit about interpolation: if $X_\theta = \{F(\theta), F \in \mathcal{F}\}$, (where $\mathcal{F}$ is the space of analytic functions adequate to the pair (X_0, X_1)) then the derived space may be described

- as $dX_\theta := \{(F'(\theta), F(\theta)), F \in \mathcal{F}\}$
- or through $\Omega_\theta(x) :=$ "$F'(\theta)$ if $F \in \mathcal{F}$ is of minimal norm such that $F(\theta) = x$"
- or through a push-out diagram.

Corrêa's solutions to Palais' problem for *OH*

Theorems (Corrêa, 2018)

- *The Rochberg-Weiss theory extends to the setting of interpolation scales of operator spaces.*
- *The identity* $OH = (\min \ell_2, \max \ell_2)_{1/2}$ *induces a "completely singular" extension sequence*

$$0 \to OH \to do(\ell_2) \to OH \to 0.$$

- *The identity* $OH = (R, C)_{1/2}$ *induces a completely singular extension sequence*

$$0 \to OH \to dOH \to OH \to 0$$

  which is not completely equivalent to the previous one.
- *Note that both are trivial at the Banach space level*

THIRD PART: entangling with group actions

This part is joint work with J.M.F. Castillo.

Fixing a group G, objects are G-spaces (X, u), i.e. Banach spaces X equiped with an action $u : G \curvearrowright X$ of G by automorphisms; and arrows are G-equivariant bounded maps (i.e. commuting with the respective actions of G). An exact sequence

$$0 \longrightarrow (X, u) \xrightarrow{\ i\ } (Z, w) \xrightarrow{\ q\ } (Y, v) \longrightarrow 0$$

means that the action of G on iX extends to an action w of G on Z inducing v on Y. For *individual* operators u, w, v this is just the classical study of “compatibility”:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \big\downarrow u & & \big\downarrow w & & \big\downarrow v & & \\
0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & Y & \longrightarrow & 0
\end{array}$$

Exact sequences of G-spaces

The notion of G-equivalence is the existence of an arrow (i.e. a G-equivariant map) T making the usual diagram commute (and therefore T is an isomorphism).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (X,u) & \longrightarrow & (Z_1,w_1) & \longrightarrow & (Y,v) & \longrightarrow & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \longrightarrow & (X,u) & \longrightarrow & (Z_2,w_2) & \longrightarrow & (Y,v) & \longrightarrow & 0 \end{array}$$

We therefore define $\mathrm{Ext}((Y,v),(X,u))$ or, keeping in mind that X, Y denote G-spaces with a fixed action of G, simply

$$\mathrm{Ext}_G(Y,X).$$

Some motivations and sources

- Riesz-Thorin theorem (1938) about interpolation scales: "if G is a group acting boundedly (appropriately) on $X_i, i = 0, 1$, then it acts boundedly on $X_\theta, 0 < \theta < 1$"
- Pytlic-Swarcz (1986) bounded non-unitarizable representation of the free group F_∞ on the Hilbert
- study of exact sequences of function spaces (Banach lattices) by Kalton, also Cabello-Sánchez (2013) ... ($G = \{\mathrm{units}\}$)
- some previous work of F.-Rosendal (2017) when the exact sequence of G-spaces splits in the Banach space category
- "compatibility of complex structures" on exact sequences of real spaces, Cuellar-Castillo-F.-Moreno (2017) ($G = \{1, i, -1, -i\}$)
- some construction of Antunes-F.-Grivaux-Rosendal (2019) related to $0 \to c_0 \to c \to \mathbb{R} \to 0$

First example: non-unitarizable representations of F_∞

Problem (Day-Dixmier)

If a countable group Γ is amenable then every bounded representation of Γ on the Hilbert is unitarizable. Is the converse true?

See the book of Pisier (2001).

Pytlic-Swarcz: Let $H = \ell_2(F_\infty)$ and $\lambda : F_\infty \curvearrowright \ell_2(F_\infty)$ the left regular unitary representation $\lambda(g)(e_h) = e_{gh}$. Let $L : \ell_1(F_\infty) \to \ell_2(F_\infty)$ be the "F_∞-left shift": $L(e_\emptyset) = 0$ and $L(e_{s_0 \ldots s_n}) = e_{s_0 \ldots s_{n-1}}$.

Then the triangular representation $g \mapsto \begin{pmatrix} \lambda(g) & [\lambda(g), L] \\ 0 & \lambda(g) \end{pmatrix}$ of F_∞ on $H \oplus H$ is bounded, non-unitarizable ($d(g) := [\lambda(g), L]$ is a "non-inner" bounded derivation)

Observation

This triangular representation may be seen as the diagonal representation $g \mapsto \begin{pmatrix} \lambda(g) & 0 \\ 0 & \lambda(g) \end{pmatrix}$ *on the trivial twisted sum* $\ell_2 \oplus_L \ell_2$.

The expression $\ell_2 \oplus_L \ell_2$ *means the completion of* $\{(x,y) \in \ell_2 \times \ell_1 : \|(x,y)\|_L := \|x - Ly\|_2 + \|y\|_2 < +\infty\}$ *under the norm* $\|.\|_L$.

Note: L is linear (unbounded with respect to the ℓ_2-norm). The associated twisted sum $\ell_2 \oplus_L \ell_2$ is trivial in the Banach space category.

