Almost Fraïssé Banach spaces

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Abstract. Continuing with the study of approximately ultrahomogeneous and Fraïssé Banach spaces introduced by V. Ferenczi, J. López-Abad, B. Mbombo and S. Todorcevic, we define formally weaker and in some aspects more natural properties of Banach spaces which we call Almost ultrahomogeneity and the Almost Fraïssé Property. We obtain relations between these different homogeneity properties of a space E and relate them to certain pseudometrics on the class $\mathrm{Age}(E)$ of finite dimensional subspaces of E. We prove that ultrapowers of an almost Fraïssé Banach space are ultrahomogeneous. We also study two properties called finitely isometrically extensible and almost finitely isometrically extensible, respectively, and prove that approximately ultrahomogeneous reflexive Banach spaces are finitely isometrically extensible.

Finally, we study oligomorphy in Banach spaces, and give a proof that a Banach space is Fraïssé if and only if it is approximately ultrahomogeneous and oligomorphic.

Contents

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1. Introduction

The standard terminology and notation of Banach space theory used in this paper may be found in [**2**]. Recall that a Banach space is called *transitive* if for any two points on the unit sphere, there is an onto isometry defined on the whole space that sends one onto the other. In 1932, S. Mazur conjectured that separable transitive Banach spaces are Hilbert [**3**, p. 151]. This is the Mazur rotation problem (two surveys on this problem and related topics are [**8**] and [**10**]). Of course Hilbert spaces are transitive, and it is known that there exist non-separable and non-hilbertian transitive Banach spaces. This paper outgrowths from the study of multidimensional aspects of the Mazur rotation problem, as initiated in [**15**] and pursued in [**10, 14**].

The notation we shall use for Fraïssé theory in Banach spaces is from [**15**] and is recalled here. We denote by $\mathrm{Age}(E)$ and $\mathrm{Age}_k(E)$ the set of all finite-dimensional subspaces of a Banach space E and the set of all k-dimensional subspaces of E, respectively. Given two Banach spaces X and E, and $\delta \geq 0$, a δ-isometry from X to E is a linear map $T\colon X \to E$ such that $\frac{1}{1+\delta}\|x\| \leq \|Tx\| \leq (1+\delta)\|x\|$ for all $x \in X$; $\mathrm{Emb}(X, E)$ ($\mathrm{Emb}_\delta(X, E)$) is the (possibly empty) set of isometries (δ-isometries, resp.) from X into E. Also, $\mathrm{Isom}(E)$ ($\mathrm{Isom}_\varepsilon(E)$) denotes the set of all surjective isometries (ε-isometries) on E.

1.1. Ultrahomogeneity, Approximately ultrahomogeneity, FIE-ness and almost FIE-ness. An approach to the Mazur rotation problem involves studying properties stronger than transitivity, satisfied by Hilbert spaces, and including non-separable non-Hilbertian examples: if we can prove that the only separable space with such a property is the Hilbert space, then arguably this is a first step towards a positive answer to Mazur problem in which the separability hypothesis was used. A fundamental example is the multidimensional version of transitivity, which is called *ultrahomogeneity*, [**15**, Definition 2.2]. A Banach space E is *ultrahomogeneous* (**UH**) when for every $X \in \mathrm{Age}(E)$ every element of $\mathrm{Emb}(X, E)$ can be extended to an element of $\mathrm{Isom}(E)$, or equivalently the group $\mathrm{Isom}(E)$ acts transitively on the metric space $\mathrm{Emb}(X, E)$ for all $X \in \mathrm{Age}(E)$. Of course, Hilbert spaces are ultrahomogeneous. Also ultrapowers of the Gurarij space or of the spaces $L_p[0, 1]$ for $p \notin 2\mathbb{N}+4$ are non-separable examples of ultrahomogeneous spaces , [**1**, Proposition 4.13] and [**15**, Corollary 2.16].

To go on, let us recall a weaker form of the notion of transitivity: namely the property of almost-transitivity. A Banach space is called *almost transitive* if the orbits of the isometry group of the space are dense in the unit sphere. A. Pełczyński and S. Rolewicz proved that the Banach space $L_p(0, 1)$ is almost transitive [**20**] when $1 \leq p < +\infty$. W. Lusky gave a multidimensional version of this result by showing that the group $\mathrm{Isom}(L_p(0, 1))$ acts almost transitively on each metric space $\mathrm{Emb}(X, L_p(0, 1))$ whenever X is a finite dimensional subspace of $L_p(0, 1)$ and $p = 2$ or $p \notin 2\mathbb{N}$ [**18**]. In the language introduced in [**15**], Lusky's result says that for those values of p, $L_p(0, 1)$ is approximately **UH**: a Banach space E is *approximately ultrahomogeneous* (in short approximately **UH**) if for each $X \in \mathrm{Age}(E)$, the group $\mathrm{Isom}(E)$ acts almost transitively on $\mathrm{Emb}(X, E)$, that is, if for each $\varepsilon > 0$ and $\phi, \psi \in \mathrm{Emb}(X, E)$, there exists $T \in \mathrm{Isom}(E)$ such that $\|T\phi - \psi\| < \varepsilon$. In Lusky's paper only a formally weaker extension property of those L_p spaces is stated, which we call here *almost ultrahomogeneity*: a Banach space E is *almost ultrahomogeneous* (almost **UH**) if for each $\varepsilon > 0$, each $X \in \mathrm{Age}(E)$ and each $\phi \in \mathrm{Emb}(X, E)$, there exists $T \in \mathrm{Isom}_\varepsilon(E)$ such that $T|_X = \phi$. Therefore almost **UH** is a natural multidimensional counterpart of almost transitivity. So, we have the following relationships:

$$\begin{array}{ccccc} \text{Ultrahomogeneous} & \Rightarrow & \text{Approx. Ultrahomogeneous} & \Rightarrow & \text{Almost Ultrahomogeneous} \\ \Downarrow & & \Downarrow & & \\ \text{Transitive} & \Rightarrow & \text{Almost transitive} & & \end{array}$$

Motivated by the three aforementioned properties, we start by introducing two properties which we call *finitely isometrically extensible* (**FIE**) and almost *finitely isometrically extensible* (**aFIE**). A Banach space E is **FIE** if any isometry defined on a finite dimensional subspace of E can be extended to a norm-one operator defined on the whole space. The **aFIE**-property is an ε-version of the **FIE**-property: a Banach space E is **aFIE** if for any $\varepsilon > 0$, any isometry defined on a finite dimensional subspace of E can be extended to an operator defined on the whole space with norm at most $1 + \varepsilon$. Contrarily to the transitivity property, the **FIE** is preserved by taking 1-complemented subspaces, and in particular the 1-dimensional part of the **FIE**-property always holds: any isometry defined on a 1-dimensional subspace of E can always be extended to a one-norm operator defined on E. It is clear that ultrahomogeneous spaces are **FIE**, but also other spaces, such as $c_0(\Gamma)$ for any non-empty set Γ, are **FIE**. Also, we prove that almost ultrahomogeneous reflexive spaces are **FIE**. We do not know if **aFIE** Banach spaces are in fact **FIE**, but we note that the two notions coincide in the case of reflexive Banach spaces.

1.2. Fraïsséness, Almost Fraïsséness and oligomorphic Banach spaces. V. Ferenczi, J. López-Abad, B. Mbombo and S. Todorcevic defined and studied Fraïssé Banach spaces (the reader can find nutritious information about this topic in [**15, 10**]). A Banach space E is *Fraïssé* if for every $\varepsilon > 0$ and every dimension k, there exists $\delta > 0$ such that if X is a k-dimensional subspace of E then the action of $\mathrm{Isom}(E)$ on $\mathrm{Emb}_\delta(X, E)$ has ε-dense orbits. Examples of Fraïssé Banach spaces are $L_p(0, 1)$ when $p \notin 2\mathbb{N} + 4$ and the Gurarij space $\mathbb{G}$. The existence of separable Fraïssé Banach spaces other than $\mathbb{G}$ or some $L_p(0, 1)$ is a main open problem.

Weaker forms of the Fraïssé property appear in recent works by several authors. M. Cuth, N. de Rancourt and M. Doucha investigate such properties in relation to genericity of separable Banach spaces [**13**]. W. Kubiś [**17**] considers "approximate Fraïssé" limits for "metric-enriched" categories, including Banach spaces. See also Chapter 6 of F. Cabello Sánchez and J. M. F. Castillo's book [**9**], regarding Fraïssé classes, including in the p-Banach space context.

Fraïssé Banach spaces are extremely important in relation to certain Ramsey properties of the classes $\mathrm{Emb}(X, E)$ and to the extreme amenability of the topological group $\mathrm{Isom}(E)$. Here we focus on the following isometric properties of Fraïssé spaces:

(1) In Fraïssé spaces E, the Banach-Mazur and a restricted version of the Kadets pseudometrics are uniformly equivalent on the class of finite dimensional subspaces of E;
(2) Separable Fraïssé spaces are isometrically determined, among separable Fraïssé spaces, by their local structure;
(3) Separable spaces who are finitely representable in a Fraïssé space E can be isometrically embedded into E;

(4) For every non-free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the ultrapower $E_{\mathcal{U}}$ of a Fraïssé space E is **UH**.

See [**15**, Theorems 2.12 and 2.19, Propositions 2.13 and 2.15]. It is natural to ask whether the full strength of the Fraïssé property is needed for these results. Finding weaker conditions could possibly lead to interesting new properties as well as shed some new light on the Fraïssé property. With this aim in mind, we introduce in this paper the *almost Fraïssé Banach spaces* (**AF** in short). A Banach space E is **AF** if for every $\varepsilon > 0$ and every dimension $k \in \mathbb{N}$, there is $\delta > 0$ such that if $X \in \mathrm{Age}_k(E)$ and $\phi \in \mathrm{Emb}_\delta(X, E)$, there is $T \in \mathrm{Isom}_\varepsilon(E)$ which extends ϕ. We also study a weakening of this property which we call here *weak almost Fraïssé* (weak **AF** shortly). A Banach space is weak **AF** if whenever $X \in \mathrm{Age}(E)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $\phi \in \mathrm{Emb}_\delta(X, E)$, then there exists $T \in \mathrm{Isom}_\varepsilon(E)$ which extends ϕ. This variation imitates the notion of weak Fraïssé space from [**15**]. As it is expected, every Fraïssé space is **AF**, every weak Fraïssé space is weak **AF** and **AF** spaces are weak **AF**. When $\mathrm{Age}_k(E)$ is compact for each k (with respect to the Banach-Mazur distance), the two above introduced classes coincide, as we prove below. This is similar to [**15**, Theorem 2.12] regarding the Fraïssé and weak Fraïssé properties. The almost and weak almost Fraïssé properties only involve ε-isometries, and in this sense, may seem more natural than their Fraïssé counterparts, where the definition involves a mixture of isometries with ε-isometries. We highlight the following facts about these classes of spaces (see Subsections 4.1 and 4.2).

(1) We introduce some analogous versions of the Kadets pseudometric on the closure of $\mathrm{Age}(E)$ with respect to the topology induced by the Banach-Mazur distance, and we prove that they are indeed pseudometrics when E is **AF**.
(2) We prove that for every non-free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the ultrapower of an **AF** space E is **AF** and **UH**.

Answering one of our questions relative to the properties of Fraïssé spaces listed above, Item (2) indicates that a formally weaker property than Fraïsséness is sufficient to obtain ultrahomogeneous ultrapowers. Also, an interesting aspect of the weak Fraïssé property is that it is characterized by properties of the compact spaces $\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$, instead of the possibly non closed $\mathrm{Age}_k(E)$.

We end the paper by giving a proof of a characterization of Fraïssé spaces through oligomorphy in Banach spaces. To fix ideas, let $n \in \mathbb{N}$ and E be a Banach space. Consider the natural action of $\mathrm{Isom}(E)$ on $S_E^n := S_E \times \overset{n}{\cdots} \times S_E$, where S_E denotes the unit sphere of E, that is,

$$(T, (x_1, \ldots, x_n)) \mapsto (Tx_1, \ldots, Tx_n). \tag{1.1}$$

We are interested in the case when for each $n \in \mathbb{N}$, the quotient (classically denoted $S_E^n /\!\!/ \mathrm{Isom}(E)$) of S_E^n by the orbit relation of the action of $\mathrm{Isom}(E)$ is compact; see the very general study of I. Ben-Yaacov et al in [**5**] about this property and its reformulation as *ω-categoricity* (in the separable case), or also [**7, 21**], where the word *approximately oligomorphic* is used. In the present paper we shall simply say that E is an *oligomorphic* Banach space in this case. All $L_p(0,1)$-spaces, $1 \leq p < +\infty$, and the Gurarij space $\mathbb{G}$ are oligomorphic in this sense [**5**, Section 17] and [**6**, Section 2], respectively.

In the last section, we establish the following result, which was claimed but not explicitly proved by I. Ben Yaacov [**4**]: a Banach space is Fraïssé if and only if it is approximately ultrahomogeneous and oligomorphic. As observed by him, this gives a model theoretic proof of the result from [**15**] that the spaces $L_p(0,1)$ are Fraïssé when $p=2$ or $p \notin 2\mathbb{N}$.

