

# Subsequential minimality in Gowers and Maurey spaces

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## ABSTRACT

We define block sequences  $(x_n)$  in every block subspace of a variant of the space of Gowers and Maurey so that the map  $x_{2n-1} \mapsto x_{2n}$  extends to an isomorphism. This implies the existence of a subsequentially minimal HI space, which solves a question in [11].

## 1. Introduction

We start this article by motivating our result with a presentation of W.T. Gowers's program of classification of Banach spaces, and its recent developments along the lines of [11, 12] and [10].

### 1.1. Gowers' classification program

W.T. Gowers' fundamental results in geometry of Banach spaces [15, 16] opened the way to a *loose classification of Banach spaces up to subspaces*, known as Gowers' program. The aim of this program is to produce a list of classes of infinite dimensional Banach spaces such that:

- the classes are *hereditary*, i.e., stable under taking subspaces (or block subspaces),
- the classes are *inevitable*, i.e., every infinite dimensional Banach space contains a subspace in one of the classes,
- the classes are mutually disjoint,
- belonging to one class gives some information about the operators that may be defined on the space or on its subspaces.

We shall refer to such a list as a *list of inevitable classes of Gowers*. The reader interested in more details about Gowers' program may consult [16] and [11]. Let us just say that the class of spaces  $c_0$  and  $\ell_p$  is seen as the most regular class, and so, the objective of this program really is the classification of those spaces which do not contain a copy of  $c_0$  or  $\ell_p$ . We shall first give a summary of the classification obtained in [11] and of the results of Gowers that led to it.

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The first classification result of Gowers was motivated by his construction with B. Maurey of a hereditarily indecomposable (or HI) space  $GM$ , i.e., a space such that no subspace may be written as the direct sum of infinite dimensional subspaces [17]. The space  $GM$  was the first known example of a space without an unconditional sequence. Gowers then proved his *first dichotomy*.

**THEOREM 1.1** First dichotomy [15]. *Every Banach space contains either an HI subspace or a subspace with an unconditional basis.*

These were the first two examples of inevitable classes.

After  $GM$  was defined, Gowers was able to apply a criterion of P.G. Casazza to prove that an unconditional variant of Gowers-Maurey's space  $G_u$  is isomorphic to no proper subspace, solving Banach's hyperplane problem [13]. Later on Gowers and Maurey proved that  $GM$  also solves Banach's hyperplane problem, but as a consequence of general properties of HI spaces, based on Fredholm theory, rather than by applying the criterion. Let us note in passing that our main result will suggest that Casazza's criterion is indeed not satisfied in Gowers-Maurey's space.

Gowers then refined the list by proving a *second dichotomy* as a consequence of his general Ramsey theorem for block sequences [16]. A space is said to be *quasi-minimal* if any two subspaces have further subspaces which are isomorphic.

**THEOREM 1.2** Second dichotomy [16]. *Every Banach space contains a quasi-minimal subspace or a subspace with a basis such that no two disjointly supported block subspaces are isomorphic.*

Finally, H. Rosenthal had defined a space to be *minimal* if it embeds into any of its subspaces. A quasi minimal space which does not contain a minimal subspace is called *strictly quasi minimal*, so Gowers again divided the class of quasi minimal spaces into the class of strictly quasi minimal spaces and the class of minimal spaces.

Gowers deduced from these dichotomies a list of four inevitable classes of Banach spaces: HI spaces, such as  $GM$ ; spaces with bases such that no disjointly supported subspaces are isomorphic, such as  $G_u$ ; strictly quasi minimal spaces with an unconditional basis, such as Tsirelson's space  $T$  [25]; and finally, minimal spaces, such as  $c_0$  or  $\ell_p$ , but also  $T^*$ , Schlumprecht's space  $S$  [2], or its dual  $S^*$  [5] (see also [19]).

In [11] several other dichotomies for Banach spaces were obtained. The first one, called the *third dichotomy*, refines the distinction between the minimality of Rosenthal and strict quasi-minimality. Given a Banach space  $X$  with a basis  $(e_n)$ , a space  $Y$  is *tight* in  $X$  if there is a sequence of successive subsets  $I_0 < I_1 < I_2 < \dots$  of  $\mathbb{N}$ , such that the support on  $(e_n)$  of any isomorphic copy of  $Y$  intersects all but finitely many of the  $I_j$ . In other words, for any infinite subset  $J$  of  $\mathbb{N}$ ,

$$Y \not\sqsubseteq \left[ e_i : i \in \mathbb{N} \setminus \bigcup_{j \in J} I_j \right],$$

where  $\sqsubseteq$  means "embeds into".

The space  $X$  itself is *tight* if all subspaces  $Y$  of  $X$  are tight in  $X$ .

As observed in [10], the tightness of a space  $Y$  in  $X$  allows the following characterization:  $Y$  is tight in  $X$  if and only if

$$\{u \in 2^\omega : Y \sqsubseteq [e_n : n \in u]\}$$

is a meager subset of the Cantor space  $2^\omega$ . Here we identify the set to subsets of  $\omega$  with the Cantor space  $2^\omega$ , equipped with its usual topology.

After observing that the tightness property is hereditary and incompatible with minimality, the authors of [11] prove:

**THEOREM 1.3** Third dichotomy [11]. *Every Banach space contains a minimal subspace or a tight subspace.*

Special types of tightness may be defined according to the way the  $I_n$  may be chosen in function of  $Y$ . It is observed in [11] that the actual known examples of tight spaces satisfy one of two stronger forms of tightness, called *by range*, and *with constants*. Thus e.g. Gowers unconditional space  $G_u$  is tight by range, and Tsirelson's space  $T$  is tight with constants, see also [12] for other examples.

We shall be mainly interested in tightness by range, which we define in the next subsection. We refer to the end of the paper for definitions and comments about tightness with constants.

## 1.2. Ranges and supports

The following distinction is essential. If  $X$  is a space with a basis  $(e_i)_i$ , then the definition of the *support*  $\text{supp } x$  of a vector  $x$  is well-known: it is the set  $\{i \in \mathbb{N} : x_i \neq 0\}$ , where  $x = \sum_{i \in \mathbb{N}} x_i e_i$ . On the other hand the *range*,  $\text{ran } x$ , of  $x$  is the smallest interval of integers containing its support. So of course, having finite range and having finite support are the same, but the range is always an interval of integers, while the support may be an arbitrary subset of  $\mathbb{N}$ .

If  $Y = [y_n : n \in \mathbb{N}]$  is a block subspace of  $X$ , then the support of  $Y$  is  $\cup_{n \in \mathbb{N}} \text{supp } y_n$ , and the range of  $Y$  is  $\cup_{n \in \mathbb{N}} \text{ran } y_n$ .

Let us now recall the criterion of Casazza, which appears in [13]. Two basic sequences  $(x_n)_{n \in \mathbb{N}}$  and  $(y_n)_{n \in \mathbb{N}}$  are said to be equivalent if the map  $x_n \mapsto y_n$  extends to an isomorphism of  $[x_n, n \in \mathbb{N}]$  onto  $[y_n, n \in \mathbb{N}]$ .

**PROPOSITION 1.4** [4]. *Let  $X$  be a Banach space with a basis. Assume that for any block sequence  $(x_n)$  in  $X$ ,  $(x_{2n})$  is not equivalent to  $(x_{2n+1})$ . Then  $X$  is isomorphic to no proper subspace.*

The criterion of Casazza leads to studying the possible isomorphisms between disjointly supported or disjointly ranged subspaces. As proved in [11], this turns out to have an essential connection with the notion of tightness. In what follows we shall say that two spaces are *comparable* if one embeds into the other.

If no two disjointly supported block-subspaces are isomorphic, then equivalently no two such subspaces are comparable. This is also equivalent to saying that for every block subspace  $Y$ , spanned by a block sequence  $(y_n)$ , the sequence of successive subsets  $I_0 < I_1 < \dots$  of  $\mathbb{N}$  witnessing the tightness of  $Y$  in  $(e_n)$  may be defined by  $I_k = \text{supp } y_k$  for each  $k$ . When this happens it is said that  $X$  is *tight by support* [11]. So Gowers' second dichotomy may be interpreted as a dichotomy between a form of tightness and a form of minimality, and  $G_u$  is tight by support.

If now for every block subspace  $Y = [y_n]$ , the sequence of successive subsets  $I_0 < I_1 < \dots$  of  $\mathbb{N}$  witnessing the tightness of  $Y$  in  $(e_n)$  may be defined by  $I_k = \text{ran } y_k$  for each  $k$ , then  $X$  is said to be *tight by range*. This is equivalent to no two block subspaces with disjoint ranges being comparable, a property which is formally weaker than tightness by support. Note that the

criterion of Casazza applies to prove that a space which is tight by range cannot be isomorphic to its proper subspaces.

The distinction between range and support is relevant here. While it is easy to check that a basis which is tight by support must be unconditional, it is proved in [12] that HI spaces may be tight by range; this is the case of an asymptotically unconditional and HI variant of Gowers-Maurey’s space, due to Gowers [14], which we denote by  $G$ .

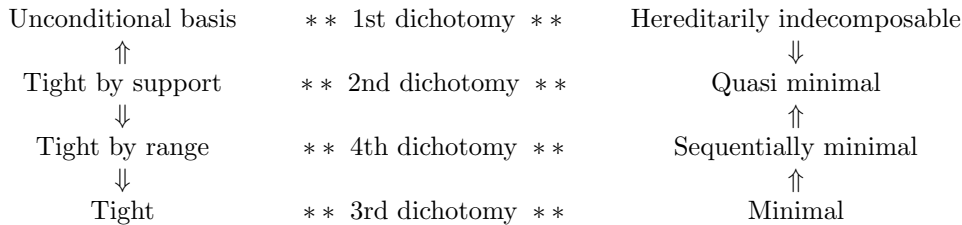
In [11] it was proved that there also exists a dichotomy relative to tightness by range. The authors define a space  $X$  with a basis  $(x_n)$  to be *subsequentially minimal* if every subspace of  $X$  contains an isomorphic copy of a subsequence of  $(x_n)$ . Tsirelson’s space  $T$  is the classical example of subsequentially minimal, non-minimal space.

**THEOREM 1.5** Fourth dichotomy [11]. *Any Banach space contains a subspace with a basis which is either tight by range or subsequentially minimal.*

The second case in Theorem 1.5 may be improved to the following hereditary property of a basis  $(x_n)$ , that is called *sequential minimality*:  $(x_n)$  is quasi minimal and every block sequence of  $[x_n]$  has a subsequentially minimal block sequence.

### 1.3. The list of 6 inevitable classes

The first four dichotomies and the interdependence of the properties involved can be visualized in the following diagram.



The easy observation that HI spaces are quasi-minimal is due to Gowers (see subsection 1.5). On the other hand it was shown in [17, Corollary 19] and [17, Theorem 21] that an HI space cannot be isomorphic to any proper subspace. This implies that an HI space cannot contain a minimal subspace.

Therefore by the third dichotomy, every HI space must contain a tight subspace, but it is unknown whether every HI space with a basis must itself be tight.

Combining the four dichotomies and the relations between them, the following list of 6 classes of Banach spaces contained in any Banach space is obtained in [11]:

**THEOREM 1.6** [11]. *Any infinite dimensional Banach space contains a subspace of one of the types listed in the following chart:*

<i>Type</i>	<i>Properties</i>	<i>Examples</i>
(1)	<i>HI, tight by range</i>	$G, G^*$
(2)	<i>HI, tight, sequentially minimal</i>	$\mathcal{X}_{GM}$
(3)	<i>tight by support</i>	$G_u, G_u^*, X_u, X_u^*, X_{abr}$
(4)	<i>unconditional basis, tight by range, quasi minimal</i>	?
(5)	<i>unconditional basis, tight, sequentially minimal</i>	$T, T^{(p)}$
(6)	<i>unconditional basis, minimal</i>	$S, S^*, T^*, c_0, \ell_p$

For information about the examples appearing in type (1) and (3)–(6) we refer to [12]. Two major open problems of [11] were whether spaces of type (2) or (4) existed. The only known proofs of sequential minimality used properties which implied unconditionality, so presumably the construction of a type (2) space would require new methods.

The main result of this paper is the existence of an example  $\mathcal{X}_{GM}$  of type (2), similar to Gowers-Maurey’s space, which is reported on the chart above.

#### 1.4. The main result

THEOREM 1.7. *There exists a version  $\mathcal{GM}$  of Gowers-Maurey’s space such that*

- (a)  $\mathcal{GM}$  does not contain an unconditional basic sequence.
- (b) Any block subspace of  $\mathcal{GM}$  contains a block sequence  $(y_n)_n$  such that  $(y_{2n})$  is equivalent to  $(y_{2n+1})$ .

The proof of Theorem will be accomplished in Section 6, Theorem 6.2. The modification leading to  $\mathcal{GM}$  is essentially technical. Note that by (a) and the first dichotomy,  $\mathcal{GM}$  contains an HI subspace. So this subspace is not isomorphic to its proper subspaces, although by Theorem 1.7 (b), it does not satisfy Casazza’s criterion. Using also the third and fourth dichotomy, we deduce that some subspace of  $\mathcal{GM}$  satisfies:

THEOREM 1.8. *There exists a tight, HI, sequentially minimal space  $\mathcal{X}_{GM}$ .*

It may be surprising to see that the answer to the existence of type (2) spaces is given by a modification of the first known example of HI space. We actually believe that  $GM$  itself satisfies Theorem 1.7 (b), and therefore fails to satisfy the criterion of Casazza.

We shall also observe that the space  $\mathcal{GM}$  is *locally minimal*, which means that all finite dimensional subspaces of  $\mathcal{GM}$  embed into all its infinite dimensional subspaces, with uniform constant. Problem 5.2 from [12] asked whether a sequentially and locally minimal should be minimal or at least contain a minimal subspace. We therefore answer this by the negative.

THEOREM 1.9. *There exists a locally minimal, sequentially minimal, tight space.*

To conclude this subsection let us mention that our results hold both in the real and in the complex setting.

### 1.5. Some comments on our construction

Let us first recall why HI spaces are quasi-minimal. Let  $X$  be an HI space with a basis. If  $\epsilon > 0$ , and two block-subspaces  $U$  and  $V$  of  $X$  are given, then one can use the HI property to obtain two normalized block-sequences  $(u_n)_n$  and  $(v_n)_n$  in  $U$  and  $V$  respectively, so that  $\|u_n - v_n\| \leq \epsilon 2^{-n}$  for all  $n \in \mathbb{N}$ . So there is a compact perturbation of the canonical injection mapping  $[u_n, n \in \mathbb{N}]$  onto  $[v_n, n \in \mathbb{N}]$ , which are therefore isomorphic. Note that if  $U$  and  $V$  are disjointly supported, or even disjointly ranged, then each  $u_n$  is disjointly supported from  $v_n$ .

If now we want to obtain a canonical isomorphism between  $[u_n, n \in \mathbb{N}]$  and  $[v_n, n \in \mathbb{N}]$ , so that  $(u_n)$  and  $(v_n)$  are disjointly ranged and seminormalized block sequences, then such a crude approach does not work. Let us explain this when  $(u_n)$  and  $(v_n)$  are *intertwined*, i.e.  $u_1 < v_1 < u_2 < v_2 < u_3 < \dots$ . By using the projection on the range of  $u_n$ , we see that the norm  $\|u_n - v_n\|$  is bounded below by a constant depending on the constant of the basis, and so the map  $u_n \mapsto u_n - v_n$  can never be compact. We may however hope to pick  $u_n$  and  $v_n$  so that this map is strictly singular. Actually in the case when  $X$  is, say, complex HI, we must do so. Indeed, we know in this case [9] that there must exist  $\lambda \in \mathbb{C}$  and a strictly singular operator  $S : [u_j : j \in \mathbb{N}] \rightarrow X$ , such that  $v_n - \lambda u_n = S(u_n)$ ; so by projecting on  $\text{ran } v_n$  we get that  $S(u_n)$  is bounded below, and that  $S$  is strictly singular non compact from  $[u_n, n \in \mathbb{N}]$  into  $X$ .

So our result of existence of two intertwined and equivalent block sequences in any subspace of  $\mathcal{GM}$  will be related to the techniques of the construction of strictly singular non-compact operators on subspaces of Schlumprecht's space  $S$  and of  $GM$  type spaces, as appears in [1] and [23]. We shall replace the condition that  $\|u_n - v_n\| \leq \epsilon 2^{-n}$  by the requirement that the sequence  $u_n - v_n$  generates a spreading model which is "largely" dominated by the spreading models of  $u_n$ ,  $v_n$  and  $u_n + v_n$ . From some techniques of [23], this will imply that the map taking  $u_n$  to  $u_n - v_n$  extends to a bounded (actually strictly singular) map, and the same for the map taking  $v_n$  to  $u_n - v_n$ . Therefore  $(u_n)$  and  $(v_n)$  will be equivalent.

Note that our estimates will imply that  $\|\sum_{i=1}^k (u_{n_i} - v_{n_i})\| \leq \epsilon \|\sum_{i=1}^k (u_{n_i} + v_{n_i})\|$  whenever  $k < n_1 < \dots < n_k$  and  $k$  is large enough with respect to  $\epsilon$ . Thus we recover the result of saturation of  $GM$  with finite block-sequence  $(y_i)_{i=1}^{2k}$  such that  $\|\sum_{i=1}^{2k} (-1)^i y_i\| \leq \epsilon \|\sum_{i=1}^{2k} y_i\|$ , for some  $k = k(\epsilon)$  large enough, but of course our result is much stronger, since we can choose  $(y_i)$  to be any finite subsequence  $(u_{n_1}, v_{n_1}, \dots, u_{n_k}, v_{n_k})$  as above. This estimate implies that  $\mathcal{GM}$  does not contain a subspace with an asymptotically unconditional basis, which means by Gowers' dichotomy that  $\mathcal{GM}$  has a subspace which is HI (and even, by [26], satisfies the HI property in a "uniform" way).

### 1.6. Some preliminary definitions

We use the usual definitions and notation for  $c_{00}$ ,  $(e_i)$ ,  $E(x)$ ,  $\text{supp}(x)$ ,  $\text{ran}(x)$ ,  $E < F$  and  $x < y$  for  $E, F \subset \mathbb{N}$ , and  $x, y \in c_{00}$ . The closed linear span of a basic sequence  $(x_n)_{n \in \mathbb{N}}$  is denoted  $[x_n, n \in \mathbb{N}]$ .

We say that two vectors  $x$  and  $y$  in  $c_{00}$  *have the same distribution* and write  $x =_{\text{dist}} y$  if there are natural numbers  $l$ ,  $m_1 < m_2 < \dots < m_l$ , and  $n_1 < n_2 < \dots < n_l$ , and a sequence  $(a_i : i = 1, 2, \dots, l) \subset \mathbb{R}$ , so that

$$x = \sum_{i=1}^l a_i e_{m_i} \text{ and } y = \sum_{i=1}^l a_i e_{n_i}.$$

We say  $x$  is the distribution of  $y$  if  $x$  and  $y$  have the same distribution and if the support of  $x$  is an initial interval of  $\mathbb{N}$ .

Note that a vector  $x \in c_{00}$  is uniquely defined by its distribution and its support.

DEFINITION 1.10. Let  $X$  be a Banach space with a basis  $(e_i)$ . We call a vector  $x$  in  $X$  an  $\ell_1^{+n}$ -average, if  $x = \frac{1}{n} \sum_{i=1}^n x_i$ , where  $(x_i)_{i=1}^n$  is a block sequence (of  $(e_i)$ ) in  $B_X$ . For  $c \in (0, 1]$  an  $\ell_1^{+n}$ -average  $x$  is called  $\ell_1^{+n}$ -average of constant  $c$ , if  $\|x\| \geq c$ .

If moreover  $(x_i)_{i=1}^n$  is  $\frac{1}{c}$ -isomorphic to the  $\ell_1^n$  unit vector basis, we say that  $x$  is an  $\ell_1^n$ -average of constant  $c$ . In particular it follows in that case that  $\|\sum_{i=1}^n \pm x_i\| \geq c$ .

REMARK 1. For a minor technical reason, we are not assuming in Definition 1.10 that the sequence  $(x_i)$  is normalized. But of course if  $x$  is supposed to be an  $\ell_1^{+n}$ - or an  $\ell_1^n$ -average, of a constant  $c$  close to 1, then the norm of most of the  $x_i$  has to be close to 1 too.

