AMALGAMATION AND RAMSEY PROPERTIES OF L_p SPACES

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Abstract. We study the dynamics of the group of isometries of L_p-spaces. In particular, we study the canonical actions of these groups on the space of δ-isometric embeddings of finite dimensional subspaces of $L_p(0,1)$ into itself, and we show that for $p \neq 4, 6, 8, \dots$ they are ε-transitive provided that δ is small enough. We achieve this by extending the classical equimeasurability principle of Plotkin and Rudin. We define the central notion of a Fraïssé Banach space which underlies these results and of which the known separable examples are the spaces $L_p(0,1)$, $p \neq 4, 6, 8, \dots$ and the Gurarij space. We also give a proof of the Ramsey property of the classes $\{\ell_p^n\}_n$, $p \neq 2, \infty$, viewing it as a multidimensional Borsuk-Ulam statement. We relate this to an arithmetic version of the Dual Ramsey Theorem of Graham and Rothschild as well as to the notion of a spreading vector of Matoušek and Rödl. Finally, we give a version of the Kechris-Pestov-Todorcevic correspondence that links the dynamics of the group of isometries of an approximately ultrahomogeneous space X with a Ramsey property of the collection of finite dimensional subspaces of X.

1. Introduction

It is a classical result of A. Pełczyński and S. Rolewicz [PelRol] that the spaces $L_p(0,1)$ are almost transitive, in the sense that the group of linear isometric surjections $\mathrm{Iso}(L_p(0,1))$ acts almost transitively on the corresponding unit sphere of $L_p(0,1)$. This was later extended by W. Lusky [Lu2] who proved that in fact, the group $\mathrm{Iso}(L_p(0,1))$ also acts almost transitively on each metric space $\mathrm{Emb}(X, L_p(0,1))$ of linear isometric embeddings from a finite dimensional subspace X of $L_p(0,1)$ into $L_p(0,1)$, but only provided that $p = 2$ or $p \notin 2\mathbb{N}$. Other Banach spaces having this property are any Hilbert space or the *Gurarij space*, and recently quasi-Banach spaces with the corresponding property have been found in [CaGaKu]. This almost "ultra" transitive' property is the metric analogue of the so-called ultrahomogeneity property of algebraic structures, the core of Fraïssé theory in model theory, and the proper context for the combinatorial characterization of the *extreme amenability* of the corresponding automorphism group, known as the *Kechris-Pestov-Todorcevic (KPT) correspondence* [KePeTo]. Recall that a topological group is extremely amenable when each of its continuous actions on a compact space has a fixed point, and that the KPT correspondence states that for ultrahomogeneous structures $\mathcal{M}$ the extreme amenability of its automorphism group $\mathrm{Aut}(\mathcal{M})$ is exactly the *Ramsey property* of its class of finitely generated substructures, called the *age* and denoted by $\mathrm{Age}(\mathcal{M})$. By this means, many new examples of extremely amenable

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groups have been given (see [KePeTo]). The theory of abstract ultrahomogeneous metric structures has been studied in [BYBeHeUs], while the KPT correspondence has been recently extended to this context by J. Melleray and T. Tsankov [MeTsa]. The KPT correspondence for metric structures was for the first time used in [BaLALuMbo1], [BaLALuMbo2] in showing, for example, that the isometry group of the Gurarij space is extremely amenable by supplying an appropriate Ramsey type result which relies on the Graham-Rothschild Theorem. We note that Gurarij space shares this this property with the infinite dimensional Hilbert spaces and the spaces $L_p(0,1)$, proved by M. Gromov and V. D. Milman [GrMi], and by T. Giordano and V. Pestov [GiPe] respectively, relying on the method of concentration of measure.

It follows from the Banach-Lamperti description of isometries of L_p spaces that the isometry groups of the spaces $L_p(0,1)$ and $L_q(0,1)$, when $1 \le p, q \neq 2 < \infty$, are topologically isomorphic. However, there are canonical actions of the same nature that have very different properties, depending on p: while for $p \notin 2\mathbb{N}$ all the canonical actions by composition $\mathrm{Iso}(L_p(0,1)) \curvearrowright \mathrm{Emb}(X, L_p(0,1))$ are almost transitive, it follows from a work of B. Randrianantoanina [Ran] based on an early result of H. P. Rosenthal [Ros], that there are finite dimensional subspaces X of $L_{2n}(0,1)$, $n \in \mathbb{N}$, $n > 1$, for which that action is far of being almost transitive, because X has well complemented and badly complemented copies on L_{2n} (see Proposition 2.10). One of the main goals of this paper is to study the canonical actions of isometry groups of the Lebesgue spaces, not only on the spaces of isometric embeddings, but also on $\mathrm{Emb}_\delta(X, L_p(0,1))$ the class of δ-isometric embeddings from X into $L_p(0,1)$. While δ-isometric embeddings were already considered by M. Lupini [Lup], in a general theory of stability including, for example, operator spaces and systems, in this paper one of our objectives is to obtain finer results based on weaker (and/or more precise) properties of homogeneity for structures, in such a way that L_p spaces are included in the classes we consider. With these examples in mind, we concentrate on the case of the Banach spaces, and develop a theory which may be specific to the Banach space setting. In particular, although our results should be extendable to the quasi-Banach setting and the case of L_p spaces for $0 < p < 1$, we shall not consider that situation.

We say that a Banach space E is *Fraïssé* when for every dimension k and $\varepsilon > 0$ there is $\delta > 0$ such that the canonical action by composition $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_\delta(X, E)$ is ε-transitive for every $X \in \mathrm{Age}_k(E)$, the collection of k-dimensional subspaces of E. We will see that $L_p(0,1)$ is Fraïssé, provided that p is not even. Other Fraïssé spaces are any Hilbert space and the Gurarij space. The interest of these $\varepsilon - \delta$ continuity properties can be appreciated by the following: First of all, they imply that for Fraïssé spaces E the Banach-Mazur and a restricted version of the *Kadets* pseudometrics are uniformly equivalent on $\mathrm{Age}(E)$; secondly, there is a characterization of the Fraïssé Banach property by passing to the ultrapower and involving homogeneity on isometric (instead of δ-isometric) embeddings in the ultrapower; thirdly, Fraïssé spaces are isometrically determined by the collection of their finite dimensional subspaces and there is a Fraïssé correspondence; finally, the spaces who are finitely representable on a Fraïssé space E can be isometrically embedded into E, and, consequently, the Hilbert space ℓ_2 is the minimal Fraïssé space and $\mathbb{G}$ is the unique Fraïssé space with trivial cotype.

The Gurarij space $\mathbb{G}$ is, by definition, an abstract inductive limit of ℓ_∞^n's. The proof of the Fraïssé property of $\mathbb{G}$ follows from a combination of the existence of general *pushouts* of finite dimensional spaces, and the fact that δ-isometric embeddings are in some precise sense 2δ-close to isometric embeddings. For the class $\mathrm{Age}(L_p(0,1))$ there is not known full pushout; instead, for $p \notin 2\mathbb{N}$ there is a restricted version stating that for every $k \in \mathbb{N}$ and $\varepsilon > 0$ there is $\delta > 0$ such that if $X, Y, Z \in \mathrm{Age}(L_p(0,1))$ with $\dim X = k$, and $\gamma \in \mathrm{Emb}_\delta(X, Y)$ and $\eta \in \mathrm{Emb}_\delta(X, Z)$, then there are $V \in \mathrm{Age}(L_p(0,1))$ and isometric embeddings $i : Y \to V$ and $j : Z \to V$ such that $\|i \circ \gamma - j \circ \eta\| \le \varepsilon$. This is exactly, by means of the Fraïssé correspondence (see Corollary 2.26), the Fraïssé property of $L_p(0,1)$. We prove this by establishing the *approximate equimeasurability principle*, the continuous statement extending the classical equimeasurability principle of Plotkin and Rudin: Suppose that μ, ν are Borel measures on $\mathbb{R}^n$ for which the coordinate functions x_j are p-integrable. Then $\widehat{\mu}^{(p)}(a) = \int |1 + \sum_{j<n} a_j x_j|^p d\mu(x) = \int |1 + \sum_{j<n} a_j x_j|^p d\nu(x) = \widehat{\nu}^{(p)}(a)$ for all $a = (a_j)_{j<n} \in \mathbb{R}^n$ then the measures μ and ν are equal. We prove that μ and ν are close, for example with respect to the Lévy-Prohorov metric, when the corresponding characteristics are close, and moreover we obtain a full characterization.

We study the approximate Ramsey property (ARP) of the classes $\{\ell_p^n\}_n$, that also can be seen as a version of a multidimensional Borsuk-Ulam Theorem (see §§§5.1.1). In general, a class of finite dimensional $\mathcal{G}$ has the (ARP) when for every $X, Y \in \mathcal{G}$ and every $\varepsilon > 0$ there is $Z \in \mathcal{G}$ such that for every 1-Lipschitz mapping $c : \mathrm{Emb}(X, Z) \to [0,1]$ there is some isometric embedding $\gamma : Y \to Z$ such that the oscillation of c in the set of compositions $\gamma \circ \mathrm{Emb}(X, Y)$ is at most ε. It is interesting to mention that, while the proof of the (ARP) of $\{\ell_\infty^n\}_n$ uses the dual Ramsey Theorem (DR) of Graham and Rothschild (see [BaLALuMbo1]), our proof of the (ARP) of $\{\ell_p^n\}_n$ $p \neq 2, \infty$ utilize an arithmetical version of (DR), namely, that for partitions of equal sized pieces.

We also analyze restricted versions of the previous notions: Given a class $\mathcal{G}$ of finite dimensional Banach spaces, we introduce what we call $\mathcal{G}$-Fraïssé spaces, those for which the natural actions on δ-embeddings are ε-transitive, provided that the embeddings have as domain an element of $\mathcal{G}$. In this way, every $L_p(0,1)$, being p even or not, is the *Fraïssé limit* of $\{\ell_p^n\}_n$. We also restrict the type of embeddings we are interested in, for example by analyzing Fraïssé lattices, where now isometries and embeddings must respect the lattice structure. We find the first Fraïssé Banach lattice, an M-space, denoted by $\mathbb{G}^\diamond$, that is the lattice version of the Gurarij space, and that has an extremely amenable group of lattice isometries, proved using a KPT correspondence for Banach lattices.

The paper is organized as follows. In §2 we introduce and study Fraïssé Banach spaces, as well as the local versions of them, meaning that the canonical actions of the rotations are restricted to embeddings defined on spaces a fixed family. For those spaces, we see the uniform equivalence of the Banach-Mazur and the Kadets pseudometrics. In §§2.1 we prove the Fraïssé correspondence for Banach spaces, including its local version, and in §§2.2 we characterize the Fraïssé property of a Banach space in terms of a uniform equivalence of metrics. In §3 we introduce the lattice versions, including the proof of the fact that every $L_p(0,1)$ is lattice Fraïssé, a fact that follows from an approximation result by G. Schechtman [Sch] on δ-isometric embeddings defined on ℓ_p^n. Section 4 is devoted to the proof of the approximate equimeasurability principle for $p \notin 2\mathbb{N}$, in §§4.1 we see how this is used to show that those $L_p(0,1)$ are Fraïssé. The proof the principle is given in §§4.2. Section 5 is devoted to the approximate Ramsey property, in particular of the class $\{\ell_p^n\}_n$, and its reformulation à la Borsuk-Ulam is given in §§§5.1.1, while its proof and the relation with an approximate Ramsey statement form equisurjections is the content of §§5.2. The last Section 6 is dedicated to the existence of a Fraïssé M-space whose group of lattice isometries is extremely amenable.

2. Fraïssé Banach spaces

We consider spaces over $\mathbb{F} = \mathbb{R}, \mathbb{C}$; given $n \in \mathbb{N}$, the *unit basis of* $\mathbb{F}^n$, denoted by $(u_j^{(n)})_{j<n}$, or simply $(u_j)_{j<n}$, is the sequence that for each j, the k^{th} coordinate of $u_j^{(n)}$ is delta of Dirac $\delta_{j,k}$. When needed, will use the set theoretical convention of identifying an integer n with $\{0, 1, \dots, n-1\}$. Given Banach spaces X and Y and $\delta \geq 0$, a *δ-isometric embedding (or δ-isometry)* $T : X \to Y$ is a linear map such that

for all $x \in X$ one has that
$$\frac{1}{1+\delta}\|x\| \le \|Tx\| \le (1+\delta)\|x\|.$$
When $\delta = 0$ we will simply use isometric embeddings to refer to 0-isometric embeddings. Let $\mathrm{Emb}_\delta(X,Y)$ be the collection of δ-isometric embeddings between X and Y, and let $\mathrm{Iso}(X)$ be the group of isometries on X. Given two families $\mathcal{H}$ and $\mathcal{G}$ of finite dimensional spaces we write $\mathcal{H} \preceq \mathcal{G}$ when for every $X \in \mathcal{H}$ there is $Y \in \mathcal{G}$ that is isometric to X; then $\mathcal{H} \equiv \mathcal{G}$ denotes that $\mathcal{H} \preceq \mathcal{G} \preceq \mathcal{H}$, and $\mathcal{H}_\equiv$ is the class of all finite dimensional spaces with an isometric copy in $\mathcal{H}$.

Definition 2.1. Given a Banach space E, let $\mathrm{Age}(E)$ be the class of all finite dimensional subspaces of E. Following standard convention (see for example [Ho, pp 324]) we will say that X and Y have the *same age* when $\mathrm{Age}(X) \equiv \mathrm{Age}(Y)$. Given a family $\mathcal{H}$ of finite dimensional spaces, let $\mathcal{H}_n$ be the subfamily of $\mathcal{H}$ consisting of those spaces of dimension n. In particular, we write $\mathrm{Age}_n(X)$ to denote $(\mathrm{Age}(X))_n$. Given a class of finite dimensional Banach spaces $\mathcal{H}$ and a Banach space E, let $\mathcal{H}_E$ be the collection of subspaces of E isometric to some element in $\mathcal{H}$.

Recall the *gap or opening* metric on $\mathrm{Age}_n(E)$ is defined by
$$\Lambda_E(X,Y) := \max\left\{\max_{x\in B_X}\min_{y\in B_Y}\|x-y\|_E, \max_{y\in B_Y}\min_{x\in B_X}\|x-y\|_E\right\};$$
in other words, $\Lambda_E(X,Y)$ is the $\|\cdot\|_E$-Hausdorff distance between the unit balls of X and Y. This induces the following *Gromov-Hausdorff* function, *E-Kadets* on $\mathrm{Age}_n(E)^2$, defined as
$$\gamma_E(X,Y) := \inf\{\Lambda_E(X_0,Y_0) \,:\, X_0, Y_0 \in \mathrm{Age}_n(E),\, X_0 \equiv X,\, Y_0 \equiv Y\}.$$
When E is universal γ_E is the original *Kadets pseudometric* (see [Kad], [KaOs]), although in general γ_E may not be a pseudometric. We will see that in other natural cases, γ_E satisfies the triangle inequality. It is easy to see that $\gamma_E(X,Y) = 0$ if and only if X and Y are isometric. There is another well-known pseudometric with this property. This is the *Banach-Mazur* pseudometric on $\mathrm{Age}_n(E)$:
$$d_{\mathrm{BM}}(X,Y) := \log(\inf_{T:X\to Y}\|T\|\cdot\|T^{-1}\|)$$
where the infimum runs over all isomorphisms $T : X \to Y$. It is well-known that d_{BM} defines a pre-compact topology on $\mathrm{Age}_n(E)$; that is, every sequence in $\mathrm{Age}_n(E)$ has a d_{BM}-convergent subsequence, not necessarily to an element of $\mathrm{Age}_n(E)$.