2. Finitely isometrically extensible Banach spaces

Even though the ultrahomogeneity properties such as **UH**, approximately **UH**, or the Fraïssé property are very powerful, these are not preserved under norm one projections. Such a regularity might be desirable for abstract results about spaces satisfying some kind of ultrahomogeneity. For this reason we consider the following:

Definition 2.1. A Banach space E is *finitely isometrically extensible* (**FIE**) if for all $X \in \mathrm{Age}(E)$ and all $\phi \in \mathrm{Emb}(X,E)$, there is an operator $T\colon E \to E$ with $\|T\| = 1$ which extends ϕ.

Note that the **FIE**-property is equivalent to the following: any isometry between finite dimensional subspaces of E can be extended to a one-norm operator from E to E.

Example 2.2. Clearly, **UH** Banach spaces enjoy the **FIE**-property. In particular Hilbert spaces are **FIE**.

Example 2.3. Any 1-universally separably injective is **FIE**. So for instance $C(K)$, where K is an extremely disconnected compact Hausdorff space, is **FIE**, by [**1**, Proposition 1.19].

The previous example is a particular case of the upcoming general statement whose proof we leave to the reader (see [**9**, proof of Lemma 8.0.1]). Recall that for a non-empty set Γ, $c_0(\Gamma, X)$ denotes the Banach space of all maps $f\colon \Gamma \to X$ with the property that for each $\varepsilon > 0$, the set $\{\gamma \in \Gamma :\ \|f(\gamma)\| \geq \varepsilon\}$ is finite, endowed with the supremum norm. When X is the scalar field, we just write $c_0(\Gamma)$.

Fact 2.4. If Λ and Γ are non-empty sets, then $\ell_\infty(\Lambda, c_0(\Gamma, X))$ is **FIE** whenever X is 1-universally separably injective.

Proposition 2.5. *Let E be a **FIE** Banach space. If F is a 1-complemented subspace of E, then F is **FIE**.*

Proof. Let $W \in \mathrm{Age}(F)$ and $T \in \mathrm{Emb}(W,F)$ be given. By the **FIE**-ness of E, there is a one-norm operator $\hat{T}\colon E \to E$ which extends T. Let $P_F\colon E \to F$ be a 1-projection. We set $\tilde{T} = P_F \circ \hat{T} \circ i_F\colon F \to F$, where $i_F\colon F \to E$ is the natural inclusion. Clearly, $\tilde{T}$ extends T and $\|\tilde{T}\| = 1$. $\square$

We conclude with a characteristic feature of **FIE** spaces.

Proposition 2.6. *Let E be a **FIE** Banach space, and $X, Y \in \mathrm{Age}(E)$. If X is C-complemented in E and Y is isometric to X, then Y is also C-complemented in E.*

Proof. If $\phi\colon Y \to X$ is an isometry, T is a norm one extension of ϕ to E, and P is a projection onto X, then the map $\phi^{-1}PT$ defines a projection onto Y with the same norm as P. $\square$

3. Almost ultrahomogeneous and almost finitely isometrically extensible Banach spaces

In order to state the next results, we introduce the weakening of ultrahomogeneity that is one of our main focus in this paper, and is closer to the original definition of the Gurarij space as well to the way the results of Lusky [**18**] regarding $L_p(0,1)$-spaces were originally formulated.

DEFINITION 3.1. A Banach space is called *almost ultrahomogeneous* (almost **UH**), if for all $\varepsilon > 0$ and all $X \in \mathrm{Age}(E)$ and all $\phi \in \mathrm{Emb}(X, E)$, there exists $T \in \mathrm{Isom}_\varepsilon(E)$ such that $T|_E = \phi$.

REMARK 3.2. Every approximately **UH** Banach space is almost **UH**. Indeed, assume that E is approximately **UH** and let $\varepsilon > 0$, $X \in \mathrm{Age}(E)$ and $\phi \in \mathrm{Emb}(X, E)$ be given. Since E is approximately **UH**, there is $U \in \mathrm{Isom}(E)$ such that $\|U|_X - \phi\| < \varepsilon/(2\dim X)$. If $P_X \colon E \to X$ is a projection with $\|P_X\| \leq \dim X$, we set $T = U - (U - \phi) \circ P_X$. It is not difficult to check that $T \in \mathrm{Isom}_\varepsilon(E)$ and $T|_X = \phi$.

To relate the almost **UH** property to the **FIE**-property, we need the following approximate version of this property.

DEFINITION 3.3. A Banach space E has the *almost finitely isometrically extensible* (**aFIE**) if it satisfies the following condition: for all $\varepsilon > 0$ and all $X \in \mathrm{Age}(E)$ and all $\phi \in \mathrm{Emb}(X, E)$, there exists $T \colon E \to E$ such that $T|_E = \phi$ and $\|T\| \leq 1+\varepsilon$.

Note that every **FIE** Banach space is **aFIE**. This notion easily relates to ultrahomogeneity properties as follows:

FACT 3.4. Any almost **UH** space is **aFIE**.

EXAMPLE 3.5. For $p \in 2\mathbb{N}$ and $p \geq 4$, $L_p(0,1)$ is not **aFIE**. Actually, for any $C > 1$, by [**15**, Proposition 2.10], there is a finite dimensional subspace such that if $T \colon X \to L_p(0,1)$ is an isometry and if $\tilde{T}$ extends T, then $\|\tilde{T}\| \geq C$.

The upcoming result is the corresponding version of Proposition 2.5 for **aFIE** Banach spaces.

PROPOSITION 3.6. *Let E be a Banach space and F be a subspace of E.*

(1) *If E is **aFIE** and F is 1-complemented in E, then F is **aFIE**.*
(2) *If E is **aFIE** and F is 1-complemented in E^{**}, then F is **FIE**. In particular, reflexive **aFIE** spaces are **FIE**.*

PROOF. (1) If $\varepsilon > 0$, $Y \in \mathrm{Age}(F)$ and $T \in \mathrm{Emb}(Y, F)$ are given, by the **aFIE**-ness of E, there is an operator $\hat{T} \colon E \to E$ such that $\hat{T}|_Y = T$ and $\|\hat{T}\| \leq 1 + \varepsilon$. Let $P_F \colon E \to F$ be a 1-projection. We set $\tilde{T} = P_F \circ \hat{T} \circ i_F \colon F \to F$, where $i_F \colon F \to E$ is the natural inclusion. Clearly, $\tilde{T}$ extends T and $\|\tilde{T}\| \leq 1 + \varepsilon$.

(2) Let $Y \in \mathrm{Age}(F)$ and $T \in \mathrm{Emb}(Y, F)$ be given. For each $Z \in \mathrm{Age}(F)$ with $Y \subset Z$, there exists $T_Z \colon Z \to E$ extending T such that $\|T_Z\| \leq 1 + \frac{1}{\dim Z}$. Let $\mathcal{U}$ be an ultrafilter on the set of finite dimensional subspaces of F containing Y and refining the Fréchet filter and define $\psi \colon F \to E^{**}$ by $\psi(y) = w^* - \lim_{\mathcal{U}} T_Z(y)$, $y \in F$. The Banach-Alaoglu theorem ensures that ψ is well-defined. If $P \colon E^{**} \to F$ is a 1-projection, the operator $\tilde{T} \colon F \to F$ given by $\tilde{T} = P \circ \psi$ satisfies the requirements. $\square$

REMARK 3.7. Since $L_p(0,1)$ is approximately **UH** (and therefore **aFIE**) for $p \notin 2\mathbb{N}+4$ [**18**], Proposition 3.6 implies that $L_p(0,1)$ is **FIE** for $p \notin 2\mathbb{N}+4$. More generally Proposition 3.6(2) implies that all separable $L_p(\mu)$ are **FIE** when $p \notin 2\mathbb{N}+4$.

Other examples of **FIE** spaces are the so called *1-uniformly finitely extensible* spaces (1-**UFO**). If $\lambda \geq 1$, a Banach space E is called λ-uniformly finitely extensible (λ-**UFO**) if for all finite dimensional subspace X of E, each operator $\tau\colon X \to E$ can be extended to an operator $T\colon E \to E$ with $\|T\| \leq \lambda\|\tau\|$. λ-**UFO** were introduced by Y. Moreno and A. Plichko in [**19**] and systematically studied in [**11**] and [**12**], see also Chapter 7 of [**9**]. It is worth mentioning that λ-**UFO** spaces satisfy the following dichotomy: every λ-**UFO** space is either an $\mathcal{L}_\infty$-space or a weak type 2 near-Hilbert space with the Maurey projection property [**12**, Theorem 5.1].

Note however that not every **FIE**-space is 1-**UFO**. Indeed, since ℓ_p is 1-complemented in $L_p(0,1)$ [**2**, Proposition 6.4.1], Proposition 2.5 entails that ℓ_p is **FIE** when $p \notin 2\mathbb{N}+4$. On the other hand, when $p \neq 2$, ℓ_p is not **UFO** by [**12**, Corollary 3.6].

The following diagram displays the basic implications between the multidimensional properties considered so far:

$$
\begin{array}{ccccc}
\text{Ultrahomogeneous} & \Rightarrow & \text{Approx. Ultrahomogeneous} & \Rightarrow & \text{Almost Ultrahomogeneous} \\
\Downarrow & & & & \Downarrow \\
\textbf{FIE} & & \Rightarrow & & \textbf{aFIE}
\end{array}
$$

While **FIE** and **aFIE** are equivalent for reflexive spaces, we do not know whether there are **aFIE**-spaces which are not **FIE**. We also do not know whether the Approximate and the Almost Ultrahomogeneities are equivalent properties. All other implications in this diagram are strict.

4. Almost Fraïssé Banach spaces

The property introduced in this section is inspired by the recent notion of Fraïsséness studied in [**15**]. We start by recalling the Fraïssé and the weak Fraïssé properties for Banach spaces.

DEFINITION 4.1. [**15**, Definition 2.2] Let E be Banach space.

(1) E is *weak Fraïssé* if for every $\varepsilon > 0$ and every $X \in \mathrm{Age}(E)$ there is $\delta > 0$ such that if $\phi, \psi \in \mathrm{Emb}_\delta(X,E)$, then there exists $T \in \mathrm{Isom}(E)$ with $\|T \circ \phi - \psi\| < \varepsilon$.
(2) E is *Fraïssé* if for every $\varepsilon > 0$ and every dimension $k \in \mathbb{N}$ there is $\delta > 0$ such that if $X \in \mathrm{Age}_k(E)$ and $\phi, \psi \in \mathrm{Emb}_\delta(X,E)$, then there exists $T \in \mathrm{Isom}(E)$ with $\|T \circ \phi - \psi\| < \varepsilon$.

In [**15**] it was proved that the Gurarij space $\mathbb{G}$ and $L_p(0,1)$ for $p \notin 2\mathbb{N}+4$ are Fraïssé Banach spaces. While the Fraïssé property obviously implies the weak Fraïssé property, it is not known whether the two properties coincide. Recall that the Banach-Mazur distance on $\mathrm{Age}_k(E)$ is defined by

$$d_{\mathrm{BM}}(X,Y) = \inf\{\log(\|T\|\|T^{-1}\|) : T\colon X \to Y \text{ is an isomorphism}\}.$$

The authors of [**15**] prove that a Banach space is Fraïssé if and only if it is weak Fraïssé and $(\mathrm{Age}_k(E), d_{\mathrm{BM}})$ is a compact metric space for each $k \in \mathbb{N}$ [**15**, Theorem 2.12]. Our guideline is now to investigate whether a similar result holds for natural variations of these properties discussed in the introduction and that we shall now define.

It will be important to recall that $(\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}, d_{\mathrm{BM}})$ is a compact metric space for each $k \in \mathbb{N}$. While the Fraïssé and weak Fraïssé properties involved only elements of $\mathrm{Age}(E)$, an interesting fact is that we shall actually consider properties where elements of $\overline{\mathrm{Age}(E)}^{\mathrm{BM}}$ can be relevant as well.

Proposition 4.2. *Let E be a Banach space. The following statements are equivalent:*

(1) *whenever $X \in \overline{\mathrm{Age}(E)}^{\mathrm{BM}}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $\phi_1, \phi_2 \in \mathrm{Emb}_\delta(X, E)$, there exists $T \in \mathrm{Isom}_\varepsilon(E)$ such that $\phi_2 = T \circ \phi_1$.*
(2) *whenever $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta > 0$, such that if $X \in \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$ and $\phi_1, \phi_2 \in \mathrm{Emb}_\delta(X, E)$, there exists $T \in \mathrm{Isom}_\varepsilon(E)$ such that $\phi_2 = T\circ\phi_1$.*
(3) *whenever $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta > 0$, such that if $X \in \mathrm{Age}_k(E)$ and $\phi_1, \phi_2 \in \mathrm{Emb}_\delta(X, E)$, there exists $T \in \mathrm{Isom}_\varepsilon(E)$ such that $\phi_2 = T\circ\phi_1$.*
(4) *whenever $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta > 0$, such that for any $X \in \mathrm{Age}_k(E)$ and any $\phi \in \mathrm{Emb}_\delta(X, E)$, there exists $T \in \mathrm{Isom}_\varepsilon(E)$ such that $T|_X = \phi$.*

Proof. The implications (2) ⇒ (1), (2) ⇒ (3) and (3) ⇒ (4) are immediate. We prove (1) ⇒ (2) , (4) ⇒ (3), and (3) ⇒ (2).