## 2. The space $S$

We recall the space introduced in [22]. We define

$$f(x) = \log_2(x + 1), \text{ for } x \geq 1. \tag{2.1}$$

The space  $S$  is the completion of  $c_{00}$  under the norm  $\|\cdot\|_S$  which satisfies the following implicit equation.

$$\|x\|_S = \max \left( \|x\|_\infty, \max_{\substack{l \in \mathbb{N} \\ E_1 < E_2 < \dots < E_l}} \frac{1}{f(l)} \sum_{j=1}^l \|E_j(x)\|_S \right) \text{ for } x \in c_{00}. \tag{2.2}$$

As observed in [22], there is a norm  $\|\cdot\|_S$  on  $c_{00}$ , which satisfies Equation (2.2), the completion  $S$  of  $(c_{00}, \|\cdot\|_S)$  is reflexive, and  $(e_i : i \in \mathbb{N})$  is a 1-subsymmetric (i.e 1-spreading and 1-unconditional) basis of  $S$ .

For  $l = 2, 3 \dots$  and  $x \in S$  we define

$$\|x\|_l = \frac{1}{f(l)} \max_{E_1 < E_2 < \dots < E_l} \sum_{j=1}^l \|E_j(x)\|_S.$$

Then  $\|\cdot\|_l$  is an equivalent norm on  $S$  and for  $x \in S$ ,

$$\frac{1}{f(l)} \|x\| \leq \|x\|_l \leq \|x\| \text{ and} \tag{2.3}$$

$$\|x\| = \max \left( \|x\|_\infty, \sup_{l \in \mathbb{N}} \|x\|_l \right). \tag{2.4}$$

### 2.1. Upper bounds of $\|\cdot\|_S$

We will need to show some upper estimates for  $\|\cdot\|_S$  and for basic sequences which have spreading models equivalent to the unit basis in  $S$ .

DEFINITION 2.1. For a bounded sequence  $(\xi_i)$  in  $\mathbb{R}$  we denote the decreasing rearrangement of  $(|\xi_i|)$  by  $(\xi_i^\#)$ .

Assume that  $g : [1, \infty) \rightarrow [1, \infty)$  is an increasing function with  $g(1) = 1$ . We define the following two norms on  $c_{00}$ . For  $x = (x_i) \in c_{00}$  we define

$$\|x\|_g = \max_{n_1 < n_2 < \dots < n_l, l \in \mathbb{N}} \frac{1}{g(l)} \sum_{i=1}^l |x_{n_i}| \text{ and } \|x\|_g = \sum_{i=1}^\infty \frac{1}{g(i)} x_i^\#.$$

It is clear that  $\|\cdot\|_g \leq \|\cdot\|_{g^p}$ . The following Lemma describes a situation in which we can bound  $\|\cdot\|_{g^p}$ ,  $0 < p < q$  by a multiple of  $\|\cdot\|_{g^q}$ .

LEMMA 2.2. *For  $0 < p < q$  there is a constant  $C(p, q)$  so that*

$$\|x\|_{f^q} \leq C(p, q)\|x\|_{f^p}, \text{ for all } x \in c_{00}.$$

Here  $f : [0, \infty) \rightarrow [0, \infty)$  is defined as in (2.1) by  $f(x) = \log_2(x+1)$ , for  $x \geq 1$ .

*Proof.* We first observe that

$$C(p, q) = \sum_{n=1}^{\infty} \frac{f^p(n) - f^p(n-1)}{f^q(n)} < \infty.$$

Indeed, by the Mean Value Theorem, there is for every  $n \in \mathbb{N}$  an  $\eta_n \in (n, n+1)$ , so that

$$f^p(n) - f^p(n-1) = (\log_2(e))^p \frac{\partial \ln^p(x)}{\partial x} \Big|_{x=\eta_n} = (\log_2(e))^p p \frac{\ln^{p-1}(\eta_n)}{\eta_n},$$

and thus

$$(\log_2(e))^{q-p} \sum_{n=2}^{\infty} \frac{f^p(n) - f^p(n-1)}{f^q(n)} = \sum_{n=2}^{\infty} \frac{1}{\eta_n} \frac{1}{\ln^{1-p}(\eta_n) \ln^q(n+1)} < \sum_{n=2}^{\infty} \frac{1}{n \ln^{1+q-p}(n)},$$

which is finite by the integral test.

Secondly we claim that for  $L \in \mathbb{N}$

$$M_L = \max\{\|x\|_{f^q} : \text{ran}(x) \subset [1, L], \text{ and } \|x\|_{f^p} \leq 1\},$$

is achieved for the vector

$$x^{(L)} = \sum_{j=1}^L (f^p(j) - f^p(j-1))e_j \text{ with } f(0) = 0,$$

which would imply that  $M_L \leq C(p, q)$ . Indeed,  $\|x^{(L)}\|_{f^p} = 1$ , and if  $z = (z_i)_{i=1}^L \in \mathbb{R}^L$ ,  $\|z\|_{f^p} = 1$ , and

$$\|z\|_{f^q} = \sum_{j=1}^L \frac{z_j^\#}{f^q(j)} = M_L,$$

we can assume without loss of generality that  $z_1 \geq z_2 \geq \dots \geq z_L \geq 0$ . Note that actually  $z_L > 0$ . Otherwise let  $l_0 = \min\{j : z_i = 0 \text{ for all } i \geq j\}$ , and note for  $l \geq l_0$  that

$$\frac{1}{f^p(l)} \sum_{j=1}^l z_j = \frac{1}{f^p(l)} \sum_{j=1}^{l_0-1} z_j < \frac{1}{f^p(l_0-1)} \sum_{j=1}^{l_0} z_j \leq 1.$$

Thus we could increase the value of  $z_{l_0}$ , and thus increase the value of  $\|z\|_{f^q}$ , without increasing the value of  $\|z\|_{f^p}$ , which contradicts the maximality of  $z$ .

We want to show now that  $z = x^{(L)}$ , which would imply our claim. If this were not true we put

$$l_o = \min\{j \in \{1, 2, \dots, L\} : z_j \neq f^p(j) - f^p(j-1)\}.$$

First we note that  $z_{l_o} < f^p(j) - f^p(j-1)$ , because otherwise  $z_{l_o} > f^p(j) - f^p(j-1)$  and thus  $\frac{1}{f^p(l_o)} \sum_{j=1}^{l_o} z_j > \frac{1}{f^p(l_o)} \sum_{j=1}^{l_o} (f^p(j) - f^p(j-1)) = 1$ ,

Note that  $l_o \neq L$  otherwise we could increase  $z_L$  to  $f^p(L) - f^p(L-1)$ , which would not increase  $\|z\|_{f^p}$ , but certainly increase  $\|z\|_{f^q}$ . If  $l_o < L$  we could increase  $z_{l_o}$  by  $\min(f^p(l_o) - f^p(l_o-1), z_{l_o}) > 0$  and decrease  $z_{l_o+1}$  by the same amount. This would not increase the  $\|\cdot\|_{f^q}$ -norm but it would increase the  $\|\cdot\|_{f^p}$ -norm of  $z$ .  $\square$



The proof of the next Lemma could be shown using [23, Theorem 1.1] and its proof. Nevertheless, since the arguments in this case are much simpler we prefer to present a self contained argument.

Recall that a seminormalized basic sequence  $(x_n)$  generates the spreading model  $E$  with basis  $(\tilde{x}_n)$  if for any  $\epsilon > 0$  and every  $k \in \mathbb{N}$ , there exists  $N \in \mathbb{N}$  such that for any  $N \leq n_1 < n_2 < \dots < n_k$ , the sequence  $(x_{n_i})_{1 \leq i \leq k}$  is  $1 + \epsilon$ -equivalent to  $(\tilde{x}_i)_{1 \leq i \leq k}$ . From this it follows that  $(\tilde{x}_n)$  is necessarily a 1-spreading basis of  $E$ . We also recall [3] that any seminormalized basic sequence contains a subsequence which generates a spreading model.

LEMMA 2.3. *Let  $(x_n)$  and  $(y_n)$  be two basic seminormalized weakly null sequences in a Banach space  $X$ , having spreading models  $E$  and  $F$  with bases  $(\tilde{x}_n)$  and  $(\tilde{y}_n)$ , respectively. Assume that for some  $0 < p < q$  and some  $0 < c, C < \infty$  it follows that*

$$\left\| \sum_{i=1}^{\infty} a_n \tilde{x}_n \right\|_E \geq c \left\| \sum_{i=1}^{\infty} a_n e_n \right\|_{f^p} \quad \text{and} \quad \left\| \sum_{i=1}^{\infty} a_n \tilde{y}_n \right\|_F \leq C \left\| \sum_{i=1}^{\infty} a_n e_n \right\|_{f^q}. \quad (2.5)$$

Then there is a subsequence  $(n_k)$  of  $\mathbb{N}$  so that the map  $x_{n_k} \mapsto y_{n_k}$  extends to a linear bounded operator.

REMARK 2. *Using the arguments in [23] one can actually show that under the assumption of Lemma 2.3 there is a subsequence  $(n_k)$  of  $\mathbb{N}$  so that the map  $x_{n_k} \mapsto y_{n_k}$  extends to a linear bounded and strictly singular operator.*

Before proving Lemma 2.3 we will need the following

LEMMA 2.4. *Assume  $0 < p < q$  and define for  $\epsilon > 0$*

$$\Delta_{(p,q)}(\epsilon) = \sup \left\{ \left\| \sum_{i=1}^{\infty} a_i e_i \right\|_{f^q} : |a_i| \leq \epsilon, i = 1, 2, \dots, \text{ and } \left\| \sum_{i=1}^{\infty} a_i e_i \right\|_{f^p} \leq 1 \right\}. \quad (2.6)$$

Then

$$\lim_{\epsilon \searrow 0} \Delta_{(p,q)}(\epsilon) = 0. \quad (2.7)$$

*Proof.* Let  $\eta > 0$  be arbitrary and choose  $n_\eta \in \mathbb{N}$  so that  $\frac{1}{f^{q-p}(n)} \leq \eta$ , for all  $n \geq n_\eta$ , and then choose

$$\epsilon = \eta \min_{n \leq n_\eta} \frac{f^q(n)}{n}.$$

For any  $(a_i) \in c_{00}$ , with  $|a_i| \leq \epsilon$ , for  $i \in \mathbb{N}$ , and  $\left\| \sum_{i=1}^{\infty} a_i e_i \right\|_{f^p} \leq 1$  it follows therefore that, for some choice of  $n \in \mathbb{N}$  and  $i_1 < i_2 < \dots < i_n$  in  $\mathbb{N}$ , we have

$$\left\| \sum_{i=1}^{\infty} a_i e_i \right\|_{f^q} = \frac{1}{f^q(n)} \sum_{s=1}^n |a_{i_s}| \leq \begin{cases} \epsilon \frac{n}{f^q(n)} \leq \eta & \text{if } n \leq n_\eta \\ \frac{1}{f^{q-p}(n)} \left\| \sum_{i=1}^{\infty} a_i e_i \right\|_{f^p} \leq \eta & \text{if } n > n_\eta, \end{cases}$$

which verifies our claim. □

*Proof of Lemma 2.3.* We can assume that  $(\tilde{y}_n)$  is not equivalent to the  $c_0$  unit vector basis. Otherwise we may replace the norm on  $[y_n : n \in \mathbb{N}]$  by

$$\left\| \sum a_i y_i \right\| = \left\| \sum a_i y_i \right\| + \left\| \sum a_i e_i \right\|_{f^q} \quad \text{if } (a_i) \in c_{00}.$$

We can therefore assume that for every  $\varepsilon > 0$  the number

$$l(\varepsilon) = \max \left\{ l : \begin{array}{l} \text{there are } (a_i)_{i=1}^l, |a_i| \geq \varepsilon, i = 1, 2, \dots, l, \text{ and} \\ n_1 < n_2 < \dots < n_l, \text{ so that } \left\| \sum_{i=1}^l a_i y_{n_i} \right\| \leq 1 \end{array} \right\}$$

exists.

Let  $r = (p + q)/2$ . By Lemma 2.4 we can choose a sequence  $(\varepsilon_n) \subset (0, 1)$  so that

$$\sum_{n \in \mathbb{N}} \Delta_{(p,r)}(c\varepsilon_n/6) \leq 1 \text{ and } \sum_{n=1}^{\infty} n\varepsilon_n \leq 1. \quad (2.8)$$

Using the *Schreier unconditionality* of basic sequences [20] (see also [6] for a more general statement), the fact that  $(\tilde{x}_n)$  is the spreading model of  $(x_n)$ , and our assumption (2.5), we can assume, after passing to simultaneous subsequences of  $(x_n)$  and  $(y_n)$ , if necessary, that for all  $(a_i) \in c_{00}$  and all finite  $F \subset \mathbb{N}$ , with  $n \leq \min F$  and  $\#F \leq l(\varepsilon_{n+1})$  we have

$$\left\| \sum_{i \in F} a_i e_i \right\|_{f^p} \leq \frac{1}{c} \left\| \sum_{i \in F} a_i \tilde{x}_i \right\| \leq \frac{2}{c} \left\| \sum_{i \in F} a_i x_i \right\| \leq \frac{6}{c} \left\| \sum_{i=1}^{\infty} a_i x_i \right\|, \quad (2.9)$$

and by using the fact that  $(\tilde{y}_n)$  is the spreading model of  $(y_n)$ , our assumption (2.5), and Lemma 2.2, we can assume that for some constant  $C_3$  and for all finite  $F \subset \mathbb{N}$ , with  $n \leq \min F$  and  $\#F \leq l(\varepsilon_{n+1})$ , and all  $(a_i)_{i \in F}$  we have

$$\left\| \sum_{i \in F} a_i y_i \right\| \leq 2 \left\| \sum_{i \in F} a_i \tilde{y}_i \right\| \leq 2C \left\| \sum_{i \in F} a_i \right\|_{f^q} \leq C_3 \left\| \sum_{i \in F} a_i e_i \right\|_{f^r}. \quad (2.10)$$

By Elton's *near unconditionality* [7] (see also [6, Theorem 6]) and the fact that  $l(\varepsilon_1)$  is finite we can assume, after passing to subsequences, if necessary, that there are constants  $C_1$  and  $C_2$  so that for every  $(a_i) \in c_{00}$ , with  $\left\| \sum a_i x_i \right\| \leq 1$ , it follows that

$$\left\| \sum_{j=1, |a_j| > \varepsilon_1}^{\infty} a_j y_j \right\| \leq C_1 \left\| \sum_{j=1, |a_j| > \varepsilon_1}^{\infty} a_j x_j \right\| \leq C_2 \left\| \sum_{j=1}^{\infty} a_j x_j \right\| \leq C_2. \quad (2.11)$$

Now let  $(a_j) \in c_{00}$  and assume that  $\left\| \sum_{i=1}^n a_i x_i \right\| = 1$ . Then, by (2.11),

$$\left\| \sum_{j=1}^{\infty} a_j y_j \right\| \leq \left\| \sum_{|a_j| \leq \varepsilon_1} a_j y_j \right\| + \left\| \sum_{|a_j| > \varepsilon_1} a_j y_j \right\| \leq \left\| \sum_{|a_j| \leq \varepsilon_1} a_j y_j \right\| + C_2 \left\| \sum_{j=1}^{\infty} a_j x_j \right\|$$

and

$$\begin{aligned} \left\| \sum_{j=1, |a_j| \leq \varepsilon_1}^{\infty} a_j y_j \right\| &\leq \sum_{n=1}^{\infty} \left\| \sum_{\varepsilon_{n+1} < |a_j| \leq \varepsilon_n} a_j y_j \right\| \\ &\leq \sum_{n=1}^{\infty} \left[ n\varepsilon_n + \left\| \sum_{n < j, \varepsilon_{n+1} < |a_j| \leq \varepsilon_n} a_j y_j \right\| \right] \\ &\leq 1 + C_3 \sum_{n=1}^{\infty} \left\| \sum_{n < j, \varepsilon_{n+1} < |a_j| \leq \varepsilon_n} a_j e_j \right\|_{f^r} \text{ (by (2.8) and (2.10))} \\ &= 1 + \frac{6C_3}{c} \sum_{n=1}^{\infty} \left\| \sum_{n < j, \varepsilon_{n+1} < |a_j| \leq \varepsilon_n} \frac{c}{6} a_j e_j \right\|_{f^r}. \end{aligned}$$

Now it follows from (2.9) that

$$\left\| \sum_{n < j, \varepsilon_{n+1} < |a_j| \leq \varepsilon_n} \frac{c}{6} a_j e_j \right\|_{f^p} \leq 1$$

and thus (2.8) and the definition of  $\Delta_{p,r}$  yield that

$$\left\| \sum_{j=1, |a_j| \leq \varepsilon_1}^{\infty} a_j y_j \right\| \leq 1 + \frac{6C_3}{c}.$$

We proved therefore that if  $\left\| \sum_{i=1}^{\infty} a_i x_i \right\| \leq 1$  then

$$\left\| \sum_{i=1}^{\infty} a_i y_i \right\| \leq C_2 + 1 + \frac{6C_3}{c},$$

which finishes the proof of our claim.

We finally want to compare the norms  $\|\cdot\|_f$  and  $\|\cdot\|_S$  and first prove the following Lemma.

LEMMA 2.5. *For every  $x^* = (\xi_j) \in B_{S^*}$  and  $n \in \mathbb{N}$ , we have that*

$$\xi_n^\# \leq \frac{1}{f(n)}. \tag{2.12}$$

*Proof.* By the 1-unconditionality of both norms in  $S$  and  $S^*$  we need to prove (2.12) only for non negative sequences  $x^* = (\xi_j)$  in  $c_{00}$ . Let  $E = \{j_1, j_2, \dots, j_n\} \subset \mathbb{N}$  have  $n$  elements, so that

$$\xi_{j_1} \geq \xi_{j_2} \geq \dots \geq \xi_{j_n} = \xi_n^\#,$$

and put

$$y^* = \xi_n^\# \sum_{s=1}^n e_{j_s}^*.$$

Since the basis of  $S^*$  is suppression 1-unconditional, it follows that  $\|y^*\| \leq 1$ , and since

$$y = \frac{f(n)}{n} \sum_{s=1}^n e_{j_s} \in S_S,$$

(see [22]) it follows that

$$1 \geq \langle y^*, y \rangle = f(n) \xi_n^\#,$$

which proves our claim. □

COROLLARY 2.6. *For  $x \in c_{00}$  we have*

$$\|x\|_S \leq \|x\|_f. \tag{2.13}$$

## 2.2. Yardstick vectors

The following type of vectors were introduced by D. Kutzarova and P.-K. Lin in [18].

DEFINITION 2.7 *Yardstick Vectors.* We call a finite or infinite sequence of natural number  $m_1, m_2, m_3, \dots$  admissible, if for any  $i$ , for which  $m_i$  exists,  $m_i$  is even and is a multiple

of the product  $\prod_{A \subset \{1,2,\dots,i-1\}} \left( \sum_{j \in A} m_j \right)$  (as usual  $\prod_{\emptyset} = 1$ ). Note that any subsequence of admissible sequences is also admissible.

By induction we define the vector  $y(m_1, m_2, \dots, m_k)$  for each  $k$  and each admissible finite sequence  $(m_1, m_2, \dots, m_k) \subset \mathbb{N}$ ; the support of  $y(m_1, m_2, \dots, m_k)$  will be the interval  $[1, \sum_{i=1}^k m_i]$ .

If  $k = 1$  we put for  $m \in \mathbb{N}$

$$y(m) = \frac{f(m)}{m} \sum_{i=1}^m e_i.$$

Assume that  $y(m_1, m_2, \dots, m_{k'})$  has been defined for each  $k' < k$  and each admissible sequence  $(m_1, m_2, \dots, m_{k'}) \subset \mathbb{N}$ .

From our induction hypothesis the support of  $y(m_1, m_2, \dots, m_{k-1})$  is  $[1, m_1 + m_2 + \dots + m_{k-1}]$  and we write  $y(m_1, m_2, \dots, m_{k-1})$  as

$$y(m_1, m_2, \dots, m_{k-1}) = \sum_{i=1}^{m_1+m_2+\dots+m_{k-1}} a_i e_i.$$

Now we define  $\tilde{y}$  to be the vector, which has the same distribution as  $y(m_1, m_2, \dots, m_{k-1})$ , and whose support is

$$\text{supp}(\tilde{y}) = \left\{ 1 + (i-1) \frac{m_k}{m_1 + m_2 + \dots + m_{k-1}} : i = 1, 2, \dots, \sum_{j=1}^{k-1} m_j \right\}$$

(i.e. we spread out the coordinates of  $y(m_1, m_2, \dots, m_{k-1})$ , so that between any two successive non zero coordinates there are  $\frac{m_k}{m_1+m_2+\dots+m_{k-1}}$  zeros).

