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On a question of Haskell P. Rosenthal

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Abstract

We consider a normalized basis in a Banach space with the following property: any normalized block sequence of the basis has a subsequence equivalent to the basis. We show that under uniformity or other natural assumptions, a basis with this property is equivalent to the unit vector basis of c_0 or ℓ_p. We also address an analogous problem concerning spreading models.

Haskell P. Rosenthal has posed the following problem on basic sequences in a Banach space:

Problem *Let X have a normalized basis $\{e_i\}$ with the property that every normalized block basis admits a subsequence equivalent to $\{e_i\}$. Is $\{e_i\}$ equivalent to the unit vector basis of l_p or c_0?*

Let us recall a well-known theorem of Zippin, which states that a normalized basis of a Banach space such that all normalized block bases are equivalent (to the original basis) must be equivalent to the unit vector basis of l_p or c_0.

The problem of Rosenthal is of particular interest as of a "mixed" Ramsey type, in the sense that it links two types of "subbases" of a given basis: namely subsequences and block sequences. An instance of a theorem which mixes a property concerning subsequences and a property concerning block bases was given by the second named author in [11]. She proved that a Banach space saturated with subsymmetric sequences must contain a minimal subspace.

Let us notice that this mixing is necessary to make Rosenthal's problem significant. Indeed, a weakening of the Rosenthal property would be to assume that every subsequence has a further subsequence equivalent to the basis. An application of Ramsey theorem would give us that the basis is subsymmetric. But obviously not every subsymmetric basis is equivalent to c_0 or l_p (take the basis of Schlumprecht's space for example).

On the other hand we may weaken the Rosenthal property by only requiring that any block sequence has a further block sequence equivalent to $\{e_i\}$. Let us call a basis with this property *block equivalence minimal*. The correct setting for such a property is Gowers' block version of Ramsey theorem. A standard diagonalization yields that for some constant $C > 0$, any block sequence has a further block sequence which is C-equivalent to $\{e_i\}$, and by Gowers' dichotomy theorem we get the existence of a winning strategy for Player 2 in Gowers' game to produce sequences $C + \epsilon$-equivalent to $\{e_i\}$. But again, Schlumprecht's space is a non-trivial example (see [12] or [2]). Actually, by the proof of Theorem 3.4 in [11], any Banach space saturated with subsymmetric sequences contains a block equivalence minimal basic sequence.

Rosenthal's problem is closely related to a problem of Argyros concerning spreading models: if all spreading models in a Banach space are equivalent, must they be equivalent to c_0 or l_p? We inspire ourselves from results of Androulakis, Odell, Schlumprecht and Tomczak-Jaegermann ([1]) about Argyros' question, to prove that the answer to Rosenthal's question is positive if uniformity is assumed (Proposition 1), or when 1 is in the Krivine's set of the basis (Corollary 5). We show that the answer is also positive if X and X^* satisfy the property of Rosenthal (Proposition 6), or when the selection of the subsequence in the definition of Rosenthal's property can be chosen to be continuous (Proposition 8). Let us notice that this is in contrast with the case of block equivalence minimality, where uniformity comes as a consequence of the definition, as well as continuity (because of the previous remark using Gowers' dichotomy theorem). Finally, after relating Rosenthal's to Argyros' question, we show how results of descriptive set theory may be used to get a dichotomy concerning the number of non-equivalent spreading models in a Banach space (Proposition 9), and we show that c_0 or l_p embeds in X if there exists a continuous way of picking subsequences generating spreading models (Proposition 10).

Let us give a definition for the main property in this paper. A normalized basis $\{e_n\}$ such that any normalized block basis of $\{e_n\}$ has a subsequence which is equivalent to $\{e_n\}$ will be said to have *Rosenthal's* property, or in short to be a *Rosenthal* basis.

First some easy remarks. We notice that a Rosenthal basis must be subsymmetric. Indeed let $\mathcal{A} \subset [\omega]^{\aleph_0}$ be the set of subsequences of ω giving a subsequence of $\{e_i\}$ equivalent to $\{e_i\}$. Then $\mathcal{A}$ is clearly Borel and therefore by the Galvin-Prikry theorem, there is an infinite subset H of ω such that either $[H]^{\aleph_0} \subset \mathcal{A}$ or $[H]^{\aleph_0} \subset \mathcal{A}^C$. Evidently the last possibility contradicts Rosenthal's property. So $\{e_i\}_\omega \sim \{e_i\}_H$ and $\{e_i\}_H$ is subsymmetric, hence $\{e_i\}_\omega$ also.