Definition 2.2. Let E be an infinite dimensional Banach space, and let $\mathcal{G} \preceq \mathrm{Age}(E)$.

(a) E is *$\mathcal{G}$-homogeneous* ($\mathcal{G}$−H) when for every $X \in \mathcal{G}$ and every and every $\gamma, \eta \in \mathrm{Emb}(X,E)$ there is some $g \in \mathrm{Iso}(E)$ such that $g \circ \gamma = \eta$; in other words, when for each $X \in \mathcal{G}$, the natural action $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}(X,E)$ by composition is transitive.
(b) E is is called *approximately $\mathcal{G}$-homogeneous* (A$\mathcal{G}$H) when for every $X \in \mathcal{G}$ and every $\varepsilon > 0$ the natural action by composition $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}(X,E)$ is ε-transitive, that is, whenever $\gamma, \eta \in \mathrm{Emb}(X,E)$ there is $g \in \mathrm{Iso}(E)$ such that $\|g \circ \gamma - \eta\| < \varepsilon$.
(c) E is is called *weak $\mathcal{G}$-Fraïssé* when for every $X \in \mathcal{G}$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that the action $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_\delta(X,E)$ is ε-transitive.
(d) E is *$\mathcal{G}$-Fraïssé* when for every dimension $k \in \mathbb{N}$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that the action $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_\delta(X,E)$ is ε-transitive for every $X \in \mathcal{G}_k$.

When $\mathcal{G} = \mathrm{Age}(E)$, then we will use *ultrahomogeneous* (uH), *approximately ultrahomogeneous* (AuH), *weak-Fraïssé* and *Fraïssé* for the corresponding $\mathcal{G}$-homogeneities. The particular case of Fraïssé property with modulus independent of the dimension was studied in [Lup] and it was named *stable* Fraïssé property.

We will say that a mapping $\varpi : S \times [0,\infty[\to [0,\infty[$, where S is an arbitrary set, is a *modulus* when $\varpi(p,\cdot)$ is increasing and continuous at zero with value zero for every $s \in S$. The following is easy to prove.

Proposition 2.3. *1) E is weak-Fraïssé if and only if for every there exists a modulus* $\varpi_{\mathcal{G},E} : \mathcal{G}\times[0,\infty[\to [0,\infty[$*, called modulus of stability of* $\mathcal{G}$ *in* E*, such that for every* $X \in \mathcal{G}$ *every* $\delta \geq 0$ *and every* $\varepsilon > 0$ *one has that* $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_\delta(X,E)$ *is* $\varpi(X,\delta)+\varepsilon$-transitive.

In this case $\varpi_{\mathcal{G},E} : \mathcal{G}\times[0,\infty[\to[0,\infty[$ *defined for* $X \in \mathcal{G}$ *and* $\delta \geq 0$ *as* $\varpi_{X,E}(\delta) = \inf\{\varepsilon > 0 : \mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_\delta(X,E)$ *is* ε*-transitive*$\}$ *is the optimal modulus of stability of* $\mathcal{G}$ *in* E.

2) E *is* $\mathcal{G}$*-Fraïssé if and only if* E *is weak* $\mathcal{G}$*-Fraïssé with a modulus that does not depend on each* $X \in \mathcal{G}$ *but in its dimension. In this case,* $\varpi_{\mathcal{G},E} : \mathbb{N}\times[0,\infty[\to[0,\infty[$, $\varpi_{\mathcal{G},E}(k,\delta) = \inf\{\varepsilon > 0 : \mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_\delta(X,E)$ *is* ε*-transitive for all* $X \in \mathcal{G}_k\}$ *is the optimal modulus of stability of* $\mathcal{G}$ *in* E.

3) If $\varpi : S\times[0,\infty[\to[0,\infty[$ *is a modulus, then*

$$\varpi^*(s,\delta) := \inf_{\delta'>\delta}\varpi(s,\delta')$$

is a continuous modulus such that $\varpi(s,\delta) \leq \varpi^*(s,\delta)$. □

Example 2.4. *A Hilbert space is* (uH) *and also is a Fraïssé Banach space with modulus* $\varpi(k,\delta) = 2\delta$.

Proof. Suppose that $\mathcal{H}$ is a Hilbert space. Clearly, a subspace of $\mathcal{H}$ is again Hilbert, so, the Banach-Mazur limit of subspaces of $\mathcal{H}$ is again (mod isometry) a subspace of $\mathcal{H}$. Suppose now that $\gamma : F \to \mathcal{H}$ is a δ-isometry, with $F \subseteq \mathcal{H}$ finite dimensional. Choose an orthonormal basis $(x_j)_{j<n}$ of F, and such that $(\gamma x_j)_{j<n}$ is an orthogonal sequence. Let $\iota : F \to \mathcal{H}$ be the isometric embedding linearly defined by $\iota(x_j) := \gamma(x_j)/\|\gamma(x_j)\|$ for all $j < n$. Then given scalars $(a_j)_{j<n}$ we have that

$$\begin{aligned}\|\sum_j a_j\gamma x_j - \sum_j a_j\iota x_j\|^2 =&\langle\sum_j a_j\left(1-\frac{1}{\|\gamma x_j\|}\right)\gamma x_j, \sum_j a_j\left(1-\frac{1}{\|\gamma x_j\|}\right)\gamma x_j\rangle = \\ =&\sum_j |a_j|^2|\|\gamma x_j\|-1|^2 \leq \delta^2\|\sum_j a_j x_j\|^2.\end{aligned}$$

So, if we extend ι to an isometry $I \in \mathrm{Iso}(\mathcal{H})$, then $\|\gamma - I\restriction F\| \leq \delta$, as desired. □

Example 2.5. *The Gurarij space* $\mathbb{G}$ [Gu] *is Fraïssé with modulus* $\varpi(k,\delta) = \delta$ *but not* (uH).

Proof. It was already known by Gurarij that $\mathbb{G}$ is not even transitive: It is well known that an isometry moves a point of differentiability of the unit sphere to a point of differentiability. Since $\mathbb{G}$ is universal, its unit sphere has points of differentiability and points of non-differentiability. On the other direction, $\mathbb{G}$ is Fraïssé with modulus $\varpi(k,\delta) = \delta$ (see [Lup, §§6.1]). □

Problem 2.6. *Are* $\mathbb{G}$ *and the Hilbert space* $\mathcal{H}$ *the only separable stable Fraïssé Banach spaces?*

Example 2.7. *For every* $1 \leq p < \infty$ *the space* $L_p[0,1]$ *is* $\{\ell_p^n\}_n$*-Fraïssé (see Proposition 3.7). In fact,* $L_p[0,1]$ *is the* Fraïssé limit *of* $\{\ell_p^n\}_n$ *(see Theorem 2.25).*

Example 2.8. *W. Lusky* [Lu2]*, using the equimeasurability theorem of A. I. Plotkin* [Plo1] *and W. Rudin* [Ru]*, proved that for* $p \notin 2\mathbb{N}$ *the space* $L_p(0,1)$ *is (AuH)*[1]*. On the other hand, the isometry group never acts transitively on the unit sphere of* $L_p(0,1)$ *if* $p \neq 2$*. One of our main results will be to show that in fact for* $p \notin 2\mathbb{N}$ *the space* $L_p(0,1)$ *is Fraïssé (Theorem 4.1).*

Problem 2.9. *Are* $\mathbb{G}$*,* $L_p(0,1)$*,* $p \neq 4,6,8,\dots$ *the only separable Fraïssé Banach spaces?*

[1]In that paper Lusky states that for every $X \in \mathrm{Age}(E)$, $\varepsilon > 0$ and $\gamma \in \mathrm{Emb}(X,L_p)$ there is a surjective $1+\varepsilon$-isomorphism of $L_p(0,1)$ extending γ, but its proof directly gives the (AuH) of $L_p(0,1)$ for those p's

It is well-known there are other almost transitive Banach spaces; for example $E := L_p(X)$ for any almost transitive Banach space X. In particular, $L_p(\mathbb{G})$ could be (AuH) or even Fraïssé. However this is not so because there are well-complemented and not well complemented isometric copies of ℓ_p^n, hence the corresponding Bochner spaces cannot be (AuH). A similar reasoning holds for $L_p(L_q)$, at least when $1 \le p, q < 2$ and $p \ne q$. Similarly, although much more complicated, we will see in next Proposition 2.10 that the spaces $L_p(0,1)$ for $p = 4, 6, 8, \dots$ cannot be (AuH). This fact was already proved by Lusky by using a counterexample of Rudin in [Ru] exposing the non-equimeasurability theorem for those p's.

Proposition 2.10. *Assume $p \in 2\mathbb{N}$, $p \ge 4$. For any $C \ge 1$ and $\delta \ge 0$, there are isometric $E, F \in$ $\mathrm{Age}(L_p(0,1))$ such that for any bounded linear mapping $T : L_p(0,1) \to L_p(0,1)$, if $T \upharpoonright E \in \mathrm{Emb}_\delta(E,F)$, then $\|T\| \ge C$.*

Proof. It is proved by B. Randrianantoanina [Ran] that for $p \in 2\mathbb{N}$, $p > 2$, the uncomplemented subspace Y_p of L_p built by H. Rosenthal in [Ros] is isometric to a certain complemented subspace Z_p of L_p spanned by 3-valued independent symmetric random variables. Since the space Z_p is the span of a sequence of independent mean zero random variables, it has an unconditional basis, [Ros, Remark 2, page 278]. Let (e_n) (resp. (f_n)) be the associated basis of Y_p (resp. Z_p), and $E_n = \langle e_j \rangle_{j<n}$, $F_n = \langle f_j \rangle_{j<n}$. On the one hand the F_n's are uniformly complemented in Z_p and therefore in $L_p(0,1)$; on the other hand the E_n's are not, otherwise by reflexivity and a weak limit argument, Y_p would be. In other words, there are projections Q_n onto F_n with uniform bound c, but $C_n := \inf\{C : E_n \text{ is } C\text{−complemented in } L_p(0,1)\}$ tends to $+\infty$. For any extension T on $L_p(0,1)$ of a δ-isometric map t between E_n and F_n, we have that $t^{-1}Q_nT$ is a projection onto E_n. Since $\|t^{-1}Q_nT\| \ge C_n$, it follows that $\|T\| \ge c^{-1}C_n/(1+\delta)$, which tends to $+\infty$. □

The terminology "homogeneous" is commonly used in classical and metric model theory. In Banach space theory, it has to be related to the concept of "disposition", for example used by V. I. Gurarij in [Gu] to define his space. A Banach space E is of *approximate disposition* when for every $X, Y \in \mathrm{Age}(E)$, $\delta > 0$, $\iota \in \mathrm{Emb}(X,Y)$ and every $\gamma \in \mathrm{Emb}(X,E)$ there is $\eta \in \mathrm{Emb}_\delta(Y,E)$ such that $\eta \circ \iota = \gamma$. It is easy to see that if E is (AuH) then it is of approximate disposition.

We have the following characterization of $\mathcal{G}$-Fraïssé Banach spaces. Note that when $\mathcal{G} \preceq \mathrm{Age}(E)$ the E-Kadets function γ_E is well defined on $\mathcal{G}$.

Theorem 2.11. *The following are equivalent for a Banach space E and $\mathcal{G} \preceq \mathrm{Age}(E)$.*

1) E is $\mathcal{G}$-Fraïssé and γ_E is a complete pseudometric on $\mathcal{G}$.

2) E is weak $\mathcal{G}$-Fraïssé and γ_E is a complete pseudometric that is uniformly equivalent to d_{BM} on $\mathcal{G}_k$ for every k.

3) E is weak $\mathcal{G}$-Fraïssé and $\mathcal{G}$ is d_{BM}-compact.

Consequently, E is Fraïssé if and only if it is weak-Fraïssé and $\mathrm{Age}(E)$ is d_{BM}-compact.

It follows from this that the Hilbert and the Gurarij spaces are very special Fraïssé spaces: Recall that a Banach space Y is *finitely representable* in X if $\mathrm{Age}_k(Y)$ is included in the d_{BM}-closure $\overline{\mathrm{Age}_k(X)}^{\mathrm{BM}}$ of $\mathrm{Age}_k(X)$ for every k.

Proposition 2.12. *Let E be a Fraïssé Banach space. The following are equivalent for a separable Banach space Y.*

1) X is finitely representable on E.

2) X can be isometrically embedded into E.

Consequently,

3) ℓ_2 *is the minimal separable Fraïssé Banach space,*

4) $\mathbb{G}$ *is the only separable Fraïssé Banach space with trivial cotype, and*

5) $\mathbb{G}$ *is the maximal separable Fraïssé Banach space.*

PROOF. *1)* implies *2)*: Suppose that Y is finitely representable in E. Let $(y_j)_{j\in\mathbb{N}}$ be a dense set of Y, and for each n, let $Y_n := \langle y_j\rangle_{j<n}$. Since Y is finitely representable and $\mathrm{Age}(E)$ is Banach-Mazur compact, we can find for each n some $\gamma_n \in \mathrm{Emb}(Y_n, E)$. By the (AuH) of E, for each $n \ge 1$ we can find $g_n \in \mathrm{Iso}(E)$ such that $\|g_n \circ \gamma_n \restriction Y_{n-1} - \gamma_{n-1}\| \le 1/2^n$. Let $\eta_0 := \gamma_0$, $\eta_n := g_1 \circ \cdots \circ g_n \circ \gamma_n$ for $n \ge 1$. It follows that $\|\eta_n \restriction Y_{n-1} - \eta_{n-1}\| \le 1/2^n$, $(\eta_l \restriction Y_n)_{l\ge n}$ is a Cauchy sequence with respect to the norm metric. Since this is a complete metric, we can define $\eta : \langle y_j\rangle_{j\in\mathbb{N}} \to E$, $\eta(y) := \lim_{l\ge n} \eta_l(y)$ if $y \in Y_n$. Clearly this is an isometric embedding that extends to Y.

3): By Dvoretzky's Theorem ℓ_2 is finitely representable in E, so ℓ_2 isometrically embeds into E if E is Fraïssé. 4): A classical result by B. Maurey and G. Pisier [MaPi] states that E has trivial cotype if and only E contains all ℓ_∞^n's uniformly; that is, there is some $C \ge 1$ such that $d_{\mathrm{BM}}(\ell_\infty^n, \mathrm{Age}_n(E)) \le C$ for all n. By the finite version of James theorem, $\ell_\infty^n \in \overline{\mathrm{Age}_n(E)}^{\mathrm{BM}}$, so it follows that $\ell_\infty^n \in \mathrm{Age}(E)$ for every n. Then E is universal for separable spaces. Since $\mathbb{G}$ is the unique, up to isometry, universal separable (AuH) (see [KuSo] or Proposition 2.21), E and $\mathbb{G}$ are isometric. □

For the proof of Theorem 2.11 we will use the next.