Then we define

$$y(m_1, m_2, \dots, m_k) = \tilde{y} + \frac{f(m_k)}{m_k} \sum_{i \in [1, m_1+m_2+\dots+m_k] \setminus \text{supp}(\tilde{y})} e_i$$

(i.e. we are replacing the zeros on the interval  $[1, m_1 + m_2 + \dots + m_k]$  by the value  $f(m_k)/m_k$ ). So, for example,  $y(m_1)$  and  $y(m_1, m_2)$  are the following vectors:

$$y(m_1) = \underbrace{\left( \frac{f(m_1)}{m_1}, \frac{f(m_1)}{m_1}, \dots, \frac{f(m_1)}{m_1} \right)}_{m_1 \text{ times}}$$

$$y(m_1, m_2) = \underbrace{\left( \frac{f(m_1)}{m_1}, \underbrace{\frac{f(m_2)}{m_2}, \dots, \frac{f(m_2)}{m_2}}_{m_2/m_1 \text{ times}}, \dots, \frac{f(m_1)}{m_1}, \underbrace{\frac{f(m_2)}{m_2}, \dots, \frac{f(m_2)}{m_2}}_{m_2/m_1 \text{ times}} \right)}_{m_1 \text{ times}}.$$

If  $\bar{x} = (x_n)_{n \in \mathbb{N}}$  is a block sequence in  $c_{00}$ , and if  $(m_1, \dots, m_k) \subset \mathbb{N}$  is admissible, we define  $y_{\bar{x}}(m_1, m_2, \dots, m_k)$  to be a linear combination of the  $x_n$ 's with the same distribution as  $y(m_1, m_2, \dots, m_k)$  has on the  $e_n$ , i.e.

$$y_{\bar{x}}(m_1, m_2, \dots, m_k) = \sum_{i \in \text{supp}(y(m_1, m_2, \dots, m_k))} a_i x_i,$$

where the  $a_i$  are such that

$$y(m_1, m_2, \dots, m_k) = \sum_{i \in \text{supp}(y(m_1, m_2, \dots, m_k))} a_i e_i.$$

It follows from the arguments in [18] that for  $k \in \mathbb{N}$  and  $\varepsilon > 0$  one can find  $m_1 < m_2 < \dots < m_k$  in  $\mathbb{N}$  so that  $\|y(m_1, m_2, \dots, m_k)\|_S \leq 1 + \varepsilon$ . Since  $y(m_1, m_2, \dots, m_k)$  is the sum of disjointly supported vectors  $z_1, z_2, \dots, z_k$ , with  $z_i$  having the same distribution as  $\frac{f(m_i)}{m_i} \sum_{j=1}^{m_i} e_j$ , for  $i = 1, 2, \dots, k$ , (and thus  $\|z_j\| = 1$ , by [22]), it follows that  $\ell_\infty^k$ ,  $k \in \mathbb{N}$  are uniformly represented in  $S$ . Something stronger is true. Using similar arguments as in [18] it is actually possible to prove under appropriate growth conditions on  $(m_i)$  that the sequence  $(y(m_1, m_2, \dots, m_k) : k \in \mathbb{N})$  is uniformly bounded in  $S$ . For completeness we will present a self contained proof of this fact. First we prove the following lemma, which will serve as the induction step for choosing the sequence  $(m_i)$ .

LEMMA 2.8. Assume we are given  $k, m \in \mathbb{N}$ , with  $k < m$ ,  $C \geq 1$  and some  $\varepsilon \in (0, f(2) - 1)/f(2))$  satisfying the following conditions:

$$m \text{ is divisible by } k \tag{2.14}$$

$$f(m) \geq C \max \left( \frac{50}{\varepsilon^2}, \frac{f(l_0)f(l_0 - 1)}{f(l_0) - f(l_0 - 1)} \right) \tag{2.15}$$

where  $l_0 = \min\{l \in \mathbb{N} : f(l) \geq 6\}$

$$\frac{f(m)}{f(m/k)} \leq 1 + \frac{\varepsilon}{6}. \tag{2.16}$$

Assume further  $(j_s)_{s=1}^m \subset \mathbb{N}$  and  $(x_s)_{s=1}^m \subset S$  have the property that

$$e_{j_1} < x_1 < e_{j_2} < x_2 \dots e_{j_{m-1}} < x_{m-1} < e_{j_m} < x_m, \text{ and} \tag{2.17}$$

$$\|x_s\| \leq \frac{C}{m}, \text{ for } s = 1, 2, \dots, m. \tag{2.18}$$

Then it follows that

$$\left\| \sum_{s=1}^m \frac{f(m)}{m} e_{j_s} + x_s \right\|_S \leq C(1 + \varepsilon) \text{ and} \tag{2.19}$$

$$\left\| \sum_{s=(i-1)(m/k)+1}^{im/k} \frac{f(m)}{m} e_{j_s} + x_s \right\|_S \leq \frac{C(1 + \varepsilon)}{k} \text{ for } i = 1, 2, 3, \dots, k. \tag{2.20}$$

In particular the vectors

$$y_i = \frac{k}{C(1 + \varepsilon)} \sum_{s=(i-1)(m/k)+1}^{im/k} \frac{f(m)}{m} e_{j_s} + x_s, \text{ for } i = 1, 2, \dots, k$$

are in  $B_S$  and  $(y_i)$  is  $C(1 + \varepsilon)$ -equivalent to the unit vector basis in  $\ell_1^k$ .

*Proof.* We note that for any scalars  $(a_i)_{i=1}^k$ , we have

$$\left\| \sum_{i=1}^k a_i y_i \right\| \geq f(m)^{-1} \left( \sum_{s=1}^m e_{j_s}^* \right) \left( \sum_{i=1}^k a_i y_i \right) = f(m)^{-1} \sum_{i=1}^k a_i \left( \sum_{s=(i-1)(m/k)+1}^{im/k} e_{j_s}^*(y_i) \right).$$

It follows easily, assuming (2.20) and using the 1-unconditionality of the basis, that  $(y_i)$  is  $C(1 + \varepsilon)$ -equivalent to the unit vector basis in  $\ell_1^k$ .

To prove (2.19) and (2.20) we put

$$x = \sum_{s=1}^m \frac{f(m)}{m} e_{j_s} + x_s.$$

We will prove by induction for each  $n \in \{1, 2, \dots, m\}$ , that whenever  $0 \leq s_0 < s_1 \leq m$ , with  $s_1 - s_0 = n$ , and  $I \subset \mathbb{N}$  is an interval with  $j_{s_0} < I < j_{s_1+1}$  (where we let  $j_0 = 0$  and  $j_{m+1} = \infty$ ), then

$$\|I(x)\| \leq \frac{f(m)}{m} \frac{n}{f(n)} C \left(1 + \frac{\varepsilon}{3}\right) + \frac{n}{m} C \frac{\varepsilon}{3}. \quad (2.21)$$

From that we deduce (2.19) by letting  $I = \mathbb{N}$  and  $n = m$ . Moreover, if we put  $I = [j_{(i-1)(m/k)+1}, j_{i(m/k)+1} - 1]$ , for  $i = 1, 2, \dots, k$ , we deduce from (2.21) and (2.16) that

$$\begin{aligned} \left\| \sum_{s=(i-1)(m/k)+1}^{im/k} \frac{f(m)}{m} e_{j_s} + x_s \right\| &= \|I(x)\| \\ &\leq \frac{f(m)}{m} \frac{m/k}{f(m/k)} C \left(1 + \frac{\varepsilon}{3}\right) + \frac{m/k}{m} C \frac{\varepsilon}{3} \\ &= \frac{1}{k} \frac{f(m)}{f(m/k)} C \left(1 + \frac{\varepsilon}{3}\right) + \frac{1}{k} C \frac{\varepsilon}{3} \leq \frac{1}{k} C (1 + \varepsilon) \end{aligned}$$

which implies (2.20).

First let  $n \in \mathbb{N}$ , so that  $f(n) \leq \frac{6}{\varepsilon}$ , and let  $I \subset \mathbb{N}$  be an interval with  $j_{s_0} < I < j_{s_1+1}$  for some choice of  $s_0, s_1 \in \{1, 2, \dots, m\}$ , and  $s_1 - s_0 = n$ ,  $l \geq 2$ . Then

$$\begin{aligned} \|I(x)\|_l &\leq \left\| \sum_{s=s_0+1}^{s_1} \frac{f(m)}{m} e_{j_s} \right\| + \left\| \sum_{s=s_0}^{s_1} x_s \right\| \\ &\leq \frac{f(m)}{m} \frac{n}{f(n)} + C \frac{n+1}{m} \\ &= \frac{f(m)}{m} \frac{n}{f(n)} \left[1 + C \frac{n+1}{n} \frac{f(n)}{f(m)}\right] \\ &\leq \frac{f(m)}{m} \frac{n}{f(n)} C \left[1 + \frac{12}{\varepsilon f(m)}\right] \leq \frac{f(m)}{m} \frac{n}{f(n)} C \left[1 + \frac{\varepsilon}{3}\right] \quad (\text{by (2.15)}) \end{aligned}$$

which implies our claim for  $n \in \mathbb{N}$ , for which  $f(n) \leq \frac{6}{\varepsilon}$ .

Fix now  $n \in \mathbb{N}$ , with  $f(n) > 6/\varepsilon$ , and assume that our induction hypothesis is true for all  $n' < n$ , that is, that (2.21) holds with  $n'$  instead of  $n$  for all intervals  $I \subset \mathbb{N}$  for which there are  $j_{s_0} < I < j_{s_1+1}$  with  $0 \leq s_0 < s_1 \leq m$  and  $s_1 - s_0 = n'$ .

Let  $l \in \mathbb{N}$ ,  $l \geq 2$ , such that  $\|x\| = \|x\|_l$  (since  $n \geq 2$  it follows that  $\|I(x)\|_\infty < \|I(x)\|_2$ , and thus  $l \geq 2$ ).

We choose numbers  $l_1$  and  $l_2$  in  $\mathbb{N} \cup \{0\}$ , with  $l = l_1 + l_2$ , and intervals  $E_1^{(1)} < E_2^{(1)}, \dots, E_{l_1}^{(1)}$  and  $E_1^{(2)} < E_2^{(2)}, \dots, E_{l_1}^{(2)}$ , so that

$$\bigcup_{t=1}^{l_1} E_t^{(1)} \cap \bigcup_{t=1}^{l_1} E_t^{(2)} = \emptyset \quad \text{and} \quad \bigcup_{t=1}^{l_1} E_t^{(1)} \cup \bigcup_{t=1}^{l_1} E_t^{(2)} = I,$$

and so that each of the  $E_t^{(1)}$  contains at least one of the  $j_s$ ,  $s_0 < s \leq s_1$  and none of the  $E_t^{(2)}$  intersects with  $\{j_{s_0+1}, j_{s_0+2}, \dots, j_{s_1}\}$ , and so that

$$\|I(x)\|_l = \frac{1}{f(l)} \left( \sum_{t=1}^{l_1} \|E_t^{(1)}(x)\| + \sum_{t=1}^{l_2} \|E_t^{(2)}(x)\| \right). \quad (2.22)$$

We note that  $l_1 \geq 2$ , otherwise it would follow that  $l_1 = 1$  and for all  $t = 1, 2, \dots, l_2$  either  $j_{s_0} < E_t^{(2)} < j_{s_0+1}$  or  $j_{s_1} < E_t^{(2)} < j_{s_1+1}$ , and thus, by (2.18)

$$\|I(x)\| = \|I(x)\|_l \leq \frac{2C}{m} + \frac{1}{f(l)} \|E_1^{(1)}(x)\| \leq \frac{2C}{m} + \frac{1}{f(2)} \|I(x)\|,$$

and thus

$$\frac{f(m)}{m} \left(1 - \frac{1}{f(l)}\right) \leq \frac{2C}{m},$$

which contradicts (2.15) and the restrictions on  $\varepsilon$ .

We can therefore apply our induction hypothesis and deduce that there are numbers  $s_0 = \tilde{s}_0 < \tilde{s}_1 < \dots < \tilde{s}_{l_1} = s_1$  so that for  $t = 1, 2, \dots, l_1$

$$\|E_t^{(1)}(x)\| \leq \frac{f(m)}{m} \frac{\tilde{s}_t - \tilde{s}_{t-1}}{f(\tilde{s}_t - \tilde{s}_{t-1})} C \left(1 + \frac{\varepsilon}{3}\right) + \frac{\tilde{s}_t - \tilde{s}_{t-1}}{m} C \frac{\varepsilon}{3}. \quad (2.23)$$

Moreover it follows that

$$\frac{1}{f(l)} \sum_{t=1}^{l_2} \|E_t^{(2)}(x)\| \leq \left\| \sum_{s=s_0}^{s_1} x_s \right\| \leq C \frac{n+1}{m}. \quad (2.24)$$

Case 1. If  $6 \leq f(l)$ , then we deduce that

$$\begin{aligned} \|I(x)\|_l &= \frac{1}{f(l)} \left( \sum_{t=1}^{l_1} \|E_t^{(1)}(x)\| + \sum_{t=1}^{l_2} \|E_t^{(2)}(x)\| \right) \\ &\leq C \frac{n+1}{m} + \frac{1}{f(l)} C \sum_{t=1}^{l_1} \left[ \frac{f(m)}{m} \frac{\tilde{s}_t - \tilde{s}_{t-1}}{f(\tilde{s}_t - \tilde{s}_{t-1})} \left(1 + \frac{\varepsilon}{3}\right) + \frac{\tilde{s}_t - \tilde{s}_{t-1}}{m} \frac{\varepsilon}{3} \right] \\ &\leq C \frac{n+1}{m} + \frac{1}{f(l)} C \left[ \frac{f(m)}{m} \frac{n}{f(n/l_1)} \left(1 + \frac{\varepsilon}{3}\right) + \frac{n}{m} \frac{\varepsilon}{3} \right] \\ &\quad (\text{by the concavity of the map } \xi \mapsto \xi/f(\xi)) \\ &= \frac{f(m)}{m} C \frac{n}{f(l)f(n/l_1)} \left(1 + \frac{\varepsilon}{3}\right) + C \frac{n}{m} \left[1 + \frac{1}{n} + \frac{\varepsilon}{3f(l)}\right] \\ &\leq \frac{f(m)}{m} C \frac{n}{f(n)} \left(1 + \frac{\varepsilon}{3}\right) + C \frac{n}{m} \left(1 + \frac{\varepsilon}{3}\right) \end{aligned}$$

where the last inequality follows from the fact that  $f(a/b)f(b) \geq f(a)$  for  $a, b \geq 2$  (see [22]) and (2.15). This finishes the proof of our induction step in this case.

Case 2. If  $f(l) < 6$  we claim that  $l_2 = 0$ . Indeed, otherwise  $l = l_1 + l_2 \geq 3$  (we already observed that  $l_1 \geq 2$ ) and

$$\begin{aligned} \|I(x)\|_{l-1} &\geq \frac{1}{f(l-1)} \left[ \sum_{t=1}^{l_1} \|E_t^{(1)}\| + \sum_{t=2}^{l_2} \|E_t^{(2)}(x)\| \right] \\ &\geq \frac{1}{f(l)} \left[ \sum_{t=1}^{l_1} \|E_t^{(1)}\| + \sum_{t=2}^{l_2} \|E_t^{(2)}(x)\| \right] + \left( \frac{1}{f(l-1)} - \frac{1}{f(l)} \right) \frac{f(m)}{m} \\ &> \frac{1}{f(l)} \left[ \sum_{t=1}^{l_1} \|E_t^{(1)}\| + \sum_{t=2}^{l_2} \|E_t^{(2)}(x)\| \right] + \frac{1}{f(l)} \|E_1^{(2)}(x)\| \quad (\text{by (2.15)}) \\ &= \|I(x)\|_l, \end{aligned}$$

which contradicts the assumption that  $\|I(x)\| = \|I(x)\|_l$ . So it follows that  $l = l_1$  and from (2.23) and the concavity of the map  $\xi \mapsto f(\xi)/\xi$ ,  $\xi \geq 1$  it follows therefore that

$$\begin{aligned} \|I(x)\|_l &\leq \frac{1}{f(l_1)} \sum_{t=1}^{l_1} \left[ \frac{f(m)}{m} \frac{\tilde{s}_t - \tilde{s}_{t-1}}{f(\tilde{s}_t - \tilde{s}_{t-1})} C \left(1 + \frac{\varepsilon}{3}\right) + \frac{\tilde{s}_t - \tilde{s}_{t-1}}{m} C \frac{\varepsilon}{3} \right] \\ &\leq \frac{f(m)}{m} \frac{n}{f(l)f(n/l)} C \left(1 + \frac{\varepsilon}{3}\right) + \frac{n}{m} C \frac{\varepsilon}{3} \leq \frac{f(m)}{m} \frac{n}{f(n)} C \left(1 + \frac{\varepsilon}{3}\right) + \frac{n}{m} C \frac{\varepsilon}{3}, \end{aligned}$$

which finishes the proof of the induction step and the proof of our lemma.  $\square$

LEMMA 2.9. Assume that  $(\varepsilon_i) \subset (0, (f(2) - 1)/f(2))$  is summable, and put  $C_i = \prod_{j \geq i} (1 + \varepsilon_j)$ , for  $i \in \mathbb{N} \cap \{0\}$ .

Assume that the sequence  $(m_i : i \in \mathbb{N} \cup \{0\}) \subset \mathbb{N}$  is an admissible sequence and satisfies the following growth conditions. For all  $i \in \mathbb{N}$  we assume that

$$f(m_i) \geq C_i \max \left( \frac{50}{\varepsilon_i^2}, \frac{f(l_0)f(l_0 - 1)}{f(l_0) - f(l_0 - 1)} \right), \quad (2.25)$$

where  $l_0 = \min\{l \in \mathbb{N} : f(l) \geq 6\}$ , and

$$\frac{f(m_i)}{f(m_i/m_{i-1})} \leq 1 + \frac{\varepsilon_i}{6}. \quad (2.26)$$

Then it follows for all  $i \leq j$  in  $\mathbb{N}$  that

$$\|y(m_i, m_{i+1}, \dots, m_j)\| \leq C_i, \text{ and} \quad (2.27)$$

$$\frac{1}{C_i} y(m_i, m_{i+1}, \dots, m_j) \text{ is an } \ell_1^{m_i-1} \text{-average of constant } 1/C_i. \quad (2.28)$$

REMARK 3. For the sequence  $(m_i)$  as chosen in Lemma 2.9 we deduce therefore that, if  $k \in \mathbb{N}$  and  $\varepsilon > 0$  and if  $i_0 \in \mathbb{N}$  is chosen so that  $k \leq m_{i_0}$  and  $\prod_{i=i_0}^{\infty} (1 + \varepsilon_i) \leq 1 + \varepsilon$ , then for all sequences  $i_0 < i_1 < i_2 < \dots < i_l$ ,  $l \in \mathbb{N}$ , it follows that

$$\|y(m_{i_1}, m_{i_2}, \dots, m_{i_l})\|_S \leq 1 + \varepsilon \text{ and} \quad (2.29)$$

$$\frac{1}{1 + \varepsilon} y(m_{i_1}, m_{i_2}, \dots, m_{i_l}) \text{ is an } \ell_1^{m_{i_0}} \text{-average of constant } \frac{1}{1 + \varepsilon}. \quad (2.30)$$

*Proof of Lemma 2.9.* Let the sequence  $(m_j : j \in \mathbb{N} \cup \{0\})$  be chosen as required. Let  $j \in \mathbb{N}$ . By induction on  $i = 0, 1, 2, \dots, j - 1$  we will show that

$$\|y(m_{j-i}, m_{j-i+1}, \dots, m_j)\| \leq C_{j-i}, \text{ and} \quad (2.31)$$

$$\frac{1}{C_{j-i}} y(m_{j-i}, m_{j-i+1}, \dots, m_j) \text{ is an } \ell_1^{m_{j-i-1}} \text{-average of constant } 1/C_{j-i}. \quad (2.32)$$

More precisely, we can write  $y = y(m_{j-i}, m_2, \dots, m_j)$  as

$$y = \frac{C_{j-i}}{m_{j-i-1}} \sum_{s=1}^{m_{j-i-1}} y_s \text{ where } y_1 < y_2 < \dots < y_{m_{j-i-1}} \text{ are in } B_S,$$

equally distributed and  $C_{j-i}$ -equivalent to the basis of  $\ell_1^{m_{j-i-1}}$ .

For  $i = 0$  it follows that  $y(m_j) = \frac{f(m_j)}{m_j} \sum_{s=1}^{m_j} e_s \in S_S$ , and the conditions of Lemma 2.8 are satisfied with  $m = m_j$ ,  $\varepsilon = \varepsilon_j$ , and  $C = 1 \leq C_{j+1}$ ,  $k = m_{j-1}$ , and  $x_s = 0$ , for  $s = 0, 1, \dots, m_j$ . Since  $C_j = (1 + \varepsilon_j)C_{j+1}$ , this implies our claim for  $i = 0$ .