(1) ⇒ (2): Suppose that (2) does not hold. So there is $k_0 \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that for each $n \in \mathbb{N}$ there exist $X_n \in \overline{\mathrm{Age}_{k_0}(E)}^{\mathrm{BM}}$ and $\phi_n^1, \phi_n^2 \in \mathrm{Emb}_{1/n}(X_n, E)$ satisfying $\phi_n^2 \neq T \circ \phi_n^1$ for any $T \in \mathrm{Isom}_{\varepsilon_0}(E)$. By compactness of $\overline{\mathrm{Age}_{k_0}(E)}^{\mathrm{BM}}$ we may assume that $X_n \overset{\mathrm{BM}}{\to} X$ for some $X \in \overline{\mathrm{Age}_{k_0}(E)}^{\mathrm{BM}}$. Now, let $\delta > 0$ be the corresponding number satisfying (1) for X and $\varepsilon_1 = \varepsilon_0/2$. Choose $0 < \xi < \delta$ and let $n_0 \in \mathbb{N}$ be such that $\frac{1}{n_0} < \frac{\delta-\xi}{1+\xi}$ and $d_{\mathrm{BM}}(X_{n_0}, X) < \xi$. If $l\colon X \to X_{n_0}$ is an isomorphism with $\|l\| = 1$ and $\|l^{-1}\| \leq 1+\xi$, then $\phi_{n_0}^1 \circ l, \phi_{n_0}^2 \circ l \in \mathrm{Emb}_\delta(X, E)$. By (1), there exists $T \in \mathrm{Isom}_{\varepsilon_1}(E)$ such that $\phi_{n_0}^2 \circ l = T \circ \phi_{n_0}^1 \circ l$. Hence $\phi_{n_0}^2 = T \circ \phi_{n_0}^1$ for some $T \in \mathrm{Isom}_{\varepsilon_0}(E)$ which is impossible.

(4) ⇒ (3): Suppose that (4) is valid and let $\varepsilon > 0$ and $k \in \mathbb{N}$ be given. Take $\delta > 0$ corresponding to (4) and fix $0 < \delta' < \delta$ such that $(1 + \delta')^2 \leq 1 + \delta$. Now let $\phi, \psi \in \mathrm{Emb}_{\delta'}(X, E)$ be given. Write $Y = \phi(X) \in \mathrm{Age}_k(E)$ and define $\eta\colon Y \to E$ by $\eta y = \psi(\phi^{-1}y)$, $y \in Y$. Thus, $\eta \in \mathrm{Emb}_\delta(Y, E)$ and by (4), there is $T \in \mathrm{Isom}_\varepsilon(E)$ such that $T|_Y = \eta$. The equation $T|_Y = \eta$ means that $T \circ \phi = \psi$.

(3) ⇒ (2): Indeed, let $\varepsilon > 0$ and $k \in \mathbb{N}$ be given, fix $\delta > 0$ as in (3) and take $0 < \xi < \frac{\delta}{2+\delta}$. If $X \in \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$, there exists an ξ-isometry $A_\xi\colon X \to X_\xi$, where $X_\xi \in \mathrm{Age}_k(E)$. Now, for $\phi, \psi \in \mathrm{Emb}_{\delta/2}(X, E)$, we have $\phi \circ A_\xi^{-1}, \psi \circ A_\xi^{-1} \in \mathrm{Emb}_\delta(X_\xi, E)$. By (3), we have $\psi \circ A_\xi^{-1} = T \circ \phi \circ A_\xi^{-1}$ for some $T \in \mathrm{Isom}_\varepsilon(E)$. Thus, $\psi = T \circ \phi$. □

Proposition 4.3. *Let E be a Banach space. The following statements are equivalent:*

(5) *whenever* $X \in \mathrm{Age}(E)$ *and* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that if* $\phi_1, \phi_2 \in \mathrm{Emb}_\delta(X, E)$*, there exists* $T \in \mathrm{Isom}_\varepsilon(E)$ *such that* $\phi_2 = T \circ \phi_1$.

(6) *whenever* $X \in \mathrm{Age}(E)$ *and* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that if* $\phi \in \mathrm{Emb}_\delta(X, E)$*, there exists* $T \in \mathrm{Isom}_\varepsilon(E)$ *such that* $T|_X = \phi$.

Proof. Suppose that (6) is valid and let $X \in \mathrm{Age}(E)$ and $\varepsilon > 0$ be given. If $\xi > 0$ satisfies $(1 + \xi)^2 \leq 1 + \varepsilon$, take $\delta > 0$ corresponding to ξ in (6). Let $\phi_1, \phi_2 \in \mathrm{Emb}_\delta(X, E)$ be given. From (6) there are $T_1, T_2 \in \mathrm{Isom}_\xi(E)$ such that $T_1|_X = \phi_1$ and $T_2|_X = \phi_2$. So, $T = T_2 \circ T_1^{-1} \in \mathrm{Isom}_\varepsilon(E)$ and $T \circ \phi_1 = \phi_2$. □

Propositions 4.2 and 4.3 together with Definition 4.1 motivate the next definitions.

Definition 4.4. Let E be a Banach space.

(1) E is *almost Fraïssé* (**AF**) if it satisfies one of conditions (and hence all of them) in Proposition 4.2.

(2) E is *weak almost Fraïssé* (weak **AF**) if it satisfies one of conditions (and hence all of them) in Proposition 4.3.

It is obvious, but worth stating, that the **AF**-property implies the weak **AF**-property. Items (2)-(3)-(4) in Proposition 4.2 indicates that some uniformity with respect to the dimension of E follows from the almost Fraïssé property, while this is not formally implied by the weak almost Fraïssé property. However, as a consequence of (1) in Proposition 4.2 and (5) in Proposition 4.3, we obtain:

Corollary 4.5. *If E is a weak **AF** Banach space and $\mathrm{Age}_k(E)$ is compact for all k, then E is **AF**.*

It seems to be open whether the **AF**-property implies that $\mathrm{Age}_k(E)$ is compact for all k, or whether the weak **AF** property implies the **AF** property in general. The next result justifies the terminology used here.

Proposition 4.6. *Any Fraïssé (resp. weak Fraïssé) Banach space is **AF** (resp. weak **AF**).*

Proof. Let $\varepsilon > 0$ and $k \in \mathbb{N}$ be given, and take $0 < \xi < 1$ such that $\frac{1+\xi}{1-\xi} < 1 + \varepsilon$. Fix $\delta > 0$ corresponding to ξ/k in the definition of Fraïssé. Also let $X \in \mathrm{Age}_k(E)$ and $\phi \in \mathrm{Emb}_\delta(X, E)$. By the Fraïssé-ness, there is $S \in \mathrm{Isom}(E)$ satisfying $\|S|_X - \phi\| < \frac{\xi}{k}$. If $\psi := S|_X - \phi$ and $P \colon E \to X$ is a projection with $\|P\| \leq k$, then by setting $T = S - \psi \circ P \colon E \to E$ we have $T|_X = \phi$ and $\max\{\|T\|, \|T^{-1}\|\} \leq \frac{1+\xi}{1-\xi} < 1 + \varepsilon$, so $T \in \mathrm{Isom}_\varepsilon(E)$. The proof of the second statement is analogous. □

It is worth noting the following property of **AF** spaces. Recall that two spaces are said to be almost isometric if they are ε-isometric for all $\varepsilon > 0$.

Fact 4.7. If E is almost isometric to F and E is **AF** (weak **AF**, respectively), then F is **AF** (weak **AF**, respectively).

Proof. Let $\varepsilon > 0$ and $k \in \mathbb{N}$ be given, and let $\delta > 0$ be the corresponding number to the **AF** definition. Take $0 < \xi < \delta/(2 + \delta)$ and let $j \colon E \to F$ be an ξ-isometry. If $Y \in \mathrm{Age}_k(F)$ and $\phi_1, \phi_2 \in \mathrm{Emb}_{\delta/2}(Y, F)$, then $\phi_1 \circ j, \phi_2 \circ j \in \mathrm{Emb}_\delta(j^{-1}(Y), E)$. Since E is **AF**, there is $T \in \mathrm{Isom}_\varepsilon(E)$ such that $T \circ \phi_1 \circ j = \phi_2 \circ j$, that is, $T \circ \phi_1 = \phi_2$. The statement for weak **AF** spaces is proved similarly. □

On the other hand, we do not know if the class of **FIE** or **aFIE** spaces is stable by almost isometries.

We end this part by displaying the relationships between the notions introduced throughout the paper.

$$\begin{array}{ccccc}
 & & \text{Fraïssé} & \Rightarrow & \text{Almost Fraïssé} \\
 & & \Downarrow & & \Downarrow \\
 & & \text{Weak Fraïssé} & \Rightarrow & \text{Weak Almost Fraïssé} \\
 & & \Downarrow & & \Downarrow \\
\text{Ultrahomogeneous} & \Rightarrow & \text{Approx. Ultrahomogeneous} & \Rightarrow & \text{Almost Ultrahomogeneous} \\
\Downarrow & & & & \Downarrow \\
\mathbf{FIE} & & \Rightarrow & & \mathbf{aFIE}
\end{array}$$

We know of no example which is almost ultrahomogeneous but not Fraïssé. So the six properties in the upper right corner could be equivalent; one important aspect of this question relates to whether $\mathrm{Age}(E)$ is closed with respect to the Banach-Mazur pseudodistance. Under this hypothesis, Fraïssé and its weak version are equivalent (Theorem 2.12 in [**15**]), and almost Fraïssé and its weak version are equivalent (Corollary 4.5).

4.1. Some pseudodistances associated to AF Banach spaces. To continue, we recall the Gromov-Hausdorff function on $\mathrm{Age}_k(E)^2$ introduced in [**15**] in order to study Fraïssé Banach spaces. If $X, Y \in \mathrm{Age}_k(E)$, the authors define

$$\gamma_E(X,Y) = \inf\{d_H(B_{X_0}, B_{Y_0}) :\ X_0 \equiv X \text{ and } Y_0 \equiv Y\}, \tag{4.1}$$

where d_H is the $\|\cdot\|_E$-Hausdorff metric and the symbol $\equiv$ means "isometric to". Another function defined on $\mathrm{Age}_k(E)^2$ and considered in [**15**] is

$$D_E(X,Y) = \inf\{d_H(B_{TX}, B_Y) :\ T \in \mathrm{Isom}(E)\}, \quad X, Y \in \mathrm{Age}_k(E).$$

It is easy to see that D_E is always a pseudometric on $\mathrm{Age}_k(E)$, that $\gamma_E \leq D_E$, and that both are invariant under the action of $\mathrm{Isom}(E)$. Also, D_E is a complete metric for every Banach space E and $\gamma_E = D_E$ when E is approximately **UH**, [**15**, Proposition 2.14] - and therefore γ_E is complete when E is approximately **UH**, as is implicitly used in [**15**]. The D_E-completeness is consequence of the following fact which is certainly well-known but for which we include a proof.

Lemma 4.8. *Let (X, d) be a complete metric space. Suppose that G is a group acting on X. If d is G-invariant, then*

$$\begin{aligned} \rho\colon X \times X &\to \mathbb{R} \\ (x,y) &\mapsto \rho(x,y) = \inf_{g\in G} d(gx, y) \end{aligned}$$

is a complete pseudometric.

Proof. It is not difficult to check that ρ is a pseudometric. Now, we prove the completeness. Let (x_n) be a sequence in X such that $\rho(x_n, x_{n+1}) < 1/2^n$ for each $n \in \mathbb{N}$. We shall contruct sequences (z_n) in X and (g_n) in G such that $g_n z_n = x_n$ and $d(z_n, z_{n+1}) < 1/2^n$ for each $n \in \mathbb{N}$. Suppose that $z_1, \ldots, z_n \in X$ and $g_1, \ldots, g_n \in G$ have already been defined. Since $\rho(x_n, x_{n+1}) < 1/2^n$, there exists $h_n \in G$ satisfying $d(h_n x_n, x_{n+1}) < 1/2^n$. By the G-invariance of d we have

$$d(h_n x_n, x_{n+1}) = d(x_n, h_n^{-1} x_{n+1}) = d(g_n z_n, h_n^{-1} x_{n+1}) \\ = d(z_n, g_n^{-1} h_n^{-1} x_{n+1}) < 1/2^n.$$

By setting $z_{n+1} = g_n^{-1} h_n^{-1} x_{n+1}$ and $g_{n+1} = h_n g_n$, we end the induction.

If $z \in X$ satisfies $z_n \to z$, then $\rho(x_n, z) \leq d(z_n, z) \to 0$ as $n \to \infty$. Therefore, ρ is complete. □

Proposition 4.9. *If E is a Banach space, then $(\mathrm{Age}(E), D_E)$ is complete. If E is approximately **UH**, then the function γ_E coincides with D_E, and is therefore a complete metric.*

Proof. The second part is [**15**, Proposition 2.14]. For the first part, let $K(E)$ be the family of nonempty compact subsets of E. Since the map $\Psi\colon X \in \mathrm{Age}(E) \mapsto B_X \in K(E)$ is injective and $(K(E), d_H)$ is complete, it suffices to check that $\Psi(\mathrm{Age}(E))$ is d_H-closed to prove that it is d_H-complete.