By renorming we can assume that $\{e_i\}$ is invariant under spreading (1-equivalent to its subsequences). Now Brunel and Sucheston have observed that for a normalized basic sequence $\{t_i\}$, invariant under spreading, the difference sequence $\{t_{2i+1} - t_{2i}\}$ is suppression unconditional (ie. the norm decreases as the support diminishes). By Rosenthal's property $\{e_{2i+1} - e_{2i}\}$ is equivalent to $\{e_i\}$, and is also invariant under spreading. So we may always assume that a Rosenthal basis is both invariant under spreading and suppression unconditional. Let us notice at this point that according to Schlumprecht's terminology, spaces with a Rosenthal basis are exactly spaces of Class 1 with a subsymmetric basis. So for once, our favorite non-trivial example S will not do: it is of Class 2! (see [13]).

We fix some notational matters: we say that a block basis $\{x_i\}$ over $\{e_i\}$ is identically distributed if there are scalars $r_0, \ldots, r_k$ and natural numbers $m_0 < m_0 + k < m_1 < m_1 + k < m_2 < m_2 + k < \ldots$ such that $x_i = r_0 e_{m_i} + \ldots + r_k e_{m_i+k}$.

We show that with a uniformity condition added in the hypothesis the answer is positive. In fact we get a bit more:

Proposition 1 *Let $\{e_i\}$ be a normalized basic sequence and $K \geq 1$ be a constant such that any identically distributed normalized block basis admits a subsequence K-equivalent to $\{e_i\}$. Then $\{e_i\}$ is equivalent to the unit vector basis of c_0 or l_p.*

Proof : The proof in the previous remark still goes to show that without loss of generality, we may assume that $\{e_i\}$ is both invariant under spreading and suppression unconditional. Now under these conditions, Krivine's theorem takes a particularly simple form:

(Krivine) Let $\{t_i\}$ be a suppression unconditional basis, invariant under spreading. Then there is a $p \in [1, \infty]$ such that for all $k < \omega, 0 < \epsilon$, there are identically distributed blocks $x_1 < x_2 < \ldots < x_k$ that are $(1+\epsilon)$-equivalent to the unit vector basis of l_p^k.

The set of p's satisfying this assertion is called the *Krivine set*. Take a p in this set for our basis $\{e_i\}$, then for any k there is a norm one block $x(k)$, such that taking k successive copies of this vector $x_1(k) < x_2(k) < \ldots < x_k(k)$ gives a sequence 2-equivalent to the unit vector basis in l_p^k.

Taking now infinitely many copies of $x(k)$: $x_0(k) < x_1(k) < x_2(k) < \ldots$, it can be observed that the sequence is identically distributed, so as before it must be K-equivalent to $\{e_i\}$. But this means that $\{e_0, \ldots, e_{k-1}\}$ is $2K$-equivalent to l_p^k, and as k was arbitrary, $\{e_i\}$ must be $2K$-equivalent to l_p or c_0 if $p = 0$. $\square$

Remark The uniformity condition is necessary in this result. Indeed take any invariant under spreading 1-unconditional basis $\{e_i\}$ (like our usual example of the unit basis of Schlumprecht's space which is not equivalent to c_0 or l_p): then any identically distributed sequence $\{x_i\} = \{r_0 e_{m_i} + \ldots + r_k e_{m_i+k}\}$ is equivalent to $\{e_i\}$, as proved by the relation

$$(\max_j |r_j|) \| \sum \lambda_i e_i \| \leq \| \sum \lambda_i x_i \| \leq (\sum_{j=0}^{k} |r_j|) \| \sum \lambda_i e_i \|,$$

for all sequences (λ_i) in c_{00}.

Notice also that this is an opposition to the property of block equivalence minimality, where uniformity is a direct consequence of the property.