Assuming (2.31) and (2.32) are true for  $i - 1$  with  $1 \leq i < j - 1$ . Using the recursive definition of  $y(m_{j-i}, m_{j-i+1}, \dots, m_j)$  one can write it as

$$y(m_{j-i}, m_{j-i+1}, \dots, m_j) = \sum_{s=1}^{m_{j-i}} \frac{f(m_{j-i})}{m_{j-i}} e_{j_s} + x_s,$$

so that the  $x_s$ ,  $s \leq m_{j-i}$ , are equally distributed vectors, and  $\sum_{s=1}^{m_{j-i}} x_s$  has the same distribution as  $y(m_{j-i+1}, \dots, m_j)$ . It follows therefore from the induction hypothesis (2.32) (for



$i - 1$ ) that  $\|x_s\| \leq C_{j-i+1}/m_{j-i}$ , for  $s = 1, 2, \dots, m_{j-1}$ . Thus Lemma 2.8 is satisfied with  $m = m_{j-i}$ ,  $k = m_{j-i-1}$ ,  $\varepsilon = \varepsilon_{j-i}$  and  $C = C_{j-i+1}$ , and we deduce that  $\|y(m_{j-i}, m_{j-i+1}, \dots, m_j) \leq (1 + \varepsilon_{j-i})C_{j-i+1} = C_{j-i}$ , which implies (2.31). Moreover, the second part of the conclusion of Lemma 2.8 yields that if we write  $y(m_{j-i}, m_{j-i+1}, \dots, m_j)$  as sum of a block of  $m_{j-i-1}$  equally distributed vectors  $\tilde{y}_1 < \tilde{y}_2 < \dots < \tilde{y}_{m_{j-i-1}}$ , we deduce that  $\|\tilde{y}_t\| \leq (1 + \varepsilon_{j-i})C_{j-i+1}/m_{j-i-1} = C_{j-i}/m_{j-i-1}$ ,  $t = 1, 2, \dots, m_{j-i-1}$ . Since the unit vector basis in  $S$  is 1-unconditional this implies that  $(y_t : 1 \leq t \leq m_{j-i-1})$ , with  $y_t = m_{j-i-1}\tilde{y}_t/C_{j-i}$ , for  $t = 1, 2, \dots, m_{j-i-1}$ , is  $C_{j-i}$ -equivalent to the  $\ell_1^{m_{j-i-1}}$  basis. Thus  $y(m_{j-i}, m_{j-i+1}, \dots, m_j)/C_{j-i}$ , is an  $\ell_1^{m_{j-i-1}}$ -average up to the constant  $1/C_{j-i}$ , in the way it is described by (2.32).

### 3. Construction of a version of Gowers Maurey space

To define the space  $\mathcal{GM}$ , which will be a version of the space  $GM$  introduced in [17], we need to choose several objects.

First, assume that  $\bar{\varepsilon} = (\varepsilon_n)_{n \geq 0} \subset (0, 1)$  satisfies the following *standard conditions*

$$\varepsilon_0 < \frac{f(2) - 1}{2}, \varepsilon_n \leq 2^{-n} \text{ and } \sum_{i > n} i^2 \varepsilon_i \leq \frac{1}{10} \varepsilon_n, \text{ for } n \in \mathbb{N}. \quad (3.1)$$

Secondly, let  $\mathbf{Q}$  be a countable set of elements of  $c_{00}$ , so that

$$\left\{ \sum_{i=1}^l a_i e_i : l \in \mathbb{N}, a_i \in \mathbf{Q}, \text{ for } i = 1, 2, \dots, l \right\} \cap [-1, 1]^{\mathbb{N}} \subset \mathbf{Q} \subset c_{00} \cap [-1, 1]^{\mathbb{N}}, \quad (3.2)$$

$$\text{if } x \in \mathbf{Q} \text{ and } E \subset \mathbb{N} \text{ is finite, then } E(x) \in \mathbf{Q}, \quad (3.3)$$

$$\text{if } (x_i)_{i=1}^l \in \mathbf{Q}^l \text{ is a finite block sequence, then } \frac{1}{f(l)} \sum_{i=1}^l x_i \text{ and } \frac{1}{\sqrt{f(l)}} \sum_{i=1}^l x_i \quad (3.4)$$

are in  $\mathbf{Q}$ .

Next we introduce a lacunary set  $J \subset \mathbb{N}$ . We write  $J$  as an increasing sequence  $\{j_1, j_2, \dots\}$ , and require the following four growth conditions

$$\sum_{i > n} \frac{2}{j_i} < \frac{1}{f(j_n)}, \text{ for all } n \in \mathbb{N}, \quad (3.5)$$

$$(j_i)_{i=1}^{\infty} \text{ is admissible, and satisfies the conditions (2.25) and (2.26) imposed on } (m_i)_{i=1}^{\infty} \text{ in Lemma 2.9 (relative to the sequence } (\varepsilon_n) \text{ as chosen above).} \quad (3.6)$$

In order to formulate the last condition on  $J$ , we first need to state an observation which is an easy consequence of James' blocking argument.

**LEMMA 3.1.** *For all  $n \in \mathbb{N}$  and all  $\varepsilon > 0$  there is an  $N = N(n, \varepsilon)$  so that the following holds:*

*Assume that  $(E, \|\cdot\|_E)$  is a Banach space with a normalized and subsymmetric basis  $(e_i)$ , and there is a  $c \in (0, 1]$  so that for all  $k \in \mathbb{N}$*

$$\left\| \sum_{j=1}^k e_j \right\|_E \geq c \frac{k}{f(k)}.$$

*Then, for all  $\varepsilon > 0$  and  $n \in \mathbb{N}$ , there is an  $m \in [n, N(n, \varepsilon)]$  which is divisible by  $n$ , and there are  $n$  subsets  $A_1 < A_2 < \dots < A_n$  of  $\{1, 2, \dots, m\}$ , all of cardinality  $m/n$ , so that  $(x_i : i = 1, 2, \dots, n)$*

is  $c^{1/n}(1 - \varepsilon)$ -equivalent to the  $\ell_1^n$ -unit vector basis, where

$$x_i = \frac{\sum_{j \in A_i} e_j}{\left\| \sum_{j \in A_i} e_j \right\|} \text{ for } i = 1, 2, \dots, n.$$

Our fourth condition on  $J = \{j_1, j_2, \dots\}$  can now be stated as follows (the first inequality being trivial):

$$j_s \leq N(j_s, \varepsilon_s) \leq \frac{1}{2} \varepsilon_{s+1} f(j_{s+1}). \quad (3.7)$$

Finally we will need an injective function  $\sigma$  from the collection of all finite sequences of elements of  $\mathbf{Q}$  to the set  $\{j_2, j_4, \dots\}$  such that if  $l \in \mathbb{N}$ ,  $z_1^*, z_2^*, \dots, z_l^* \in \mathbf{Q}$  and  $N = \max(\cup_{s=1}^l \text{supp}(z_s^*))$ , then

$$\varepsilon_N f(\sigma(z_1^*, \dots, z_l^*)) \geq N. \quad (3.8)$$

Depending on our choice of  $\varepsilon$ ,  $\mathbf{Q}$ ,  $J$  and  $\sigma$  we can now define recursively subsets  $\mathcal{GM}_m^*$  in  $c_{00} \cap [-1, 1]^{\mathbb{N}}$ , for each  $m \in \mathbb{N}_0$ , which will serve as a set of normalizing functionals of  $\mathcal{GM}$ .

Let

$$\mathcal{GM}_0^* = \{\lambda e_n : n \in \mathbb{N}, |\lambda| \leq 1\}.$$

Assume that  $\mathcal{GM}_m^*$  has been defined for some  $m \in \mathbb{N}_0$ . Then  $\mathcal{GM}_{m+1}^*$  is the set of all functionals of the form  $E(z^*)$  where  $E \subseteq \mathbb{N}$  is an interval and  $z^*$  has one of the following three forms (3.9), (3.10) or (3.11):

$$z^* = \sum_{i=1}^l \alpha_i z_i^* \quad (3.9)$$

where  $\sum_{i=1}^l |\alpha_i| \leq 1$  and  $z_i^* \in \mathcal{GM}_m^*$  for  $i = 1, \dots, l$ .

$$z^* = \frac{1}{f(l)} \sum_{i=1}^l z_i^*, \quad (3.10)$$

where  $z_i^* \in \mathcal{GM}_m^*$  for  $i = 1, \dots, l$ , and  $z_1^* < \dots < z_l^*$ .

$$z^* = \frac{1}{\sqrt{f(k)}} \sum_{i=1}^k z_i^* \text{ and } z_i^* = \frac{1}{f(n_i)} \sum_{j=1}^{n_i} z_{i,j}^* \quad (3.11)$$

where

- a)  $z_{1,1}^* < \dots < z_{1,n_1}^* < z_{2,1}^* < \dots < z_{l,n_l}^*$ ,
- b)  $z_{i,j}^* \in \mathcal{GM}_m^* \cap \mathbf{Q}$ , for  $1 \leq i \leq k$  and  $1 \leq j \leq n_i$  (and thus  $z_i^* \in \mathbf{Q}$ , for  $i = 1, 2, \dots, k$ ), and
- c)  $n_1 = j_{2k'}$ , for some  $k' \geq k$ , and  $n_{i+1} = \sigma(z_1^*, \dots, z_i^*)$ , for  $i = 1, \dots, k-1$ .

Finally, the norm of  $\mathcal{GM}$  is defined by

$$\|x\|_{\mathcal{GM}} = \sup\{z^*(x) : z^* \in \cup_{m=0}^{\infty} \mathcal{GM}_m^*\}.$$

**REMARK 4.** *There are two main technical differences between the original space  $GM$  defined in [17] and the space  $\mathcal{GM}$  defined here:*

- (i) we allow in (3.11)  $k$  to take any value in  $\mathbb{N}$ , while in [17]  $k$  had to be chosen out of the very lacunary set  $\{j_{2s+1}, s \in \mathbb{N}\}$  and  $\sigma$  in [17] could only take values in  $\{j_{2s} : s \in \mathbb{N}\}$ .
- (ii) in (3.11) we allow that  $n_1$  is of the form  $n_1 = j_{2k'}$ , with  $k' \geq k$ , while in [17], it is required that  $k' = k$ .

*The point is that it is not enough to use the coding procedure of [17] to obtain as they do, given  $\varepsilon > 0$ , some  $k$  and two intertwined finite sequences  $u_1 < v_1 < \dots < u_k < v_k$  such that*

$\|\sum_{i=1}^k(u_i - v_i)\| \leq \epsilon \|\sum_{i=1}^k(u_i + v_i)\|$ . To deduce estimates about spreading models, we need this to be valid for any  $k$  large enough and for any initial vector  $u_1$  far enough along the basis.

The proof that our construction still does not contain an unconditional basis becomes therefore a bit harder. Nevertheless the main ideas of the proof stay the same.

**Notation.** For  $m \in \mathbb{N}$ , and if  $X$  is a Banach space with a normalized basis  $(e_i)$  (we will use this notation for  $S$  as well as for  $\mathcal{GM}$ ).

$$A_m^*(X) = \left\{ \frac{1}{f(l)} \sum_{i=1}^l x_i^* : x_1^* < x_2^* < \dots < x_l^* \text{ in } B_{X^*} \cap c_{00} \right\}.$$

Note that  $A_m^*(\mathcal{GM}) \subset B_{\mathcal{GM}^*}$  and  $A_m^*(S) \subset B_{S^*}$ .

We define for  $x \in X$  and  $m \in \mathbb{N}$

$$\|x\|_m = \sup_{x^* \in A_m^*} |x^*(x)| = \max_{E_1 < E_2 < \dots < E_m} \frac{1}{f(m)} \sum_{i=1}^m \|E_i(x)\|.$$

and observe that

$$\frac{1}{f(m)} \|x\|_S \leq \|x\|_m \leq \|x\|_S \leq \|x\|_{\mathcal{GM}}.$$

For  $k \in \mathbb{N}$  we also define

$$\Gamma_k^* = \left\{ \frac{1}{\sqrt{f(k)}} \sum_{i=1}^k x_i^* : \begin{array}{l} x_1^* < x_2^* < \dots < x_k^* \text{ in } B_{\mathcal{GM}^*} \cap \mathbf{Q} \\ x_i^* \in A_{m_i}^*(\mathcal{GM}) \text{ with } m_1, m_2, \dots, m_k \in M \\ m_1 = j_{2k'}, k' \geq k, \text{ and } m_{i+1} = \sigma(x_1^*, \dots, x_i^*) \text{ if } i < k \end{array} \right\},$$

and put for  $x \in \mathcal{GM}$

$$\|x\|_{G_m^*} = \sup_{x^* \in \Gamma_m^*} |x^*(x)|.$$

#### 4. Some technical observations concerning the space $\mathcal{GM}$

In this section we prove several properties of the space  $\mathcal{GM}$ , as defined in the previous section. In particular we will conclude that also this version does not contain any unconditional basic sequences. In this section we will abbreviate  $\|\cdot\|_{\mathcal{GM}}$  by  $\|\cdot\|$ .

The following observation follows from James' blocking argument (See Lemma 3.1).

**LEMMA 4.1.** *The space  $\ell_1$  is finitely block represented in every infinite dimensional block subspace of  $\mathcal{GM}$ .*

The next Lemma is easy to show (c.f. [22] or [17])

**LEMMA 4.2** Action of  $\|\cdot\|_l$  on  $\ell_1^+$  averages. Assume that  $x \in B_{\mathcal{GM}}$  is an  $\ell_1^n$ -average and  $l \in \mathbb{N}$ . Then

$$\|x\|_l \leq \frac{1}{f(l)} \left(1 + \frac{l}{n}\right). \quad (4.1)$$

**DEFINITION 4.3** (Rapidly Increasing Sequences). We call a block sequence  $(x_n) \subset B_{\mathcal{GM}}$  rapidly increasing sequence of constant  $c$  or  $c$ -RIS, with  $c \in (0, 1]$  if the following two conditions

(4.2) and (4.3) are satisfied (recall that the sequence  $\varepsilon_n$  is given by (3.1)):

For  $n \in \mathbb{N}$ ,  $x_n$  is an  $\ell_1^{k_n}$ -average of constant  $c$ , if  $c < 1$ , or of constant  $1/(1 + \varepsilon_n)$ , if  $c = 1$ , and the following two inequalities are satisfied: (4.2)

$\max\left(\frac{2n}{f(k_n)}, \frac{f(k_n)}{k_n}\right) < \varepsilon_n^2$  and  $f(\varepsilon_n \sqrt{k_n}) \geq \frac{1}{\varepsilon_n^2} \max \text{supp}(x_{n-1})$ , if  $n \geq 2$ , (4.3)

$(x_n)$  has a spreading model  $E$  with a 1-unconditional and seminormalized basis  $(e_i)$  and for  $l \in \mathbb{N}$  and  $(a_i)_{i=1}^l \subset \mathbb{R}$  and  $l \leq n_1 < n_2 < \dots < n_l$  in  $\mathbb{N}$

$$\frac{1}{1 + \varepsilon_l} \left\| \sum_{i=1}^l a_i e_i \right\|_E \leq \left\| \sum_{i=1}^l a_i x_{n_i} \right\| \leq (1 + \varepsilon_l) \left\| \sum_{i=1}^l a_i e_i \right\|_E.$$

We say that a sequence  $(x_n)$  is an RIS, if it is an  $c$ -RIS for some constant  $c$ . If  $c = 1$ , we say that  $(x_n)$  is an asymptotically isometric RIS.

We note that from Lemma 4.1 it follows immediately that any infinite dimensional block subspace  $Y$  of  $\mathcal{GM}$  contains an asymptotically isometric RIS.

REMARK 5. Let  $(x_n)$  be a  $c$ -RIS, and  $(E, \|\cdot\|)$  be the spreading model of  $(x_n)$ . Define for  $l \in \mathbb{N}$

$$g(l) = \frac{l}{\left\| \sum_{i=1}^l e_i \right\|_E} = \lim_{n_1 < n_2 < \dots < n_l} \frac{l}{\left\| \sum_{i=1}^l x_{n_i} \right\|}. \quad (4.4)$$

From the construction of  $\mathcal{GM}$  it follows that

$$g(l) \leq f(l)/c \text{ for all } l \in \mathbb{N}, \quad (4.5)$$

in particular the spreading model  $E$  of  $(x_n)$  satisfies the conditions of Lemma 3.1.

It follows therefore that for  $n \in \mathbb{N}$  and  $\varepsilon > 0$  we can choose an appropriate  $m \in [n, N(n, \varepsilon)]$  and  $m$  elements from  $(x_j)$  so that their sum is up to a scalar multiple, which is as close to  $\frac{g(m)}{m}$ , as we wish, an  $\ell_1^n$  average of constant  $c - \varepsilon$ .

Thus it is justified to introduce the following notion of *Special Rapidly Increasing Sequences*.

DEFINITION 4.4 *Special Rapidly Increasing Sequences.* A block sequence  $(x_n)$  in  $B_{\mathcal{GM}}$  is called a *Special Rapidly Increasing Sequence* of constant  $c$ , with  $c \in (0, 1]$ , or  *$c$ -SRIS*, if there is a  $c$ -RIS  $(\tilde{x}_n) \subset B_{\mathcal{GM}}$ , so that for each  $n \in \mathbb{N}$  there is  $\tilde{p} = \tilde{p}(n) \in P$ ,  $\tilde{p} \geq n$ ,  $\tilde{m}(n) \in [j_{\tilde{p}(n)}, N(j_{\tilde{p}(n)}, \varepsilon_{\tilde{p}(n)})]$  (here  $N(\cdot, \cdot)$  is chosen as in Lemma 3.1) and natural numbers  $\tilde{m}(n) \leq \tilde{s}(n, 1) < \tilde{s}(n, 2) < \dots < \tilde{s}(n, \tilde{m}(n))$ , so that

$$x_n = \frac{\tilde{g}(\tilde{m}(n))}{\tilde{m}(n)} \sum_{r=1}^{\tilde{m}(n)} \tilde{x}_{\tilde{s}(n,r)}, \quad (4.6)$$

where

$$\tilde{g}(l) = \frac{l}{\left\| \sum_{i=1}^l \tilde{e}_i \right\|_{\tilde{E}}} = \lim_{n_1 < n_2 < \dots < n_l} \frac{l}{\left\| \sum_{i=1}^l \tilde{x}_{n_i} \right\|_{\tilde{E}}}$$

and  $\tilde{E}$  is the spreading model of  $(\tilde{x}_n)$  with semi normalized 1-unconditional basis  $(\tilde{e}_n)$ ,

$$x_n \text{ is an } \ell_1^{j_{\tilde{p}(n)}}\text{-average of constant } c, \text{ if } c < 1, \text{ or } 1/(1 + \varepsilon_{\tilde{p}(n)}), \text{ if } c = 1, \quad (4.7)$$

$$(x_n) \text{ is an RIS of constant } c, \text{ with } k_n = j_{\tilde{p}(n)} \text{ in condition (4.2)}. \quad (4.8)$$

REMARK 6. From the remark before Definition 4.4 it follows that every block subspace contains special rapidly increasing sequences. The point of Definition 4.4 is that we may regard the  $x_n$  at the same time as  $\ell_1^{k_n}$ -averages for fast increasing  $k_n$ , but also, up to some factor, sums of elements of an RIS. We shall use this to prove that SRIS generate spreading models equivalent to the unit vector basis of  $S$ . Note that every normalized block basis of  $\mathcal{GM}$  dominates the unit basis of  $S$ .