Proposition 2.13. *Let E be a Banach space, and suppose that $\mathcal{G} \preceq \mathrm{Age}(E)$.*

1) If E is (A$\mathcal{G}$H), then γ_E is defined on $\mathcal{G}_E$ by the formula $\gamma_E(X,Y) = \inf_{g\in\mathrm{Iso}(E)} \Lambda_E(gX, Y)$. Consequently, γ_E is a pseudometric on each $\mathcal{G}_k$.

2) In general, $d_{\mathrm{BM}}(X,Y) \le 4k\Lambda_E(X,Y)$ for every $X, Y \in \mathrm{Age}_k(E)$ such that $\Lambda_E(X,Y) \le 1/(2k)$, and consequently, if E is (A$\mathcal{G}$H), the identity $(\mathcal{G}_k, \gamma_E) \to (\mathcal{G}_k, d_{\mathrm{BM}})$ is uniformly continuous for each k.

3) If E is weak $\mathcal{G}$-Fraïssé, then γ_E is a pseudometric on each $\mathcal{G}_k$ topologically equivalent to d_{BM}.

PROOF. *1):* We use the following.

Claim 2.13.1. For $\gamma, \eta \in \mathrm{Emb}_\delta(X, E)$ one has that $\Lambda_E(\gamma X, \eta X) \le 2(1+\delta)\|\gamma - \eta\|$.

Proof of Claim: Given $\gamma x \in X$ of norm one,

$$\begin{aligned}\|\gamma x - \frac{\eta x}{\|\eta x\|}\| \le &\|\gamma x - \eta x\| + \|\eta x - \frac{\eta x}{\|\eta x\|}\| = \|\gamma x - \eta x\| + |\|\eta x\| - 1| = \|\gamma x - \eta x\| + |\|\eta x\| - \|\gamma x\|| \le \\ \le &2\|\gamma x - \eta x\| \le 2(1+\delta)\|\gamma - \eta\|.\end{aligned}$$

□

Fix $X, Y \in \mathcal{G}_E$ with $\dim X = \dim Y$. Fix also isometric embeddings $\gamma : X \to E$, $\eta : Y \to E$ and $\varepsilon > 0$, let $g, h \in \mathrm{Iso}(E)$ be such that $\|g \restriction X - \gamma\|, \|h \restriction Y - \eta\| \le \varepsilon$. Since Λ_E is invariant for the natural action of $\mathrm{Iso}(E)$ on $\mathrm{Age}(E)$,

$$\begin{aligned}\Lambda_E(h^{-1}gX, Y) = &\Lambda_E(gX, hY) \le \Lambda_E(gX, \gamma X) + \Lambda_E(hY, \eta Y) + \Lambda_E(\gamma X, \eta Y) \le \\ \le &2\|g \restriction X - \gamma\| + 2\|h \restriction Y - \eta\| + \Lambda_E(\gamma X, \eta Y) \le \Lambda_E(\gamma X, \eta Y) + 4\varepsilon.\end{aligned}$$

2): Let $(x_j)_{j<k}$ be an *Auerbach* basis of X, that is, $(x_j)_j$ is normalized and $\max_j |a_j| \le \|\sum_j a_j x_j\|$ for every $(a_j)_j$. Suppose that $\Lambda_E(X,Y) \le 1/(2k)$. For each $j < k$, let $y_j \in B_Y$ be such that $\|x_j - y_j\| \le \Lambda_E(X,Y)$. Given $(a_j)_j$, we have that $\|\sum_j a_j y_j\| \le \|\sum_j a_j x_j\| + \max_j |a_j| \sum_j \|x_j - y_j\| \le (1 + k\Lambda_E(X,Y))\|\sum_j a_j x_j\|$, and similarly, $\|\sum_j a_j y_j\| \ge (1 - k\Lambda_E(X,Y))\|\sum_j a_j x_j\|$. Since $\Lambda_E(X,Y) < 1/k$, it follows that $\theta : X \to Y$ defined linearly by $x_j \mapsto y_j$ is an isomorphism between X and Y, and since $\Lambda_E(X,Y) \le 1/(2k)$ we obtain that $(1 - k\Lambda_E(X,Y)) \ge 1/(1+2k\Lambda_E(X,Y))$. This implies that $\|\theta\|, \|\theta^{-1}\| \le 1 + 2k\Lambda_E(X,Y)$, and so

$$d_{\mathrm{BM}}(X,Y) \le \log(\|\theta\| \cdot \|\theta^{-1}\|) \le \log((1+2k\Lambda_E(X,Y))^2) \le 4k\Lambda_E(X,Y).$$

It follows from this inequality and the fact that d_{BM} is $\mathrm{Iso}(E)$-invariant that $\mathrm{Id} : (\mathcal{G}_k, \gamma_E) \to (\mathcal{G}_k, d_{\mathrm{BM}})$ is uniformly continuous for each k.

3): For suppose that E is weak $\mathcal{G}$-Fraïssé. To simplify the notation, we may assume that $\mathcal{G} \subseteq \mathrm{Age}(E)$. Suppose that $(X_n)_n$ in $\mathcal{G}_k$ d_{BM}-converges to $X \in \mathcal{G}_k$. Fix $\varepsilon > 0$, and let $0 < \delta \le 1$ be such that $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_\delta(X,E)$ is $\varepsilon/4$-transitive. Let n_0 be such that $d_{\mathrm{BM}}(X,X_n) \le \log(1+\delta)$, and for each $n \ge n_0$ choose $\theta_n : X \to X_n$ so that $\|\theta_n\| \cdot \|\theta_n\|^{-1} \le 1+\delta$. Observe that $\theta_n \in \mathrm{Emb}_\delta(X,X_n)$. It follows that, for such n's, there is $g_n \in \mathrm{Iso}(E)$ such that $\|\theta_n - g_n \upharpoonright X\| \le \varepsilon/4$, and by Claim 2.13.1, $\gamma_E(X,X_n) \le \Lambda_E(X_n, g_n(X)) \le 2(1+\delta)\varepsilon/4 \le \varepsilon$. □

Proof of Theorem 2.11. Obviously, the statement we have to prove is equivalent to the corresponding one for any other family $\mathcal{H}$ such that $\mathcal{H} \equiv \mathcal{G}$, so for convenience we assume that $\mathcal{G} = \mathcal{G}_E$ and that it is Λ_E-closed in $\mathrm{Age}(E)$.

1) $\implies$ *2)*: We already know by Proposition 2.13 *2)* that $\mathrm{Id} : (\mathcal{G}_k, \gamma_E) \to (\mathcal{G}_k, d_{\mathrm{BM}})$ is uniformly continuous for each k. The other part readily follows from the next.

Claim 2.13.2. Suppose that $\varepsilon > 0$ and $0 < \delta \le 1$. If $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_\delta(X,E)$ is ε-transitive, then $d_{\mathrm{BM}}(X,Y) \le \delta/2$ implies that $\gamma_E(X,Y) \le 4\varepsilon$ for every $Y \in \mathrm{Age}(E)$.

Proof of Claim: We assume the hypotheses, and suppose that $d_{\mathrm{BM}}(X,Y) \le \delta/2$. Let $\theta : X \to Y$ be an isomorphism such that $\|\theta\|\|\theta^{-1}\| \le e^{\delta/2} \le 1+\delta$. Hence, there is some $g \in \mathrm{Iso}(E)$ such that $\|g \upharpoonright X - \theta\| \le \varepsilon$. By Claim 2.13.1, $\Lambda_E(gX, \theta X) \le 4\varepsilon$, so $\gamma_E(X,Y) \le 4\varepsilon$. □

2) $\implies$ *3)*: Suppose that γ_E is a complete pseudometric that is uniformly equivalent to d_{BM} on $\mathcal{G}_k$ for every k. We have to prove that $\mathcal{G}_k$ is compact, so let $(X_n)_n$ be a sequence in $\mathcal{G}_k$. Since d_{BM} is a compact pseudometric on the class of all k-dimensional Banach spaces, we can extract a d_{BM}-Cauchy subsequence $(X_n)_{n\in N}$. Since $\mathrm{Id} : (\mathcal{G}_k, d_{\mathrm{BM}}) \to (\mathcal{G}_k, \gamma_E)$ is uniformly continuous, $(X_n)_{n\in N}$ is γ_E-Cauchy. Since γ_E is complete in $\mathcal{G}$, it follows that $(X_n)_{n\in N}$ γ_E-converges to some $X \in \mathcal{G}_k$. Finally, the identity $\mathrm{Id} : (\mathcal{G}_k, \gamma_E) \to (\mathcal{G}_k, d_{\mathrm{BM}})$ is continuous, hence, $(X_n)_{n\in N}$ d_{BM}-converges to X.

3) $\implies$ *1)* : It follows from Proposition 2.13 *3)* and the fact that we are supposing that $\mathcal{G}_k$ is d_{BM}-compact that γ_E is also a compact pseudometric, hence complete. Fix now k and $\varepsilon > 0$, and for each $X \in \mathcal{G}_k$, choose $0 < \delta_X \le 1$ such that $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_{\delta_X}(X,E)$ is $\varepsilon/2$-transitive. By compactness of $(\mathcal{G}_k, d_{\mathrm{BM}})$, we can find $(X_j)_{j<n}$ such that $\mathcal{G}_k \subseteq \bigcup_{j<n} B_{\mathrm{BM}}(X_j, \delta_{X_j}/3)$. We claim that $\delta := (1/3)\min_j \delta_{X_j}$ works: For suppose that $\gamma, \eta \in \mathrm{Emb}_\delta(X,E)$ with $X \in \mathcal{G}_k$. Let $j < n$ and let $\theta : X_j \to X$ be an isomorphism such that $\|\theta\|\|\theta^{-1}\| \le e^{\delta_{X_j}/3}$. Since $\delta_{X_j} \le 1$, it follows that

$$\|\theta\|, \|\theta^{-1}\| \le \|\theta\|\|\theta^{-1}\| \le 1 + \frac{\delta_{X_j}}{2}.$$

Since $(1+\delta)(1+\delta_{X_j}/2) \le (1+\delta_{X_j}/3)(1+\delta_{X_j}/2) \le 1+\delta_{X_j}$, it follows that $\gamma\circ\theta, \eta\circ\theta \in \mathrm{Emb}_{\delta_{X_j}}(X_j,E)$, so there is $g \in \mathrm{Iso}(E)$ such that $\|g\circ\gamma\circ\theta - \eta\circ\theta\| \le \varepsilon/2$. Hence, $\|g\circ\gamma - \eta\| \le (\varepsilon/2)\|\theta^{-1}\| \le \varepsilon$.

□

It is interesting to note that being Fraïssé is an ultra property. Recall that given a Banach space E, and given a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $E_{\mathcal{U}}$ to denote the *ultrapower* $E^{\mathbb{N}}/\mathcal{U}$. For each $n \in \mathbb{N}$, let $\pi_n : E^{\mathbb{N}} \to E$ be the n^{th} projection, $\pi_n((x_m)_m) = x_n$. We denote by $\mathrm{Iso}(E)_{\mathcal{U}}$ the subgroup of $\mathrm{Iso}(E_{\mathcal{U}})$

consisting of all isometries of the ultrapower $E_{\mathcal{U}}$ of the form $[(x_n)_n]_{\mathcal{U}} \mapsto [(g_n(x_n))_n]_{\mathcal{U}}$ for some sequence $(g_n)_n \in \mathrm{Iso}(E)^{\mathbb{N}}$. It is well known that $\mathrm{Age}(E_{\mathcal{U}}) \equiv \overline{\mathrm{Age}(E)}^{\mathrm{BM}}$.

Proposition 2.14. *Let E be a Banach space, and let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. The following are equivalent.*

1) E is Fraïssé.

2) $E_{\mathcal{U}}$ is Fraïssé and $(\mathrm{Iso}(E))_{\mathcal{U}}$ is dense in $\mathrm{Iso}(E_{\mathcal{U}})$ with respect to the SOT.

3) For every $X \in \mathrm{Age}(E_{\mathcal{U}})$ one has that $(\mathrm{Iso}(E))_{\mathcal{U}} \curvearrowright \mathrm{Emb}(X, E_{\mathcal{U}})$ is almost transitive.

4) For every $X \in \mathrm{Age}(E_{\mathcal{U}})$ one has that $(\mathrm{Iso}(E))_{\mathcal{U}} \curvearrowright \mathrm{Emb}(X, E_{\mathcal{U}})$ is transitive.

5) For every separable $X \subset E_{\mathcal{U}}$ one has that $(\mathrm{Iso}(E))_{\mathcal{U}} \curvearrowright \mathrm{Emb}(X, E_{\mathcal{U}})$ is transitive.

6) $E_{\mathcal{U}}$ is (uH) and $(\mathrm{Iso}(E))_{\mathcal{U}}$ is dense in $\mathrm{Iso}(E_{\mathcal{U}})$ with respect to the SOT.

Moreover, if any of the previous conditions hold we have that

$$\varpi_{E_{\mathcal{U}}}(k,\delta) = \varpi_E^*(k,\delta). \tag{1}$$

In particular, it follows that if E is Fraïssé, then its ultrapowers are Fraïssé and (uH).

Proof. *1) $\implies$ 2)*:

Claim 2.14.1. For every $k \in \mathbb{N}$, $X \in \mathrm{Age}_k(E_{\mathcal{U}})$ and $0 < \xi < \delta$ we have that $(\mathrm{Iso}(E))_{\mathcal{U}}$ acts $(\varpi_E(k,\delta)+\xi)$-transitively on $\mathrm{Emb}_{\delta-\xi}(X, E_{\mathcal{U}})$. In particular, the action $(\mathrm{Iso}(E))_{\mathcal{U}} \curvearrowright \mathrm{Emb}(X, E_{\mathcal{U}})$ is approximately transitive for every $X \in \mathrm{Age}(E_{\mathcal{U}})$, and $\varpi_{E_{\mathcal{U}}}(k,\delta) \le \inf_{\delta'>\delta} \varpi_E(k,\delta')$.