We now study Rosenthal's problem without the uniformity condition. First we notice that the only relevant case is the reflexive one, as showed by the next lemma. We need some notation about spreading models. A sequence $\{x_i\}$ in a Banach space X is called seminormalized if there are real numbers $0 < c < C$ such that $c < \|x_i\| < C$, $\forall i$. Let $\{x_i\}$ in a Banach space

X be a seminormalized basic sequence. Suppose that

$$\forall r_0, \ldots, r_k \in \mathbb{R}\ \exists t \in \mathbb{R}\ \forall \epsilon > 0\ \exists N\ \forall N < l_0 < \ldots < l_k,$$
$$\Big|\ \|r_0 x_{l_0} + \ldots + r_k x_{l_k}\| - t\ \Big| < \epsilon$$

(or more intuitively $\lim_{l_0<\ldots<l_k,\, l_0\to\infty} \|r_0 x_{l_0} + \ldots + r_k x_{l_k}\|$ exists), then we say that $\{x_i\}$ generates a spreading model $\{\tilde{x_i}\}$ with the norm defined as follows:

$$\|r_0\tilde{x_0} + \ldots + r_k\tilde{x_k}\| := \lim_{l_0<\ldots<l_k,\, l_0\to\infty} \|r_0 x_{l_0} + \ldots + r_k x_{l_k}\|$$

The spreading model $\{\tilde{x_i}\}$ is then a basic sequence, invariant under spreading. Furthermore it is easily seen that the basic constant of $\{\tilde{x_i}\}$ is majorized by that of $\{x_i\}$. Moreover any subsequence of $\{x_i\}$ generates the same spreading model.

Lemma 2 *Let $\{e_i\}$ be a Rosenthal basis for a Banach space X. Then $\{e_i\}$ is equivalent to c_0 or l_1 or X is reflexive. In the last case, all spreading models in X are equivalent to $\{e_i\}$.*

Proof : As before we can assume that $\{e_i\}$ is suppression unconditional and invariant under spreading. Now by James's theorem, X is reflexive or contains a subspace isomorphic to c_0 or l_1. In the last case, we may assume c_0 or l_1 is equivalent to a block subspace of X, so that $\{e_i\}$ itself is equivalent to c_0 or l_1. If now X is reflexive, any spreading model is generated by a weakly null sequence, so by a block basic sequence, so once again by Rosenthal's property, is equivalent to $\{e_i\}$. □

Let $\{x_i\}$ and $\{y_i\}$ be basic sequences and $K > 0$. $\{x_i\}$ K-dominates $\{y_i\}$ (written $\{x_i\} \geq^K \{y_i\}$), if

$$K\|\sum a_i x_i\| \geq \|\sum a_i y_i\| \quad \forall (a_i) \in c_{00}$$

We will need a recent result of Androulakis, Odell, Schlumprecht, Tomczak-Jaegermann:

Proposition 3 (Androulakis, Odell, Schlumprecht, Tomczak-Jaegermann) *Let $\{x_i^n\}_i$, $n \in \omega$ be a sequence of normalized basic weakly null sequences in a Banach space X which have spreading models $\{\tilde{x}_i^n\}_i$, then there exists a seminormalized basic weakly null sequence $\{y_i\}$ in X with spreading model $\{\tilde{y}_i\}$ such that*

$$2^n\|\sum a_i\tilde{y}_i\| \geq \|\sum a_i\tilde{x}_i^n\| \quad \forall n\ \forall (a_i) \in c_{00}$$

Corollary 4 *Suppose X is a Banach space with a basis $\{e_i\}$ with Rosenthal's property. Then there exists $K > 0$ such that $\{e_i\}$ K-dominates any identically distributed normalized block basis of $\{e_i\}$.*

Proof : We may assume that $\{e_i\}$ is a normalized suppression unconditional basic sequence, invariant under spreading. Assume that for any $n < \omega$ there is a normalized basic weakly null sequence $\{x_i^n\}$ in X with spreading model $\{\tilde{x}_i^n\}$ such that $\{e_i\}$ does not 4^n-dominate $\{\tilde{x}_i^n\}$. Take $\{y_i\}$ as in Proposition 3 and notice that its spreading model must be equivalent to $\{e_i\}$ by Lemma 2. So there is some K such that $\{e_i\} \geq^K \{\tilde{y}_i\} \geq^{2^n} \{\tilde{x}_i^n\}$; taking n large enough you get a contradiction. So there is some K such that $\{e_i\}$ K-dominates any spreading model generated by a normalized basic weakly null sequence. Now any identically distributed normalized block basis is invariant under spreading, so is its own spreading model, giving the result. □