LEMMA 4.5 Action of  $A_l^*$  on sums of elements of an RIS. Assume that  $(x_n)$  is a  $c$ -RIS,  $c \in (0, 1]$ . Let  $m \leq n_1 < n_2 < \dots < n_m$  be in  $\mathbb{N}$  and  $(a_i)_{i=1}^m \in \mathbb{R}^m$ . Put  $y = \sum_{i=1}^m a_i x_{n_i}$ .

a) If  $f(l) \leq 2m/\varepsilon_{n_1}$  then there are numbers  $l_1$  and  $l_2$  in  $\mathbb{N}$ , with  $l_1 + l_2 \leq \min(2l, m)$ , intervals  $I_1 < I_2 < \dots < I_{l_1}$  in  $\{1, 2, \dots, m\}$ , so that  $l_2 = \#I_0$ , with  $I_0 = \{1, 2, \dots\} \setminus \bigcup_{j=1}^{l_1} I_j$  and

$$\|y\|_l \leq \frac{1}{f(l)} \left[ \sum_{j=1}^{l_1} \left\| \sum_{s \in I_j} a_s x_{n_s} \right\| + \sum_{s \in I_0} |a_s| \frac{1 + 2\varepsilon_{n_s}}{c} \|x_{n_s}\| \right], \quad (4.9)$$

b) If  $f(l) > 2m/\varepsilon$ , for some  $\varepsilon \in [\varepsilon_{n_1}, 1)$ , then

$$\|y\|_l \leq \max_{i \leq m} |a_i| \left[ 2\varepsilon + \max_{i=1, 2, \dots, m} \|x_{n_i}\|_l \right] \leq \max_{i \leq m} |a_i| [2\varepsilon + 1]. \quad (4.10)$$

Proof. We put  $z_s = x_{n_s}$  for  $s = 1, 2, \dots, m$ . In order to prove (a) we choose finite intervals  $E_1 < E_2 < \dots < E_l$  of  $\mathbb{N}$ , so that

$$\|y\|_l = \frac{1}{f(l)} \sum_{t=1}^l \|E_t(y)\|.$$

Without loss of generality we can assume that

$$\bigcup_{t=1}^l E_t = \text{ran}(y) = [\min \text{supp}(z_1), \max \text{supp}(z_m)].$$

For  $t = 1, 2, \dots, l$  we divide  $E_t$  in three intervals  $E_t^{(1)}, E_t^{(2)}, E_t^{(3)}$  (some of them possibly empty) as follows: we let, if it exists,  $m(t)$  be the unique number in  $\{1, 2, \dots, m\}$  so that  $\min \text{ran}(z_{m(t)}) < \min E_t \leq \max \text{ran}(z_{m(t)})$  and put

$$E_t^{(1)} = E_t \cap [1, \max \text{ran}(z_{m(t)})].$$

If  $m(t)$  does not exist we let  $E_t^{(1)} = \emptyset$ . Then we let  $m'(t)$  be the unique number  $m'(t)$ , if it exists, so that  $\min \text{ran}(z_{m'(t)}) < \max E_t \leq \max \text{ran}(z_{m(t)})$ , and put

$$E_t^{(3)} = (E_t \cap [\min \text{ran}(z_{m'(t)}), \infty)) \setminus E_t^{(1)}.$$

If  $m'(t)$  does not exist we let  $E_t^{(3)} = \emptyset$ . Finally we let  $E_t^{(2)} \setminus (E_t^{(1)} \cup E_t^{(3)})$ . Let  $\tilde{\mathcal{E}}$  be the non empty elements of  $\{E_t^{(1)}, E_t^{(2)}, E_t^{(3)} : t \leq l\}$  and  $\tilde{l}$  the cardinality of  $\tilde{\mathcal{E}}$ .

We note that  $\tilde{\mathcal{E}}$  consists of pairwise disjoint intervals which can be ordered into  $\tilde{E}_1 < \tilde{E}_2 < \dots < \tilde{E}_{\tilde{l}}$ , and that for any  $i \leq m$  and any  $j \leq \tilde{l}$ , either  $\tilde{E}_j$  contains  $\text{ran}(z_i)$ , or is contained in  $\text{ran}(z_i)$ , or  $\tilde{E}_j$  and  $\text{ran}(z_i)$  are disjoint.

For  $s \in \{1, 2, \dots, m\}$  we deduce from our condition on  $f(l)$  and (4.2) that

$$\frac{l}{k_{n_s}} \leq \frac{f(l)}{f(k_{n_s})} \leq \frac{2m}{\varepsilon_{n_1} f(k_{n_s})} \leq \frac{2n_s}{\varepsilon_{n_1} f(k_{n_s})} \leq \frac{\varepsilon_{n_s}^2}{\varepsilon_{n_1}} \leq \varepsilon_{n_s}.$$

We let

$$I_0 = \{s = 1, 2, \dots, m : \#\{t : \tilde{E}_t \subset \text{ran}(x_s)\} \geq 2\},$$

Lemma 4.2 yields that for every  $s \in I_0$

$$\sum_{t, \tilde{E}_t \subset \text{ran}(z_s)} \|\tilde{E}_t(z_s)\| \leq 1 + \frac{l}{k_{n_s}} \leq 1 + \varepsilon_{n_s} \leq \frac{\|z_s\| + 2\varepsilon_{n_s}}{c}.$$

We reorder the set  $\tilde{\mathcal{E}}'$  of all sets  $\tilde{E}_t$ ,  $t \in \{1, 2, \dots, \tilde{l}\}$ , which contain the range of at least one  $x_{n_s}$ , into  $E'_1 < \dots < E'_{l_1}$  and we define  $t = 1, \dots, l_1$

$$I_t = \{s \in \{1, 2, \dots, m\} : \text{ran}(x_s) \subset \tilde{E}_t\},$$

and conclude that  $l_1 + l_2 \leq \min(m, 2l)$ , where  $l_2 = \#I_0$ , and

$$\begin{aligned} \frac{1}{f(l)} \sum_{t=1}^l \|E_t(y)\| &\leq \frac{1}{f(l)} \sum_{t=1}^{\tilde{l}} \|\tilde{E}_t(y)\| \\ &= \frac{1}{f(l)} \left[ \sum_{t=1}^l \left\| \sum_{s \in I_t} a_s z_s \right\| + \sum_{s \in I_0} |a_s| \sum_{t=1}^{\tilde{l}} \|\tilde{E}_t(z_s)\| \right] \\ &\leq \frac{1}{f(l)} \left[ \sum_{t=1}^l \left\| \sum_{s \in I_t} a_s z_s \right\| + \sum_{s \in I_0} |a_s| \frac{\|z_s\| + \varepsilon_{n_s}}{c} \right] \end{aligned}$$

which implies (a).

In order to prove our claim (b) let  $\varepsilon \in [\varepsilon_{n_1}, 1]$  and define

$$i_0 = \max \{i = 1, 2, \dots, k : \max(\text{supp}(z_{i-1})) < f(l)\varepsilon\}$$

(with  $\max(\text{supp}(z_0)) := 0$ ).

Then by (4.2) it follows for  $i \in \{i_0 + 1, i_0 + 2, \dots, m\}$  that

$$f(k_{n_i}) > \frac{1}{\varepsilon_{n_i}} \max \text{supp}(z_{i-1}) \geq \frac{1}{\varepsilon} \max \text{supp}(z_{i_0}) \geq f(l)$$

and thus,

$$\begin{aligned} \left\| \sum_{i=1}^m a_i z_i \right\|_l &\leq \sum_{i=1}^{i_0-1} |a_i| \|z_i\|_l + |a_{i_0}| \|z_{i_0}\|_l + \sum_{i=i_0+1}^m |a_i| \|z_i\|_l \\ &\leq \max_{i \leq m} |a_i| \left[ \frac{\max \text{supp}(z_{i_0-1})}{f(l)} + \|z_{i_0}\|_l + \frac{1}{f(l)} \sum_{i=i_0+1}^m \left(1 + \frac{l}{k_{n_i}}\right) \right] \\ &\quad \text{(by Lemma 4.2)} \\ &\leq \max_{i \leq m} |a_i| \left[ \varepsilon + \|z_{i_0}\|_l + \frac{2m}{f(l)} \right] < \max_{i \leq m} |a_i| [2\varepsilon + \|z_{i_0}\|_l] \end{aligned}$$

which proves part (b).  $\square$

LEMMA 4.6 Action of  $\Gamma_k^*$  on sums of elements of an RIS. Assume that  $(x_n)$  is a block-sequence in  $\mathcal{GM}$ ,  $c \in (0, 1]$ , and let  $z^* \in \Gamma_k^*$ . Let  $m \leq n_1 < n_2 < \dots < n_m$  be in  $\mathbb{N}$  and  $(a_i)_{i=1}^m \in \mathbb{R}^m$ . Put  $y = \sum_{i=1}^m a_i x_{n_i}$ .

a) If  $(x_n)$  is a  $c$ -RIS, then

$$|z^*(y)| \leq \frac{|z_{t_0}^*(y)|}{\sqrt{f(k)}} + \frac{\max_{s \leq m} |a_s|}{\sqrt{f(k)}} \left[ 2m\varepsilon_{n_1} + \sum_{t \in T_0} \max_{s \in S_t} \|x_{n_s}\|_{l_t} \right] \quad (4.11)$$

$$+ \max_{s \leq m} |a_s| + \frac{1}{\sqrt{f(k)}} \sum_{s=s_0+1}^m |a_s| \left| \sum_{t \in T_s} z_t^*(x_{n_s}) \right|,$$

where  $z_t^*$ ,  $t = 1, \dots, k$ ,  $s_0 \in \{1, \dots, m\}$ ,  $t_0 \in \{1, 2, \dots, k\}$ ,  $T_s \subset \{t_0 + 1, t_0 + 2, \dots, k\}$ ,  $s = 0, 1, 2, \dots, m$ , and  $S_t \subset \{1, 2, \dots, m\}$ ,  $t \in T_0$  are defined as follows:

$$z^* = \frac{1}{\sqrt{f(k)}} \sum_{t=1}^k z_t^* \in \Gamma_k^*, \text{ with } z_1^* \in A_{l_1}^* \text{ where } l_1 = j_{2k'} \text{ for some } k' \geq k, \text{ and } z_i^* \in A_{l_i}^*,$$

with  $l_i = \sigma(z_1^*, z_2^*, \dots, z_{i-1}^*)$ , for  $i = 2, \dots, k$ ,

$$s_0 = \min \{s = 1, 2, \dots, m : \max \text{supp}(x_{n_{s-1}}) > \varepsilon_{n_1} \sqrt{f(k)}\}$$

(with the usual convention that  $\max \text{supp}(x_{n_0}) = 0$ ),

$$t_0 = \min \{t = 1, \dots, k : z_t^*(y) \neq 0\}$$

(we assume that  $t_0$  exists, otherwise  $z^*(y) = 0$ ),

$$T_s = \{t = t_0 + 1, \dots, k : \text{supp}(z_t^*) \subset [\min \text{ran}(x_{n_s}), \min \text{ran}(x_{n_{s+1}})]\}, \text{ if } s = 1, \dots, m$$

(with  $\min \text{ran}(x_{n_{m+1}}) := \infty$ )

$$T_0 = \{t_0 + 1, \dots, k\} \setminus \bigcup_{s=1}^k T_s$$

$$S_t = \{s \in \{1, 2, \dots, m\} : \text{ran}(z_t^*) \cap \text{ran}(z_s) \neq \emptyset\}, \text{ for } t \in T_0.$$

b) If  $(x_n)$  is a  $c$ -SRIS, then

$$|z^*(y)| \leq \frac{|z_{t_0}^*(y)|}{\sqrt{f(k)}} + \frac{\max_{s \leq m} |a_s|}{\sqrt{f(k)}} \min(m, k) \max_{s \leq m, t_0 < t \leq k} \|x_s\|_{l_t} + 3 \max_{s \leq m} |a_s|. \quad (4.12)$$

*Proof.* We first assume that  $(x_n)$  is only a  $c$ -RIS. For  $m \leq n_1 < n_2 < \dots < n_m$  in  $\mathbb{N}$  and  $(a_s)_{s=1}^m \in \mathbb{R} \setminus \{0\}$  we put

$$y = \sum_{s=1}^m a_s x_{n_s}.$$

Secondly since

$$z^* = \frac{1}{\sqrt{f(k)}} \sum_{t=1}^k z_t^* \in \Gamma_k^*,$$

we note that for  $t_0$  as defined in the statement it follows that

$$\max(\text{supp}(z_{t_0}^*)) \geq \min(\text{supp}(x_{n_1})).$$

We abbreviate  $z_s = a_s x_{n_s}$ , for  $s = 1, 2, \dots, m$ . Since for  $t \in T_0$ ,

$$S_t = \{s \in \{1, 2, \dots, m\} : \text{ran}(z_t^*) \cap \text{ran}(z_s) \neq \emptyset\},$$

$S_t$  is an interval in  $\{1, 2, \dots, m\}$  and each  $s \in \{1, 2, \dots, m\}$  may be element in at most two of the sets  $S_t$ ,  $t \in T_0$ . Using these notations, we can now write  $z^*(y)$  as

$$z^*(y) = \frac{1}{\sqrt{f(k)}} z_{t_0}^*(y) + \frac{1}{\sqrt{f(k)}} \sum_{t \in T_0} \sum_{s \in S_t} z_t^*(z_s) + \frac{1}{\sqrt{f(k)}} \sum_{s=1}^m \sum_{t \in T_s} z_t^*(z_s). \quad (4.13)$$

In order to estimate the second term in (4.13) we first deduce from (3.8) and the trivial estimate  $\min \text{supp}(z_1) > 2n_1$ , that for  $t \in T_0$

$$f(l_t) \geq \frac{\max \text{supp}(z_{t_0}^*)}{\varepsilon_{\max \text{supp}(z_{t_0}^*)}} \geq \frac{\min \text{supp}(z_1)}{\varepsilon_{n_1}} > \frac{2n_1}{\varepsilon_{n_1}} \geq \frac{2m}{\varepsilon_{n_1}},$$

and Lemma 4.5 (b) yields therefore that

$$\left| z_t^* \left( \sum_{s \in S_t} z_s \right) \right| \leq \max_{s \in S_t} |a_s| [2\varepsilon_{n_1} + \max_{s \in S_t} \|x_{n_s}\|_{l_t}], \text{ whenever } t \in T_0. \quad (4.14)$$

In order to estimate the third term in (4.13) we recall that

$$s_0 = \min \{s = 1, 2, \dots, m : \max \text{supp}(z_{s-1}) > \varepsilon_{n_1} \sqrt{f(k)}\}. \quad (4.15)$$

We first note that if  $s_0 \geq 2$

$$\frac{1}{\sqrt{f(k)}} \sum_{s=1}^{s_0-1} \sum_{t \in T_s} |z_t^*(z_s)| \leq \frac{1}{\sqrt{f(k)}} \max \text{supp}(z_{s_0-1}) \max_{s \leq m} |a_s| \leq \varepsilon_{n_1} \max_{s \leq m} |a_s|. \quad (4.16)$$

Secondly, if  $T_{s_0} \neq \emptyset$  we let

$$I = \left[ \min \bigcup_{t \in T_{s_0}} \text{ran}(z_t^*), \max \bigcup_{t \in T_{s_0}} \text{ran}(z_t^*) \right],$$

and deduce that

$$\frac{1}{\sqrt{f(k)}} \left| \sum_{t \in T_{s_0}} z_t^*(z_{s_0}) \right| = |I(z^*)(z_{s_0})| \leq |a_{s_0}| \cdot \|x_{n_{s_0}}\| \leq \max_{s \leq m} |a_s|. \quad (4.17)$$

Adding up the estimates obtained in (4.16) and (4.17) and inserting them into (4.13), we obtain

$$\begin{aligned} |z^*(y)| &\leq \frac{|z_{t_0}^*(y)|}{\sqrt{f(k)}} + \frac{1}{\sqrt{f(k)}} \sum_{t \in T_0} \max_{s \in S_t} |a_s| [\max_{s \in S_t} \|x_{n_s}\|_{l_t} + 2\varepsilon_{n_1}] \\ &\quad + \max_{s \leq m} |a_s| + \frac{1}{\sqrt{f(k)}} \sum_{s=s_0+1}^m |a_s| \left| \sum_{t \in T_s} z_t^*(x_{n_s}) \right| \\ &\leq \frac{|z_{t_0}^*(y)|}{\sqrt{f(k)}} + \frac{\max_{s \leq m} |a_s|}{\sqrt{f(k)}} \left[ 2m\varepsilon_{n_1} + \sum_{t \in T_0} \max_{s \in S_t} \|x_{n_s}\|_{l_t} \right] \\ &\quad + \max_{s \leq m} |a_s| + \frac{1}{\sqrt{f(k)}} \sum_{s=s_0+1}^m |a_s| \left| \sum_{t \in T_s} z_t^*(x_{n_s}) \right|, \end{aligned} \quad (4.18)$$

which proves (4.11).

In order to prove part (b) we now assume that  $(x_n)$  is an SRIS of constant  $c$ , and want for  $s = s_0 + 1, \dots, m$ , with  $T_s \neq \emptyset$ , to find an upper estimate for

$$\frac{1}{\sqrt{f(k)}} \left| \sum_{t \in T_s} z_t^*(x_{n_s}) \right|.$$

Thus, we assume that there is an RIS  $(\tilde{x}_n) \subset B_{\mathcal{G}, \mathcal{M}}$  of constant  $c$ , and for each  $n \in \mathbb{N}$  numbers  $\tilde{p}(n) \in \mathbb{N}$ ,  $\tilde{p}(n) \geq n$ ,  $\tilde{m}(n) \in [j_{\tilde{p}(n)}, N(j_{\tilde{p}(n)}, \varepsilon_{\tilde{p}(n)})]$  and  $\tilde{m}(n) \leq \tilde{s}_1(n) < s_2(n) < \dots < \tilde{s}_{\tilde{m}(n)}(n)$  so



that

$$x_n = \frac{\tilde{g}(\tilde{m}(n))}{\tilde{m}(n)} \sum_{r=1}^{\tilde{m}(n)} \tilde{x}_{\tilde{s}_r(n)},$$

where  $\tilde{g} : [1, \infty) \rightarrow [1, \infty)$  is an increasing function with  $\tilde{g}(\xi) \leq f(\xi)/c$ ,  $\xi \geq 1$  and so that  $k_n = j_{\tilde{p}(n)}$  (thus  $x_n$  is an  $\ell_1^{\tilde{p}(n)}$ -average of constant  $c$ ).

We fix  $s = s_0 + 1, s_0 + 2, \dots, m$ , with  $T_s \neq \emptyset$ , and apply now our estimate (4.18) to  $x_{n_s} = \frac{\tilde{g}(\tilde{m}(n_s))}{\tilde{m}(n_s)} \sum_{r=1}^{\tilde{m}(n_s)} \tilde{x}_{\tilde{s}_r(n_s)}$  instead of  $y$  and to  $\tilde{z}^* = \frac{1}{\sqrt{f(k)}} \sum_{t \in T_s} z_t^*$ , instead of  $z^*$ . Strictly speaking  $\tilde{z}^*$  is not in  $\Gamma_k^*$ , but it is of the form  $I(z^*)$ , where  $I \subset \mathbb{N}$  is an interval, and it is easy to see that it satisfies the same estimates (4.18). From the definition of  $s_0$  and by the second condition on  $k_{n_s}$  in (4.3) we deduce that

$$\sqrt{f(k)} < \frac{1}{\varepsilon_{n_1}} \text{supp}(z_{s-1}) < \sqrt{f(\varepsilon_{n_s} k_{n_s}^{1/2})},$$

and thus

$$k \leq \varepsilon_{n_s} k_{n_s}^{1/2} = \varepsilon_{n_s} j_{\tilde{p}(n_s)}^{1/2} \leq \varepsilon_{n_s} \tilde{m}_s^{1/2}(n_s),$$

which yields

$$\frac{k \tilde{g}(\tilde{m}(n_s))}{\tilde{m}(n_s)} \leq \varepsilon_{n_s} \frac{\tilde{g}(\tilde{m}(n_s))}{\tilde{m}^{1/2}(n_s)} \leq \varepsilon_{n_s}.$$

Put  $t_s = \min T_s$  (which takes the role of  $t_0$ ). If we apply (4.18) to  $x_{n_s}$ , the sets  $T_0$  and  $T_s$  will be replaced by sets  $\tilde{T}_0$  and  $\tilde{T}_s$ ,  $\tilde{s} \leq \tilde{m}(n_s)$ , for which we know (and only will need to know) that

$$\#\tilde{T}_0 \leq k \text{ and } \sum_{\tilde{s}=1}^{\tilde{m}(n_s)} \#\tilde{T}_{\tilde{s}} \leq k.$$

Using also the estimate  $2m\varepsilon_{n_1} + \|\tilde{x}_{\tilde{s}_r(n_s)}\| \leq 2$ , we obtain from (4.18), applied to  $\tilde{z}^*(x_{n_s})$ , that

$$|\tilde{z}^*(x_{n_s})| \leq \frac{|z_{t_s}^*(x_{n_s})|}{\sqrt{f(k)}} + \frac{\tilde{g}(\tilde{m}(n_s))}{\tilde{m}(n_s)} \frac{2k}{\sqrt{f(k)}} + \frac{\tilde{g}(\tilde{m}(n_s))}{\tilde{m}(n_s)} + \frac{\tilde{g}(\tilde{m}(n_s))}{\tilde{m}(n_s)} \frac{k}{\sqrt{f(k)}} \leq \frac{|z_{t_s}^*(x_{n_s})|}{\sqrt{f(k)}} + 4\varepsilon_{n_s}.$$

Inserting this estimate back into (4.18) for  $|z^*(y)|$  we get

$$\begin{aligned} |z^*(y)| &\leq \frac{|z_{t_0}^*(y)|}{\sqrt{f(k)}} + \frac{\max_{s \leq m} |a_s|}{\sqrt{f(k)}} \left[ 2m\varepsilon_{n_1} + \sum_{t \in T_0} \max_{s \in S_t} \|x_{n_s}\|_{l_t} \right] \\ &\quad + \max_{s \leq m} |a_s| + \frac{1}{\sqrt{f(k)}} \sum_{s=1, T_s \neq \emptyset}^m |a_s| [ |z_{t_s}^*(x_{n_s})| + 4\varepsilon_{n_s} ] \\ &\leq \frac{|z_{t_0}^*(y)|}{\sqrt{f(k)}} + \frac{\max_{s \leq m} |a_s|}{\sqrt{f(k)}} \left[ \sum_{t \in T_0} \max_{s \in S_t} \|x_{n_s}\|_{l_t} + \sum_{s=1, T_s \neq \emptyset}^m \|x_{n_s}\|_{l_{t_s}} + 2 \right] + \max_{s \leq m} |a_s| \\ &\leq \frac{|z_{t_0}^*(y)|}{\sqrt{f(k)}} + \frac{\max_{s \leq m} |a_s|}{\sqrt{f(k)}} \min(m, k) \max_{s \leq m, t_0 < t \leq k} \|x_s\|_{l_t} + 3 \max_{s \leq m} |a_s|, \end{aligned} \tag{4.19}$$

which proves our claim (b).  $\square$

LEMMA 4.7. Assume that  $(x_n)$  is an SRIS of constant  $c$ ,  $c > 0$  and assume  $z^* \in \Gamma^*$ . As before write  $z^*$  as

$$z^* = \frac{1}{\sqrt{f(k)}} \sum_{t=1}^k z_t^* \in \Gamma_k^*,$$

with  $z_1^* \in A_{l_1}^*$ , and  $l_1 = j_{2k'}$ , for some  $k' \geq k$ , and  $z_i^* \in A_{l_i}^*$ , with  $l_i = \sigma(z_1^*, z_2^*, \dots, z_{i-1}^*)$ , for  $i = 2, \dots, k$  and assume that

$$t_0 = \min\{t = 1, \dots, k : z_t^*(y) \neq 0\},$$

exists (otherwise  $z^*(y) = 0$ ). Let  $m$  and  $m \leq n_1 < n_2 < \dots < n_m$  be in  $\mathbb{N}$ ,  $(a_s)_{s=1}^m \subset \mathbb{R}$  and assume that the numbers  $j_{\tilde{p}(n_s)}$  (as chosen in Definition 4.4) are all different from the numbers  $l_{t_0+1}, l_{t_0+2}, \dots, l_k$ .