Proof of Claim: For suppose that $X \in \mathrm{Age}_k(E_{\mathcal{U}})$. Since we know that $\mathrm{Age}_k(E_{\mathcal{U}}) \equiv \overline{\mathrm{Age}_k(E)}^{\mathrm{BM}} \equiv \mathrm{Age}_k(E)$, we may assume that $X \in \mathrm{Age}_k(E)$. Let $\gamma \in \mathrm{Emb}_{\delta-\xi}(X, E_{\mathcal{U}})$. It follows that the set $A = \{n \in \mathbb{N} \,:\, \pi_n \circ \gamma \in \mathrm{Emb}_\delta(X,E)\} \in \mathcal{U}$, so for each $n \in A$, choose $g_n \in \mathrm{Iso}(E)$ such that $\|g_n \restriction X - \pi_n \circ \gamma\| \le \varpi_k(\delta) + \xi/2$. For each $n \notin A$, let $g_n := \mathrm{Id}_E$. Define $I := [(g_n)_n]_{\mathcal{U}}$; then $A \subseteq \{n \in \mathbb{N} \,:\, \|\pi_n \circ I \restriction X - \pi_n \circ \gamma\| \le \varpi_k(\delta) + \xi\}$, so, this set is in $\mathcal{U}$. This means that $\|I \restriction X - \gamma\| \le \varpi_k(\delta) + \xi$. □

It follows from this, and the characterization in Theorem 2.11 that $E_{\mathcal{U}}$ is Fraïssé. Moreover, the Claim also easily implies that $(\mathrm{Iso}(E))_{\mathcal{U}}$ is dense in $\mathrm{Iso}(E_{\mathcal{U}})$.

2) $\implies$ 3): Since $E_{\mathcal{U}}$ is Fraïssé, it is (AuH), and this together with the fact we are assuming that $(\mathrm{Iso}(E))_{\mathcal{U}}$ is dense in $\mathrm{Iso}(E_{\mathcal{U}})$, readily implies *3)*.

3) $\implies$ 4): Suppose that $X \in \mathrm{Age}(E_{\mathcal{U}})$ and $\gamma \in \mathrm{Emb}(X, E_{\mathcal{U}}))$. For each $k \in \mathbb{N}$, choose $I_k := [(g_n^{(k)})]_{\mathcal{U}} \in (\mathrm{Iso}(E))_{\mathcal{U}}$ such that $\|I_k \restriction X - \gamma\| < 1/k$. For each k, let $A_k := \{n \in \mathbb{N} \,:\, \|g_n^{(k)} \restriction X - \pi_n \circ \gamma\| \le 1/k\} \in \mathcal{U}$. By making a small perturbation if needed, we may assume that for every $n \in \mathbb{N}$ one has that $\pi_n \circ \gamma$ is not an isometry. So, in particular, $\bigcap_k A_k = \emptyset$. For each $n \in \bigcup_k A_k$, let $k(n)$ be the maximal k such that $n \in A_k$, and define $g_n := g_n^{(k(n))}$, while for $n \notin \bigcup_k A_k$, let $g_n := \mathrm{Id}_E$. Let $I := [(g_n)]_{\mathcal{U}}$. Then for every $k \in \mathbb{N}$, $A_k \subseteq \{n \,:\, \|g_n \restriction X - \pi_n \circ \gamma\| \le 1/k\}$, so, this set is in $\mathcal{U}$, and as a consequence we have that $I \restriction X = \gamma$.

4) $\implies$ 5): the proof is very similar to the previous one of that *3) $\implies$ 4)*. We leave the details to the reader.

5) $\implies$ 6): the fact that $E_{\mathcal{U}}$ is (uH) is trivial, and the second fact is a direct consequence of *4)*.

In order to obtain (1) we will not prove *6) $\implies$ 1)* directly, but *6) $\implies$ 2)* and then *2) $\implies$ 1)*.

6) $\implies$ 1): Suppose otherwise that E is not Fraïssé; by Theorem 2.11, there is some k, $\varepsilon > 0$ and for every n some $X_n \in \mathrm{Age}_k(E)$ and $\gamma_n \in \mathrm{Emb}_{1/n}(X_n, E)$ such that

$$\inf_{g \in \mathrm{Iso}(E)} \|g \restriction X_n - \gamma_n\| > \varepsilon. \tag{2}$$

For each n, let $(x_j^{(n)})_{j<k}$ be an Auerbach basis of X_n, and for each $j<k$ let $x_j := [(x_j^{(n)})]_{\mathcal{U}}$, $X := \langle x_j \rangle_{j<k}$, and $\gamma : X \to E_{\mathcal{U}}$, $\gamma(\sum_j a_j x_j) := [(\gamma_n(\sum_j a_j x_j^{(n)}))]_{\mathcal{U}}$. Then, $\gamma \in \mathrm{Emb}(X, E_{\mathcal{U}})$; By *6)*, there is $I = [(g_n)]_{\mathcal{U}} \in (\mathrm{Iso}(E))_{\mathcal{U}}$ such that $\|I \restriction X - \gamma\| < \varepsilon/2$. In particular, $A := \{n : \|\pi_n \circ I \restriction X - \pi_n \circ \gamma\| \le \varepsilon/2\} \in \mathcal{U}$. Let now $B := \{n : (x_j^{(n)})_{j<k}$ and $(x_j)_{j<k}$ are 2-equivalent$\} \in \mathcal{U}$. It follows that given $n \in A \cap B$, and given $x = \sum_j a_j x_j^{(n)} \in X_n$,

$$\|g_n(x) - \gamma_n(x)\| = \|\pi_n \circ I(\sum_j a_j x_j) - \pi_n \gamma(\sum_j a_j x_j)\| \le \frac{\varepsilon}{2} \|\sum_j a_j x_j\| \le \varepsilon \|x\|.$$

This contradicts (2)

Finally, suppose that $E_{\mathcal{U}}$ is Fraïssé with modulus $\varpi_{E_{\mathcal{U}}}$, and that $(\mathrm{Iso}(E))_{\mathcal{U}}$ is dense in $\mathrm{Iso}(E_{\mathcal{U}})$.

Claim 2.14.2. $\varpi_{E_{\mathcal{U}}}(k,\delta) = \varpi^*_{E_{\mathcal{U}}}(k,\delta) = \varpi^*_{E\mathcal{U}}(p,\delta)$.

Proof of Claim: Fix $\gamma \in \mathrm{Emb}_\delta(X, E_{\mathcal{U}})$, for $X \in \mathrm{Age}_k(E_{\mathcal{U}})$, and $\varepsilon > 0$. Without loss of generality, we assume that $\pi \circ \gamma \notin \mathrm{Emb}(X, E_{\mathcal{U}})$. For each k, let $\delta_k := \delta + 1/k$, and let $A_k := \{n \in \mathbb{N} : \pi_n \gamma \in \mathrm{Emb}_{\delta_k}(X, E_{\mathcal{U}})\}$, that belongs to $\mathcal{U}$. Let $\gamma_k : X \to E_{\mathcal{U}}$ be such that $\pi_n \circ \gamma_k = \pi_n \circ \gamma$ for every $n \in A_k$. Then $\gamma_k \in \mathrm{Emb}_{\delta_k}(X, E_{\mathcal{U}})$, so there is some $I_k := [(g_n^{(k)})_n]_{\mathcal{U}} \in (\mathrm{Iso}(E))_{\mathcal{U}}$ such that the set $B_k = \{n \in A_k : \|g_n \restriction X - \pi_n \circ \gamma_k\| \le \varpi_{E_{\mathcal{U}}}(k, \delta_k) + \varepsilon\} \in \mathcal{U}$. For each $n \in \bigcup_k B_k$, let $k(n)$ be the maximum of those k's such that $n \in B_k$. For such n, let $g_n := g_n^{(k(n))}$, and for $n \notin \bigcup_k B_k$, let $g_n := \mathrm{Id}_E$. It follows that $I := [(g_n)_n]_{\mathcal{U}} \in (\mathrm{Iso}(E))_{\mathcal{U}}$ satisfies that $\|I \restriction X - \gamma\| \le \inf_{\delta'} \varpi(k, \delta') + \varepsilon$. □

Now let us see that $\varpi_E(k,\delta) \le \varpi_{E_{\mathcal{U}}}(k,\delta)$: given $\gamma \in \mathrm{Emb}_\delta(X, E_{\mathcal{U}})$, we define $\bar{\gamma} \in \mathrm{Emb}_\delta(X, E_{\mathcal{U}})$, $\bar{\gamma}(x) := [(\gamma(x))]_{\mathcal{U}}$. Let $I = [(g_n)_n]_{\mathcal{U}} \in (\mathrm{Iso}(E))_{\mathcal{U}}$ be such that $\|I \restriction X - \bar{\gamma}\| \le \varpi_{E_{\mathcal{U}}}(k,\delta) + \varepsilon/2$, so $A = \{n \in \mathbb{N} : \|g_n \restriction X - \gamma\| \le \varpi_{E_{\mathcal{U}}}(k,\delta) + \varepsilon\} \in \mathcal{U}$; if we choose $n \in A$, then $\|g_n \restriction X - \gamma\| \le \varpi_{E_{\mathcal{U}}}(k,\delta) + \varepsilon$.

From this, and the Claim 2.14.1 and Claim 2.14.2 we have that $\varpi_{E_{\mathcal{U}}}(k,\delta) = \inf_{\delta' > \delta} \varpi_E(k, \delta')$. □

Corollary 2.15. *For $p \neq 4, 6, 8, \dots$, any non-trivial ultrapower of $L_p(0,1)$ is an* (uH) *L_p-space.* □

It had been already observed in [AvCSCaGoMo, Proposition 4.13] that any non-trivial ultrapower of $\mathbb{G}$ is ultrahomogeneous. We shall now see that Fraïssé Banach spaces are locally determined. The following is a slight modification of a similar concept introduced in [Lup].

Definition 2.16. Given a family $\mathcal{G}$ of finite dimensional spaces, let $[\mathcal{G}]$ be the class of all Banach spaces E such that the collection of subspaces of elements of $\mathcal{G}_E$ is Λ_E-dense in $\mathrm{Age}(E)$.

Remark 2.17. It is easily see that if $E \in [\{\ell_p^n\}_n]$, then E is a $\mathcal{L}_{p,1+}$-space, that is for every $\delta > 0$ and every $X \in \mathrm{Age}(E)$ there is some $X \subseteq Y \in \mathrm{Age}(E)$ such that $d_{\mathrm{BM}}(Y, \ell_p^{\dim Y}) \le \delta$. Conversely, if E is an $\mathcal{L}_{p,1+}$-space that is in addition a *stable* $\{\ell_p^n\}_n$-Fraïssé space, then $E \in [\{\ell_p^n\}_n]$. Stable $\mathcal{G}$-Fraïssé spaces are those for which their moduli does not depend on the dimensions. Given $X \in \mathrm{Age}(E)$ and $\varepsilon > 0$, let $0 < \delta \le 1$ be such that $\varpi_{\mathcal{G},E}(\delta) \le \varepsilon/4$, and let $X \subseteq Y \in \mathrm{Age}(E)$, $\gamma \in \mathrm{Emb}_\delta(\ell_p^n, Y)$ and $\eta \in \mathrm{Emb}(\ell_p^n, E)$ for some n. Find $g \in \mathrm{Iso}(E)$ such that $\|\gamma - g \circ \eta\| \le \varepsilon/4$. Let $Z := \mathrm{Im}(g \circ \eta)$. It follows from Claim 2.13.1 that $\Lambda_E(Y, Z) \le 2(1+\delta)\|\gamma - g \circ \eta\| \le \varepsilon$.

Theorem 2.18. *Suppose that X and Y are $\mathcal{G}$-Fraïssé Banach spaces, with $\mathcal{G} \preceq \mathrm{Age}(X), \mathrm{Age}(Y)$ and $X \in [\mathcal{G}]$. The following are equivalent.*

1) $Y \in [\mathcal{G}]$.

2) X is isometric to Y.

This result motivates the next notion.

Definition 2.19. Let $\mathcal{G}$ be a class of finite dimensional Banach spaces. The *Fraïssé limit* of $\mathcal{G}$, denoted by $\operatorname{Flim}\mathcal{G}$ is, if exists, the unique separable $\mathcal{G}$-Fraïssé Banach space E such that $E \in [\mathcal{G}]$.

We have the following interesting fact.

Proposition 2.20. *If* $\operatorname{Flim}\mathcal{G}$ *and* $\operatorname{Flim}\overline{\mathcal{G}}^{\mathrm{BM}}$ *exists, then they are isometric.*

Proof. Set $X = \operatorname{Flim}\mathcal{G}$, $Y = \operatorname{Flim}\overline{\mathcal{G}}^{\mathrm{BM}}$ and $\mathcal{H} := \overline{\mathcal{G}}^{\mathrm{BM}}$. Since Y is $\mathcal{G}$-Fraïssé, by Theorem 2.18, it suffices to prove that $Y \in [\mathcal{G}]$: fix $F \in \operatorname{Age}(Y)$ and $\varepsilon > 0$, and let $G \in (\overline{\mathcal{G}}^{\mathrm{BM}})_Y$ be such that $F \subseteq_{\varepsilon/2} G$. Let $0 < \delta \le 1$ be such that $\varpi_{\mathcal{H},Y}(\dim G, \delta) \le \varepsilon/8$, and let $G_0 \in \mathcal{G}$, $\gamma \in \operatorname{Emb}_\delta(G_0, G)$ and $\eta \in \operatorname{Emb}(G_0, Y)$. Find $g \in \operatorname{Iso}(Y)$ such that $\|\gamma - g\circ\eta\| \le \varepsilon/8$. It follows that $G_1 := \operatorname{Im}(g\circ\eta) \in \mathcal{G}_Y$ and by Claim 2.13.1, $\Lambda_Y(G, G_1) \le \varepsilon/2$. Hence, $F \subseteq_{\varepsilon/2} G \subseteq_{\varepsilon/2} G_1$ and so $F \subseteq_\varepsilon G_1$. □

Proof of Theorem 2.18. *2)* $\implies$ *1)* is trivial. Let us prove that *1)* $\implies$ *2)*: Given two subspaces F, G of a given space E, and $\varepsilon > 0$, we write $X \subseteq_\varepsilon Y$ when there is some $Z \subseteq \operatorname{Age}(Y)$ of dimension $\dim Z = \dim X$ such that $\Lambda_E(X, Z) \le \varepsilon$. Fix two rapidly decreasing strictly positive sequences $(\delta_k)_k$ and $(\varepsilon_k)_k$ with

i) $\delta_k \le \varepsilon_k \le 1$ for every k.
ii) $\prod_k (1+\delta_k) \le \sqrt{2}$, $\sum_{k>l} \varepsilon_k \le \varepsilon_l/4$.
iii) $\varpi_X(\dim X_k, \delta_k), \varpi_Y(\dim Y_k, \delta_k) \le \varepsilon_k/4$

Fix now sequences $(X_k)_k$ in $\mathcal{G}_X$ and $(Y_k)_k$ in $\mathcal{G}_Y$ whose respective unions are dense, and such that $\sum_{k<l} X_k \subseteq_{\varepsilon_l} X_l$ and $\sum_{k<l} Y_k \subseteq_{\varepsilon_l} Y_l$. Using that both X and Y are $\mathcal{G}$-Fraïssé Banach spaces and that $\bigcup_k X_k$ and $\bigcup_k Y_k$ are dense in X and Y, respectively, we are going to find two sequences of integers $(m_k)_k$ and $(n_k)_k$ with $m_k \le n_k < m_{k+1}$, and $\gamma_k \in \operatorname{Emb}_{\delta_{m_k}}(X_{m_k}, Y_{n_k})$, $\eta_k \in \operatorname{Emb}_{\delta_{n_k}}(Y_{n_k}, X_{m_{k+1}})$ such that

$$\|\eta_k \circ \gamma_k - i_{X_{m_k},X}\| \le \varepsilon_{m_k} \text{ and } \|\gamma_{k+1}\circ\eta_k - i_{Y_{n_k},Y}\| \le \varepsilon_{n_k}. \tag{3}$$

Suppose that $\gamma_k : X_{m_k} \to Y_{n_k}$ is defined. Since $Y_{n_k} \in \mathcal{G}_Y$, we can find $f \in \operatorname{Emb}(Y_{n_k}, X)$ such that $\|f\circ\gamma_k - i_{X_{m_k},X}\| \le \varpi_X(\dim X_{m_k}, \delta_{m_k}) + \delta_{m_k}$. Let $m_{k+1} > n_k$ be large enough so that there is some linear mapping $\iota : f(Y_{n_k}) \to X_{m_{k+1}}$ such that $\|\iota - i_{f(Y_{n_k}),X}\| \le \delta_{n_k}/2$. Then it follows that $\eta_k := \iota\circ f \in \operatorname{Emb}_{\delta_{n_k}}(Y_{n_k}, X_{m_{k+1}})$ and $\|\eta_k\circ\gamma_k - \operatorname{Id}_{X_{m_k}}\| \le \varepsilon_{m_k}$. Similarly one finds n_{k+1} and γ_{k+1}. We re-enumerate as follows: Let $\widehat{X}_k := X_{m_k}$, $\widehat{Y}_k := Y_{n_k}$, $\widehat{\varepsilon}_k := \varepsilon_{m_k}$, $\widehat{\delta}_k := \delta_{m_k}$ for every k.