Let (X_n) be a sequence in $\mathrm{Age}(E)$ such that $\Psi(X_n) = B_{X_n} \overset{d_H}{\to} A$. In particular there exists k such that $X_n \in \mathrm{Age}_k(E)$ for each $n \in \mathbb{N}$. Note that the limit A is a non-empty balanced compact convex subset of E. So $X_0 = \bigcup_{\lambda > 0} \lambda A$ is a subspace of E and $\Psi(X_0) = B_{X_0} = A$. Finally, by taking $n_0 \in \mathbb{N}$ such that $d_H(B_{X_{n_0}}, B_{X_0}) < 1/2k$, we obtain that $\dim X_0 = k$, that is, $X_0 \in \mathrm{Age}_k(E)$.

Now, since d_H is $\mathrm{Isom}(E)$-invariant, the conclusion of the Proposition follows from Lemma 4.8. □

Now, following the above ideas, we introduce "almost" versions D_E^a of D_E, and γ_E^a of γ_E. These functions are intended to be relevant to the case when E is almost **UH** (and not necessarily **UH**), or weak **AF** (and not necessarily weak Fraïssé). While D_E^a will be defined on $\mathrm{Age}_k(E)$, an interesting new feature of γ_E^a is that it will be defined on $\overline{\mathrm{Age}(E)}^{\mathrm{BM}}$.

Definition 4.10. If $X, Y \in \mathrm{Age}(E)$, we let

$$D_\delta(X, Y) = \inf\{d_H(T(B_X), U(B_Y)) : \ T, U \in \mathrm{Isom}_\delta(E)\}$$

and

$$D_E^a(X, Y) \coloneqq \lim_{\delta \to 0} D_\delta(X, Y) = \sup_{\delta > 0} D_\delta(X, Y).$$

We note the following:

Fact 4.11. Let E be a Banach space. Then $D_E^a(X, Y)$ is a pseudometric on $\mathrm{Age}(E)$ such that $D_E^a(X, Y) \leq D_E(X, Y)$.

Proof. We use the immediate fact: if $A, B \subset E$ are compact and $T \in \mathrm{Isom}_\delta(E)$, then $d_H(T(A), T(B)) \leq (1+\delta) d_H(A, B)$. From this we obtain for each $\varepsilon, \delta > 0$

$$D_\delta(X, Y) + D_\delta(Y, Z) + \varepsilon \geq (1+\delta)^{-1} D_{\delta^2 + 2\delta}(X, Z).$$

The triangle inequality follows when ε, δ tends to zero. □

In contrast to D_E, we have no reason to think that D_E^a is a complete pseudometric in general. We now turn to the definition of the function γ_E^a.

Definition 4.12. Let E be a Banach space. If $X, Y \in \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$, we set

$$d_\delta(X,Y) = \inf\{d_H(t(B_X), t'(B_Y)) :\ t \in \mathrm{Emb}_\delta(X,E), t' \in \mathrm{Emb}_\delta(Y,E)\}$$

and

$$\gamma_E^a(X,Y) := \lim_{\delta\to 0} d_\delta(X,Y) = \sup_{\delta>0} d_\delta(X,Y).$$

Lemma 4.13. *The following functions give alternative definitions of γ_E^a: for $X, Y \in \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$, we set*

$$d_\delta^1(X,Y) = \inf\{d_H(B_{tX}, B_{sY}) :\ t \in \mathrm{Emb}_\delta(X,E), s \in \mathrm{Emb}_\delta(Y,E)\}, \quad \textit{and}$$
$$d_\delta^2(X,Y) = \inf\{d_H(t(B_X), B_{sY}) :\ t \in \mathrm{Emb}_\delta(X,E), s \in \mathrm{Emb}_\delta(Y,E)\}.$$

Then

$$\gamma_E^a(X,Y) = \lim_{\delta\to 0} d_\delta^1(X,Y) = \lim_{\delta\to 0} d_\delta^2(X,Y).$$

Moreover, if $X, Y \in \mathrm{Age}(E)$ and

$$D_\delta^1(X,Y) = \inf\{d_H(B_{UX}, B_{VY}) :\ U, V \in \mathrm{Isom}_\delta(E)\},$$

then

$$D_E^a(X,Y) = \lim_{\delta\to 0} D_\delta^1(X,Y) = \sup_{\delta>0} D_\delta^1(X,Y).$$

Proof. If $u\colon E \to F$ is a δ-isometry, then $\dfrac{1}{1+\delta} B_{uE} \subset u(B_E) \subset (1+\delta) B_{uE}$. So, $d_H(u(B_E), B_{uE}) \le \delta$. Hence if $t \in \mathrm{Emb}_\delta(X,E)$ and $s \in \mathrm{Emb}_\delta(Y,E)$, then

$$d_H(t(B_X), s(B_Y)) \le 2\delta + d_H(B_{tX}, B_{sY}), \quad \text{and}$$
$$d_H(B_{tX}, B_{sY}) \le 2\delta + d_H(t(B_X), s(B_Y)).$$

Thus $\gamma_E^a(X,Y) = \lim_{\delta\to 0} d_\delta^1(X,Y)$. The other statements are proved similarly. □

Before comparing this function to the more classical ones, we observe an easy consequence of its definition. It is inspired from [**15**, Proposition 2.14], where it was proved that $d_{\mathrm{BM}}(X,Y) \le 4k d_H(B_X, B_Y)$ for each $X, Y \in \mathrm{Age}_k(E)$ such that $d_H(B_X, B_Y) < 1/2k$.

Lemma 4.14. *Let E be a Banach space and $X, Y \in \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$.*

(1) *If $X, Y \in \mathrm{Age}_k(E)$ and $d_H(X,Y) \le d < 1/2k$, then there exists an isomorphism $\lambda\colon X \to Y$ such that $\|\lambda\| \le 1 + kd$, $\|\lambda^{-1}\| \le 1/(1-kd)$ and $\|\lambda - \mathrm{Id}\| \le kd$.*
(2) *Suppose that $X, Y \in \mathrm{Age}_k(E)$ and let $\sigma > 0$ be given. If $\lambda\colon X \to Y$ is an ε-perturbation of* Id *with $k\varepsilon \le \sigma/(2+\sigma)$, there exists $T \in \mathrm{Isom}_\sigma(E)$ such that $T|_X = \lambda$.*
(3) *$d_{\mathrm{BM}}(X,Y) \le 4k\gamma_E^a(X,Y)$ whenever $\gamma_E^a(X,Y) < 1/2k$, and $\gamma_E^a(X,Y) = 0$ if and only if $X \equiv Y$.*
(4) *γ_E^a is* BM*-lower semicontinuous on $\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$, in the sense that if $\lim_n d_{\mathrm{BM}}(X_n, X) = 0$ and $\lim_n d_{\mathrm{BM}}(Y_n, Y) = 0$, then*

$$\gamma_E^a(X,Y) \le \liminf \gamma_E^a(X_n, Y_n).$$

PROOF. (1) Fix an Auerbach basis of X, $\{x_1,\dots,x_k\}$. From definition of d_H, for each $j=1,\dots,k$ there are $y_j\in Y$ satisfying $\|x_j-y_j\|\le d$. If $a_1,\dots,a_k\in\mathbb{K}$, we have

$$(1-kd)\left\|\sum_{j=1}^k a_jx_j'\right\|\le\left\|\sum_{j=1}^k a_jy_j'\right\|\le(1+kd)\left\|\sum_{j=1}^k a_jx_j'\right\|.$$

Thus, $\lambda\colon X\to Y$ defined linearly by $x_j\in X\mapsto y_j\in Y$ is an isomorphism with $\|\lambda\|\le 1+kd$, $\|\lambda^{-1}\|\le 1/(1-kd)$ and $\|\lambda-\mathrm{Id}\|\le kd$.

(2) If $P_X\colon E\to X$ is a projection with $\|P_X\|\le k$, let $T\colon E\to E$ be defined by $T=\mathrm{Id}-(\mathrm{Id}_X-\lambda)\circ P_X$. Note that $T|_X=\lambda$, $\|T\|\le 1+k\varepsilon\le 1+\sigma$ and $\|T^{-1}\|\le 1/(1-k\varepsilon)\le 1+\sigma$. Hence, $T\in\mathrm{Isom}_\sigma(E)$.

(3) Let $d>0$ be such that $\gamma_E^a(X,Y)<d<1/2k$. Also let $\delta>0$ be such that $d_\delta^1(X,Y)<d<1/2k$. So there are $t\in\mathrm{Emb}_\delta(X,E)$ and $s\in\mathrm{Emb}_\delta(Y,E)$ satisfying $d_H(B_{tX},B_{sY})<d$.

Write $X'=tX$ and $Y'=sY$. By Item (1), there exists an isomorphism $\lambda\colon X'\to Y'$ such that $\|\lambda\|\le 1+kd$ and $\|\lambda^{-1}\|\le 1/(1-kd)$. Hence, $d_{\mathrm{BM}}(X,Y)\le 4\log(1+\delta)+\log(\frac{1+kd}{1-kd})\le 4\log(1+\delta)+4kd$. Since δ,d were arbitrary, we conclude that $d_{\mathrm{BM}}(X,Y)\le 4k\gamma_E^a(X,Y)$.

(4) Note that if $X,Y,X',Y'\in\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$, and $s\colon X\to X'$, $t\colon Y\to Y'$ are $1+\varepsilon$-isometric maps, then

$$\begin{aligned}d_\delta(X',Y')&=\inf\{d_H(B_{uX'},B_{vY'}):\ u\in\mathrm{Emb}_\delta(X,E),v\in\mathrm{Emb}_\delta(Y,E)\}\\&=\inf\{d_H(B_{usX},B_{vtY}):\ u\in\mathrm{Emb}_\delta(X,E),v\in\mathrm{Emb}_\delta(Y,E)\}\\&\ge d_{\delta+\varepsilon+\delta\varepsilon}(X,Y).\end{aligned}$$

In particular, if $\lim_n d_{\mathrm{BM}}(X_n,X)=0$ and $\lim_n d_{\mathrm{BM}}(Y_n,Y)=0$, then

$$d_{\delta+\varepsilon+\delta\varepsilon}(X,Y)\le\liminf d_\delta(X_n,Y_n)\le\liminf\gamma_E^a(X_n,Y_n)$$

and since δ and ε were arbitrary,

$$\gamma_E^a(X,Y)\le\liminf\gamma_E^a(X_n,Y_n).$$ □

Now we list some relationships between the above defined functions and the different forms of ultrahomogeneity.

PROPOSITION 4.15. *Let E be a Banach space and $X,Y\in\mathrm{Age}(E)$. Then*

(1) $\gamma_E^a(X,Y)\le\gamma_E(X,Y)\le D_E(X,Y)$ *and* $\gamma_E^a(X,Y)\le D_E^a(X,Y)\le D_E(X,Y)$.
(2) *If E is almost **UH**, then* $\gamma_E^a(X,Y)\le D_E^a(X,Y)\le\gamma_E(X,Y)\le D_E(X,Y)$.
(3) *If E is weak **AF**, then* $\gamma_E^a(X,Y)=D_E^a(X,Y)\le\gamma_E(X,Y)\le D_E(X,Y)$, *and particular γ_E^a is a pseudometric on* $\mathrm{Age}(E)$.
(4) *If E is approximately **UH**, then* $\gamma_E^a(X,Y)\le D_E^a(X,Y)\le\gamma_E(X,Y)=D_E(X,Y)$, *and in particular γ_E is a pseudometric on* $\mathrm{Age}(E)$.
(5) *If E is weak Fraïssé, then the four maps $\gamma_E^a,\gamma_E,D_E^a,D_E$ are pseudometrics which coincide on* $\mathrm{Age}(E)$.

PROOF. (1) is obvious and (4) was observed in [**15**].

(2) if t,t' are isometric embeddings of X and Y into E, and $\delta>0$ is given, let $T,T'\in\mathrm{Isom}_\delta(E)$ be extensions of t and t' respectively. Then $D_\delta(X,Y)\le d_H(TB_X,T'B_Y)=d_H(tB_X,t'B_Y)$. Taking the supremum over δ and the infimum over t,t' gives that $D_E^a(X,Y)\le\gamma_E(X,Y)$.

(3) If $\varepsilon > 0$ is given, let $\delta > 0$ be the corresponding number of definition of weak **AF** to X and Y. From its definition there are $t \in \mathrm{Emb}_\delta(X,E)$ and $s \in \mathrm{Emb}_\delta(Y,E)$ such that $d_H(t(B_X), s(B_Y)) \leq \gamma_E^a(X,Y)$. Since E is weak **AF**, there are $T, S \in \mathrm{Isom}_\varepsilon(E)$ which extend t and s, respectively. Thus $D_\varepsilon(X,Y) \leq \gamma_E^a(X,Y)$ and the arbitrariness of $\varepsilon > 0$ yields $D_E^a(X,Y) \leq \gamma_E^a(X,Y)$.