Then it follows that

$$|z_t^*(x_{n_s})| \leq \varepsilon_{n_1}, \text{ for } t = t_0 + 1, t_0 + 2, \dots, k \text{ and } s = 1, 2, \dots, m, \text{ and} \quad (4.20)$$

$$|z^*(y)| \leq \frac{|z_{t_0}^*(y)|}{\sqrt{f(k)}} + 4 \max_{s \leq m} |a_s|, \text{ where } y = \sum_{s=1}^m a_s x_{n_s}. \quad (4.21)$$

REMARK 7. The assumption of Lemma 4.7 are for example satisfied if in Definition 4.4 the numbers  $\tilde{p}(n)$ ,  $n \in \mathbb{N}$ , are chosen to be odd numbers (since the image of  $\sigma$  is a subset of  $\{j_{2i} : i \in \mathbb{N}\}$ ).

*Proof of Lemma 4.7.* We need to estimate  $\|x_{n_s}\|_{l_t}$  for  $s \in \{1, 2, \dots\}$  and  $t = t_0 + 1, t_0 + 2, \dots, k$  and then apply (4.12). Recall that

$$x_{n_s} = \frac{\tilde{g}(\tilde{m}(n_s))}{\tilde{m}(n_s)} \sum_{r=1}^{\tilde{m}(n_s)} \tilde{x}_{\tilde{s}_r(n_s)}, \quad (4.22)$$

where  $\tilde{g} : [1, \infty) \rightarrow [1, \infty)$  is an increasing function with  $\tilde{g}(\xi) \leq f(\xi)/c$ ,  $\xi \geq 1$  and so that  $k_{n_s} = j_{\tilde{p}(n_s)}$ ,  $(\tilde{x}_n)$  is an RIS of constant  $c$ , and  $\tilde{m} \in [j_{\tilde{p}(n_s)}, N(j_{\tilde{p}(n_s)}, \varepsilon_{\tilde{p}(n_s)})]$ , and  $\tilde{m}(n_s) \leq \tilde{s}_1(n_s) < \dots < \tilde{s}_{\tilde{m}}(n_s)$ .

We note that either  $l_t < k_{n_s} = j_{\tilde{p}(n_s)}$ , then, since  $z_s$  is an  $\ell_1^{j_{\tilde{p}(n_s)}}$ -average, we deduce from Lemma 4.2, and (3.8) that

$$\|x_{n_s}\|_{l_t} \leq \frac{2}{f(l_t)} \leq \frac{2}{f(\sigma(z_1^*, z_2^*, \dots, z_{t-1}^*))} \leq \varepsilon_{n_1}. \quad (4.23)$$

Or we have that  $l_t > k_{n_s} = j_{\tilde{p}(n_s)}$ . This implies by (3.7) that

$$f(l_t) \geq f(j_{\tilde{p}(n_s)+1}) \geq \frac{2}{\varepsilon_{\tilde{p}(n_s)+1}} N(j_{\tilde{p}(n_s)}, \varepsilon_{\tilde{p}(n_s)}) \geq \frac{2\tilde{m}(n_s)}{\varepsilon_{\tilde{p}(n_s)+1}} \geq \frac{2\tilde{m}(n_s)}{\varepsilon_{n_1}}.$$

But then it follows from (4.22), Lemma 4.5 (b), and (4.2) that

$$\|x_{n_s}\|_{l_t} \leq 2 \frac{\tilde{g}(\tilde{m}(n_s))}{\tilde{m}(n_s)} \leq 2 \frac{f(\tilde{m}(n_s))}{\tilde{m}(n_s)} \leq 2 \frac{f(k_{n_s})}{k_{n_s}} \leq \varepsilon_{n_1}. \quad (4.24)$$

Thus, (4.12) yields

$$|z^*(y)| \leq \frac{|z_{t_0}^*(y)|}{\sqrt{f(k)}} + \frac{\max_{s \leq m} |a_s|}{\sqrt{f(k)}} \min(m, k) \varepsilon_{n_1} + 3 \max_{s \leq m} |a_s| \leq \frac{|z_{t_0}^*(y)|}{\sqrt{f(k)}} + 4 \max_{s \leq m} |a_s|,$$

which proves our claim.

We now can formulate and prove our Key Lemma.

LEMMA 4.8. For each  $c \in (0, 1]$  there is a constant  $C = C_c > 0$  so that the following holds.

Let  $(x_n)$  be a  $c$ -SRIS, and assume that the  $\tilde{p}(n)$ ,  $n \in \mathbb{N}$ , as in Definition 4.4 are chosen to be odd numbers. Let  $m \leq n_1 < n_2 < \dots < n_m$  be in  $\mathbb{N}$  and put  $y = \sum_{s=1}^m x_{n_s}$ . Then

- a)  $c \frac{m}{f(m)} \leq \|y\| \leq C \frac{m}{f(m)}$ , if  $c < 1$  and  $(1 - \varepsilon_{n_1}) \frac{m}{f(m)} \leq \|y\| \leq C \frac{m}{f(m)}$ , if  $c = 1$ .  
 b) For  $l \in \mathbb{N}$ ,  $l \geq 2$

$$\|y\|_l \leq \begin{cases} \frac{C}{f(l)} \frac{m}{f(m/\min(2l,m))} & \text{if } f(l) \leq m/\varepsilon_{n_1}, \\ 2C & \text{if } f(l) \geq m/\varepsilon_{n_1}. \end{cases}$$

REMARK 8. Of course we can (and will later) replace  $2C$  in the second case of (b) in Lemma 4.8 by an another constant. Nevertheless the “ $2C$ ” is needed so that the induction argument in the proof will work out.

*Proof.* The first inequality in (a) follows from the fact that

$$\|y\| \geq \|y\|_m \geq \frac{1}{f(m)} \sum_{i=1}^m \|x_{n_s}\| \geq c \frac{m}{f(m)},$$

if  $c < 1$ . A similar argument works for  $c = 1$ .

Using the first condition in (3.1) it is easy to see that one can choose  $m_0 \in \mathbb{N}$  so that

$$f(m) \leq \frac{f(l)f(m/2l)}{1 + \varepsilon_0}, \quad 2m\varepsilon_m < c\varepsilon_{m-1} \text{ and } \frac{f(m)}{f(m/4)} \leq 2, \quad (4.25)$$

whenever  $m \geq m_0$  and  $2 \leq l \leq m/4$

(note that the second inequality is satisfied as long as  $c \geq 1/m$ , by the third condition in (3.1)). Put  $C = 4m_0$ .

We will prove the second inequality in (a) and (b) by induction for each  $m \in \mathbb{N}$ . If  $m \leq m_0$  (a) and (b) are trivial.

So assume (a) and (b) are true for all  $m' < m$ , for some  $m \geq m_0$ . Let  $m \leq n_1 < n_2 < \dots < n_m$  be in  $\mathbb{N}$  and put

$$y = \sum_{s=1}^m x_{n_s}.$$

For  $l \in \mathbb{N}$ ,  $l \geq 2$ , we first estimate  $\|y\|_l$ .

If  $f(l) \geq m/\varepsilon_{n_1}$  then Lemma 4.5 (b) implies that  $\|y\|_l \leq 2 \leq C$ .

If  $f(l) \leq m/\varepsilon_{n_1}$  it follows from the second part of (4.25) and Lemma 4.5 (a) that there are natural numbers  $0 = s_0 < s_1 < \dots < s_{l'} = m$ , with  $l' = \min(2l, m)$  so that

$$\|y\|_l \leq \frac{\varepsilon_{n_1-1}}{f(l)} + \frac{1}{f(l)} \sum_{j=1}^{l'} \left\| \sum_{s=s_{j-1}+1}^{s_j} x_{n_s} \right\|.$$

If  $l \geq m/4$  then by the third part of (4.25)

$$\|y\|_l \leq \frac{\varepsilon_{n_1-1}}{f(l)} + \frac{m}{f(l)} \leq \frac{\varepsilon_{n_1-1}}{f(l)} + \frac{m}{f(m/4)} \leq \frac{\varepsilon_{n_1-1}}{f(l)} + 2 \frac{m}{f(m)} \leq C \frac{m}{f(m)}.$$

If  $l \leq m/4$  we are using the induction hypothesis and the fact that the map  $[1, \infty) \ni x \mapsto x/f(x)$  is concave to obtain

$$\begin{aligned} \|y\|_l &\leq \frac{\varepsilon_{n_1-1}}{f(l)} + \frac{1}{f(l)} \sum_{j=1}^{l'} \left\| \sum_{s=s_{j-1}+1}^{s_j} x_{n_s} \right\| \\ &\leq \frac{\varepsilon_{n_1-1}}{f(l)} + \frac{C}{f(l)} \sum_{j=1}^{l'} \frac{s_j - s_{j-1}}{f(s_j - s_{j-1})} \\ &\leq \frac{\varepsilon_{n_1-1}}{f(l)} + \frac{C}{f(l)} l' \frac{m/l'}{f(m/l')} \\ &\leq \frac{\varepsilon_0}{f(2)} + \frac{C}{1 + \varepsilon_0} \frac{m}{f(m)} \quad (\text{by first condition in (4.25)}) \\ &\leq C \frac{m}{f(m)}, \end{aligned}$$

for the last inequality note that

$$\frac{\varepsilon_0}{f(2)} \leq C \frac{\varepsilon_0}{1 + \varepsilon_0} \frac{2}{f(2)} = C \left[1 - \frac{1}{1 + \varepsilon_0}\right] \frac{2}{f(2)} \leq C \left[1 - \frac{1}{1 + \varepsilon_0}\right] \frac{m}{f(m)}.$$

This proves that  $\|y\|_l \leq Cm/f(m)$ , for every  $l \geq 2$ . Together with Lemma 4.7 (which estimates  $\|y\|_{\Gamma_k^*}$  for  $k \in \mathbb{N}$ ) this yields that  $\|y\| \leq Cm/f(m)$ . That finishes the induction step and the proof of (a).

Part (b) follows if  $f(l) \geq m/\varepsilon_{n_1}$  directly from Lemma 4.5 (b). If  $f(l) \leq m/\varepsilon_{n_1}$  we apply Lemma 4.5 (a), the concavity of the map  $[1, \infty) \ni x \mapsto x/f(x)$  and part (a) of this lemma, to obtain for some choice of natural numbers  $0 = s_0 < s_1 < \dots < s_{l'} = m$ , with  $l' = \min(2l, m)$  so that

$$\begin{aligned} \|y\|_l &\leq \frac{\varepsilon_{n_1-1}}{f(l)} + \frac{1}{f(l)} \sum_{j=1}^{l'} \left\| \sum_{s=s_{j-1}+1}^{s_j} x_{n_s} \right\| \\ &\leq \frac{\varepsilon_{n_1-1}}{f(l)} + \frac{C}{f(l)} \sum_{j=1}^{l'} \frac{s_j - s_{j-1}}{f(s_j - s_{j-1})} \\ &\leq \frac{\varepsilon_{n_1-1}}{f(l)} + \frac{C}{f(l)} \frac{m}{f(m/l')} \leq \frac{2C}{f(l)} \frac{m}{f(m/l')}, \end{aligned}$$

which proves our claim.  $\square$

**REMARK 9.** *Following now the proof in [17] (from [17, Lemma 7] on) one deduces that  $\mathcal{GM}$ , as defined here has also no unconditional basis sequence. In Section 6 (see Theorem 6.1) we will prove that in every block subspace of  $\mathcal{GM}$  there are two seminormalized block sequence  $(u_n)$  and  $(v_n)$ , which are intertwined, i.e.  $u_1 < v_1 < u_2 < v_2 < \dots$ , with the property that for some constants  $0 < c, C < \infty$*

$$\left\| \sum_{s=1}^l u_{n_s} + v_{n_s} \right\| \geq c \frac{l}{\sqrt{f(l)}} \quad \text{and} \quad \left\| \sum_{s=1}^l u_{n_s} - v_{n_s} \right\| \leq C \frac{l}{f(l)}.$$

for all  $l \in \mathbb{N}$  and all choices of  $l \leq n_1 < n_2 < \dots < n_l$  in  $\mathbb{N}$ . This certainly implies that  $\mathcal{GM}$  has no unconditional block sequence.

We do not know whether or not  $\mathcal{GM}$  is HI, but we suspect it is. The point is that to use spreading models and other refinements, we needed to pass to subsequences of the Rapidly Increasing Sequences as defined in GM. Therefore we lost the freedom to pick the vectors of

an RIS-sequence in arbitrary subspaces, as would be needed to repeat Gowers-Maurey's proof that GM is HI.

Nevertheless, Gowers' first dichotomy yields that  $\mathcal{GM}$  contains at least an infinite dimensional block subspace which is HI.

### 5. Yardstick Vectors in $\mathcal{GM}$

We will prove that every block basis in  $\mathcal{GM}$  has a further block basis whose spreading model is equivalent to the unit vector basis of  $S$ . Thus, we can define in  $\mathcal{GM}$  the yardsticks as introduced in Section 2. The following observation follows from Lemmas 4.5 and 4.8, and an argument in [2].

PROPOSITION 5.1. Assume that  $(x_n)$  is a  $c$ -RIS in  $\mathcal{GM}$ , for which the following condition is satisfied:

There exists a constant  $C' \geq 1$ , so that for all  $m, k \in \mathbb{N}$ ,  $m \leq n_1 < n_2 < \dots < n_m$  (5.1)

in  $\mathbb{N}$ , all  $(a_i)_{i=1}^m$  and all  $z^* \in \Gamma_k^*$ , it follows that

$$\left| z^* \left( \sum_{s=1}^m a_s x_{n_s} \right) \right| \leq \frac{1}{\sqrt{f(k)}} \max_{j \in J} \left\| \sum_{s=1}^m a_s x_{n_s} \right\|_j + C' \max_{s \leq m} |a_s|.$$

Then the spreading model of  $(x_n)$  is equivalent to the unit vector basis of  $S$ . More precisely there is a constant  $C$  so that for every  $c$ -SRIS  $(x_n)$  in  $\mathcal{GM}$

$$c' \left\| \sum_{s=1}^m a_s e_s \right\|_S \leq \left\| \sum_{s=1}^m a_s x_{n_s} \right\|_{\mathcal{GM}} \leq C \left\| \sum_{s=1}^m a_s e_s \right\|_S, \tag{5.2}$$

whenever  $m \leq n_1 < n_2 < \dots < n_m$  are in  $\mathbb{N}$  and  $(a_s)_{s=1}^m \subset \mathbb{R}$ , where  $c' = c$  if  $c < 1$  and  $c' = 1 - \varepsilon_{n_1}$  if  $c = 1$ .

REMARK 10. Note that Lemma 4.7 and the remark thereafter establishes a case in which the assumption (5.1) is satisfied.

Proof of Proposition 5.1. Consider the norm  $\langle \cdot \rangle$  on  $c_{00}$  given by the implicit equation

$$\langle x \rangle = \max \left( \|x\|_\infty, \max_{\substack{l \in \mathbb{N}, l \geq 3 \\ E_1 < E_2 < \dots < E_l}} \frac{1}{f(l/2)} \sum_{j=1}^l \langle E_j(x) \rangle \right), \quad x \in c_{00}$$

and recall [2, Lemma 3.3] which states that  $\langle \cdot \rangle$  is an equivalent norm on  $S$ .

We put

$$C'' = C' \frac{\sqrt{f(2)}}{\sqrt{f(2)} - 1}.$$

By induction we will show for each  $m \in \mathbb{N}$  and all choices of  $(a_s)_{s=1}^m \subset \mathbb{R}$  and  $m \leq n_1 < n_2 < \dots < n_m$  in  $\mathbb{N}$ , that

$$c' \left\| \sum_{s=1}^m a_s e_s \right\|_S \leq \left\| \sum_{s=1}^m a_s x_{n_s} \right\|_{\mathcal{GM}} \leq C'' \left\langle \sum_{s=1}^m a_s e_s \right\rangle \left( 1 + \frac{2}{c} \sum_{s=1}^m \varepsilon_{n_s} \right). \tag{5.3}$$

This will, together with the above cited result from [2], prove our claim. The first inequality in (5.3) is clear, and it is also clear that (5.3) holds for  $m = 1$ . So assume that (5.3) holds for

all  $m' < m$ ,  $m \geq 2$ ,  $m' \leq n_1 < n_2 < \dots < n_{m'}$  in  $\mathbb{N}$ , and  $(a_s)_{s=1}^{m'} \subset \mathbb{R}$ . Let  $m \leq n_1 < n_2 < \dots < n_m$ ,  $(a_s)_{s=1}^m \subset \mathbb{R}$  and put  $y = \sum_{s=1}^m a_s x_{n_s}$ . We distinguish between two cases: If

$$C'' \max_{s \leq m} |a_s| \geq \max_{l \in \mathbb{N}, l \geq 2} \|y\|_l,$$

then we note that for all  $l \in \mathbb{N}$ ,  $l \geq 2$ ,

$$\|y\|_l \leq C'' \max_{s \leq m} |a_s| \leq C'' \left\langle \sum_{s=1}^m a_s e_s \right\rangle,$$

and, thus, for any  $k \in \mathbb{N}$ ,  $k \geq 2$ , and  $z^* \in \Gamma_k^*$ , it follows from our assumption (5.1) that

$$|z^*(y)| \leq \frac{1}{\sqrt{f(2)}} \max_{j \in J} \|y\|_j + C' \max_{s \leq m} |a_s| \leq \left[ \frac{C''}{\sqrt{f(2)}} + C' \right] \max_{s \leq m} |a_s| = C'' \max_{s \leq m} |a_s|.$$

If

$$C'' \max_{s \leq m} |a_s| < \max_{l \in \mathbb{N}, l \geq 2} \|y\|_l,$$

we proceed as follows.