Corollary 5 *Suppose that $\{e_i\}$ is a normalized suppression unconditional basic sequence, invariant under spreading, with Rosenthal's property. If* 1 *is in $\{e_i\}$'s Krivine set, then $\{e_i\} \sim l_1$.*

Proof : For any k take some norm one block $x(k)$ on $\{e_i\}$, such that taking k successive copies $x_1(k) < x_2(k) < \ldots < x_k(k)$ of it you get a sequence 2-equivalent to the unit vector basis in l_1^k. An infinite sequence of successive copies of this vector $x_0(k) < x_1(k) < \ldots$ must be K-dominated by $\{e_i\}$, where K is the constant given by Corollary 4. So for any k, $\{e_i\}_{i=0}^{k-1}$ $2K$-dominates l_1^k, but must itself, by the triangle inequality, be 1-dominated by l_1^k. Hence $\{e_i\}$ is equivalent to l_1. □

The following proposition states that the answer to Rosenthal question is positive if we also assume Rosenthal property in the dual.

Proposition 6 *Let X be a Banach space with a Rosenthal basis, and such that X^* has a Rosenthal basis. Then X is isomorphic to c_0 or $l_p, p > 1$ (and any Rosenthal basis of X is equivalent to the unit vector basis of c_0 or l_p).*

Proof : By Lemma 2 we may assume that X is reflexive. Let $\{e_i\}$ be a Rosenthal basis of X. By renorming we may assume that the basis is suppression unconditional and invariant under spreading. The biorthogonal basis $\{e_i^*\}$ satisfies these properties as well; in particular it is its own spreading model, so by Lemma 2, it is equivalent to any Rosenthal basis of X^*; so it has Rosenthal's property. By Corollary 4, there exists $K > 0$ such that any normalized identically distributed block basis in X (resp. X^*) is K-dominated by $\{e_n\}$ (resp. $\{e_n^*\}$). Given a normalized identically distributed block basis $\{x_n\}$ in X, denote by $\{x_n^*\}$ a normalized identically distributed block basis in X^* such that each x_n^* satisfies $x_n^*(x_n) = 1$ and has support no larger than the support of x_n (this is possible by 1-unconditionality and 1-subsymmetry): $\{x_n^*\}$ is K-dominated by $\{e_n^*\}$. It follows that $\{x_n\}$ $1/K$-dominates $\{e_n\}$. Indeed, for $(a_i) \in c_{00}$,

$$K\|\sum a_i x_i\| \geq K \sup_{(b_i)\in c_{00}} \frac{(\sum b_i x_i^*)(\sum a_i x_i)}{\|\sum b_i x_i^*\|} \geq \sup_{(b_i)\in c_{00}} \frac{\sum b_i a_i}{\|\sum b_i e_i^*\|} = \|\sum a_i e_i\|.$$

Hence any identically distributed normalized block $\{x_i\}$ of $\{e_i\}$ is K^2-equivalent to $\{e_i\}$. By Proposition 1, $\{e_i\}$ must be equivalent to the unit basis of l_p for some $p > 1$. □

We now prove that if the selection of the subsequence in Rosenthal's property is continuous then the answer to the problem is also positive; in fact we get more, it is enough to find a continuous selection of subsequences dominating the basis. We let $bb(\{e_i\})$ (or $bb(X)$) be the set of normalized block bases of $\{e_i\}$, denote by $bb_D(X)$ the same set equipped with the product of the discrete topology on X, by $bb_E(X)$ the same set equipped with the "Ellentuck-Gowers" topology: basic open sets are of the form $[a, A]$ with $a < A$ for $a = (a_1, \ldots, a_n)$ a finite normalized block sequence and A an infinite normalized block sequence, where

$$[a, A] = \{a^\frown x, x \in bb(A)\}.$$

Here $a^\frown x \in bb(X)$ denotes the concatenation of a and x, and $bb(A)$ denotes the set of normalized block bases of A. Proposition 8 uses the weakest notion of continuity combining the two topologies. We first prove a Lemma.