Claim 2.20.1. Fix $x \in X$ and $\varepsilon > 0$. Let m be such that $\{x\} \subseteq_{\varepsilon/2} \widehat{X}_m$ and $\widehat{\varepsilon}_{m-1} \le \varepsilon/2$. Then for every $k, l \ge m$ and $v \in X_k$, $w \in X_l$ such that $\|v - x\|, \|w - x\| \le \varepsilon\|x\|$ we have that

$$\|\gamma_k(v) - \gamma_l(w)\| \le 3\varepsilon\|x\|.$$

Once this is established, we define $\gamma : X \to Y$ as follows: Give $x \in X$, let $x[n] \in X_n$ be such that $\|x - x[n]\| = d(x, X_n)$. By the choice of the sequences $(X_n)_n$, $x[n] \to_n x$. By the previous claim, $(\gamma_n(x[n]))_n$ is a Cauchy sequence. Let $\gamma(x) := \lim_n \gamma_n(x[n])$. It is easily seen that γ is a bounded linear mapping $\gamma : X \to Y$ such that $\|\gamma\| \le 1$. Similarly one defines $\eta : Y \to X$ and proves that $\gamma\circ\eta = \operatorname{Id}_Y$, $\eta\circ\gamma = \operatorname{Id}_X$.

Proof of Claim: Fix all data, and suppose that $k \ge l$. If $k = l$, then $\|\gamma_k(v) - \gamma_k(w)\| \le \|\gamma_k\|\|v - w\| \le 4\varepsilon\|x\|$. Suppose that $k = l + n$ with $n > 0$. Set $w_0 := w$, and for each $1 \le j < n$, set $w_{j+1} :=$

$\eta_{l+j}(\gamma_{l+j}(w_j)) \in \widehat{X}_{l+j+1}$. Notice that $\|\gamma_{l+j}(w_j)\| \le \prod_k (1+\delta_k)^2 \|w_0\| \le 4\|x\|$. Now, it follows from (3),

$$\begin{aligned}\|\gamma_k(w_n)-\gamma_l(w)\| =& \|\gamma_k(w_n)-\gamma_l(w_0)\| \le \sum_{j=0}^{n-1} \|\gamma_{l+j+1}(w_{j+1})-\gamma_{l+j}(w_j)\| = \\ =& \sum_{j=0}^{n-1} \|\gamma_{l+j+1}(\eta_{l+j}(\gamma_{l+j}(w_j)))-\gamma_{l+j}(w_j)\| \le \sum_{j=0}^{n-1} \widehat{\varepsilon}_{l+j} \|\gamma_{l+j}\| \|w_j\| \le \\ \le& 4\|x\| \sum_{j=0}^{n-1} \widehat{\varepsilon}_{l+j} \le \frac{\varepsilon}{2}\|x\|. \end{aligned} \tag{4}$$

On the other hand, since $\|w_{j+1}-w_j\| = \|\eta_{l+j}(\gamma_{l+j}(w_j)) - w_j\| \le \widehat{\varepsilon}_{l+j}\|w_j\|$, again invoking (3),

$$\|w_n - v\| \le \sum_{j=0}^{n-1} \|w_{j+1}-w_j\| + \|w_0 - v\| = \sum_{j=0}^{n-1} \widehat{\varepsilon}_{l+j}\|w_j\| + 2\varepsilon\|x\| \le \widehat{\varepsilon}_{l-1}\|x\| + 2\varepsilon\|x\| \le \frac{5}{2}\varepsilon\|x\| \tag{5}$$

Combining (4) and (5) we obtain that $\|\gamma_k(v)-\gamma_l(w)\| \le 3\varepsilon\|x\|$. □

Note now that given $x \in X_{m_k}$ of norm 1, and $l \ge k$, we have that $\|\gamma_l(x) - \gamma_{l+1}\circ\eta_l\circ\gamma_l(x)\| \le \varepsilon_l \|\gamma_l(x)\| \le \varepsilon_l(1+\delta_l) \le 2\varepsilon_l$. Also, $\|\gamma_{l+1}(x) - \gamma_{l+1}\circ\eta_l\circ\gamma_l(x)\| \le \|\gamma_{l+1}\|\varepsilon_l \le 2\varepsilon_l$. So, given $l \ge k$ and $m \in \mathbb{N}$, we have that $\|\gamma_{l+m}(x)-\gamma_l(x)\| \le 4\sum_{j=l}^{l+m-1}\varepsilon_j \le \varepsilon_{l-1}$. This proves that $(\gamma_l(x))_{l\ge k}$ is a Cauchy sequence, and similarly, $(\eta_l(y))_{l\ge k}$ is also Cauchy for $y \in Y_{n_k}$. we define $\gamma : \bigcup_k X_{m_k} \to \bigcup_k Y_{n_k}$, and $\eta : \bigcup_k Y_{n_k} \to \bigcup_k X_{m_k}$, by $\gamma(x) := \lim_{l\ge k}\gamma_k(x)$ and $\eta(y) := \lim_{l\le k}\eta_k(y)$ for $x \in X_{m_k}$ and $y \in Y_{n_k}$. The extensions of γ and η to X and Y, respectively are isometric embedding and one if the inverse of the other, so X and Y are isometric. *2)* $\implies$ *1)* is trivial. □

In particular, if X and Y are separable Fraïssé spaces such that $\mathrm{Age}(X) \equiv \mathrm{Age}(Y)$, then $X \equiv Y$. However we have the following stronger characterization.

Proposition 2.21 (Uniqueness)**.** *For separable* (AuH) *spaces X and Y the following are equivalent.*

(a) $X \equiv Y$.

(b) $\mathrm{Age}(X) \equiv \mathrm{Age}(Y)$.

Proof. We will find $\subseteq$-increasing sequences $(X_n)_n$ and $(Y_n)_n$ and isometric embeddings $\gamma_n : X_n \to Y_n$ and $\eta_n : Y_n \to X_{n+1}$ such that

1) $X_n \in \mathcal{F}$ and $Y_n \in \mathcal{G}$ for every n, and $\bigcup_n X_n$ and $\bigcup_n Y_n$ are dense in X and Y, respectively.
2) $\|\eta_n\circ\gamma_n - \mathrm{Id}_{X_n}\| \le 2^{-n} := \varepsilon_n$.
3) $\|\gamma_{n+1}\circ\eta_n - \mathrm{Id}_{Y_n}\| \le 2^{-n}$.

Once this is done, given $x \in X_n$ we have that that $\|\gamma_{n+1}(x)-\gamma_n(x)\| \le 2^{-n+1}$. So, $(\gamma_n(x))_{m\ge n}$ is a Cauchy sequence. We define then $g_n : X_n \to Y$ by $g_n(x) := \lim_{m\ge n}\gamma_m(x)$. Obviously, $g_n \restriction X_n = \gamma_n$, so we can define $g : \bigcup_n X_n \to Y$ piece-wise, and we extend it to $g : X \to Y$. Similarly one defines $h : Y \to X$. It is easy to see that $h\circ g = \mathrm{Id}_X$ and $g\circ h = \mathrm{Id}_Y$. Let us argue that the sequences above exist: We fix $\{x_n\}_n$ and $\{y_n\}_n$ dense subsets of X and Y, respectively with $x_0 = y_0 = 0$. Let $X_0 = Y_0 := \{0\}$, $\gamma_0(0) = 0$. Suppose defined $X_0 \subseteq \cdots \subseteq X_n$ with $\{x_k\}_{k\le j} \subseteq X_j$, $Y_0 \subseteq \cdots \subseteq Y_n$ with $\{y_k\}_{k\le j} \subseteq Y_j$, $\gamma_j : X_j \to Y_j$ for $j \le n$ and $\eta_j : Y_j \to X_{j+1}$ for $j < n$ such that $\|\gamma_{j+1}\circ\eta_j - \mathrm{Id}\restriction Y_j\|, \|\eta_j\circ\gamma_j - \mathrm{Id}\restriction X_j\| \le \varepsilon_j$ for every $j < n$. We choose X_{n+1} and $\eta_n : Y_n \to X_{n+1}$ as follows. First fix $\theta \in \mathrm{Emb}(Y_n, X)$. This is possible since $Y_n \in \mathrm{Age}(Y) \equiv \mathrm{Age}(X)$. Now let $g \in \mathrm{Iso}(X)$ be such that $\|g\circ\theta\circ\gamma_n - \mathrm{Id}_{X_n}\| \le \varepsilon_n$, and set

$\eta_n := g \circ \theta \in \mathrm{Emb}(Y_n, X)$. Let $X_{n+1} := X_n + \mathrm{Im}\eta_n + \langle x_j \rangle_{j \le n+1}$. Then $\eta_n : Y_n \to X_{n+1}$ satisfies what we want. Similarly one can find the desired Y_{n+1} and $\gamma_{n+1} : X_{n+1} \to Y_{n+1}$. □

2.1. **Classes of finite dimensional spaces.** The question addressed now is for which families $\mathcal{G}$ of finite dimensional spaces there is some separable Fraïssé Banach space E such that $\mathcal{G} \equiv \mathrm{Age}(E)$, or more generally, such that $\mathcal{G} \preceq \mathrm{Age}(E)$ and $E \in [\mathcal{G}]$. In the discrete algebraic case, this is the content of the classical result by R. Fraïssé [Fra] characterizing ultrahomogeneous countable first order structures in terms of properties of their classes of finitely generated substructures. In the cases of Banach spaces there is a similar characterization, that we pass to expose (see also [Lup] for similar results for *stable* Fraïssé operator spaces and systems, or [BY] in general for metric structures).

Definition 2.22 (Fraïssé classes)**.** Let $\mathcal{G}$ be a class of finite dimensional Banach spaces.

(a) $\mathcal{G}$ has the *hereditary property* (HP) when for every $X \in \mathcal{G}$ and every Y, if $\mathrm{Emb}(Y, X) \neq \emptyset$, then $Y \in \mathcal{G}_\equiv$,
(b) $\mathcal{G}$ has the *amalgamation property* (AP) when $\{0\} \in \mathcal{G}$ and for every $X, Y, Z \in \mathcal{G}$ and every isometric embeddings $\gamma : X \to Y$ and $\eta : X \to Z$ there is $H \in \mathcal{G}$ and isometries $i : Y \to H$ and $j : Z \to H$ such that $i \circ \gamma = j \circ \eta$.
(c) $\mathcal{G}$ has the *near amalgamation property* (NAP) when $\{0\} \in \mathcal{G}$ and for every $\varepsilon > 0$, $X, Y, Z \in \mathcal{G}$ and every isometric embeddings $\gamma : X \to Y$ and $\eta : X \to Z$ there is $H \in \mathcal{G}$ and isometries $i : Y \to H$ and $j : Z \to H$ such that $\|i \circ \gamma - j \circ \eta\| \le \varepsilon$.
(d) $\mathcal{G}$ is a *weak amalgamation class* when $\{0\} \in \mathcal{G}$ and for every $\varepsilon > 0$ and $X \in \mathcal{G}$ there is $\delta > 0$ such that for every $Y, Z \in \mathcal{G}$ and δ-isometric embeddings $\gamma : X \to Y$ and $\eta : X \to Z$ there is $H \in \mathcal{G}$ and isometries $i : Y \to H$ and $j : Z \to H$ such that $\|i \circ \gamma - j \circ \eta\| \le \varepsilon$;
(e) $\mathcal{G}$ is an *amalgamation class* when $\{0\} \in \mathcal{G}$ and for every dimension $k \in \mathbb{N}$ and $\varepsilon > 0$ there is $\delta > 0$ such that if $X \in \mathcal{G}_k$, $Y, Z \in \mathcal{G}$ and $\gamma \in \mathrm{Emb}_\delta(X, Y)$, $\eta \in \mathrm{Emb}_\delta(X, Z)$, then there is $H \in \mathcal{G}$ and isometries $i : Y \to H$ and $j : Z \to H$ such that $\|i \circ \gamma - j \circ \eta\| \le \varepsilon$.
(f) $\mathcal{G}$ is a *Fraïssé class* when it is hereditary d_{BM}-compact amalgamation class.

The modulus of a (weak) amalgamation class is defined similarly to the modulus of a (weak) $\mathcal{G}$-Fraïssé space: the class $\mathcal{G}$ has modulus $\varpi : \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}^+$ when $\{0\} \in \mathcal{G}$ and for every $\delta > 0$ every k and every $\varepsilon > 0$, if $X \in \mathcal{G}_k$, $Y, Z \in \mathcal{G}$ and $\gamma \in \mathrm{Emb}_\delta(X, Y)$, $\eta \in \mathrm{Emb}_\delta(X, Z)$, then there is $H \in \mathcal{G}$ and isometries $i : Y \to H$ and $j : Z \to H$ such that $\|i \circ \gamma - j \circ \eta\| \le \varpi(k, \delta) + \varepsilon$. We have the following interesting implication.

Proposition 2.23. *1) The amalgamation and the near amalgamation properties are equivalent for compact and hereditary families.*
2) The Banach-Mazur closure of an hereditary amalgamation class is a Fraïssé class.