If  $l \in \mathbb{N}$ , with  $f(l) \geq 2m/\varepsilon_{n_1}$ , then Lemma 4.5 (b) implies that

$$\|y\|_l \leq 2 \max_{i \leq m} |a_i| \leq 2 \left\langle \sum_{i=1}^m a_i e_i \right\rangle \leq C'' \left\langle \sum_{i=1}^m a_i e_i \right\rangle.$$

If  $l \geq 2$  and  $f(l) \leq 2m/\varepsilon_{n_1}$ , then Lemma 4.5 (a) yields for some choice of  $0 = s_0 < s_1 < s_2 < \dots < s_{l'}$  with  $l' = \min(m, 2l)$ , that

$$\begin{aligned} \|y\|_l &\leq \frac{1}{f(l)} \left[ \sum_{t=1}^{l'} \left\| \sum_{s=s_{t-1}+1}^{s_t} a_s x_{n_s} \right\|_{\mathcal{G}\mathcal{M}} + \frac{2}{c} \sum_{t \leq l', s_t=1+s_{t-1}} \varepsilon_{n_{s_t}} |a_{s_t}| \right] \\ &\leq \frac{C''}{f(l)} \left[ \sum_{t=1}^{l'} \left\langle \sum_{s=s_{t-1}+1}^{s_t} a_s e_s \right\rangle \left( 1 + \frac{2}{c} \sum_{s=1}^m \varepsilon_{n_s} \right) \right] \\ &\quad \text{(By applying the induction hypothesis in cases that } s_t \geq 2 + s_{t-1}) \\ &\leq C'' \left\langle \sum_{s=1}^m a_s e_s \right\rangle \left( 1 + \frac{2}{c} \sum_{s=1}^m \varepsilon_{n_s} \right). \end{aligned} \tag{5.4}$$

Our assumption (5.1) yields for  $k \in \mathbb{N}$ ,  $k \geq 2$  and  $z^* \in \Gamma_k^*$ , that

$$|z^*(y)| \leq \frac{1}{\sqrt{f(k)}} \max_{j \in \mathbb{N}} \|y\|_j + C' \max_{s \leq m} |a_s| \leq \left[ \frac{1}{\sqrt{f(2)}} + \frac{C'}{C''} \right] \max_{j \in \mathbb{N}} \|y\|_j = \max_{j \in \mathbb{N}} \|y\|_j,$$

which together with (5.4), finishes the proof of our induction step.

Lemma 3.1 and Proposition 5.1 imply therefore

**COROLLARY 5.2.** *There is a constant  $D$ , so that for every asymptotically isometric SRIS  $\bar{x} = (x_n)$  in  $\mathcal{G}\mathcal{M}$ , for which  $\tilde{p}(n)$  is odd for all  $n \in \mathbb{N}$ , and every  $l \in \mathbb{N}$  and any  $s_1 < s_2 < \dots < s_l$  in  $\mathbb{N}$  we have*

$$\frac{1}{2} \leq \|y_{\bar{x}'}(j_{s_1}, j_{s_2}, \dots, j_{s_l})\|_{j_{s_i}} \leq \|y_{\bar{x}'}(j_{s_1}, j_{s_2}, \dots, j_{s_l})\|_{\mathcal{G}\mathcal{M}} \leq \frac{D}{2}, \text{ for all } i = 1, 2, \dots, l \tag{5.5}$$

where  $\bar{x}'$  is a far enough out starting tail subsequence of  $\bar{x}$ .

6. Construction of two equivalent intertwined sequences

We now want to construct in a given block subspace  $Y$  of  $\mathcal{GM}$  two seminormalized block sequences  $(u_n)$  and  $(v_n)$ , which are equivalent and so that  $u_1 < v_1 < u_2 < \dots$ .

Let  $\bar{x} = (x_i)$  be any asymptotically isometric SRIS in  $Y$ , so that  $\tilde{p}(n)$  is odd for  $n \in \mathbb{N}$ . Using Proposition 5.1 and the remark thereafter, it follows that the spreading model of  $\bar{x}$  is equivalent to the unit vector basis of  $S$ , and, since Corollary 5.2 applies we let  $D < \infty$  be chosen so that (5.5) holds true.

By induction we choose a block sequence  $(z_n)$  of  $\bar{x}$ . The vectors  $u_n$  and  $v_n$  will then be chosen so that  $u_n < v_n$  and  $z_n = u_n + v_n$ .

For  $n = 1$  we first choose  $k'_1$ , so that  $f(k'_1)/k'_1 < \varepsilon_1^2$  (which means that  $k'_1$  satisfies condition (4.2) for  $n = 1$ ), and then let

$$z_1 = \frac{1}{D} y_{\bar{x}^{(1)}}(j_{2q_1(1)}) = \frac{1}{D} \frac{f(j_{2q_1(1)})}{j_{2q_1(1)}} \sum_{t=1}^{j_{2q_1(1)}} \bar{x}_t^{(1)},$$

where  $q_1(1) \in \mathbb{N}$  is chosen large enough so that  $y(j_{2q_1(1)})$  is an  $\ell_1^{k_1}$ -average of constant  $1 - \varepsilon_1$ , with  $k_1 \geq k'_1$  (using Lemma 2.9),  $\bar{x}^{(1)}$  is a tail subsequence of  $\bar{x}$ , which starts far enough out so that  $\|z_1\| \leq 1$  and so that  $z_1$  is an  $\ell_1^{k_1}$ -average of constant  $\frac{1}{D}$  (using Proposition 5.1 and Corollary 5.2). Finally we choose

$$u_1 = \frac{1}{D} \frac{f(j_{2q_1(1)})}{j_{2q_1(1)}} \sum_{t=1}^{j_{2q_1(1)}/2} \bar{x}_t^{(1)} \text{ and } v_1 = \frac{1}{D} \frac{f(j_{2q_1(1)})}{j_{2q_1(1)}} \sum_{t=1+(j_{2q_1(1)}/2)}^{j_{2q_1(1)}} \bar{x}_t^{(1)},$$

(recall that the elements of  $J$  are even).

Assume now that for some  $n \geq 2$ , we have chosen  $z_1 < z_2 < \dots < z_{n-1}$  in  $B_{\mathcal{GM}}$ , and assume that the following conditions are satisfied:

- for each  $i < n$ ,  $z_i$  is an  $\ell_1^{k_i}$ -average of constant  $1/D$ , so that

$$f(k_i)/k_i < \varepsilon_i^2 \text{ and } f(\varepsilon_i \sqrt{k_i}) > \frac{1}{\varepsilon_i^2} \max \text{supp}(z_{i-1}), \text{ if } i \geq 2 \tag{6.1}$$

(in other words  $z_1 < z_2 < \dots < z_{n-1}$  satisfies the condition (4.2) of the first  $n - 1$  elements of an RIS).

- secondly  $z_i$  is of the form

$$z_i = u_i + v_i = \frac{1}{D} y_{\bar{x}^{(i)}}(j_{2q_i(1)}, j_{2q_i(2)}, \dots, j_{2q_i(l_i)}), \tag{6.2}$$

where  $l_i$ , and  $q_i(1) < q_i(2) < \dots < q_i(l_i)$  are in  $\mathbb{N}$  and  $\bar{x}^{(i)}$  is a tail subsequence of  $\bar{x}$ , starting far enough out to ensure that  $(z_i)_{i=1}^{n-1}$  is a block sequence and is in  $B_{\mathcal{GM}}$  (using Corollary 5.2). By Definition 2.7 of the yardstick vectors in Section 2 we can write  $z_i$  as  $z_i = \sum_{r=1}^{l_i} z(i, r)$  where the  $z(i, r)$ ,  $r = 1, 2, \dots, l_i$ , have pairwise disjoint support and so that for each  $r \leq l_i$  the vector  $z(i, r)$  is of the form

$$z(i, r) = \frac{1}{D} \frac{f(j_{2q_i(r)})}{j_{2q_i(r)}} \sum_{s=1}^{j_{2q_i(r)}} x(i, r, s), \tag{6.3}$$

where  $x(i, r, s)$ ,  $s = 1, 2, \dots, j_{2q_i(r)}$  are elements of the sequence  $\bar{x}^{(i)}$ , and we have

$$u_i = \frac{1}{D} \sum_{r=1}^{l_i} \frac{f(j_{2q_i(r)})}{j_{2q_i(r)}} \sum_{s=1}^{j_{2q_i(r)}/2} x(i, r, s) \text{ and} \quad (6.4)$$

$$v_i = \frac{1}{D} \sum_{r=1}^{l_i} \frac{f(j_{2q_i(r)})}{j_{2q_i(r)}} \sum_{s=1+(j_{2q_i(r)}/2)}^{j_{2q_i(r)}} x(i, r, s). \quad (6.5)$$

- moreover we assume that so far the following condition is satisfied:

For each sequence  $\bar{i} = (i_t : t = 1, 2, \dots, l) \subset \{1, 2, \dots, n-1\}$ , with  $1 \leq i_1 < i_2 < \dots < i_l \leq n-1$ , and for each  $\bar{\rho} = (\rho_t : t = 1, 2, \dots, l) \in \{-1, 1\}^l$  there is a sequence of functionals  $\bar{z}^*(\bar{i}, \bar{\rho}) = (z_t^* : t = 1, 2, \dots, l) = (z_{(\bar{i}, \bar{\rho})}^*(t) : t = 1, 2, \dots, l)$  in  $B_{\mathcal{GM}^*}$  so that for all  $t = 1, 2, \dots, l-1$ :

- (a)  $\text{supp}(z_t^*) \subset \bigcup_{s=1}^{j_{2q_i(r_t)}} \text{ran}(x(i_t, r_t, s))$ , for some  $r_t \in \{1, 2, \dots, l_{i_t}\}$ ,
- (b)  $z_t^* \in A_{j_{2q_i(r_t)}}^* \cap \mathbf{Q}$ ,
- (c)  $z_t^*(z(i_t, r_t))\rho_t \geq \frac{1}{2D}$ ,  $z_t^*(u(i_t)) = z_t^*(v(i_t)) = \frac{1}{2}z_t^*(z(i_t, r_t))$ , and,
- (d) if  $t \geq 2$ , then  $2j_{2q_i(r_t)} = \sigma(\rho_1 z_1^*, \rho_2 z_2^*, \dots, \rho_{t-1} z_{t-1}^*)$ .

In order to choose  $z_n$  we proceed as follows. We first choose  $k'_n \in J$  so that  $f(k'_n)/k'_n < \varepsilon_n^2$  and  $f(\varepsilon_n \sqrt{k'_n}) \geq \frac{1}{\varepsilon_n^2} \max \text{supp}(z_{n-1})$ . Assume that  $q_n \in \mathbb{N}$  satisfies the following properties:

$$j_{q_n} \geq k'_n \text{ and } \sqrt{f(j_{q_n} \varepsilon_n)} > \frac{\max \text{supp}(z_{n-1})}{\varepsilon_n}. \quad (6.7)$$

For each increasing sequence  $\bar{i} = (i_t : t = 1, 2, \dots, l) \subset \{1, 2, \dots, n-1\}$ , and each  $\bar{\rho} = (\rho_t : t = 1, 2, \dots, l) \subset \{\pm 1\}$  we can assume that

$$\sigma(z_{(\bar{i}, \bar{\rho})}^*(1), z_{(\bar{i}, \bar{\rho})}^*(2), \dots, z_{(\bar{i}, \bar{\rho})}^*(l)) \geq j_{q_n+1}, \quad (6.8)$$

for any  $\bar{i} = (i_t : t = 1, 2, \dots, l) \subset \{1, 2, \dots, n-1\}$ , with  $1 \leq i_1 < i_2 < \dots < i_l \leq n-1$ , and for each  $\bar{\rho} = (\rho_t : t = 1, 2, \dots, l) \subset \{\pm 1\}$ . Note that this can be accomplished by only perturbing the last element  $z_{(\bar{i}, \bar{\rho})}^*(l)$ , and thus still satisfying condition (6.6) (c) (and all the other conditions of (6.6)). Then we consider the set

$$\Sigma_n = \left\{ \sigma(z_{(\bar{i}, \bar{\rho})}^*(1), z_{(\bar{i}, \bar{\rho})}^*(2), \dots, z_{(\bar{i}, \bar{\rho})}^*(l)) : \begin{array}{l} \bar{i} = (i_t : t \leq l) \subset \{1, \dots, n-1\} \text{ increasing} \\ \bar{\rho} = (\rho_t : t = 1, 2, \dots, l) \subset \{\pm 1\} \end{array} \right\}$$

and order it into

$$j_{2q_n(1)} < j_{2q_n(2)} < \dots < j_{2q_n(l_n)}. \quad (6.9)$$

We then choose a tail subsequence  $\bar{x}^{(n)}$  of  $\bar{x}$  whose first element starts after  $z_{n-1}$  and so that its first  $\sum_{r=1}^{l_n} j_{2q_n(r)}$  elements are  $(1 + \varepsilon_n)C$  ( $C$  as in Proposition 5.1) equivalent to the first  $\sum_{r=1}^{l_n} j_{2q_n(r)}$  elements of  $S$ , and then put

$$z_n = \frac{1}{D} y_{\bar{x}}(j_{2q_n(1)}, j_{2q_n(2)}, \dots, j_{2q_n(l_n)}). \quad (6.10)$$

Then  $z_n > z_{n-1}$  and  $\|z_n\|_{\mathcal{GM}} \leq 1$  by Proposition 5.1, Lemma 2.9, Proposition 5.1 and Corollary 5.2 yield that  $z_n$  is an  $\ell_1^{k_n}$  average of constant  $\frac{1}{D}$ , with  $k_n \geq k'_n$ .

As before we can, by the definition of the yardstick vectors, write  $z_n$  as

$$z_n = \sum_{r=1}^{l_n} z(n, r), \quad (6.11)$$



where the  $z(n, r)$  have pairwise disjoint support and  $z(n, r)$  is for each  $r \leq l_n$  of the form

$$z(n, r) = \frac{1}{D} \frac{f(j_{2q_n(r)})}{j_{2q_n(r)}} \sum_{s=1}^{j_{2q_n(r)}} x(n, r, s),$$

where  $x(n, r, s)$ ,  $s = 1, 2, \dots, j_{2q_n(r)}$  are elements of the sequence  $\bar{x}^{(n)}$ , and we let

$$\begin{aligned} u_n &= \frac{1}{D} \sum_{r=1}^{l_n} \frac{f(j_{2q_n(r)})}{j_{2q_n(r)}} \sum_{s=1}^{j_{2q_n(r)}/2} x(n, r, s) \text{ and} \\ v_n &= \frac{1}{D} \sum_{r=1}^{l_n} \frac{f(j_{2q_n(r)})}{j_{2q_n(r)}} \sum_{t=s+(j_{2q_n(r)}/2)}^{j_{2q_n(r)}} x(n, r, s). \end{aligned} \tag{6.12}$$

It is now easy to find for every  $\bar{l} = (i_t : t = 1, 2, \dots, l) \subset \{1, 2, \dots, n\}$ , with  $1 \leq i_1 < i_2 < \dots < i_l \leq n$ , and for each  $\bar{\rho} = (\rho_t : t = 1, 2, \dots, l) \in \{-1, 1\}^l$  a sequence of functionals  $\bar{z}^*(\bar{l}, \bar{\rho}) = (z_t^* : t = 1, 2, \dots, l) = (z_{(\bar{l}, \bar{\rho})}^*(t) : t = 1, 2, \dots, l) \in B_{\mathcal{GM}^*}$  so that (6.6) holds. Indeed, if the last element  $i_l < n$ , then we already have chosen  $\bar{z}^*(\bar{l}, \bar{\rho})$ . So let us assume  $i_l = n$ . Let  $\bar{l}' = (i_t : t = 1, 2, \dots, l-1)$  and  $\bar{\rho}' = (\rho_t : t = 1, 2, \dots, l-1)$ . If  $\bar{l}'$  and  $\bar{\rho}'$  are empty we choose  $r = 1$ . Otherwise we choose  $r = r(\bar{l}, \bar{\rho}) \in \{1, 2, \dots, l_n\}$  so that

$$j_{2q_n(r)} = \sigma(z_{(\bar{l}', \bar{\rho}')}^*(1), z_{(\bar{l}', \bar{\rho}')}^*(2), \dots, z_{(\bar{l}', \bar{\rho}')}^*(l-1))$$

(by choice of  $q_n(i)$ ,  $i = 1, 2, \dots, l_n$ , this is possible). Then choose for every  $s = 1, 2, \dots, j_{2q_n(r)}$ , a functional  $x_s^* = x^*(n, s, \bar{l}, \bar{\rho}) \in B_{\mathcal{GM}^*} \cap \mathbf{Q}$ , so that

$$\begin{aligned} x_s^*(x(r, n, s)) &= x_{s'}^*(x(r, n, s')) \geq 1 - 2\varepsilon_n, \text{ for } s \neq s' \text{ in } \{1, 2, \dots, j_{2q_n(r)}\} \\ \text{and } \text{supp}(x_q^*) &\subset \text{ran}(x(r, n, q)). \end{aligned}$$

Let

$$z_{(\bar{l}, \bar{\rho})}^*(l) = \rho_l \frac{1}{f(j_{2q_n(r)})} \sum_{q=1}^{j_{2q_n(r)}} x_q^*,$$

and

$$\bar{z}_{(\bar{l}, \bar{\rho})}^* = (\bar{z}_{(\bar{l}', \bar{\rho}')}^*, z_{(\bar{l}, \bar{\rho})}^*(l)).$$

It follows therefore that (6.6) is satisfied, which finishes our recursive definition of  $z_n$ ,  $u_n$  and  $v_n$ .

Note that the construction of the  $z_n$  accomplishes the following:

If  $m \in \mathbb{N}$  and  $m \leq n_1 < n_2 < \dots < n_m$  are in  $\mathbb{N}$  and  $(a_s)_{s=1}^m \subset \mathbb{R}$ , we can choose  $\bar{l} = (n_s)_{s=1}^m$  and  $\bar{\rho} = (\text{sign}(a_s))_{s=1}^m$ , and conclude that

$$z^* = \frac{1}{\sqrt{f(k)}} \sum_{t=1}^m z_{(\bar{l}, \bar{\rho})}^*(t) \in \Gamma_m^*,$$

and

$$\left\| \sum_{s=1}^m a_s z_{n_s} \right\| \geq z^* \left( \sum_{s=1}^m a_s z_{n_s} \right) \geq \frac{1}{\sqrt{f(k)}} \frac{1}{D} \sum_{s=1}^m |a_s|. \tag{6.13}$$

After passing to a subsequence we can assume that  $(z_n)$  has a spreading model and that it is a  $\frac{1}{D}$ -RIS. We define  $w_n = u_n - v_n$ . Then  $(w_n)$  also satisfies (4.2) of the definition of  $\frac{1}{D}$ -RIS, and passing to a subsequence, we may also assume that  $(w_n)$  has a spreading model satisfying (4.3) and is therefore also a  $\frac{1}{D}$ -RIS. We claim that  $(w_n)$  satisfies the condition (5.1) of Proposition 5.1 and it follows therefore that  $(w_n)$  has a spreading model which is equivalent to the unit vector basis in  $S$ .