Lemma 7 *Assume X is a Banach space with a Rosenthal basis $\{e_i\}$ not equivalent to c_0 or l_p, and let $\phi : bb(X) \to bb(X)$ map any $x \in bb(X)$ to a subsequence of x. Then for any $bb_E(X)$-open set $[a, A]$ in X and all $n > 0$, there exists a normalized block basis x in $bb(A)$ such that $\phi(a^\frown x)$ does not n-dominate $\{e_i\}$.*

Proof : Otherwise passing to a further block, we may assume that $A = \{A_i\}$ is C-equivalent to $\{e_i\}$ for some C; by Corollary4 there exists K such that $\{A_i\}$ K-dominates any of its identically distributed blocks; furthermore by the assumption any block x of A is such that $\phi(a^\frown x)$ n-dominates $\{e_i\}$, so by 1-subsymmetry of $\{e_i\}$, some subsequence of x n-dominates $\{e_i\}$, thus nC-dominates $\{A_i\}$; so by Proposition 1, A would be equivalent to c_0 or l_p. $\square$

Proposition 8 *Assume X is a Banach space with a Rosenthal basis $\{e_i\}$, and that $\phi : bb(X) \to bb(X)$ is a $bb_E(X) - bb_D(X)$ continuous map such that for any normalized basic sequence $x = (x_i)$ in $bb(X)$, the sequence $\phi(x)$ is a subsequence of x which dominates $\{e_i\}$. Then $\{e_i\}$ is equivalent to the unit vector basis of c_0 or l_p.*

Proof : Otherwise we build a block sequence $z = \{z_i\}$ with $\phi(z)$ not dominating $\{e_n\}$ by induction, using Lemma 7. Let $X^1 = \{x_i^1\}$ be a block such that $\phi(X^1)$ does not 2-dominate $\{e_i\}$. There exists an integer N_1 such that $\{\phi(X^1)_i\}_{1 \leq i \leq N_1}$ does not 2-dominate $\{e_i\}_{1 \leq i \leq N_1}$. By continuity of ϕ, there exists n_1 in $\mathbb{N}$ and A_1 in $bb(X)$, with $x^1 = (x_1^1, \ldots, x_{n_1}^1) < A_1$, such that if $y = \{y_i\}$ is in $bb(A_1)$ then $(\phi(x^{1\frown} y))_j = (\phi(X^1))_j$ for all $1 \leq j \leq N_1$. We let $z_i = x_i^1$ for $1 \leq i \leq n_1$.

Now let $X^2 = \{x_i^2\}$ be a block in A_1 such that $\phi(x^{1\frown} X^2)$ does not 4-dominate $\{e_i\}$. There exists an integer $N_2 > N_1$ such that the sequence $\{\phi(x^{1\frown} X^2)_i\}, 1 \leq i \leq N_2$ does not 4 dominate $\{e_i\}_{1 \leq i \leq N_2}$. By continuity of ϕ, there exists n_2 and $A_2 \in bb(A_1)$,with $x^2 = (x_1^2, \ldots, x_{n_2}^2) < A_2$ such that if $y \in bb(A_2)$ then $(\phi(x^{1\frown} x^{2\frown} y))_j = (\phi(x^{1\frown} X^2))_j$ for all $1 \leq j \leq N_2$. We let $z_{n_1+i} = x_i^2$ for $1 \leq i \leq n_2$. Repeating this procedure, we obtain by induction a normalized block sequence $z = \{z_i\}$, an increasing sequence of integers $\{N_i\}$, a sequence of integers $\{n_i\}$, finite blocks x^i, such that for all k,

$$\{(\phi(z))_i\}_{1 \leq i \leq N_k} = \{\phi(x^{1\frown} x^{2\frown} \ldots ^\frown x^k)_i\}_{1 \leq i \leq N_k},$$

so that $(\phi(z)_i)_{1 \leq i \leq N_k}$ does not 2^k dominate $\{e_i\}_{1 \leq i \leq N_k}$, and so $\phi(z)$ does not 2^k dominate $\{e_i\}$. As k is arbitrary this contradicts the definition of ϕ. $\square$