(5) Because of (2), it is enough to prove that $D_E \leq \gamma_E^a$. Let $\varepsilon > 0$ and $\delta > 0$ be associated number by the weak Fraïssé property in X and Y. If $t \in \mathrm{Emb}_\delta(X,E)$ and $s \in \mathrm{Emb}_\delta(X,E)$ are given, let $T, S \in \mathrm{Isom}(E)$ be such that $\|T|_X - t\| \leq \varepsilon$ and $\|S|_Y - s\| \leq \varepsilon$. Then

$$\begin{aligned} d_H(T(B_X), S(B_Y)) &\leq d_H(T(B_X), t(B_X)) + d_H(S(B_Y), s(B_Y)) + d_H(t(B_X), s(B_Y)) \\ &\leq 2\varepsilon + d_H(t(B_X), s(B_Y)). \end{aligned}$$

Thus, $D_E(X,Y) \leq 2\varepsilon + d_\delta(X,Y) \leq 2\varepsilon + \gamma_E^a(X,Y)$. By taking $\varepsilon \to 0^+$ we get the result. □

We now turn to characterizations of almost Fraïssé spaces among weak almost Fraïssé spaces. As we shall see, the situation is more involved than for Fraïssé spaces, which by [**15**, Theorem 2.12] are exactly the weak Fraïssé spaces for which $\mathrm{Age}_k(E)$ is BM-compact for all k.

We consider another natural pseudometric, which is only defined on $\mathrm{Age}(E)$.

Definition 4.16. Let E be a Banach space and $X, Y \in \mathrm{Age}(E)$. We let

$$D_{\mathrm{BM}}^E(X,Y) = \inf\{\log(\|T\|\|T^{-1}\|) : T\colon E \to E \text{ is an isomorphism and } T(X) = Y\}.$$

We have the following facts:

Lemma 4.17. *Let E be a Banach space.*

1. *D_{BM}^E is apseudometric on* $\mathrm{Age(E)}$ *dominating* d_{BM}.
2. *There is a constant $c(\delta,k) > 0$ such that for any $X, Y \in \mathrm{Age}_k(E)$, we have $D_E^a(X,Y) \leq \delta < 1/2k^2 \Rightarrow D_{\mathrm{BM}}^E(X,Y) \leq c(\delta,k)$.*
3. *For any $X, Y \in \mathrm{Age}(E)$, $D_E^a(X,Y) = 0 \Leftrightarrow D_{\mathrm{BM}}^E(X,Y) = 0$.*
4. *If E is weak **AF**, then* Id *is an homeomorphism between* $(\mathrm{Age}(E), d_{\mathrm{BM}})$ *and* $(\mathrm{Age}(E), D_{\mathrm{BM}}^E)$.
5. *If E is **AF**, then* Id *is a uniform homeomorphism between* $(\mathrm{Age}(E), d_{\mathrm{BM}})$ *and* $(\mathrm{Age}(E), D_{\mathrm{BM}}^E)$.

Proof. (1) follows from definition. (2) By Lemma 4.13 there are $U, V \in \mathrm{Isom}_\delta(E)$ satisfying $d_H(B_{UX}, B_{VY}) \leq \delta$. If $X' = UX$ and $Y' = VY$, then by Lemma 4.14(1) there is $\lambda\colon X' \to Y'$ which is a $k\delta$-perturbation of Id. Once again by Lemma 4.14(2), there exists $\tilde{\lambda} \in \mathrm{Isom}_{\sigma(k,\delta)}(E)$ such that $\tilde{\lambda}|_X = \lambda$, where $\sigma(k,\delta) = 2k^2\delta/(1-k^2\delta)$. If $T = V^{-1}\tilde{\lambda}U$, then $T\colon E \to E$ is an isomorphism with $TX = Y$ and

$$D_{\mathrm{BM}}^E(X,Y) \leq \log(\|T\|\|T^{-1}\|) \leq 2\log(1+\delta) + \log\left(\frac{1+k^2\delta}{1-k^2\delta}\right) := c(\delta,k).$$

(3) follows from (2) since $\lim\limits_{\delta\to 0} c(\delta,k) = 0$. (4) and (5) are obvious implications of the definition. □

Question 4.18. When is D_{BM}^E a complete pseudometric?

Note that if D^E_{BM} is complete and E is **AF**, then $\mathrm{Age}(E)$ is BM-compact.

Recall that for $X, Y \in \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$ we have $\gamma^a_E(X,Y) = 0$ if and only if $X \equiv Y$ (Lemma 4.14), and if E is weak **AF**, then we have that $\gamma^a_E = \sup_{\delta>0} d_\delta$ is a pseudometric on $\mathrm{Age}(E)$ coinciding with $D^a_E = \sup_{\delta>0} D_\delta$ (Item 3 in Proposition 4.15).

Note that although γ^a_E is defined on $\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$, it is not clear whether it is a pseudometric there. We start with a lemma which implies that when E is **AF**, then γ^a_E is indeed a complete pseudometric on $\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$.

We denote by $(C_k(E), \overline{\gamma^a_E})$ the γ^a_E-completion of $\mathrm{Age}_k(E)$, and by j the map $C_k(E) \to \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$, defined by $j(\overline{(X_n)_n}) = \mathrm{BM} - \lim_n X_n$. Note that j is well defined by uniform continuity of the map Id on $\mathrm{Age}(E)$ with respect to the γ^a_E and BM pseudometrics.

Lemma 4.19. *Let E be a Banach space. Then we have the relation, for $X, Y \in C_k(E)$,*

$$\gamma^a_E(jX, jY) \leq \overline{\gamma^a_E}(X,Y).$$

Furthermore, consider the properties:

1. *E is **AF**.*
2. *γ^a_E is a pseudometric on $\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$.*
3. *The map $j\colon C_k(E) \to \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$ is surjective and satisfies $\gamma^a_E(jX, jY) = \overline{\gamma^a_E}(X,Y)$.*
4. *$\mathrm{Id}\colon (\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}, \gamma^a_E) \to (\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}, \mathrm{BM})$ is uniformly continuous for each k.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4).

*In particular, if E is **AF**, then γ^a_E is a complete pseudometric on $\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$.*

Proof. Write $X = (X_n)_n, Y = (Y_n)_n$, where X_n, Y_n are γ^a_E-Cauchy sequences. Then X_n and Y_n tend to jX and jY respectively with respect to d_{BM}. By Lemma 4.14(3), it follows that $\gamma^a_E(jX, jY) \leq \liminf \gamma^a_E(X_n, Y_n) = \lim_n \gamma^a_E(X_n, Y_n) = \overline{\gamma^a_E}(X,Y)$.

Also, if (2) holds, i.e. γ^a_E is a complete pseudometric on $\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$, and admitting for now (2) $\Rightarrow$ (3), then j is a surjective isometry between $(C_k(E), \overline{\gamma^a_E})$ and $(\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}, \gamma^a_E)$, so γ^a_E is necessarily a complete pseudometric on $\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$, proving the last affirmation of the lemma.

(2) $\Rightarrow$ (4) is an immediate consequence of Lemma 4.14(3). (3) $\Rightarrow$ (2) is also clear, since if (3) holds then the γ^a_E-completion $(C_k(E), \overline{\gamma^a_E})$ of $\mathrm{Age}_k(E)$ coincides with $(\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}, \gamma^a_E)$, through the map j.

(1) $\Rightarrow$ (2): We prove the triangular inequality. Let $X, Y, Z \in \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$ and $r > 0$ be fixed. If $0 < \varepsilon < r$ is given, let $\delta > 0$ be the corresponding value in the definition of **AF**. Also, fix $\delta' > 0$ such that $(1+\delta')(1+\varepsilon) < 1 + r$ and $0 < \delta' < \delta$. Then

$$d_{\delta'}(X,Y) \leq \gamma^a_E(X,Y) \quad \text{and} \quad d_{\delta'}(Y,Z) \leq \gamma^a_E(Y,Z).$$

From definition there are $u \in \mathrm{Emb}_{\delta'}(X,E)$, $v \in \mathrm{Emb}_{\delta'}(Y,E)$, $t \in \mathrm{Emb}_{\delta'}(Y,E)$ and $s \in \mathrm{Emb}_{\delta'}(Z,E)$ such that

$$d_H(u(B_X), v(B_Y)) \leq \gamma_E^a(X,Y) \quad \text{and} \quad d_H(t(B_Y), s(B_Z)) \leq \gamma_E^a(Y,Z). \tag{4.2}$$

Since E is **AF**, by (2) of Theorem 4.2 there is $T \in \mathrm{Isom}_\varepsilon(E)$ with $v = T \circ t$. So,

$$d_H(Tt(B_Y), Ts(B_Z)) \leq (1+\varepsilon) d_H(t(B_Y), s(B_Z)). \tag{4.3}$$

By combining (4.2) and (4.3) we obtain

$$\frac{1}{1+\varepsilon} d_H(v(B_Y), Ts(B_Z)) = \frac{1}{1+\varepsilon} d_H(Tt(B_Y), Ts(B_Z)) \leq \gamma_E^a(Y,Z).$$

By adding the previous inequality and (4.2) it follows that

$$\frac{1}{1+\varepsilon} d_H(u(B_X), Ts(B_Z)) \leq \frac{1}{1+\varepsilon} \gamma_E^a(X,Y) + \gamma_E^a(Y,Z).$$

Since $T \circ s \in \mathrm{Emb}_r(Z,E)$ and $u \in \mathrm{Emb}_{\delta'}(X,E) \subset \mathrm{Emb}_r(X,E)$, we have

$$\frac{1}{1+\varepsilon} d_r(X,Z) \leq \frac{1}{1+\varepsilon} \gamma_E^a(X,Y) + \gamma_E^a(Y,Z).$$

Since $r, \varepsilon > 0$ were arbitrary, we obtain $\gamma_E^a(X,Z) \leq \gamma_E^a(X,Y) + \gamma_E^a(Y,Z)$.

(2) $\Rightarrow$ (3): Let $(X_n)_n$ and $(Y_n)_n$ be γ_E^a-Cauchy sequences in $\mathrm{Age}(E)$ and $X = j(\overline{(X_n)_n})$, $Y = j(\overline{(Y_n)_n})$. By using the triangular inequality and the lower semicontinuity of γ_E^a (Lemma 4.14) we have

$$\begin{aligned} |\gamma_E^a(X,Y) - \gamma_E^a(X_n,Y_n)| &\leq \gamma_E^a(X_n,X) + \gamma_E^a(X_n,X) \\ &\leq \liminf_k \gamma_E^a(X_n,X_k) + \liminf_k \gamma_E^a(X_n,X_k). \end{aligned}$$

Since $(X_n)_n$ and $(Y_n)_n$ are γ_E^a-Cauchy, the last inequality implies that

$$\gamma_E^a(X,Y) = \lim_n \gamma_E^a(X_n,Y_n) = \overline{\gamma}_E^a(\overline{(X_n)_n}, \overline{(Y_n)_n}).$$

From the completeness of $(C_k(E), \overline{\gamma_E^a})$ it follows that $j(C_k(E))$ is closed and since it is dense (it contains $\mathrm{Age}_k(E)$), j is surjective. □

We finally obtain a list of equivalent sufficient conditions regarding weak **AF** spaces.

PROPOSITION 4.20. *Let E be a weak **AF** Banach space such that γ_E^a is a pseudometric on $\overline{\mathrm{Age}(E)}^{\mathrm{BM}}$. The following statements are equivalent:*

(1) $\mathrm{Id}\colon (\mathrm{Age}_k(E), \gamma_E^a) \to (\mathrm{Age}_k(E), \mathrm{BM})$ *is a uniform homeomorphism for each k.*
(2) *The map*
$$\mathrm{Id}\colon (\overline{\mathrm{Age}(E)}^{\mathrm{BM}}, \gamma_E^a) \to (\overline{\mathrm{Age}(E)}^{\mathrm{BM}}, \mathrm{BM})$$
is an homeomorphism.
(3) *The map*
$$\mathrm{Id}\colon (\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}, \gamma_E^a) \to (\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}, \mathrm{BM})$$
is a uniform homeomorphism for each k.
(4) γ_E^a *is a compact pseudometric on $\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$ for each k.*
(5) *The set $(\mathrm{Age}_k(E), \gamma_E^a)$ is totally bounded for each k.*

PROOF. (4) $\Rightarrow$ (3) : if (4) holds then by Lemma 4.14 (2), Id is a uniformly continuous bijection between the compact spaces $(\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}, \gamma_E^a)$ and $(\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}, \mathrm{BM})$ and therefore a uniform homeomorphism.

(3) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (1): if (2) is valid, the map Id is a homeomorphism between compact spaces, and therefore a uniform homeomorphism, implying (1).

(1) $\Rightarrow$ (5): By (1), the completions of $\mathrm{Age}_k(E)$ with respect to BM and γ_E^a coincide. In particular the γ_E^a-completion of $(\mathrm{Age}_k(E), \gamma_E^a)$ is compact, and so $\mathrm{Age}_k(E)$ is totally bounded for γ_E^a.