We first estimate  $\|z_n\|_l$  for  $n \in \mathbb{N}$  and  $l \in J$ . Let

$$r_0 = \max\{r = 1, 2, \dots, l_n : j_{2q_n(r-1)} \leq l\} \text{ with } j_{2q_n(0)} := 0.$$

Then by the definition of  $z_n$  and the  $z(n, r)$ , by condition (3.6) on the sequence  $(j_i)$ , and by Lemma 2.9, it follows that  $\sum_{r=r_0}^{l_n} z(n, r)$  is an  $\ell_1^k$  average for some  $k \geq j_{2q_n(r_0)-1} > l$ . It follows therefore from Lemma 4.2 that

$$\left\| \sum_{r=r_0}^{l_n} z(n, r) \right\|_l \leq \frac{2}{f(l)}. \quad (6.14)$$

If  $r = 1, 2, \dots, r_0 - 1$ , and thus  $j_{2q_n(r)} < l$ , we deduce from (3.7) that

$$f(l) \geq f(j_{2q_n(r)+1}) \geq \frac{2}{\varepsilon_{2q_n(r)+1}} j_{2q_n(r)}$$

and thus, by Lemma 4.5 (b)

$$\|z(n, r)\|_l \leq \frac{2f(j_{2q_n(r)})}{j_{2q_n(r)}}. \quad (6.15)$$

It follows therefore from (6.14), (6.15) and (3.5) that for  $l \in J \setminus \{q_n(1), q_n(2), \dots, q_n(l_n)\}$

$$\|z_n\|_l \leq \sum_{r=1}^{l_n} \frac{2f(j_{2q_n(r)})}{j_{2q_n(r)}} + \frac{2}{f(l)} \leq \frac{1}{f(j_{2q_n(1)-1})} + \frac{2}{f(l)} \leq \frac{1}{j_{q_n}} + \frac{2}{f(l)}. \quad (6.16)$$

Using the same argument we observe similar inequalities for  $u_n, v_n, w_n$  :

$$\|u_n\|_l \leq \frac{1}{j_{q_n}} + \frac{2}{f(l)}, \quad \|v_n\|_l \leq \frac{1}{j_{q_n}} + \frac{2}{f(l)} \text{ and } \|w_n\|_l \leq \frac{1}{j_{q_n}} + \frac{2}{f(l)}. \quad (6.17)$$

In order to verify condition (5.1) of Proposition 5.1 let  $m \leq n_1 < n_2 < \dots < n_s$  be in  $\mathbb{N}$  and  $(a_s)_{s=1}^m \subset \mathbb{R} \setminus \{0\}$  we put  $y = \sum_{s=1}^m a_s w_{n_s}$ . Secondly let  $k \in \mathbb{N}$  and  $z^* \in \Gamma_k^*$ . As before we write  $z^*$  as

$$z^* = \frac{1}{\sqrt{f(k)}} \sum_{t=1}^k z_t^* \in \Gamma_k^*,$$

with  $z_1^* \in A_{l_1}^*$ , and  $l_1 = j_{2k'}$ , for some  $k' \geq k$ , and  $z_i^* \in A_{l_i}^*$ , with  $l_i = \sigma(z_1^*, z_2^*, \dots, z_{i-1}^*)$ , for  $i = 2, \dots, k$  and assume that

$$t_0 = \min\{t = 1, \dots, k : z_t^*(y) \neq 0\},$$

exists (otherwise  $z^*(y) = 0$ ). Note that the equalities in condition (6.6)(c) imply that  $z^*(\bar{v}, \bar{\rho})(w_j) = 0$  for every  $j \in \mathbb{N}$ , every increasing sequence  $\bar{v} = (i_t : t = 1, 2, \dots, l) \subset \{1, 2, \dots, \}$ , for each  $\bar{\rho} = (\rho_t : t = 1, 2, \dots, l) \in \{-1, 1\}^l$ , and for every  $t = 1, 2, \dots, l$ . So it follows that the sequence  $(z_1^*, z_2^*, \dots, z_{t_0}^*)$  cannot be one of the sequences  $z^*(\bar{v}, \bar{\rho})(t)$ , where  $\bar{v} = (i_t : t = 1, 2, \dots, l) \subset \{1, 2, \dots, \}$ , is increasing and  $\bar{\rho} = (\rho_t : t = 1, 2, \dots, l) \in \{-1, 1\}^l$ . From the injectivity of  $\sigma$  it follows therefore that the sets  $\{l_t : t > t_0\}$  and the set  $\{j_{2q_n(r)} : n \in \mathbb{N}, r \leq l_n\}$  are disjoint.

We can now apply (4.11) and (6.17) to deduce that

$$\begin{aligned}
 |z^*(y)| &\leq \frac{|z_{t_0}^*(y)|}{\sqrt{f(k)}} + \frac{\max_{s \leq m} |a_s|}{\sqrt{f(k)}} \left[ \sum_{t \in T_0} \max_{s \in S_t} \|w_{n_s}\|_{l_t} + 2m\varepsilon_{n_1} \right] \\
 &\quad + \max_{s \leq m} |a_s| + \frac{1}{\sqrt{f(k)}} \sum_{s=s_0+1}^m |a_s| \left| \sum_{t \in T_s} z_t^*(w_{n_s}) \right| \\
 &\leq \frac{|z_{t_0}^*(y)|}{\sqrt{f(k)}} + \frac{\max_{s \leq m} |a_s|}{\sqrt{f(k)}} \left[ 1 + \sum_{t=t_0+1}^k \frac{2}{f(l_t)} + \sum_{s=1}^m \frac{1}{j_{q_{n_s}}} \right] \\
 &\quad + \max_{s \leq m} |a_s| + \frac{\max_{s \leq m} |a_s|}{\sqrt{f(k)}} \left[ \sum_{s=s_0+1}^m \sum_{t \in T_s} \left( \frac{1}{j_{q_{n_s}}} + \frac{2}{f(l_t)} \right) \right] \\
 &\leq \frac{|z_{t_0}^*(y)|}{\sqrt{f(k)}} + \max_{s \leq m} |a_s| + \frac{\max_{s \leq m} |a_s|}{\sqrt{f(k)}} \left[ 1 + \sum_{t=t_0+1}^k \frac{4}{f(l_t)} + \sum_{s=1}^m \frac{1}{j_{q_{n_s}}} + \sum_{s=s_0+1}^m \frac{k}{j_{q_{n_s}}} \right]
 \end{aligned}$$

where by (4.15)  $s_0 = \min \{s = 1, 2 \dots \max \text{supp}(w_{s-1}) < \varepsilon_{n_1} \sqrt{f(k)}\}$ . It follows for  $s > s_0$  from (6.7) that

$$\sqrt{f(k)} < \frac{\max \text{supp}(z_{s_0})}{\varepsilon_{n_1}} \leq \sqrt{f(\varepsilon_{n_s} j_{q_{n_s}})}$$

and thus, that  $k/j_{q_{n_s}} < \varepsilon_{n_s} \leq \varepsilon_{n_1}$  which yields,

$$|z^*(y)| \leq \frac{|z_{t_0}^*(y)|}{\sqrt{f(k)}} + 8 \max_{s \leq m} |a_s|$$

and allows us to conclude from Proposition 5.1 that the spreading model of  $(w_n)$  is equivalent to the unit vector basis in  $S$ .

Together with (6.13) we therefore proved the following result.

**THEOREM 6.1.** *There is a constant  $c > 1$  so that in every block subspace of  $\mathcal{GM}$  we can find block sequences  $u_n$  and  $v_n$ , with  $u_1 < v_1 < u_2 < v_2 < \dots$ , so that  $(u_n - v_n)$  has a spreading model which is  $c$ -equivalent to the unit vector basis of  $S$ , and the sequences  $(u_n), (v_n)$  ( $u_n + v_n$ ) have spreading models which  $c^{-1}$ -dominate the norm  $\|\cdot\|_{f^{1/2}}$  which was introduced in Section 2. I.e. if we put  $x_n = u_n, v_n$  or  $u_n + v_n$ , for  $n \in \mathbb{N}$ , and we denote by  $(E, \|\cdot\|_E)$  the spreading model of  $(x_n)$  and its basis by  $(e_j)$  then*

$$c \left\| \sum_{s=1}^{\infty} a_s e_s \right\|_E \geq \|(a_s)\|_{f^{1/2}} = \max_{m \in \mathbb{N}, s_1 < s_2, \dots, s_m} \frac{1}{\sqrt{f(m)}} \sum_{i=1}^m |a_{s_i}| \text{ for } (a_s) \in c_{00}. \quad (6.18)$$

Thus, Corollary 2.6 and Lemma 2.5 yield our final result:

**THEOREM 6.2.** *Let  $(u_n)$  and  $(v_n)$  be as in Theorem 6.1. Then there is a subsequence  $(n_k)$  of  $\mathbb{N}$  so that  $(u_{n_k})$  and  $(v_{n_k})$  are equivalent.*

*Proof.* Using Corollary 2.6 and Lemma 2.3 twice, we may assume that

$$\begin{aligned}
 S_1 : [u_n : n \in \mathbb{N}] &\rightarrow [u_n - v_n : n \in \mathbb{N}], \quad \text{defined by } u_n \mapsto u_n - v_n, \text{ and} \\
 S_2 : [v_n : n \in \mathbb{N}] &\rightarrow [u_n - v_n : n \in \mathbb{N}], \quad \text{defined by } v_n \mapsto u_n - v_n, \text{ for } n \in \mathbb{N}
 \end{aligned}$$

are bounded. So the bounded map  $\text{Id}|_{[u_n]} - S_1$  defines  $u_n \mapsto v_n$ , while  $\text{Id}|_{[v_n]} + S_2$  defines  $v_n \mapsto u_n$ , proving the claim.  $\square$

REMARK 11. *It is worth noting that while  $(u_n)$  and  $(v_n)$  are intertwined and equivalent in  $\mathcal{GM}$ , the sequences  $(v_n)$  and  $(u_{n+1})$  are not, in general, equivalent. Otherwise the shift on  $[u_n : n \in \mathbb{N}]$  would be an isomorphism and we would obtain an isomorphism of a subspace of  $\mathcal{GM}$  with its hyperplanes. But this is impossible if  $(u_n)$  was picked inside an HI subspace of  $\mathcal{GM}$ .*

## 7. Consequences of the main result

### 7.1. Asymptotic unconditionality

Recall that a seminormalized basis  $(e_n)$  is said to be *asymptotically unconditional* if there exists a constant  $C \geq 1$  such that for any  $k \in \mathbb{N}$  and any successive blocks  $k < x_1 < \dots < x_k$  on the basis, the sequence  $(x_1, \dots, x_k)$  is  $C$ -unconditional. The following is an easy consequence of Theorem 6.1.

PROPOSITION 7.1. *The space  $\mathcal{GM}$  does not contain any asymptotically unconditional block sequence.*

We recall that the asymptotically unconditional HI space  $G$  of Gowers is tight by range [12] and therefore contains no intertwined and equivalent block sequences.

The sequences  $(u_n)$  and  $(v_n)$  are chosen in an arbitrary, but fixed subspace  $Y$  of  $\mathcal{GM}$ , and this is why our techniques do not seem to imply that  $\mathcal{GM}$  is HI (although we suspect it is). This restriction is essentially technical, however, since as we shall now see, by using Gowers' Ramsey theorem, it disappears when passing to an appropriate subspace of  $\mathcal{GM}$ .

### 7.2. Applications of Gowers' Theorem

Recall that Gowers' game  $\mathcal{G}_X$  in a space  $X$  with a basis is a game between two players, where Player 1 plays block subspaces  $Y_n$  of  $X$  and Player 2 successive blocks  $y_n \in Y_n$ , the outcome of the game being the block-sequence  $(y_n)$ .

The set  $b(X)$  of block-sequences of  $X$  is seen as a subset of  $X^\omega$  equipped with the product of the norm topology on  $X$ . Also for  $\Delta = (\delta_n)_n$  a sequence of positive numbers, and  $\mathbb{A} \subset b(X)$ , the set  $\mathbb{A}_\Delta$  is defined as

$$\mathbb{A}_\Delta = \{(x_n) \in b(X) \mid \exists (y_n) \in \mathbb{A}, \|y_n - x_n\| \leq \delta_n \forall n\}.$$

THEOREM 7.2 Gowers' Ramsey Theorem, [16]. *Let  $X$  be a space with a basis, and  $\mathbb{A}$  an analytic subset of  $b(X)$ . Let  $\Delta > 0$ . Then there exists a block-subspace  $Y$  of  $X$  such that  $\mathbb{A} \cap b(Y) = \emptyset$ , or such that Player 2 has a winning strategy in Gowers' game  $\mathcal{G}_Y$  to produce an outcome in  $\mathbb{A}_\Delta$ .*

Given  $\rho \geq 1$ , consider the set  $\mathbb{A}_\rho$  of block sequences  $(x_n)_n$  in  $\mathcal{GM}$  such that for all  $k \in \mathbb{N}$  and all  $k \leq n_1 < n_2 < \dots < n_k$  in  $\mathbb{N}$   $(x_{2n_i} - x_{2n_i+1})_{i=1}^k$  is  $\rho$ -equivalent to the unit vector basis of  $S$ , and the sequences  $(x_{2n_i})_{i=1}^k, (x_{2n_i+1})_{i=1}^k, (x_{2n_i} + x_{2n_i+1})_{i=1}^k$   $\rho^{-1}$  dominate the norm  $\|\cdot\|_{f^{1/2}}$  (as introduced in Definition 2.1). It is easily checked that  $\mathbb{A}_\rho$  is closed (with respect to the product

of the discrete topology on  $\mathcal{GM}$ ). Theorem 6.1 implies that for  $\rho < c$ , every block subspace of  $\mathcal{GM}$  contains a block sequence in  $\mathbb{A}_\rho$ . So up to modifying the constant  $c$  to take into account a small enough perturbation  $\Delta$ , we may apply Gowers' Theorem to find a block-subspace  $Y$  of so that the vectors  $u_n$  and  $v_n$  of Theorem 6.1 may be chosen in arbitrary block-subspaces of  $Y$  prescribed by Player 1.

**PROPOSITION 7.3.** *There exists  $c \geq 1$  and a block subspace of  $\mathcal{GM}$  in which Player 2 has a winning strategy to produce  $u_1 < v_1 < u_2 < v_2 < \dots$ , so that  $(u_n - v_n)$  has a spreading model which is  $c$ -equivalent to the unit vector basis of  $S$ , and the sequences  $(u_n), (v_n)$   $(u_n + v_n)$  have spreading models which  $c^{-1}$  dominate the norm  $\|\cdot\|_{f^{1/2}}$ .*

Note that  $\|\sum_{i=1}^m e_i\|_{f^{1/2}} = mf(m)^{-1/2}$  while  $\|\sum_{i=1}^m e_i\|_S = mf(m)^{-1}$ . So for any  $\epsilon > 0$ , one can find  $m \in \mathbb{N}$  with following property: for any  $U, V$  block subspaces of  $Y$ , there exist  $u_1 < v_1 < \dots < u_m < v_m$ , with  $u_i \in U, v_i \in V$  for each  $i$ , such that  $\|\sum_{i=1}^m u_i - v_i\| < \epsilon \|\sum_{i=1}^m u_i + v_i\|$ , which of course implies the HI property. This property is actually the uniform version of the HI property which appears as the counterpart of asymptotic unconditionality in the dichotomy proved by Wagner [26].

The third dichotomy implies that we may assume that the space of Proposition 7.3 is tight, and the fourth dichotomy that it is subsequentially minimal. Actually slightly more may be observed.

**THEOREM 7.4.** *There exists a tight HI block-subspace  $\mathcal{X}_{GM}$  of  $\mathcal{GM}$  with a normalized basis which is subsequentially minimal. More precisely, there exists  $c \geq 1$ , such that for any block-subspace  $Y$  of  $\mathcal{X}_{GM}$ , there exists a block-sequence  $(y_k)$  of  $Y$  and a subsequence  $(f_k)$  of the basis of  $\mathcal{X}_{GM}$  such that*

- (a)  $y_1 < f_1 < y_2 < f_2 < \dots$
- (b)  $(y_k), (f_k), (y_k + f_k)$  have spreading models which  $c^{-1}$  dominate the norm  $\|\cdot\|_{f^{1/2}}$ ,
- (c)  $(y_k - f_k)$  has a spreading model which is  $c$ -equivalent to the unit vector basis of  $S$ ,
- (d) consequently,  $(f_k)$  is equivalent to  $(y_k)$ .

This is a variation on [11, Proposition 6.5]. Since the proof is much shorter than the demonstration of the fourth dichotomy, we give a sketch of it.

*Proof.* Let  $\mathbb{A}_\rho \subset b(\mathcal{GM})$ , for small enough  $\rho$ , be defined as after Theorem 7.2. Using Gowers' first dichotomy (see Theorem 1.1), the fact that no HI space has a minimal subspace, and the third dichotomy proven in [11] (see Theorem 1.3) we may pass to an HI tight subspace. By Theorem 6.2 and Gowers' Ramsey Theorem (Theorem 7.2) we can, after choosing  $\rho$  small enough, assume that Player 2 has a winning strategy in Gowers' game to play inside  $\mathbb{A}_\rho$  on some further infinite dimensional subspace; also we may and shall only use blocks with rational coordinates in this proof (and assume Gowers' game is played with such blocks). Then the finite block-sequences of initial moves prescribed by the winning strategy of Player 2 form a non-empty tree  $T$  which does not have any maximal elements. We denote by  $[T]$  the infinite block sequences  $(x_j)$  for which all the initial segments  $(x_j)_{j=1}^n, n \in \mathbb{N}$ , lie in  $T$ . Since  $\mathbb{A}_\rho$  is closed it follows that  $[T] \subseteq \mathbb{A}_\rho$  and for all  $(y_1, \dots, y_m) \in T$  and all block sequences  $(z_n)$ , there is a block  $y_{m+1}$  of  $(z_n)$  such that  $(y_1, \dots, y_m, y_{m+1}) \in T$ , [11, Lemma 6.4]. Since  $T$  is countable, we can construct inductively a block sequence  $(v_n)$  such that for all  $(u_1, \dots, u_m) \in T$  there is some  $n \in \mathbb{N}$  with  $(u_1, \dots, u_m, v_n) \in T$ .

We claim that  $\mathcal{X}_{GM} := [v_n, n \in \mathbb{N}]$  works. Indeed, if  $(z_n)$  is any block sequence of  $(v_n)$ , we may construct inductively a block-sequence  $(y_i)$  of  $(z_n)$  and a subsequence  $(f_i)$  of  $(v_n)$  such that  $(y_1, f_1, \dots, y_n, f_n)$  belongs to  $T$  for all  $n$ . Therefore, since  $\mathbb{A}_\rho$  is closed  $(y_1, f_1, y_2, f_2, \dots)$  belongs to  $\mathbb{A}_\rho$ . Finally the normalized basis  $(v_n/\|v_n\|)$  of  $\mathcal{X}_{GM}$  satisfies the conclusion of Theorem 7.4.  $\square$

Since this construction can be done in any block-subspace of  $\mathcal{GM}$ , we may assume that  $\mathcal{X}_{GM}$  is actually sequentially minimal.

### 7.3. Local minimality

We briefly expose the fifth dichotomy obtained in [11], which is related to the second general kind of tightness called tightness with constants. A space  $X = [e_n]$  is tight *with constants* when for every infinite dimensional space  $Y$ , the sequence of successive subsets  $I_0 < I_1 < \dots$  of  $\mathbb{N}$  witnessing the tightness of  $Y$  in  $X$  may be chosen so that  $Y \not\subseteq_K [e_n \mid n \in \mathbb{N} \setminus I_K]$  for each  $K$ . Equivalent no infinite dimensional space embeds uniformly into the tail subspaces of  $X$  [11, Proposition 4.1]. This is the case for Tsirelson's space  $T$  or its  $p$ -convexified version  $T^{(p)}$ .

On the other hand we already mentioned that a space  $X$  is said to be *locally minimal* if there exists a constant  $K \geq 1$  such that every finite dimensional subspace of  $X$   $K$ -embeds into every infinite dimensional subspace of  $X$ .

**THEOREM 7.5** Fifth dichotomy [11]. *Any Banach space contains a subspace with a basis which is either tight with constants or locally minimal.*

Since  $S$  contains  $\ell_\infty^n, n \in \mathbb{N}$ , uniformly and since  $\mathcal{GM}$  is saturated with sequences with spreading model  $c$ -equivalent to the basis of  $S$ ,  $\mathcal{GM}$  also contains  $\ell_\infty^n, n \in \mathbb{N}$ , uniformly in every subspace. So by the universal properties of these spaces,  $\mathcal{GM}$  is locally minimal.

**THEOREM 7.6.** *There exists a locally and sequentially minimal HI Banach space.*

Since an HI space does not contain a minimal subspace, this answers [12, Problem 5.2], that is, the space  $\mathcal{GM}$  demonstrates that there are other forms of tightness than tightness by range or with constants.

The fifth dichotomy and a dichotomy due to A. Tcaciuc [24] are used in [11] to refine the types (1)–(6) into subclasses. In their terminology,  $\mathcal{X}_{GM}$  is of type (2b).

### 7.4. Open problems

The most important problem which remains open in Gowers' classification program is whether there exist spaces of type (4). Note that such a space would satisfy the criterion of Casazza, and therefore would not be isomorphic to its proper subspaces.

**PROBLEM 7.7.** *Find a space with an unconditional basis, tight by range and quasi-minimal.*

The nature of the tightness of  $\mathcal{X}_{GM}$  remains to be understood. This property is a consequence of the non-minimality of HI spaces and of the third dichotomy, with no information on how the sequence  $(I_n)$  of subsets of  $\mathbb{N}$  depends on the subspace  $Y$ .

PROBLEM 7.8. Find information on the sequences  $(I_n)$  in the definition of the tightness of  $\mathcal{X}_{GM}$ . Is  $\mathcal{GM}$  or  $GM$  itself tight?

C. Rosendal [21] defined notions of  $\alpha$ -minimality and  $\alpha$ -tightness, where  $\alpha < \omega_1$  is an ordinal. Local minimality implies that  $\mathcal{X}_{GM}$  is  $\omega^2$ -minimal and not  $\omega$ -tight. On the other hand, being tight, it must be  $\alpha$ -tight for some  $\alpha < \omega_1$ , [21, Theorem 3].

PROBLEM 7.9. Find  $\min\{\alpha \in \omega_1 \mid \mathcal{X}_{GM} \text{ is } \alpha\text{-tight}\}$ .

It is unknown whether an HI space may be tight with constants. With the exception of the uniformly convex HI space of [8], examples of the Gowers-Maurey family usually contain  $\ell_\infty^n$ ,  $n \in \mathbb{N}$ , uniformly - and therefore are locally minimal.

PROBLEM 7.10. Find an HI space which is tight with constants.

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