Proof. *1)*: Fix a compact and hereditary family $\mathcal{G}$ with the (NAP), and fix also $X, Y, Z \in \mathcal{G}$ and isometric embeddings $\gamma : X \to Y$ and $\eta : X \to Z$. Choose a sequence $(\varepsilon_n)_n$ of strictly positive real numbers and decreasing to zero, and for each n, let $V_n \in \mathcal{G}$ and $i_n \in \mathrm{Emb}(Y, V_n)$ and $j_n \in \mathrm{Emb}(Z, V_n)$ such that $\|i_n \circ \gamma - j_n \circ \eta\| \le \varepsilon_n$. For each n, let $W_n := \mathrm{Im} i_n + \mathrm{Im} j_n \subseteq V_n$. Since $\mathcal{G}$ is hereditary, $W_n \in \mathcal{G}$. By passing to a subsequence if needed, we assume that the all W_n have the same dimension, and that W_n converge with respect to the Banach-Mazur pseudometric to W, that belongs to $\mathcal{G}$. For each n, let $\theta_n : W_n \to W$ be such that $\lim_n \max\{\|\theta_n\|, \|\theta_n^{-1}\|\} = 1$. Let $i : Y \to W$, $j : Z \to W$ be accumulation points of $(\theta_n \circ i_n)_n$ and of $(\theta_n \circ j_n)_n$, respectively. It follows that i and j are isometries and $i \circ \gamma = j \circ \eta$.

2): Fix an hereditary amalgamation class $\mathcal{G}$, and $\mathcal{H}$ be the Banach-Mazur closure of $\mathcal{G}$. Notice that $\mathcal{H}$ is also hereditary. Now fix a dimension k and $\varepsilon > 0$. Let $\delta > 0$ be witnessing that $\mathcal{G}$ is an amalgamation class for the parameters k and $\varepsilon/2$. We claim that $\delta/2$ works for $\mathcal{H}$: Fix $X \in \mathcal{H}_k$, $Y, Z \in \mathcal{H}$ and $\gamma \in \mathrm{Emb}_{\delta/2}(X,Y)$, $\eta \in \mathrm{Emb}_{\delta/2}(X,Z)$. For each $n \geq 1$ let $\delta_n := \delta/(12n)$, choose $X_0, Y_n, Z_n \in \mathcal{G}$ and $\theta^X : X_0 \to X$, $\theta_n^Y : Y \to Y_n$, $\theta_n^Z : Z \to Z_n$, surjective isomorphisms such that $\|(\theta^X)^{-1}\| = 1$, $\|\theta^X\| \leq 1+\delta_1$ and $\theta_n^Y \in \mathrm{Emb}_{\delta_n}(Y, Y_n)$ and $\theta_n^Z \in \mathrm{Emb}_{\delta_n}(Z, Z_n)$. Notice that given $n \geq 1$, $\gamma_n := \theta_n^Y \circ \gamma \circ \theta^X \in \mathrm{Emb}_\delta(X_n, Y_n)$ and $\eta_n := \theta_n^Z \circ \eta \circ \theta^X \in \mathrm{Emb}_\delta(X_n, Z_n)$ so we can choose $V_n \in \mathcal{G}$ and $i_n \in \mathrm{Emb}(Y_n, V_n)$, $j_n \in \mathrm{Emb}(Z_n, V_n)$ such that $\|i_n \circ \gamma_n - j_n \circ \eta_n\| \leq \varepsilon/2$. Since $\mathcal{G}$ is hereditary, as before in *1)* we may assume that $\dim V_n$ is constant, and the sequence $(V_n)_n$ converges in the Banach-Mazur norm to $V \in \overline{\mathcal{G}}^{\mathrm{BM}}$. Choose $\theta_n^V : V_n \to V$ be such that $\theta_n^V \in \mathrm{Emb}_{\delta_n}(V, V_n)$. Choose a convergent subsequence of $(\theta_n^V \circ i_n \circ \theta_n^Y)_n$ and of $(\theta_n^V \circ j_n \circ \theta_n^Z)_n$ with limits $i \in \mathrm{Emb}(Y,V)$ and $j \in \mathrm{Emb}(Z,V)$, respectively. Then $\|i \circ \gamma - j \circ \eta\| \leq \varepsilon$. □

In the previous, the condition of being hereditary seems necessary: For example, Let $\mathcal{F}$ consists of all 2-dimensional polyhedral spaces together with the spaces ℓ_∞^n of any dimension n. Then $\mathcal{F}$ is an amalgamation class and its closure is the class of all 2-dimensional normed spaces together with ℓ_∞^n of any dimension, that does not have the near amalgamation property.

In classical Fraïssé theory for discrete algebraic structures, Fraïssé classes $\mathcal{G}$ have in addition the joint embedding property (JEP): For every $X, Y \in \mathcal{G}$ there is some $Z \in \mathcal{G}$ such that X and Y can be isomorphically embedded into Z. However it is easy to see that this property in the context of Banach spaces is a property of amalgamation classes (because $\{0\} \in \mathcal{G}$).

Amalgamation families with modulus not depending on the dimension k where introduced by M. Lupini in [Lup]; they are said to have the *stable* near amalgamation property and in this case Fraïssé classes are called stable Fraïssé classes. Examples of them are the class of all finite dimensional Hilbert, and Banach spaces, corresponding to the two known stable Fraïssé spaces: the Hilbert and the Gurarij, respectively. We do not know if the class $\mathrm{Age}(L_p(0,1))$ is a Fraïssé class for $p \neq 4, 6, 8, \dots$. It is easy to see that $\mathrm{Age}(E)$ has always (HP), (JEP) and when E is (AuH), weak Fraïssé, Fraïssé then $\mathrm{Age}(E)$ has (NAP), is a weak amalgamation class or is an amalgamation class, respectively, and in the (weak) Fraïssé case the respective moduli are the same (when E is Fraïssé, the compactness of $\mathrm{Age}(E)$ follows from Theorem 2.11). We call Fraïssé correspondence the reverse implication, presented in Corollary 2.26. Other limits and correspondences of this type have been studied in [BY] and in [Ku].

As for the Fraïssé and the weak Fraïssé properties, there is also a metric explanation of the relation between the weak amalgamation and the amalgamation. Given a family $\mathcal{G}$ of finite dimensional spaces with the (JEP) and the (NAP), we can define a "Kadets-like" pseudometric on each $\mathcal{G}_k$,

$$\gamma_{\mathcal{G}}(X,Y) := \inf\{\Lambda_Z(X_0,Y_0) \,:\, X_0, Y_0 \in \mathrm{Age}_k(Z),\, Z \in \mathcal{G},\, X_0 \equiv X,\, Y_0 \equiv Y\}.$$

The following characterization is proved similarly as the equivalences in Theorem 2.11. We leave the details to the reader.

Proposition 2.24. *The following are equivalent.*

(a) $\mathcal{G}$ is an amalgamation class and $\gamma_{\mathcal{G}}$ is a complete pseudometric.

(b) $\mathcal{G}$ is a weak amalgamation class and $\gamma_{\mathcal{G}}$ is a complete pseudometric that is uniformly equivalent to d_{BM} on $\mathcal{G}_k$ for every k.

(c) $\mathcal{G}$ is a d_{BM}-compact weak amalgamation class. □

Theorem 2.25. *If $\mathcal{G}$ is an amalgamation class with modulus ϖ, then* $\mathrm{Flim}\,\mathcal{G}$ *exists and it has $\mathcal{G}$-Fraïssé modulus ϖ^*.*

The existence of these spaces have been proved for stable Fraïssé Banach spaces, and for other structures in functional analysis in [Lup].

Corollary 2.26 (Fraïssé correspondence)**.** *The following are equivalent for a class $\mathcal{G}$ of finite dimensional Banach spaces.*

(1) $\mathcal{G}$ is a Fraïssé class.

(2) $\mathcal{G} \equiv \mathrm{Age}(E)$ of a unique separable Fraïssé Banach space $E = \mathrm{Flim}\,\mathcal{G}$.

Proof. Suppose that $\mathcal{G}$ is a Fraïssé class, i.e. $\mathcal{G}$ is a compact, hereditary amalgamation class. It follows from Theorem 2.25 that $\mathrm{Flim}\,\mathcal{G}$ exists. Let us see that $\mathrm{Age}(\mathrm{Flim}\,\mathcal{G}) \equiv \overline{\mathcal{G}}^{\mathrm{BM}}$: We already know that $\mathcal{G} \preceq \mathrm{Age}(\mathrm{Flim}\,\mathcal{G})$. Let us see that $\mathrm{Age}(\mathrm{Flim}\,\mathcal{G}) \preceq \mathcal{G}$: Since $\mathrm{Flim}\,\mathcal{G} \in [\mathcal{G}]$ and $\mathcal{G}$ is hereditary, we obtain that $\mathrm{Age}(\mathrm{Flim}\,\mathcal{G}) \preceq \overline{\mathcal{G}}^{\mathrm{BM}} = \mathcal{G}$. □

Corollary 2.27. *Let $\mathcal{G}$ be a class of finite dimensional Banach spaces.*

(1) If $\mathcal{G}$ is an hereditary amalgamation class, then $\mathrm{Flim}\,\mathcal{G}$ is the unique separable Fraïssé Banach space E such that $\mathrm{Age}(E) \equiv \overline{\mathcal{G}}^{\mathrm{BM}}$.

(2) If E is a separable Banach space such that $\mathrm{Age}(E)$ is an amalgamation class, then $\mathrm{Flim}\,\mathrm{Age}(E)$ is the unique separable Fraïssé Banach space X with an isometric copy of E such that $\mathrm{Age}(X) = \overline{\mathrm{Age}(E)}^{\mathrm{BM}}$.

Proof. *(1)*: If $\mathcal{G}$ is an hereditary amalgamation class, then by Proposition 2.23 *2)*, $\overline{\mathcal{G}}^{\mathrm{BM}}$ is a Fraïssé class, so $\mathrm{Flim}\,\overline{\mathcal{G}}^{\mathrm{BM}}$ exists and it is a Fraïssé Banach space. Since $\mathrm{Flim}\,\mathcal{G}$ also exists, it follows by Proposition 2.20 that $\mathrm{Flim}\,\mathcal{G} = \mathrm{Flim}\,\overline{\mathcal{G}}^{\mathrm{BM}}$. *(2)*: Set $X := \mathrm{Flim}\,\mathrm{Age}(E)$. We know from *(1)* that X is the unique Fraïssé Banach space with $\mathrm{Age}(X) = \overline{\mathrm{Age}(E)}^{\mathrm{BM}}$, and it follows from Proposition 2.12 that E can be isometrically embedded into X. □

Before we present a proof of Theorem 2.25, we give some examples of classes of spaces and limits.

Example 2.28. *For $1 \le p \le \infty$, the family $\mathcal{F} = \{\ell_p^k\}_{k\in\mathbb{N}}$ it is clearly compact, and it is an amalgamation class. For $1 \le p \neq 2 < \infty$, this is done in Proposition 3.7 using a work of G. Schechtman in* [Sch]. *In this case, the Fraïssé limit* $\mathrm{Flim}\{\ell_p^n\}_n$ *is* $L_p[0,1]$ *(see Proposition 3.7).*

For $p = \infty$, it is rather easy to see that $\{\ell_\infty^n\}_n$ is an amalgamation class, whose Fraïssé limit is the Gurarij space $\mathbb{G}$.

When $p = 2$, $\{\ell_2^k\}_k$ is also hereditary, because a subspace of a Hilbert space is a Hilbert space; so $\{\ell_2^k\}_k$ is a Fraïssé class, whose Fraïssé limit is $\mathrm{Flim}\{\ell_2^n\}_n = \ell_2$ *and it is automatically a Fraïssé Banach space.*

Example 2.29. $\mathrm{Age}(C[0,1])$ *is a Fraïssé class, and* $\mathrm{Flim}\,\mathrm{Age}(C[0,1]) = \mathbb{G}$

Proof. Being both $C[0,1]$ and the Gurarij space $\mathbb{G}$ universal spaces, we obtain that $\mathrm{Age}(C[0,1]) = \mathrm{Age}(\mathbb{G})$ is a Fraïssé class because $\mathbb{G}$ is Fraïssé. □

Example 2.30. *For $p \neq 4, 6, 8, \dots$, the class* $\mathrm{Age}(L_p(0,1))$ *is Fraïssé, whose Fraïssé limit is* $L_p(0,1)$. *We will see in Theorem 4.1 that for these p's, $L_p(0,1)$ is weak-Fraïssé; moreover, for all p's* $\mathrm{Age}(L_p(0,1))$ *is compact: Observe that in general, given a Banach space, the closure* $\overline{\mathrm{Age}(E)}^{\mathrm{BM}}$ *of* $\mathrm{Age}(E)$ *under the Banach-Mazur pseudometric is exactly, modulo $\equiv$, equal to* $\mathrm{Age}(E_{\mathcal{U}})$ *for every non-trivial ultrafilter $\mathcal{U}$ over $\mathbb{N}$. It follows from the S. Kakutani characterization of abstract L_p spaces (see* [Kak] *or* [LiTza, Theorem 1.b.2. of Vol. 2]*) that each $E_{\mathcal{U}}$ is isometric to some $L_p(\mu)$ (in fact, W. Henson proved in* [He, Theorem 2.4] *that this particular ultrapower is isometric to the ℓ_p-sum of $\mathfrak{c}$ copies of $L_p([0,1]^{\mathfrak{c}})$, $\mathfrak{c}$ being the cardinality of the continuum). Hence* $\overline{\mathrm{Age}(L_p(0,1))}^{\mathrm{BM}} \equiv \mathrm{Age}(L_p(\mu))$. *Finally, it is easy to see that* $\mathrm{Age}(L_p(\mu)) \preceq \mathrm{Age}(L_p(0,1))$.

We continue with the proof of the Fraïssé correspondence. Recall that given a sequence $(X_n,\gamma_n)_n$ where each X_n is a normed space and $\gamma_n \in \mathrm{Emb}(X_n, X_{n+1})$, the inductive limit $\lim_n(X_n,\gamma_n)$ of $(X_n,\gamma_n)_n$ is defined as the following normed space: We define isometric embeddings $\gamma_{m,n}\in \mathrm{Emb}(X_m,X_n)$ for $m\le n$ as follows; $\gamma_{m,m} := \mathrm{Id}_{X_m}$ and $\gamma_{m,n+1} := \gamma_n\circ\gamma_{m,n}$; let V be the subspace of the product space $\prod_n X_n$ defined as

$$V := \{(x_n)_n \,:\, \text{there is some } m \text{ such that } x_n=\gamma_{m,n}(x_m) \text{ for all } n\ge m\}.$$

In V one defines the seminorm

$$\|(x_n)_n\| := \|x_m\|_{X_m}$$

where m is such that $\gamma_{m,n}(x_m)=x_n$ for all $n\ge m$. It is easy to see that $\|\cdot\|$ is well defined, and that

$$N := \{(x_n)_n\in V \,:\, \text{there is some } m \text{ such that } x_n=0 \text{ for all } n\ge m\} = \{(x_n)_n\in V \,:\, \|(x_n)_n\|=0\}.$$

Let V_0 be the quotient space V/N endowed with the norm $\|\cdot\|$. Finally, let $\lim_n(X_n,\gamma_n)$ be the completion of V/N. For each m, let $\gamma_m^{(\infty)} : X_m\to \lim_n(X_n,\gamma_n)$ be defined for $x\in X_m$ by

$$\gamma_m^{(\infty)}(x) := (\overbrace{0,\dots,0}^{(m-1)}, x, \gamma_{m,m+1}x, \gamma_{m,m+2}x,\dots) + N.$$

Obviously, $\gamma_m^{(\infty)}$ is an isometric embedding, and the sequence $(X_n^\infty)_n$, $X_n^\infty := \gamma_n^{(\infty)}(X_n)$, is increasing with dense union in $\lim_n(X_n,\gamma_n)$.