(5) $\Rightarrow$ (4): since γ_E^a is a pseudometric on $\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$, it follows from (2)$\Leftrightarrow$(3) in Lemma 4.19 that the map $j : C_k(E) \to (\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}, \gamma_E^a)$ given by $\overline{(X_n)_n} \mapsto \mathrm{BM} - \lim_n X_n$ is a surjective isometry. Therefore $(\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}, \gamma_E^a)$ is compact. $\square$

Since it is not clear how the above properties relate to the **AF** property, we ask:

QUESTION 4.21. Does the **AF**-property imply (1) to (5) of Proposition 4.20?

What we know is that the **AF**-property implies that γ_E^a is a pseudometric on $\overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$ and that this pseudometric is complete ((2) and (3) of Lemma 4.19).

To conclude this section, we comment on the relation between **AF** spaces and Kubiś' work [**17**], in which "approximate" Fraïssé limits are considered for "metric-enriched" categories. His theory includes the class of separable Banach spaces [**17**, Example 2.2] and allows $1+\varepsilon$-isometric embeddings to be considered. The Gurarij space is the approximate Fraïssé limit of the class of finite dimensional normed spaces in this sense [**17**, Section 4.1]; it is probable (but remains to be proved) that the Lebesgue spaces also appear as approximate Fraïssé limits in Kubiś' sense. The almost Fraïssé property studied in our paper seems formally more general than this notion. Indeed, while Kubiś considers $1+\varepsilon$-isometries (or isometric embeddings) which are perturbations of isometries (or isometric embeddings), the concept of an almost Fraïssé space does not, at least formally, require the existence of any non-trivial isometries on the space.

4.2. Ultrapowers of AF Banach spaces. Now we proceed to proving that ultrapowers of **AF** Banach spaces are **AF** and **UH**, and obtaining characterizations of **AF** for ultrapowers. The **UH**-property of ultrapowers of E was proven in [**15**] under the formally stronger assumption that E is Fraïssé. Recall that for a Banach E and a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, $E_{\mathcal{U}}$ denotes the ultrapower $E^{\mathbb{N}}/\mathcal{U}$. For $\varepsilon \geq 0$, We denote by $(\mathrm{Isom}^{\varepsilon}(E))_{\mathcal{U}}$ the set of maps T acting on $E_{\mathcal{U}}$ by $T([(x_n)]_{\mathcal{U}}) = [(T_n x_n)]_{\mathcal{U}}$ for each $[(x_n)]_{\mathcal{U}} \in E_{\mathcal{U}}$, where $(T_n)_n$ is a sequence of elements of $\mathrm{Isom}_{\varepsilon_n}(E)$ with $\lim_n \varepsilon_n = \varepsilon$, and we note that $(\mathrm{Isom}^{\varepsilon}(E))_{\mathcal{U}} \subseteq \mathrm{Isom}_{\varepsilon}(E_{\mathcal{U}})$.

LEMMA 4.22. *Let E be an **AF** Banach space and $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Then for each $\varepsilon > 0$ and $k \in \mathbb{N}$, there is $\delta > 0$ with the following property: if $X \in \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$ and $\phi_1, \phi_2 \in \mathrm{Emb}_\delta(X, E_{\mathcal{U}})$, there exists $T \in (\mathrm{Isom}^{\varepsilon}(E))_{\mathcal{U}}$ such that $T \circ \phi_1 = \phi_2$.*

PROOF. Let $\varepsilon > 0$ and $k \in \mathbb{N}$ be given and $\xi > 0$ satisfying $(1+\xi)^2 \leq 1+\varepsilon$. Fix $\delta_0 > 0$ corresponding to the definition of **AF** and set $\delta = \delta_0/2$. Also, let $X \in \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$ and $\phi_1, \phi_2 \in \mathrm{Emb}_{\delta'}(X, E_{\mathcal{U}})$. Since $\dim X < \infty$, there are two sequences (ϕ_n^1) and (ϕ_n^2) of linear operators from X to E such that $\phi_1 x = [(\phi_n^1(x))]_{\mathcal{U}}$ and $\phi_2 x = [(\phi_n^2(x))]_{\mathcal{U}}$ for all $x \in X$, $\|\phi_1\| = \lim_{\mathcal{U}} \|\phi_n^1\|$ and $\|\phi_2\| = \lim_{\mathcal{U}} \|\phi_n^2\|$. Let $A \in \mathcal{U}$ be such that $\phi_n^1, \phi_n^2 \in \mathrm{Emb}_\delta(X, E)$ for all $n \in A$. By definition of **AF**, for each $n \in A$, there exists $T_n \in \mathrm{Isom}_\xi(E)$ such that $T_n \circ \phi_n^1 = \phi_n^2$. Define $T\colon E_{\mathcal{U}} \to E_{\mathcal{U}}$ by $[(x_n)]_{\mathcal{U}} \mapsto [(y_n)]_{\mathcal{U}}$, where

$$y_n = \begin{cases} T_n(x_n), & n \in A; \\ x_n, & n \notin A. \end{cases}$$

Suppose that $[(x_n)]_{\mathcal{U}} = [(x'_n)]_{\mathcal{U}}$, then $\{n \in \mathbb{N} : \|x_n - x'_n\| < r/(1+\xi)\} \in \mathcal{U}$ for all $r > 0$. Thus, $\{n \in A : \|x_n - x'_n\| < r/(1+\xi)\} \in \mathcal{U}$ and hence, $\{n \in A : \|T_n(x_n) - T_n(x'_n)\| < r\} \in \mathcal{U}$. So, T is well defined. Also, $\|T\| \leq 1 + \xi$. Now, assume that $T([(x_n)]_{\mathcal{U}}) = [(y_n)]_{\mathcal{U}} = \mathbf{0}$. Thus, for each $r > 0$, $\{n \in \mathbb{N} : \|y_n\| < r/(1+\xi)\} \in \mathcal{U}$ and it follows that $\{n \in A : \|y_n\| < r/(1+\xi)\} \in \mathcal{U}$. Then, $\{n \in A : \|x_n\| < r\} \in \mathcal{U}$, i.e., $[(x_n)]_{\mathcal{U}} = \mathbf{0}$. From its definition we have $\|T^{-1}\| \leq 1 + \xi$. Whence $T \in (\mathrm{Isom}^\varepsilon(E))_{\mathcal{U}}$. Finally, since $A \subset \{n \in \mathbb{N} : \|\phi_n^2 x - (T_n \circ \phi_n^1)x\| < r\}$ for all $r > 0$ and $x \in X$, we have $T \circ \phi_1 = \phi_2$. □

Note that the previous proof also works under the assumption that E is weak **AF** and that $X \in \mathrm{Age}(E)$. So it is worth noting the next result.

Lemma 4.23. *Let E be a weak **AF** Banach space and $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Then for each $\varepsilon > 0$ and $X \in \mathrm{Age}(E)$, there is $\delta > 0$ with the following property: if $\phi_1, \phi_2 \in \mathrm{Emb}_\delta(X, E_{\mathcal{U}})$, there exists $T \in (\mathrm{Isom}^\varepsilon(E))_{\mathcal{U}}$ such that $T \circ \phi_1 = \phi_2$.*

We also have:

Lemma 4.24. *Let E be an **AF** Banach space and $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Then for every $k \in \mathbb{N}$, $X \in \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$ and $\phi_1, \phi_2 \in \mathrm{Emb}(X, E_{\mathcal{U}})$, there exists $T \in \mathrm{Isom}(E_{\mathcal{U}})$ such that $T \circ \phi_1 = \phi_2$.*

Proof. Let $k \in \mathbb{N}$ be fixed and also let $X \in \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$ and $\phi_1, \phi_2 \in \mathrm{Emb}(X, E_{\mathcal{U}})$. For each $m \in \mathbb{N}$, let $\delta_m > 0$ be the corresponding number to $1/m$ in the definition of **AF** and assume that $\delta_m \to 0$. Since $\dim X < \infty$, there are two sequences (ϕ_n^1) and (ϕ_n^2) of linear operators from X to E and a null sequence of positive numbers (α_n) such that $\phi_1 x = [(\phi_n^1 x)]_{\mathcal{U}}$ and $\phi_2 x = [(\phi_n^2 x)]_{\mathcal{U}}$ for all $x \in X$, and for $i = 1, 2$ we have $\|\phi_i\| = \lim_{\mathcal{U}} \|\phi_n^i\|$ and ϕ_n^i is an α_n-isometry for each $n \in \mathbb{N}$. We may suppose that $\alpha_m < \delta_m$ for each $m \in \mathbb{N}$. So, the set $A_m = \{n \in \mathbb{N} : \phi_n^1, \phi_n^2 \in \mathrm{Emb}_{\delta_m}(X, E)\}$ is in $\mathcal{U}$ for every $m \in \mathbb{N}$. By taking small perturbations, we assume that ϕ_n is not an isometry for each $n \in \mathbb{N}$. Thus, $\bigcap_S A_m = \emptyset$ for each $S \subset \mathbb{N}$ infinite. By definition of **AF**, for each $m \in \mathbb{N}$ and $n \in A_m$, there is $T_n^m \in \mathrm{Isom}_{1/m}(E)$ with $T_n^m \circ \phi_n^1 = \phi_n^2$. If $n \in \mathbb{N}$, let $k(n) \in \mathbb{N}$ be the maximal k satisfying $n \in A_k$ and define $T\colon E_{\mathcal{U}} \to E_{\mathcal{U}}$ by $[(x_n)]_{\mathcal{U}} \mapsto [(y_n)]_{\mathcal{U}}$, where

$$y_n = \begin{cases} T_n^{k(n)}(x_n), & n \in \bigcup A_m; \\ x_n, & n \notin \bigcup A_m. \end{cases}$$

Since $k(n) \geq n$ for each $n \in \mathbb{N}$, T is an isometry and by following the proof of Lemma 4.22 we conclude that $T \circ \phi_1 = \phi_2$. □

It is worth noting the following statement for weak **AF** spaces, which is obtained through a similar proof.

LEMMA 4.25. *Let E be a weak* ***AF*** *Banach space and $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Then for every $X \in \mathrm{Age}(E)$ and $\phi_1, \phi_2 \in \mathrm{Emb}(X, E_{\mathcal{U}})$, there exists $T \in \mathrm{Isom}(E_{\mathcal{U}})$ such that $T \circ \phi_1 = \phi_2$.*

THEOREM 4.26. *Let E be a Banach space and $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. If E is* ***AF****, then $E_{\mathcal{U}}$ is* ***AF*** *and* ***UH****.*

PROOF. Since $\mathrm{Age}_k(E_{\mathcal{U}}) \equiv \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$ for each $k \in \mathbb{N}$ the **AF**-result follows from Lemma 4.22. The **UH** follows from Lemma 4.24 and from the fact that $\mathrm{Age}_k(E_{\mathcal{U}}) \equiv \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}}$ for each $k \in \mathbb{N}$. □

Recall from the beginning of this subsection that $\mathrm{Isom}^0(E)_{\mathcal{U}}$ denotes the set of all elements $T \in \mathrm{Isom}(E_{\mathcal{U}})$ satisfying the next condition: there are a null sequence $(a_n) \in (0, \infty)^{\mathbb{N}}$ and a sequence of operators (T_n) with $T_n \in \mathrm{Isom}_{a_n}(E)$ for each $n \in \mathbb{N}$ such that $T([(x_n)]_{\mathcal{U}}) = [(T_n x_n)]_{\mathcal{U}}$ for all $[(x_n)]_{\mathcal{U}} \in E_{\mathcal{U}}$. The set $\mathrm{Isom}^0(E)_{\mathcal{U}}$ can be compared with the set $\mathrm{Isom}(E)_{\mathcal{U}}$ of all elements of $\mathrm{Isom}(E_{\mathcal{U}})$ of the form $[(x_n)]_{\mathcal{U}} \mapsto [(g_n(x_n))]_{\mathcal{U}}$ for some sequence $(g_n) \in \mathrm{Isom}(E)^{\mathbb{N}}$, which was considered in [**15**, Proposition 2.15]. We have the obvious inclusion $\mathrm{Isom}(E)_{\mathcal{U}} \subseteq \mathrm{Isom}^0(E)_{\mathcal{U}}$.

PROPOSITION 4.27.

(1) *if $E_{\mathcal{U}}$ is approximately* ***UH*** *and the set $\mathrm{Isom}^0(E)_{\mathcal{U}}$ is dense with respect to the strong operator topology in $\mathrm{Isom}(E_{\mathcal{U}})$, then E is* ***AF****.*
(2) *if $E_{\mathcal{U}}$ is almost* ***UH*** *and the set $(\mathrm{Isom}^\delta(E))_{\mathcal{U}}$ is dense with respect to the strong operator topology in $\mathrm{Isom}_\delta(E_{\mathcal{U}})$, then E is* ***AF****.*

PROOF. Suppose that E is not **AF**. Then there are $k_0 \in \mathbb{N}$ and $0 < \varepsilon_0 < 1/4$ such that for each $n \in \mathbb{N}$, there exist $X_n \in \mathrm{Age}_{k_0}(E)$ and $\phi_n \in \mathrm{Emb}_{1/n}(X_n, E)$ with $A|_{X_n} \neq \phi_n$ for all $A \in \mathrm{Isom}_{\varepsilon_0}(E)$. Let $\hat{X}$ be the natural finite dimensional subspace of $E_{\mathcal{U}}$ associated to the sequence (X_n) and $\phi\colon \hat{X} \to E_{\mathcal{U}}$ be defined as $\phi(\hat{x_n}) = [(\phi_n x_n)]_{\mathcal{U}}$, where $x_n \in X_n$ for each $n \in \mathbb{N}$.