Second example: SOT-discrete vs discrete orbits

Theorem (Antunes-F.-Grivaux-Rosendal, 2019)

There exists a bounded group of automorphisms on c_0 which is SOT-discrete but every non-zero orbit is indiscrete.

A solution is the group generated by $\{T_n\}$ where T_n is defined on $c_0 \oplus \mathbb{R}^2$ by

$$\begin{pmatrix} U_n & d_n \\ 0 & Id \end{pmatrix},$$

where U_n is diagonal on c_0 changing just the sign of e_n and $d_n(x) :=< x_n, x > e_n$, ($(x_n)_n$ dense in S_1)

Digression: apparently open for reflexive spaces.

Second example, reinterpreted

The group is $G = \{-1, 1\}^{<\omega}$, u is its canonical action on c_0, v its action on $\mathbb{R}^2$ by the identity, and the G-sequence is

$$0 \to (c_0, u) \to (c_0 \oplus \mathbb{R}^2, w) \to (\mathbb{R}^2, v) \to 0.$$

Here w is the bounded representation on $c_0 \oplus \mathbb{R}^2$ defined as

$$w : g \mapsto \begin{pmatrix} u(g) & d(g) \\ 0 & v(g) \end{pmatrix},$$

where $d(g) : \mathbb{R}^2 \to c_0$ is given by

$$d(g) := [u(g), R, v(g)] := u(g)R - Rv(g),$$

where $R : \mathbb{R}^2 \to \ell_\infty$ is the linear map defined by $R(x) = (< x_n, x >)_{n \in \mathbb{N}}$.

Second example, reinterpreted

This means equivalently that

$$g \mapsto \begin{pmatrix} u(g) & 0 \\ 0 & v(g) \end{pmatrix}$$

is a bounded (diagonal) representation of G on $c_0 \oplus_R \mathbb{R}^2$, i.e. on $\{(x, y) \in \ell_\infty \times \mathbb{R}^2 : x - Ry \in c_0\}$ with the norm $\|x - Ry\|_\infty + \|y\|_2$.

Generalizing: G-sequences through quasi-linear maps

An exact sequence

$$0 \longrightarrow (X,u) \stackrel{i}{\longrightarrow} (Z,w) \stackrel{q}{\longrightarrow} (Y,v) \longrightarrow 0$$

of G-spaces may be associated to a pair $(\Omega, g \mapsto d(g))$ where

- $\Omega : Y \curvearrowright X$ is defined from some dense linear G-subspace Y_0 of Y with values in some linear G-superspace X_∞ of X.
- $d(g) : Y \curvearrowright X$ is a derivation, meaning $d(gh) = u(g)d(h) + d(g)v(h)$
- the map $g \mapsto [u(g), \Omega, v(g)] + d(g)$ is uniformly bounded.

Equivalence of G-sequences

1) Two exact sequences of G-spaces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (X,u) & \longrightarrow & (Z_1,w_1) & \longrightarrow & (Y,v) & \longrightarrow & 0 \\ & & \| & & \downarrow ? & & \| & & \\ 0 & \longrightarrow & (X,u) & \longrightarrow & (Z_2,w_2) & \longrightarrow & (Y,v) & \longrightarrow & 0. \end{array}$$

associated to (Ω_1, d_1) and (Ω_2, d_2) are G-equivalent if and only if there exists a linear map $L : Y \curvearrowright X$ such that

- $\Omega_1 - \Omega_2 + L$ is bounded (with respect to the Y and X norms)
- $d_1(g) - d_2(g) = [u(g), L, v(g)]$ for all $g \in G$

2) In particular an exact sequence of G-spaces is G-trivial if and only there is a linear L such that $\Omega - L$ is bounded and $d(g) = -[u(g), L, v(g)]$ for all $g \in G$ **(mistake corrected here)**.

Equivalence of G-sequences

Theorem

If G is amenable and X is G-complemented in its bidual (e.g, if X is reflexive), then two G-sequences $0 \to X \to \diamond \to Y \to 0$ are G-equivalent if and only if they are equivalent in the Banach space category.

- the F_∞-sequence $0 \to \ell_2 \to \ell_2 \oplus \ell_2 \to \ell_2 \to 0$ associated to Pytlic-Swarcz does not split, although it splits in the Banach space category. *(Note: if $X \oplus_\Omega Y$ is the Hilbert, G-splitting is the same as the representation of G being unitarizable).*

- if $G = \{-1, 1\}^{<\omega}$, the previous G-sequence $0 \to c_0 \to c_0 \oplus_R \mathbb{R}^2 \to \mathbb{R}^2 \to 0$ does not split. Therefore $\mathrm{Ext}_G(\mathbb{R}^2, c_0) \neq \{0\}$ while $\mathrm{Ext}(\mathbb{R}^2, c_0) = \{0\}$.

Some other consequences

- Extension of the canonical action of $\mathrm{Isom}(L_p)$ on L_p to a bounded action on $L_p \oplus_{Kalton-Peck} L_p$, for $1 \leq p < +\infty, p \neq 2$.
- the previous item is false if $p = 2$ (through a result on complex structures of Castillo-Cuellar-F.-Moreno)
- An observation that any bounded derivation $d(g)$ associated to the unitary representation in the Pytlic-Swarcz example is necessarily of the form $d(g) = [\mathcal{L}, g]$ for some linear (unbounded) map $\mathcal{L}$.
- Some (converse of) characterizations of operators acting boundedly on a scale of interpolation (as in Riesz-Thorin) through commutator relations with Ω_θ
- ...

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