Lemma 2.31. *Suppose that $\mathcal{G}$ has the* (JEP). *Then $\mathcal{G}$ is an amalgamation class with modulus of stability ϖ if and only if for every $\Delta\subseteq\mathbb{R}^+$ finite, $\varepsilon>0$ and $\mathcal{H}\cup\{Y\}\subseteq\mathcal{G}$ finite there is $Z\in\mathcal{G}$ and some $I\in\mathrm{Emb}(Y,Z)$ such that for every $X\in\mathcal{G}$ and every $\delta\in\Delta$ if $\gamma,\eta\in\mathrm{Emb}_\delta(X,Y)$ then there is $J\in\mathrm{Emb}(Y,Z)$ such that $\|I\circ\gamma - J\circ\eta\|\le\varpi(\dim X,\delta)+\varepsilon$.*

Proof. Fix all data. Order $\mathcal{H} := \{X_j\}_{j=1}^m$, $\Delta := \{\delta_j\}_{j=1}^n$. For each $1\le j\le m$ and $1\le l\le n$, let $\{\gamma_k^{(j,l)}\}_{k=1}^s$ be a $\varepsilon/3$-dense subsets of $\mathrm{Emb}_{\delta_l}(X_j,Y)$. Inductively we find a sequence $(V_k)_{k=1}^{s^2mn+1}$ in $\mathcal{G}$, $V_1 := Y$ and isometric embeddings $I_k\in\mathrm{Emb}(V_k,V_{k+1})$, $1\le k\le s^2mn$, such that for every $1\le k_0,k_1\le n$, $1\le j\le m$ and $1\le l\le n$, setting $k := s^2(j-1)(l-1)+k_0k_1+1$ there is some $J\in\mathrm{Emb}(Y,V_{k+1})$ such that $\|J\circ\gamma_{k_0}^{(j,l)} - I_k\circ I_{k-1}\circ\cdots\circ I_1\circ\gamma_{k_1}^{(j,l)}\|\le\varpi(\dim X_j,\delta_l)+\varepsilon$. Then $V := V_{s^2mn+1}$ and $I := I_{s^2mn}\circ I_{s^2mn-1}\circ\cdots\circ I_1$ work. □

Proof of Theorem 2.25. The proof is an standard back-and-forth argument. Suppose that $\mathcal{G}$ is an amalgamation class, with modulus of stability ϖ. Let $\{\delta_n\}_n := \mathbb{Q}\cap[0,1]$, $\delta_0=0$, $(\varepsilon_n)_n$ be a positive sequence such that $\sum_{n>m}\varepsilon_n\le\varepsilon_m$, and let $\mathcal{A}=\{Z_n\}_n\subseteq\mathcal{G}$ be a countable d_{BM}-dense subset of $\mathcal{G}$. We use Lemma 2.31 to find a sequence $(X_n,I_n)_n$, where

(a) $X_n\in\mathcal{G}$ and $I_n\in\mathrm{Emb}(X_n,X_{n+1})$;
(b) for every $n\in\mathbb{N}$ and every $X\in\{Z_j\}_{j\le n}\cup\{X_j\}_{j\le n}$ and every $k\le n$, if $\gamma,\eta\in\mathrm{Emb}_{\delta_k}(X,X_n)$ then there is $J\in\mathrm{Emb}(X_n,X_{n+1})$ such that $\|I_n\circ\gamma - J\circ\eta\|\le\varpi(\dim X,\delta_k)+\varepsilon_n$;
(c) $\mathrm{Emb}(Z_m,X_{n+1})\ne\emptyset$ for every $m\le n$.

For (c) we use the (JEP) of $\mathcal{G}$, that we know is true for amalgamation classes. We claim that the inductive limit $E := \lim_n(X_n,I_n)$ is $\mathcal{G}$-Fraïssé with modulus ϖ, $E\in[\mathcal{G}]$ and that $\mathcal{G}\preceq\mathrm{Age}(E)$.

Claim 2.31.1. E is $\mathcal{G}$-Fraïssé.

Proof of Claim: Fix $\delta'>\delta\ge 0$, $\varepsilon>0$, and $X\in\mathcal{G}$ and fix $\gamma,\eta\in\mathrm{Emb}_\delta(X,E)$. Choose n large enough such that

i) $\varepsilon_{n-1}<\varepsilon$;

ii) there are $j, k \le n$, $\delta'' < \varepsilon/2$ such that
 ii.1) $\delta < \delta_k < \delta'$ and $(\varpi(\dim X, \delta_k) + \varepsilon)\delta'' < \varepsilon/3$,
 ii.2) there is an onto map $\theta \in \mathrm{Emb}_{\delta''}(X, Z_j)$ and $\gamma_0, \eta_0 \in \mathrm{Emb}_{\delta_k}(Z_j, X_n)$ such that $\|I_n^\infty \circ \gamma_0 \circ \theta - \gamma\|, \|I_n^\infty \circ \eta_0 \circ \theta - \eta\| \le \varepsilon/2$.

For each r and s set $I_{r,r} := \mathrm{Id}_{X_n}$; $I_{r,r+s+1} := I_{r+s} \circ I_{r,r+s}$. We find now $J_s \in \mathrm{Emb}(X_{n+2s}, X_{n+2s+1})$ and $L_s \in \mathrm{Emb}(X_{n+2s+1}, X_{n+2s+2})$ such that

(d) $\|J_0 \circ \eta_0 - I_n\gamma_0\| \le \varpi(d, \delta_k) + \varepsilon_n$, being $d := \dim X$
(e) $\|L_{s+1} \circ J_s - I_{n+2s,n+2s+2}\| \le \varepsilon_{n+2s+1}$ for every $s \ge 0$.
(f) $\|J_s \circ L_s - I_{n+s-1,n+2s+1}\| \le \varepsilon_{n+2s}$ for every $s \ge 1$.

Setting $\bar{\varepsilon}_n := \varpi(d, \delta_k) + \varepsilon_n$, we have the following commutative infinite diagram.

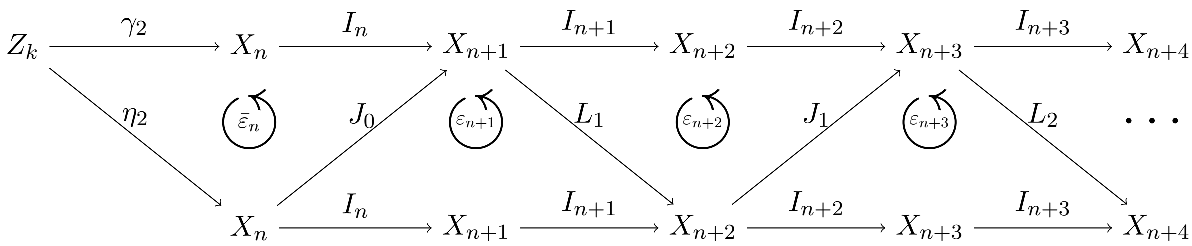

FIGURE 1

Given a sequence $x = (x_k)_k \in \bigcup_l X_l^\infty$, we define for each k, $\overline{y}_k(x) := I_{n+2k+1}^\infty J_k(x_{n+2k}) \in X_{n+2k+1}^\infty$. It follows from the (e) and (f) that $(\overline{y}_k(x))_k$ is a Cauchy sequence, so we define $J : \bigcup_l X_l^\infty \to E$, $J(x) = \lim_k \overline{y}_k(x)$, and then we extend it to $J : E \to E$. Similarly, we define, given $y = (y_k)_k \in \bigcup_l X_l^\infty$, one defines the Cauchy sequence $\overline{x}_k(y) := I_{n+2k+2}^\infty L_k(x_{n+2k+1}) \in X_{n+2k+2}^\infty$, and the corresponding $L : E \to E$. It is easy to see that $L \circ J = J \circ L = \mathrm{Id}_E$.

Set now $\gamma_1 := I_n^\infty \circ \gamma_0$ and $\eta_1 := I_n^\infty \circ \eta_0$. It follows that $J(\eta_1(z)) = \lim_s I_{n+2s+1}^\infty J_k(I_{n,n+2s}(\eta_0(z)))$ for every $z \in Z_k$. Now for each s one has that $\|J_s(I_{n,n+2s}(\eta_0(z))) - I_{n,n+2s+1}(\gamma_0(z))\| \le \varpi(\dim X, \delta_k) + \sum_{l \le 2k} \varepsilon_{n+l} \le \varpi(\dim X, \delta_k) + \varepsilon_{n-1}$. This means that $\|J \circ \eta_1 - \gamma_1\| \le \varpi(\dim X, \delta_k) + \varepsilon_{n-1}$. Consequently,

$$\begin{aligned}\|J \circ \eta - \gamma\| \le & \|J \circ \eta_1\theta - \gamma_1\theta\| + \|J \circ \eta_1\theta - J \circ \eta\| + \|\gamma_1\theta - \gamma\| \le (\varpi(\dim X, \delta_k) + \varepsilon_{n-1})\|\theta\| + 2\frac{\varepsilon}{3} \le \\ \le & \varpi(\dim X, \delta_k) + \varepsilon \le \varpi(\dim X, \delta') + \varepsilon.\end{aligned}$$ □

This proves that E is $\mathcal{G}$-Fraïssé with modulus ϖ^*. By definition of E, the sequence $(X_n^\infty)_n$ witnesses that $E \in [\mathcal{G}]$.

Claim 2.31.2. $\mathcal{G} \preceq \mathrm{Age}(E)$.

Proof of Claim: For suppose that $Z \in \mathcal{G}_k$. Recall that $(\varepsilon_n)_n$ was chosen so that $\sum_{n \ge m+1} \varepsilon_n \le \varepsilon_m$. Find a decreasing positive sequence $(\delta_n)_n$ such that $\varpi_{\mathcal{G},E}(k, \delta_n) \le \varepsilon_n$ for every n. Since $Z \in \overline{\mathcal{A}}^{\mathrm{BM}}$ and $\mathcal{A} \subseteq \mathrm{Age}(E)_\equiv$, we can find for each n some $\gamma_n \in \mathrm{Emb}_{\delta_n}(Z, E)$. Now for each n, let $g_n \in \mathrm{Iso}(E)$ be such that $\|g_n \circ \gamma_{n+1} - \gamma_n\| \le \varepsilon_n$, and set $\eta_0 := \gamma_0$; $\eta_{n+1} := g_0 \circ \cdots \circ g_n \circ \gamma_{n+1}$. Then $\|\eta_{n+l} - \eta_n\| \le \sum_{j=n}^{n+l-1} \varepsilon_j \le \varepsilon_{n-1}$, so $(\eta_n)_n$ is a Cauchy sequence of δ_n-embeddings from Z into E, so its limit $\eta : Z \to E$ is an isometric embedding, and consequently $Z \in \mathrm{Age}(E)_\equiv$. □

□

2.2. **Characterization of homogeneities.** We rephrase the homogeneity properties introduced in terms of algebraic, topological and metric properties of some list of canonical functions. Let $\mathcal{N}_k$ be the collection of all norms on the vector field $\mathbb{F}^k$, let $\mathrm{emb}(\mathbb{F}^k, E)$ be the family of all 1-1 linear mappings from $\mathbb{F}^k$ into E. Given a collection $\mathcal{G}$ of normed spaces over $\mathbb{F}$, let $\mathcal{N}_k(\mathcal{G})$ be the collection of those norms $M \in \mathcal{N}_k$ such that $(\mathbb{F}^k, M) \in \mathcal{G}_\equiv$; we define $\mathrm{emb}(\mathcal{G}_k) := \bigcup_{M\in\mathcal{N}_k(\mathcal{G})} \mathrm{Emb}((\mathbb{F}^k, M), E)$; we consider the natural action $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}(X, E)$, $g \cdot \gamma := g \circ \gamma$, and the consequent $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}(\mathcal{G}_k, E)$. We have now $\widehat{\nu}_k : \mathrm{emb}(\mathcal{G}_k, E) \to \mathcal{N}_k(\mathcal{G})$, defined as the norm $\widehat{\nu}_k(\gamma)(x) := \|\gamma x\|_E$; obviously, $\widehat{\nu}_k$ is $\mathrm{Iso}(E)$-invariant, so we have naturally defined the quotient mapping

$$\nu_k : \mathrm{emb}(\mathbb{F}^k, E) \,/\!/\, \mathrm{Iso}(E) \to \mathcal{N}_k(E),$$

$\nu_k([\gamma]) := \widehat{\nu}_k(\gamma)$. In general, $\nu_k(\mathrm{emb}(\mathcal{G}_k, E)) \subseteq \mathcal{N}_k(\mathcal{G})$, and when $\mathcal{G} \preceq \mathrm{Age}(E)$ one has $\nu_k(\mathrm{emb}(\mathcal{G}_k, E)) = \mathcal{N}_k(\mathcal{G})$

The following rephrases the approximate $\mathcal{G}$-homogeneity of E in terms of an algebraic property of each ν_k.

Proposition 2.32. *Suppose that $\mathcal{G} \preceq \mathrm{Age}(E)$. Then E is approximately $\mathcal{G}$-homogeneous if and only if each ν_k is 1-1 on $\mathrm{emb}(\mathcal{G}_k, E) \,/\!/\, \mathrm{Iso}(E)$.* □

The next is characterize E being weak $\mathcal{G}$-Fraïssé in terms of topological properties of each ν_k. So, we naturally endow $\mathcal{N}_k$ and $\mathrm{emb}(\mathbb{F}^k, E)$ with the topology of point-wise convergence, that coincides with the compact-open topology. The following is easy to prove.

Proposition 2.33. *Both $\widehat{\nu}_k$ and ν_k are continuous.* □

We introduce two metrics; let

$$\omega(M, N) := \log\left(\max\{\|\mathrm{Id}\|_{(\mathbb{F}^k,M),(\mathbb{F}^k,N)}, \|\mathrm{Id}\|_{(\mathbb{F}^k,N),(\mathbb{F}^k,M)}\}\right),$$

that defines the point-wise topology on $\mathcal{N}_k$; the second one $d(\eta, \gamma) := \|\eta - \gamma\|_{\ell_1,E} := \max_{j=1}^n \|\eta(u_j) - \gamma(u_j)\|_E$ defines the pointwise topology on $\mathrm{emb}(\mathbb{F}^k, E)$; notice that d is $\mathrm{Iso}(E)$-invariant for the natural action $g \cdot \gamma := g \circ \gamma$ introduced above, and that the quotient metric $\widehat{d}([\gamma], [\eta]) := \inf_{g\in\mathrm{Iso}(E)} \|g \circ \gamma - \eta\|_{\ell_1^k,E}$ defines the quotient topology on $\mathrm{emb}(\mathbb{F}^k, E) \,/\!/\, \mathrm{Iso}(E)$.