In case (1), since $E_{\mathcal{U}}$ is approximately **UH** and $\phi \in \mathrm{Emb}(\hat{X}, E_{\mathcal{U}})$, there is $T \in \mathrm{Isom}(E_{\mathcal{U}})$ satisfying $\|T|_{\hat{X}} - \phi\| < \varepsilon_0/4k_0$. By the density of $\mathrm{Isom}^0(E)_{\mathcal{U}}$, there is $S \in \mathrm{Isom}^0(E)_{\mathcal{U}}$ such that $\|T - S\|_{\hat{X}} < \varepsilon_0/4k_0$. In case (2), since $E_{\mathcal{U}}$ is almost **UH** and $\phi \in \mathrm{Emb}(\hat{X}, E_{\mathcal{U}})$, there is $T \in \mathrm{Isom}_{\varepsilon/4k_0}(E_{\mathcal{U}})$ satisfying $T|_{\hat{X}} = \phi$. By the density of $\mathrm{Isom}^\delta(E)_{\mathcal{U}}$, there is $S \in \mathrm{Isom}^\delta(E)_{\mathcal{U}}$ such that $\|T - S\|_{\hat{X}} < \varepsilon_0/4k_0$, where $\delta := \varepsilon_0/4k_0$.

So $\|\phi - S|_{\hat{X}}\| < \varepsilon_0/2k_0$ which means that $\lim_{\mathcal{U}} \|\phi_n - S_n|_{X_n}\| < \varepsilon_0/2k_0$, where (S_n) is a sequence of a_n-isometries from E onto E with $a_n \to 0$ in Case 1 or $a_n \to \delta$ in Case 2. Thus $A = \{n \in \mathbb{N} : \|\phi_n - S_n|_{X_n}\| < \varepsilon_0/2k_0\} \in \mathcal{U}$. Choose $n \in \mathbb{N}$ such that $\|\phi_n - S_n|_{X_n}\| < \varepsilon_0/2k_0$ and S_n is a $\epsilon_0/3$-isometry. If $P_{X_n}\colon E \to X_n$ is a projection with $\|P_{X_n}\| \leq k_0$, then $A_n := S_n + (\phi_n - S_n) \circ P_{X_n}$ extends ϕ_n and a small computation shows that it is a ε_0-isometry, which is absurd. Hence E is **AF**. □

This is to compare with [**15**, Proposition 2.15] stating (implicitely) that if $E_{\mathcal{U}}$ is approximately **UH** and the set $\mathrm{Isom}(E)_{\mathcal{U}}$ is dense in $\mathrm{Isom}(E_{\mathcal{U}})$, then E is Fraïssé. So for example, under the weaker hypothesis of density of $\mathrm{Isom}^0(E)_{\mathcal{U}}$, we obtain the weaker **AF** property for E.

5. Oligomorphy and the Fraïssé property

5.1. Some pseudometrics on S_E^n. Fixing a Banach space E and $n \in \mathbb{N}$, we denote by $(T, x) \mapsto T \cdot x$ the usual action of $\mathrm{Isom}(E)$ on E^n defined in (1.1), i.e.

$$(T, (x_1, \ldots, x_n)) \mapsto (Tx_1, \ldots, Tx_n). \tag{5.1}$$

It is usual to fix an equivalent norm $\|\cdot\|$ on E^n such that this action is isometric on E^n. Because of Definition 5.4, we shall additionally need that the similar action induced by δ-isometric maps on E is δ'-isometric, for small enough δ and δ'. For this reason we choose to fix on E^n the usual ℓ_2-norm.

Definition 5.1. If $x, y \in S_E^n$, we set

$$d(x, y) = \inf_{T \in \mathrm{Isom}(E)} \|T \cdot x - y\|.$$

If necessary we denote by $\tilde{d}$ the induced metric on the quotient Q_n of S_E^n by the relation $x \sim y \iff d(x, y) = 0$. We may also use the classical notation $S_E^n /\!/ \mathrm{Isom}(E)$ for this quotient, see e.g. [**21**]. From Lemma 4.8 we have:

Fact 5.2. d is a complete pseudometric. Consequently, $\tilde{d}$ is a complete metric.

On S_E^n, by analogy with the Banach-Mazur pseudometric, we also consider the following pseudometric.

Definition 5.3. If $x, y \in S_E^n$, we set

$$d_{\mathrm{BM}}(x, y) = \log \|A\| \|A^{-1}\|,$$

if there exists (a necessarily unique) linear invertible map A from $[x]$ to $[y]$ with $A(x_i) = y_i$ for all $i = 1, \ldots, n$, and $d_{\mathrm{BM}}(x, y) = +\infty$ otherwise.

Let $\tilde{d}_{\mathrm{BM}}$ denote the distance induced on the quotient of S_E^n by the relation $x \sim_{\mathrm{BM}} y \iff d_{\mathrm{BM}}(x, y) = 0$. Finally we also consider a third and less classical pseudometric:

Definition 5.4. If $x, y \in S_E^n$, we set

$$d^a(x, y) = \sup_{\delta > 0} \inf_{T \in \mathrm{Isom}_\delta(E)} \|T \cdot x - y\|.$$

Lemma 5.5. *The function d^a is a pseudometric on S_E^n.*

Proof. Let us define $d_a^\delta(x, y) := \inf_{T \in \mathrm{Isom}_\delta(E)} \|T \cdot x - y\|$. By using that $\mathrm{Isom}_\delta(E)$ is invariant under taking inverses, we obtain that $(1+\delta)^{-1} d_a^\delta(x, y) \leq d_a^\delta(y, x) \leq (1+\delta) d_a^\delta(x, y)$ and therefore that d_a is symmetric. The triangle inequality follows from the estimate

$$d_a^\delta(x, y) + d_a^\delta(z, y) \geq (1+\delta)^{-1} d_a^{\delta^2 + 2\delta}(x, z),$$

which we leave to the reader as an exercise. □

When necessary we denote by $\tilde{d}_a$ the induced distance on the quotient Q_n^a of S_E^n by the relation $x \sim^a y \Leftrightarrow d_a(x, y) = 0$.

We observe the following immediate relations between these pseudometrics and corresponding ultrahomogeneity properties.

Fact 5.6.

(i) If $x, y \in S_E^n$ then $d_a(x,y) \leq d(x,y)$.
(ii) If $x, y \in S_E^n$ and $d_a(x,y) = 0$, then $d_{\mathrm{BM}}(x,y) = 0$.
(iii) E is approximately **UH** if and only if whenever $n \in \mathbb{N}$ and $x, y \in S_E^n$ satisfy $d_{\mathrm{BM}}(x,y) = 0$, we have $d(x,y) = 0$.
(iv) E is almost **UH** if and only if whenever $n \in \mathbb{N}$ and $x, y \in S_E^n$ satisfy $d_{\mathrm{BM}}(x,y) = 0$, we have $d_a(x,y) = 0$.

Notation 5.7. Let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be any list of relations of linear dependence between the elements of an n-uple $(x_1, \ldots, x_n)$ of S_E^n such that none of the relations of the list is consequence of the others. We denote by $(S_E^n)_{\mathcal{R}}$ the set of n-uples of S_E^n satisfying $\mathcal{R}$ and no additional relation of linear dependence. In particular $(S_E^n)_\emptyset$ is the set of n-uples of S_E^n which are linearly independent.

The point of this technical notation is that in order that $d_{\mathrm{BM}}(x,y) < +\infty$, x and y must belong to a same set $(S_E^n)_{\mathcal{R}}$. So this formalization will help us deal with discontinuities of d_{BM} with respect to d. As an example one may think of a sequence of couples $(x_1, x_k)_k$ with $(x_k)_k$ tending to x_1 and $x_k \neq x_1$. Then $(x_1, x_k)_k$ tends to (x_1, x_1) with respect to d but not with respect to d_{BM}.

Fact 5.8. Given any $\mathcal{R}$ as above, the map $\mathrm{Id}\colon ((S_E^n)_{\mathcal{R}}, d) \to ((S_E^n)_{\mathcal{R}}, d_{\mathrm{BM}})$ is continuous.

Proof. Let $\varepsilon > 0$ be given. Fix $(x_i)_{i \in I}$ a basis of $[x_1, \ldots, x_n]$ with constant K. If $d(x,y) < \alpha$, without loss of generality we may assume $\|x - y\| \leq \alpha$. Classical estimates guarantee that if α was small enough, then $(y_i)_{i \in I}$ is a $2K$-basis of $[y_i]$ and that the map t defined by $x_i \mapsto y_i$, $i \in I$, is a $(1+\varepsilon)$-isomorphism. Since both x and y belong to $(S_E^n)_{\mathcal{R}}$, this maps sends x_i to y_i for all the other value of i as well, so $d_{\mathrm{BM}}(x,y) \leq \varepsilon$. $\square$

We note that uniform continuity holds if we restrict to a subset where we control the basis constant of $(x_i)_{i \in I}$: for $K \geq 1$, let $(S_E^n)_K$ be the set of $x \in S_E^n$ which are a basic sequence with constant at most K. Note that $(S_E^n)_K$ is d_{BM}-closed and therefore d-closed. Hence:

Fact 5.9. Given $n \in \mathbb{N}$, given any $K \geq 1$, the map $\mathrm{Id}\colon ((S_E^n)_K, d) \to ((S_E^n)_K, d_{\mathrm{BM}})$ is uniformly continuous.

For n integer, we use the well-known fact that every n-dimensional space has a basis with constant $\sqrt{n}$ (since every n-dimensional space is $\sqrt{n}$ isomorphic to ℓ_2^n).

Fact 5.10. The following statements are equivalent:
(1) The space E is weak Fraïssé.
(2) For any $n \in \mathbb{N}$, the map $\mathrm{Id}\colon ((S_E^n)_{2\sqrt{n}}, d) \to ((S_E^n)_{2\sqrt{n}}, d_{\mathrm{BM}})$ is an homeomorphism.
(3) For any $n \in \mathbb{N}$, for any $K \geq 1$ the map $\mathrm{Id}\colon ((S_E^n)_K, d) \to ((S_E^n)_K, d_{\mathrm{BM}})$ is an homeomorphism.
(4) For any $n \in \mathbb{N}$, the map $\mathrm{Id}\colon ((S_E^n)_\emptyset, d) \to ((S_E^n)_\emptyset, d_{\mathrm{BM}})$ is an homeomorphism.

Proof. (3) $\Rightarrow$ (2) is obvious. (4) $\Rightarrow$ (3) also holds because $(S_E^n)_\emptyset = \cup_{K \geq 1} (S_E^n)_K$.
(2) $\Rightarrow$ (1): Fix $X \in \mathrm{Age}(E)$ and $x = (x_1, \ldots, x_n) \in S_E^n$ a $\sqrt{n}$-basis of X. If t is an $(1+\alpha)$-isometric embedding of $[x]$ into E, let $y = (y_1, \ldots, y_n) = (tx_1, \ldots, tx_n)$, which belongs to $(S_E^n)_{2\sqrt{n}}$ if α was chosen small enough. Then we deduce from

$d_{\mathrm{BM}}(x,y) < \alpha$ that $d(x,y) < \varepsilon$, i.e. there exists $T \in \mathrm{Isom}(E)$ so that $\|T\cdot x - y\| < \varepsilon$, so $\|T|_{[x]} - t\|$ is small if ε was well chosen.

(1) $\Rightarrow$ (4): because of Fact 5.8 for $\emptyset$, we just need to prove that for any $\varepsilon > 0$, for any $x = (x_1, \ldots, x_n) \in (S_E^n)_\emptyset$, there exists $\alpha > 0$, such that $d_{\mathrm{BM}}(x,y) < \alpha \Rightarrow d(x,y) < \varepsilon$. If $d_{\mathrm{BM}}(x,y) < \alpha$, i.e. there is t an $(1+\alpha)$-isometric embedding of $[x]$ into E defined by $tx_i = y_i$ and if α was chosen small enough, then there exists $T \in \mathrm{Isom}(E)$ so that $\|T|_{[x]} - t\| < \varepsilon$, therefore $\|Tx_i - y_i\| < \varepsilon$ for all i, and $\|T \cdot x - y\| \leq n\varepsilon$, so $d(x,y) < n\varepsilon$. □

FACT 5.11. The following assertions are equivalent:

(1) The space E is Fraïssé.
(2) For any $n \in \mathbb{N}$, the map $\mathrm{Id}\colon ((S_E^n)_{2\sqrt{n}}, d) \to ((S_E^n)_{2\sqrt{n}}, d_{\mathrm{BM}})$ is a uniform homeomorphism.
(3) For any $n \in \mathbb{N}$, for any $K \geq 1$, the map $\mathrm{Id}\colon ((S_E^n)_K, d) \to ((S_E^n)_K, d_{\mathrm{BM}})$ is a uniform homeomorphism.
(4) For any $n \in \mathbb{N}$, the map $\mathrm{Id}\colon ((S_E^n)_\emptyset, d) \to ((S_E^n)_\emptyset, d_{\mathrm{BM}})$ is a homeomorphism with uniformly continuous inverse.