Let us remark, that once again, there is an opposition between Rosenthal's property and block equivalence minimality. Indeed, for any block equivalence minimal basis, Gowers' theorem implies the existence of a winning strategy to produce block sequences (C-) equivalent to $\{e_i\}$; in Gowers' game defined by Bagaria and Lopez-Abad, which is actually equivalent to the original game defined by Gowers ([3]), Player 1 plays block vectors and Player 2 sometimes chooses a vector in the finite dimensional space defined by the blocks played by Player 1. The winning strategy then defines a continuous map from block sequences to further block sequences (C-) equivalent to $\{e_i\}$. Notice also that Gowers-Maurey constructions ([4]) yield winning strategies in the previous sense: technically, l_1^n-averages used to build interesting vectors in their space may at each step of the construction be chosen in an arbitrary block-subspace. Roughly speaking, this means that if one tried to adapt their ideas to build a non-trivial Rosenthal basis, not only one would have to find a way to pass from selecting further (finite) blocks to selecting (infinite) subsequences, but also one would probably have to add new methods to suppress the continuity of the selection map.

Finally, we investigate the relation between Rosenthal's question and a problem of S. Argyros.

Problem (S. Argyros) *Let X be a Banach space such that all spreading models in X are equivalent. Must these spreading models be equivalent to the unit vector basis or l_p for some $p \geq 1$?*

For example, spaces l_p have unique spreading model up to equivalence. Indeed, in the reflexive case, all spreading models are generated by weakly null sequences; and in l_1, any spreading model is generated by a l_1-sequence, or by Rosenthal's theorem, by a weakly Cauchy sequence. In the second case, the difference sequence is weakly null, so generates l_1, and it follows that the spreading model is equivalent to l_1. But this does not generalize to the case of c_0, since the unit basis of c_0 and the summing basis generate non-equivalent spreading models; however all spreading models generated by weakly null sequences are clearly equivalent to c_0.

Lemma 2 shows that a positive answer to the problem of Argyros implies a positive answer to the problem of Rosenthal. Actually Androulakis, Odell, Schlumprecht and Tomczak-Jaegermann proved that the answer to Argyros' Problem is positive under the additional assumption of uniformity or that 1 is in the Krivine set of some basic sequence. Our methods are inspired from their results. A natural generalization of Argyros' question is mentioned in their article: if a Banach space contains only countably many spreading models up to equivalence, must one of them be equivalent to c_0 or l_p? In the other direction, the following remark about Banach spaces with more than countably many spreading models is a straightforward consequence of a well-known result of Silver.

Proposition 9 *Let X be a separable Banach space. Then either X contains continuum many non-equivalent spreading models, or X contains at most countably many non-equivalent spreading models. When X^* is separable, the same dichotomy holds for spreading models generated by weakly null basic sequences; when X has a Schauder basis, it holds for spreading models generated by block basic sequences.*

Proof : In the following, $\sim^C$ denotes the usual C-equivalence between basic sequences. We consider the set $\mathcal{S}$ of semi-normalized basic sequences generating spreading models, which can be described as the set of semi-normalized basic sequences $\{x_i\}$ such that: $\forall k \in \mathbb{N}, \forall \epsilon > 0, \exists N : \forall N < l_0 < \ldots < l_k, \forall N < l'_0 < \ldots < l'_k, (x_{l_i})_{i=0}^k \sim^{1+\epsilon} (x_{l'_i})_{i=0}^k$. This set is clearly a Borel subset of the Polish space X^ω. Now consider the equivalence relation $\simeq$ on $\mathcal{S}$ meaning that the two sequences generate spreading models which are equivalent in the usual $\sim$ sense. That is $(y_n) \simeq (z_n)$ iff

$$\exists C > 0, \forall k \in \mathbb{N}, \exists N : \forall N < l_0 < \ldots < l_k, (y_{l_i})_{i=0}^k \sim^C (z_{l_i})_{i=0}^k.$$