PROOF. (3) $\Rightarrow$ (2) is obvious. (4) $\Rightarrow$ (3) holds because of Fact 5.9 and because $(S_E^n)_\emptyset = \cup_{K\geq 1}(S_E^n)_K$. (2) $\Rightarrow$ (1): Fix $X \in \mathrm{Age}(E)$ and $x = (x_1, \ldots, x_n) \in S_E^n$ a $\sqrt{n}$-basis of X. If t is an $(1+\alpha)$-isometric embedding of $[x]$ into E, let $y = (y_1, \ldots, y_n) = (tx_1, \ldots, tx_n)$. Then y is a $2\sqrt{n}$-basis if α was chosen small enough. Then we deduce from $d_{\mathrm{BM}}(x,y) < \alpha$ that $d(x,y) < \varepsilon$, i.e. there exists $T \in \mathrm{Isom}(E)$ so that $\|T \cdot x - y\| < \varepsilon$, so $\|T|_{[x]} - t\|$ is small if ε was well chosen.

(1) $\Rightarrow$ (2): because of Fact 5.8 for $\emptyset$, we just need to prove that for any n and any $\varepsilon > 0$, there exists $\alpha > 0$, such that for any $x, y \in (S_E^n)_\emptyset$, $d_{\mathrm{BM}}(x,y) < \alpha \Rightarrow d(x,y) < \varepsilon$. Fix some n, and fix $\varepsilon > 0$. If $d_{\mathrm{BM}}(x,y) < \alpha$, i.e. there is t an $(1+\alpha)$-isometric embedding of $[x]$ into E defined by $tx_i = y_i$, if α was chosen small enough then there exists $T \in \mathrm{Isom}(E)$ so that $\|T|_{[x]} - t\| < \varepsilon$, therefore $\|Tx_i - y_i\| < \varepsilon$ for all i, and $\|T \cdot x - y\| \leq n\varepsilon$, so $d(x,y) < n\varepsilon$. □

5.2. Oligomorphic Banach spaces.

DEFINITION 5.12. Let E be a Banach space. We shall say that E is an oligomorphic Banach space when for each $n \in \mathbb{N}$ the quotient $S_E^n /\!\!/ \mathrm{Isom}(E)$ of S_E^n by the orbit relation of the action of $\mathrm{Isom}(E)$ is $\tilde{d}$-compact (the pseudometric $\tilde{d}$ was defined in Definition 5.1).

A few commentaries are necessary here. In the separable case, this property is equivalent to the model theoretic notion of ω-categoricity; we shall not use this name since we have no reason to restrict our results to separable spaces. In [**5, 7, 21**] in a more general model theory context, the name "approximately oligomorphic" action is used. Here the word "approximately" may be misleading since already used among ultrahomogeneity properties. So in the present paper we choose the simple terminology "oligomorphic Banach space".

All $L_p(0,1)$-spaces, $1 \leq p < +\infty$, and the Gurarij space $\mathbb{G}$ are oligomorphic [**5**, Section 17] and [**6**, Section 2], respectively. Moreover, separable oligomorphic Banach spaces contain isometric copies of ℓ_2 [**16**, Corollary 5.13]. See also [**13**] about the relations between oligomorphy, G_δ-classes of Banach spaces, and restricted forms of the Fraïssé property for separable Banach spaces.

Theorem 5.13. *A Banach space is Fraïssé if and only if it is approximately* ***UH*** *and oligomorphic.*

Proof. Fix $K \geq 1$ and $n \in \mathbb{N}$. Let E be an approximately **UH** Banach space and consider the map

$$\mathrm{Id}\colon ((S_E^n)_K, \tilde{d}) \to ((S_E^n)_K, \tilde{d}_{\mathrm{BM}}).$$

By Facts 5.6 and 5.9 it defines a (uniformly) continuous bijective map between metric spaces.

If E is oligomorphic, and since $(S_E^n)_K$ is $\tilde{d}$-closed in S_E^n, the domain of this Id-map is compact. Therefore Id is a uniform homeomorphism. We conclude that E is Fraïssé by Fact 5.11(3).

Assume conversely that E is Fraïssé. We prove compactness of $S_E^n /\!\!/ \mathrm{Isom}(E)$ by letting $y_k = (x_1^k, \dots, x_n^k)$ be a sequence in S_E^n and describing how to find a $\tilde{d}$-converging subsequence of $(y_k)_k$.

We first look at $[x_1^k, x_2^k]$. Passing to a subsequence either this is basic with a fixed constant K_2, or $d(x_2^k, \mathbb{R}x_1^k)$ tends to 0. In the second case, we may find λ such that $d(x_2^k, \lambda x_1^k)$ tends to 0, and therefore assume wlog that $x_2^k = \lambda x_1^k$. Repeating this at each step we obtain $I \subseteq \{1, \dots, n\}$, K and $\mathcal{R}$ such that we may assume without loss of generality that for all k

- $\{x_i^k : i \in I\}$ is a K-basic sequence
- $[y_k] = [x_i^k : i \in I]$
- $\{x_1^k, \dots, x_n^k\} \in (S_E^n)_{\mathcal{R}}$.

Without loss of generality we may also assume that $\{x_i^k : i \in I\}$ is d_{BM}-convergent and therefore $\{x_1^k, \dots, x_n^k\}$ as well. The d_{BM}-limit $\{z_1, \dots, z_n\}$ of $\{x_1^k, \dots, x_n^k\}$ will satisfy the three conditions that $\{z_i : i \in I\}$ is a K-basic sequence, $[z_i : 1 \leq i \leq n] = [z_i : i \in I]$, and $\{z_1, \dots, z_n\} \in (S_E^n)_{\mathcal{R}}$. Now, from Fact 5.11, it follows that $\{x_1^k, \dots, x_n^k\}$ is d-Cauchy, and hence d-convergent in S_E^n by Fact 5.2. So, the limit has to coincide with $\{z_1, \dots, z_n\}$. In the end we have that $\{x_i^k : i \in I\}$ is d-convergent to $\{z_i : i \in I\}$. Since they all belong to $(S_E^n)_{\mathcal{R}}$, we finally have d-convergence of y_k to $\{z_1, \dots, z_n\}$. □

Spaces $L_p[0,1]$ for $p = 4, 6, \dots$ are examples of oligomorphic spaces which are not Fraïssé. The proof from [**15**, Chapter 4] that for $p \notin 2\mathbb{N} + 4$, $L_p[0,1]$ is Fraïssé, relies on technical estimates regarding approximate versions of the equimeasurability theorem of Plotkin and Rudin. This proof is constructive in the sense that one could expect to obtain some explicit estimates regarding the parameters ε and δ of the Fraïssé property. However as Ben Yaacov has mentioned to us [**4**], one can use Theorem 5.13 to obtain an abstract, model theoreric proof of the result of [**15**]. Just combine the result of Lusky that $L_p(0,1)$ is approximately ultrahomogeneous for $p \notin 2\mathbb{N} + 4$, with the oligomorphic property of spaces $L_p[0,1]$.

6. Final remarks and Questions

In this section we propose some questions that emerged throughout this work. Answering those could in our view give a better understanding of the theory.

Question 6.1. Is the ℓ_∞-sum of **FIE** (**aFIE**) Banach spaces a **FIE** (**aFIE**, respectively) Banach space?

By Fact 2.4 the answer is positive for the **FIE**-property of arbitrary ℓ_∞-sums of $c_0(\Gamma)$.

Question 6.2. Are the properties **aFIE** and **FIE** equivalent?

We know that answer is positive for reflexive spaces (see (2) of Proposition 3.6).

Corollary 4.5 motivates the next two questions.

Question 6.3. When does the **AF**-property imply that $\mathrm{Age}_k(E)$ is compact for all k?

Question 6.4. When does the weak **AF** property imply the **AF** property?

Another natural question is:

Question 6.5. If X and Y are almost Fraïssé and almost isometric, must they be isometric?

In [**15**, Problem 2.9], it is asked what other separable spaces different from $\mathbb{G}$ and $L_p(0,1)$, $p \notin 2\mathbb{N}+4$, are Fraïssé. Another related question is the following:

Question 6.6. Are $\mathbb{G}$, $L_p(0,1)$ for $p \notin 2\mathbb{N}+4$ and the Hilbert space ℓ_2 the only separable almost ultrahomogeneous Banach spaces?

The notion of oligomorphy could suggest defining and studying the next weaker property:

Definition 6.7. We say that a Banach space E is *almost oligomorphic* if for each $n \in \mathbb{N}$ the quotient of S_E^n by the orbit relation of the action of $\mathrm{Isom}(E)$ is $\tilde{d}_a$-compact.

Since $d_a \leq d$, the identity map on S_E^n with respect to d and d_a is continuous and therefore any space with the oligomorphic property also satisfies the almost oligomorphic property. Inspired by Theorem 5.13 we ask:

Question 6.8. Is a Banach space almost Fraïssé if and only if it is almost ultrahomogeneous and almost oligomorphic?

For a positive answer, there seem to be technical difficulties with the lack of completeness of d_a. We conjecture that answer is positive if we add as hypothesis that the age of the space is BM-closed.

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References

[1] Avilés, A.; Cabello Sánchez, F.; Castillo, J. M. F.; González, M.; Moreno, Y. *Separably injective Banach spaces.* Lecture Notes in Mathematics, 2132. Springer, [Cham], 2016.

[2] Albiac, F.; Kalton, N. J. *Topics in Banach space theory.* Second edition. With a foreword by Gilles Godefroy. Graduate Texts in Mathematics, 233. Springer, [Cham], 2016.

[3] Banach, S. *Théorie des opérations linéaires.* Editions Jacques Gabay, Sceaux, 1993. Reprint of the 1932 original.

[4] Ben-Yaacov, I. personal communication.
[5] Ben-Yaacov, I.; . Berenstein, A.; Henson, C. W.; Usvyatsov, A. *Model theory for metric structures*, in: Model Theory with Applications to Algebra and Analysis, Vol. 2, edited by Z. Chatzidakis, D. Macpherson, A. Pillay, and A. Wilkie, London Mathematical Society Lecture Note Series Vol. 350 (Cambridge University Press, 2008), pp. 315–427.
[6] Ben-Yaacov, I.; Henson, C. W. *Generic orbits and type isolation in the Gurarij space*, Fund. Math. 237, 47–82 (2017).
[7] Ben-Yaacov, I.; Tsankov, T. *Weakly almost periodic functions, model-theoretic stability, and minimality of topological groups*, Transactions of the American Mathematical Society, Volume 368, Number 11, November 2016, Pages 8267–8294. http://dx.doi.org/10.1090/tran/6883 Article electronically published on June 29, 2016.
[8] Cabello Sánchez, F. *Regards sur le problème des rotations de Mazur*. Extracta Math. 12 (1997), 97-116.
[9] Cabello Sánchez, F.; Castillo, J. M. F. *Homological methods in Banach Space Theory*, Cambridge Studies in Advanced Mathematics, 2021. Online ISBN. 9781108778312.
[10] Cabello Sánchez, F.; Ferenczi, V.; & Randrianantoanina, B. *On Mazur rotations problem and its multidimensional versions.* São Paulo J. Math. Sci. 16, 406–458 (2022).
[11] Castillo, J. M. F.; Plichko, A. *Banach spaces in various positions.* J. Funct. Anal. 259 (2010), no. 8, 2098–2138.
[12] Castillo, J. M. F.; Ferenczi, V.; Moreno, Y. *On uniformly finitely extensible Banach spaces.* J. Math. Anal. Appl. 410 (2014), no. 2, 670–686.
[13] Cuth, M; de Rancourt, N.; Doucha, M. *G_δ-Banach spaces.* Preprint 2024.
[14] Ferenczi, V.; Lopez-Abad, J.; *Envelopes in Banach spaces.* Preprint arXiv 2112.11795.
[15] Ferenczi, V.; Lopez-Abad, J.; Mbombo, B.; Todorcevic, S. *Amalgamation and Ramsey properties of L_p spaces.* Adv. Math. 369 (2020), 107190, 76 pp.
[16] Khanaki, K. *$\aleph_0$-categorical Banach spaces contain ℓ_p or c_0*. MLQ Math. Log. Q. 67 (2021), no. 4, 469–488.
[17] Kubiś, W. *Metric-enriched categories and approximate Fraïssé limits*, arXiv 1210.6506v3, 2018.
[18] Lusky, W. *Some consequences of W. Rudin's paper: "L_p-isometries and equimeasurability"*. Indiana Univ. Math. J. 27 (1978), no. 5, 859–866.
[19] Moreno, Y.; Plichko, A. *On automorphic Banach spaces.* Israel J. Math. 169 (2009), 29–45.
[20] Pełczyński, A; Rolewicz, S. *Best norms with respect to isometry groups in normed linear spaces,* in: Short Communication on International Math. Congress in Stockholm, 1962, p. 104.
[21] Tsankov, T. *Groupes d'automorphismes et leurs actions*, Thèse présentée en vue de l'obtention de l'Habilitation à Diriger des Recherches, Paris Diderot, 2014.

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