This equivalence relation is Borel as well. Now by a Theorem of Silver (Th 35.20 in [6]), a Borel (even coanalytic) equivalence relation on a Borel subset of a Polish space has either only countably many classes or there exists a Cantor set of mutually non-equivalent elements. As two spreading models are $\sim$-equivalent if and only if any two semi-normalized basic sequences which generate them are $\simeq$-equivalent, the result follows. When X^* is separable, the same proof holds for the set of weakly null semi-normalized basic sequences generating spreading models, which is also Borel in X^ω; or when X has a basis, for the set of block basic sequences generating spreading models. $\square$

Remark It is also a consequence of the Theorem of Silver that a Schauder basis of a Banach space X has continuum many non-equivalent subsymmetric block basic sequences or only countably many classes of equivalence of them. Indeed, the set of of normalized block bases equipped with the product topology on X is Polish, and the set of subsymmetric normalized block basic sequences is F_σ in it.

A result analogous to Proposition 8 holds also for spreading models. It turns out that the continuity of a map, which picks subsequences generating spreading models in a strong sense described below, is strong enough to imply that there is actually a copy of ℓ_p or c_0 in the space.

First some terminology: given a sequence $\epsilon = \{\epsilon_i\}$, $\epsilon_i \searrow 0$, and a basic sequence $\{x_i\}$ in a Banach space we say that $\{x_i\}$ ϵ-generates a spreading model $\{\tilde{x}_i\}$, if for any $k < n_1 < n_2 < \ldots < n_k$ we have $(x_{n_1}, \ldots, x_{n_k}) \sim^{1+\epsilon_k} (\tilde{x}_1, \ldots, \tilde{x}_k)$. Obviously every basic sequence for any sequence ϵ of non-zero scalars has a subsequence ϵ-generating a spreading model. We will use the notation introduced before Lemma 7.

We say that a Banach space X contains almost isometric copies of c_0 (resp. ℓ_p), if for any $\delta > 0$, X has a subspace $(1+\delta)$-isomorphic to c_0 (resp. ℓ_p).

Proposition 10 *Let X be a Banach space with a basis $\{e_i\}$. Fix a sequence $\epsilon = \{\epsilon_i\}$, $\epsilon_i \searrow 0$. Assume there is a continuous map $\phi : bb_D(X) \to bb_D(X)$ such that for any normalized basic sequence $x = \{x_i\}$ in $bb(X)$, the sequence $\phi(x)$ is a subsequence of x which ϵ-generates a spreading model. Then X contains almost isometric copies of c_0 or l_p for some $1 \leq p < \infty$.*

Proof : We recall the notion of asymptotic spaces as presented in [10]. Let X be a Banach space with a basis $\{e_i\}$. A tail subspace means here a block subspace of X of a finite codimension.

We say that a normalized basic sequence $\{a_i\}_{i=1}^n$ is asymptotic in X, if

$$\forall \delta > 0 \;\; \forall k_1 \;\; \exists x_1 \in \langle e_i \rangle_{i>k_1} \;\; \forall k_2 \;\; \exists x_2 \in \langle e_i \rangle_{i>k_2} \;\; \ldots \;\; \forall k_n \;\; \exists x_n \in \langle e_i \rangle_{i>k_n}$$

so that $\{x_i\}_{i=1}^n$ is a normalized block sequence $(1+\delta)$-equivalent to $\{a_i\}_{i=1}^n$. In other words, if we consider the asymptotic game, in which player I picks tail subspaces and player II picks block vectors from the subspaces chosen by player I, then a normalized basic sequence $\{a_i\}_{i=1}^n$ is asymptotic iff player II for any δ has a winning strategy in choosing a normalized block sequence of vectors $(1+\delta)$-equivalent to $\{a_i\}_{i=1}^n$.

Since a block sequence has a subsequence generating an unconditional spreading model, by Krivine's theorem, there is some $1 \leq p \leq \infty$ such that ℓ_p^n or c_0^n (in case $p = \infty$) is asymptotic in X for any $n \in \mathbb{N}$ (i.e. the unit basic vectors in ℓ_p^n or c_0^n form asymptotic sequences).

Let X satisfy the assumption of the proposition. We will use in the proof only asymptotic sequences of length 2. Pick $1 \leq p \leq \infty$ such that ℓ_p^2 or c_0^2 (in case $p = \infty$) is asymptotic in X.

We will show that any asymptotic pair (a_1, a_2) of X is 1-equivalent to the unit basic vectors of suitable ℓ_p^2 or c_0^2.

Fix $\delta > 0$ and pick any asymptotic pair (a_1, a_2). Pick $n \in \mathbb{N}$, $n > 1$ such that $(1 + \epsilon_{n-1})^3 < 1 + \delta$. Consider the asymptotic game for $(1 + \epsilon_{n-1})$ and (a_1, a_2). Let $x = (x_1, x_2, \ldots)$ be a block sequence consisting of vectors picked by player II in the first move in some game (you can produce such a sequence by letting player I choose in the first move tail subspaces of arbitrary large codimension). Let $\phi(x) = (x_{j_1}, x_{j_2}, \ldots)$. By the continuity of ϕ there is some $J > j_n$ such that $\phi([(x_1, \ldots, x_J), X]) \subset [(x_{j_1}, \ldots, x_{j_n}), X]$.

Now consider sequence $y = (x_1, \ldots x_J, y_1, y_2, \ldots)$, where $(y_1, y_2, \ldots)$, with $x_J < y_1$, is a block sequence of vectors chosen by player II in the second move in some game for $(1+\epsilon_{n-1})$ and (a_1, a_2), in which player II picked in the first move the vector x_{j_n} (again you produce such sequence by letting player I choose in the second move tail subspaces of arbitrary large codimension). Let

$$\phi(y) = (x_{j_1}, \ldots, x_{j_n}, \ldots x_{j'}, y_{k_1}, y_{k_2}, \ldots)$$

Again by the continuity of ϕ there is some $K > k_1$ such that

$$\phi([(x_1, \ldots, x_J, y_1, \ldots, y_K), X]) \subset [(x_{j_1}, \ldots, x_{j_n}, \ldots, x_{j'}, y_{k_1}), X].$$

Now consider the asymptotic game for $(1+\epsilon_{n-1})$ and asymptotic ℓ_p^2 or c_0^2. Repeating the previous procedure for such ℓ_p or c_0 we extend the finite sequence $(x_1, \ldots, x_J, y_1, \ldots, y_K)$ by suitable block sequences $(1+\epsilon_{n-1})$-realizing ℓ_p^2 or c_0^2 in X. In this way we obtain finite block sequences

$$b = (x_1, \ldots, x_J, y_1, \ldots y_K, v_1, \ldots v_L, z_1, \ldots, z_M)$$

and

$$c = (x_{j_1}, \dots, x_{j_n}, \dots, x_{j'}, y_{k_1}, \dots, y_{k'}, v_{l_1}, \dots, v_{l'}, z_{m_1})$$

with $(x_{j_n}, y_{k_1}) \sim^{1+\epsilon_{n-1}} (a_1, a_2)$, $(v_{l_1}, z_{m_1}) \sim^{1+\epsilon_{n-1}} \ell_p$ or c_0, and $\phi([b, X]) \subset [c, X]$.

By definition of ϕ, $(x_{j_n}, y_{k_1}) \sim^{1+\epsilon_{n-1}} (v_{l_1}, z_{m_1})$, Hence, by the choice of n, we have $(a_1, a_2) \sim^{1+\delta} \ell_p^2$ or c_0^2. Since δ was arbitrary small, (a_1, a_2) is 1-equivalent to the unit vector basis of ℓ_p^2 or c_0^2.

Now by a standard procedure we produce a block subspace X_0 of X such that for any $\delta > 0$ there is a N_δ such that for any $N_\delta < x_1 < x_2$, $x_1, x_2 \in X_0$ we have $(x_1, x_2) \sim^{(1+\delta)} \ell_p^2$ or c_0^2, and then produce an almost isometric copy of ℓ_p or c_0 in the space (cf. e.g. [9], [8]), which finishes the proof. □

Remark Notice that by this proposition in ℓ_p, for $1 < p < \infty$, endowed with a distorting norm one cannot pick sequences producing spreading models (in the sense defined above) in a continuous way, however any block sequence is subsymmetric.

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