Proposition 2.34. *Suppose that $\mathcal{G} \preceq \mathrm{Age}(E)$.*

1) *E is weak $\mathcal{G}$-Fraïssé if and only if for every k one has that ν_k is 1-1 on $\mathrm{emb}(\mathcal{G}, E) \,/\!/\, \mathrm{Iso}(E)$, and for every $\gamma \in \mathrm{emb}(\mathcal{G}_k, E) \,/\!/\, \mathrm{Iso}(E)$ the mapping ν_k is continuous and open at $[\gamma]$* [2]
2) *Consequently, if E is weak $\mathcal{G}$-Fraïssé, then each restriction $\nu_k : \mathrm{emb}(\mathcal{G}, E) \,/\!/\, \mathrm{Iso}(E) \to \mathcal{N}_k(\mathcal{G})$ is an homeomorphism.*
3) *E is weak Fraïssé if and only ν_k is an homeomorphism for every $k \in \mathbb{N}$.*

Proof. *1)*: Suppose that E is weak $\mathcal{G}$-Fraïssé, and fix $k \in \mathbb{N}$. We know that ν_k is 1-1, so let $\gamma \in \mathrm{emb}(\mathcal{G}, E)$, and we see that ν_k is open at $[\gamma]$. Fix $\varepsilon > 0$; we have to prove that there is some $\delta > 0$ such that if $\eta \in \mathrm{emb}(\mathbb{F}^k, E)$ satisfies that $\omega(\nu_k([\gamma]), \nu_k([\eta])) < \delta$, then there is some $g \in \mathrm{Iso}(E)$ such that $d(g\eta, \gamma) < \varepsilon$. We set $M := \nu_k(\gamma)$ and $X := (\mathbb{F}^k, M)$. Let $\xi > 0$ be such that $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_\xi(X, E)$ is $\varepsilon/\|\mathrm{Id}\|_{X,\ell_1^k}$-transitive. We claim that $\delta := \log(1 + \xi)$ works. For suppose that $\eta \in \mathrm{emb}(\mathbb{F}^k, E)$ is such that $\omega(M, N) < \delta$, where $N := \nu_k(\eta)$. Then $\|\eta\|_{X,E} \le \|\eta\|_{(\mathbb{F}^k,N),E} \cdot e^{\omega(M,N)} \le (1 + \xi)$ and similarly

[2] recall that $f : X \to Y$ is open at x when for every open neighborhood U of x there is some open neighborhood V of $f(x)$ such that $V \subseteq f(U)$.

one has that $\|x\|_E \geq (1+\xi)^{-1}N(x)$, so $\eta \in \mathrm{Emb}_\xi(X,E)$. Hence there is some $g \in \mathrm{Iso}(E)$ such that $\|g\gamma - \eta\|_{X,E} \leq \varepsilon / \|\mathrm{Id}\|_{X,\ell_1^k}$, and consequently $\|g\gamma-\eta\|_{\ell_1^k,E} \leq \varepsilon$.

Suppose now that for every k one has that ν_k is $1-1$ on $\mathrm{emb}(\mathcal{G},E) \,/\!\!/\, \mathrm{Iso}(E)$ and open at each $[\gamma]$ with $\gamma \in \mathrm{emb}(\mathcal{G},E)$. Fix $X \in \mathcal{G}$ and $\varepsilon > 0$; let $M \in \mathcal{N}_k(\mathcal{G})$ be such that $X \equiv (\mathbb{F}^k, M)$ and let $\theta : (\mathbb{F}^k, M) \to X$ be an isometry, and $\gamma \in \mathrm{emb}(\mathcal{G},E)$ be such that $\nu_k(\gamma) = M$; let δ be such that if $\eta \in \mathrm{emb}(\mathbb{F}^k, E)$ is such that $\omega(M, \nu_k(\eta)) \leq \delta$, then $\widehat{d}([\gamma],[\eta]) < \widehat{\varepsilon} := \varepsilon/(2\|\theta^{-1}\|_{X,\ell_1^k})$. We claim that $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_\delta(X,E)$ is ε-transitive; for suppose that $\eta_0, \eta_1 \in \mathrm{Emb}_\delta(X,E)$; we set $M_j := \tau_k(\eta_j \circ \theta) \in \mathcal{N}_k(E)$ for $j = 0,1$. Since $\omega(M,M_0), \omega(M,M_1) \leq \delta$, we get that $\widehat{d}([\eta_0 \circ \theta],[\gamma]), \widehat{d}([\eta_1 \circ \theta],[\gamma] < \widehat{\varepsilon}$, hence $\widehat{d}([\eta_0 \circ \theta],[\eta_1 \circ \theta]) < 2\widehat{\varepsilon}$; this means that there is some $g \in \mathrm{Iso}(E)$ such that $\|g \circ \eta_0 \circ \theta - \eta_1 \circ \theta\|_{\ell_1^k,E} < 2\widehat{\varepsilon}$, and hence

$$\|g \circ \eta_0 - \eta_1\|_{X,E} \leq \|g \circ \eta_0 \circ \theta, \eta_1 \circ \theta\|_{\ell_1^k,E} \cdot \|\theta^{-1}\|_{X,\ell_1^k} \leq \varepsilon.$$

2) and *3)* is a direct consequence of *1)* using that continuous bijection is an homeomorphism exactly when it is open at each point. □

We pass now to reformulate E being $\mathcal{G}$-Fraïssé as a metric property of each ν_k. So, we introduce a metric on $\mathrm{emb}(\mathbb{F}^k,E)$ in a way that bounded sets are sent to ω-bounded sets by the mappings ν_k. Given $\gamma, \eta \in \mathrm{emb}(\mathbb{F}^k, E)$, let

$$\mathfrak{d}(\gamma,\eta) := \max\left\{\|\gamma-\eta\|_{\ell_1^k,E}, \left|\omega(\nu_k(\gamma), \|\cdot\|_{\ell_1^k}) - \omega(\nu_k(\eta), \|\cdot\|_{\ell_1^k})\right|\right\};$$

this is an $\mathrm{Iso}(E)$-invariant metric that defines the pointwise convergence topology on $\mathrm{emb}(\mathbb{F}^k,E)$, and the quotient topology is defined by the quotient metric

$$\widehat{\mathfrak{d}}([\gamma],[\eta]) := \max\left\{\inf_{g \in \mathrm{Iso}(E)} \|\gamma - g\eta\|_{\ell_1^k,E}, \left|\omega(\nu_k(\gamma), \|\cdot\|_{\ell_1^k}) - \omega(\nu_k(\eta), \|\cdot\|_{\ell_1^k})\right|\right\};$$

We have the following

Proposition 2.35. $\nu_k : (\mathrm{emb}(\mathbb{F}^k,E) \,/\!\!/\, \mathrm{Iso}(E), \widehat{\mathfrak{d}}) \to (\mathcal{N}_k(E), \omega)$ *is uniformly continuous on bounded sets.*

Proof. Fix $\gamma_0 \in \mathrm{emb}(\mathbb{F}^k,E)$ and $K > 0$, and let us see that ν_k is uniformly continuous on the $\widehat{\mathfrak{d}}$-ball with center $[\gamma_0]$ and radius K. Given $x \in \mathbb{F}^k$ we have that

$$\begin{aligned}\|\eta(x)\| \leq & \|\gamma(x)\| + \|\gamma-\eta\|_{\nu_k(\gamma),E} \cdot \|\gamma(x)\| \leq \|\gamma(x)\| + \|\gamma-\eta\|_{\ell_1^k,E} \cdot \|\mathrm{Id}\|_{\nu_k(\gamma),\ell_1^k} \|\gamma(x)\| \\ & \leq \left(1 + \widehat{\mathfrak{d}}([\gamma],[\eta]) \cdot \|\mathrm{Id}\|_{\nu_k(\gamma),\ell_1^k}\right) \|\gamma(x)\|\end{aligned}$$

and similarly $\|\gamma(x)\| \leq (1+\widehat{\mathfrak{d}}([\gamma],[\eta]) \cdot \|\mathrm{Id}\|_{\nu_k(\gamma),\ell_1^k})\|\eta(x)\|$. This means that

$$\omega(\nu_k([\gamma]), \nu_k([\eta])) \leq \log\left(1 + \widehat{\mathfrak{d}}([\gamma],[\eta]) \cdot \|\mathrm{Id}\|_{\nu_k(\gamma),\ell_1^k}\right) \leq \|\mathrm{Id}\|_{\nu_k(\gamma),\ell_1^k} \widehat{\mathfrak{d}}([\gamma],[\eta]) \leq e^K \widehat{\mathfrak{d}}([\gamma],[\eta]). \qquad \square$$

This is the metric characterization of being $\mathcal{G}$-Fraïssé.

Proposition 2.36. *Suppose that* $\mathcal{G} \preceq \mathrm{Age}(E)$.

1) *E is $\mathcal{G}$-Fraïssé if and only if for every $k \in \mathbb{N}$ the mapping $\nu_k : (\mathrm{emb}(\mathbb{F}^k,E) \,/\!\!/\, \mathrm{Iso}(E), \widehat{\mathfrak{d}}) \to (\mathcal{N}_k(E),\omega)$ is uniformly open on bounded subsets of $\mathcal{N}_k(\mathcal{G})$, i.e., given $A \subseteq \mathcal{N}_k(\mathcal{G})$ that is ω-bounded and given $\varepsilon > 0$, there is some $\delta > 0$ such that if $\nu_k([\gamma]) \in A$ and $\eta \in \mathrm{emb}(\mathbb{F}^k,E)$ is such that $\omega(\nu_k([\gamma]),\nu_k([\eta])) < \delta$, then $\widehat{d}([\gamma],[\eta]) < \varepsilon$.*

Consequently, if E is $\mathcal{G}$-Fraïssé then every restriction $\nu_k : (\mathrm{emb}(\mathcal{G},E) \,/\!\!/\, \mathrm{Iso}(E), \widehat{\mathfrak{d}}) \to (\mathcal{N}_k(\mathcal{G}),\omega)$ is an homeomorphism that is uniform when restricted to bounded sets.

2) *E is Fraïssé if and only for every $k \in \mathbb{N}$ one has that $\nu_k : (\mathrm{emb}(\mathbb{F}^k,E) \,/\!\!/\, \mathrm{Iso}(E), \widehat{\mathfrak{d}}) \to (\mathcal{N}_k(E),\omega)$ is an homeomorphism that is uniform when restricted to bounded sets.*

Proof. *1)* is proved following the same ideas as in the proof of Proposition 2.34 *1)*. *2)* readily follows from *1)*. □

3. Lattice and ℓ_p^n-homogeneity

In the next Section we will prove that $L_p(0,1)$ for p not even are Fraïssé Banach spaces. As we mentioned before this not the case for $p = 4, 6, 8, \dots$. However "partial" homogeneity properties are valid for all $L_p(0,1)$. This is the case when dealing with partial isometries defined on ℓ_p^n's. The aim of this section is to study those. In trying to understand these embeddings and isometries of $L_p(0,1)$ the setting of *lattices* is very natural. In this context we consider lattice isometries and lattice embeddings on a Banach lattice X, i.e., linear isometries or isometric embeddings which preserve the lattice structure. In the important case of a finite dimensional sublattice F, i.e. $F = \langle f_j \rangle_{j<n}$ where the $(f_j)_j$'s is a pairwise disjoint sequence of positive elements of X, a lattice isometric embedding is an isometric embedding T such that $(T(f_i))_j$ is a pairwise disjoint sequence of positive elements. For a complete information of them we refer the reader to [LiTza, Vol. 2].

With this in mind, we may consider lattice versions of homogeneity, by replacing isometries (resp. isometric embeddings) by lattice isometries (resp. lattice isometric embeddings) and (finite dimensional) subspaces by (finite dimensional) sublattices. So, given a sublattice Y of X, let $\mathrm{Emb}_\diamond(Y,X)$ be the space of lattice isometric embeddings from Y into X, let $\mathrm{Iso}_\diamond(X)$ be the topological group of surjective lattice isometries on X with its SOT, and let $\mathrm{Iso}_\diamond(X) \curvearrowright \mathrm{Emb}_\diamond(Y,X)$ be the canonical action $g \cdot \gamma := \gamma \circ g$. In particular we shall be interested in the following definitions:

Definition 3.1. Let X be a Banach lattice.

(a) X is *lattice ultrahomogeneous* ($\diamond$ −uH) if $\mathrm{Iso}_\diamond(X) \curvearrowright \mathrm{Emb}_\diamond(F,X)$ is transitive for every finite dimensional sublattice F of X;
(b) X is *approximately lattice ultrahomogeneous* ($\diamond$ −AuH) if $\mathrm{Iso}_\diamond(X) \curvearrowright \mathrm{Emb}_\diamond(F,X)$ is ε-transitive for every finite dimensional sublattice F of X.

There is a third natural notion in the case of lattices which is as follows. We say that an isometry (resp. isometric embedding) is disjoint preserving (or d.p.) if it sends disjoint vectors to disjoint vectors, and that a subspace is disjointly generated if it is generated by a sequence of disjoint vectors. So, the difference between these versions and the lattice ones is that now we do not impose positivity. Then a Banach lattice X is *disjointly homogeneous* if any d.p. isometric embedding defined on a disjointly generated finite dimensional subspace of X extends to a global d.p. isometry. It is *approximately disjointly homogeneous* if for any d.p. isometry t defined on a disjointly generated finite dimensional subspace F of X and for any $\epsilon > 0$, there is a global d.p. isometry T such that $\|T \restriction F - t\| \le \epsilon$. These properties will not be as relevant as the previous ones because of the following observation.

Proposition 3.2. *Assume X is a Köthe function space (see* [LiTza, Vol. 2]*). If X is (resp. approximately) lattice homogeneous then it is (resp. approximately) disjointly homogeneous.*

Proof. Assume X is lattice homogeneous. Let $F = \langle f_j \rangle_{j<n}$ with $(f_j)_j$ pairwise disjoint, and t an isometric d.p. map from F onto $G = \langle t f_j \rangle_{j<n}$. There exist isometric maps u and v on X acting by changes of signs, and sending each f_j to $|f_j|$, and $t f_j$ to $|t f_j|$, respectively. The map $v t u^{-1}$ is a lattice isometric map from $\langle |f_j| \rangle_{j<n}$ onto $\langle |t f_j| \rangle_{j<n}$. If T is a lattice isometry on X extending $v t u^{-1}$ then $v^{-1} T u$ is a d.p. isometry on X extending t. The same proof holds for approximate lattice homogeneity. □

3.1. **Stable approximate $\{\ell_p^n\}_n$-homogeneity of $L_p(0,1)$.** We start with a few classical definitions. Let $1 \leq p < +\infty$. A *simple* space is a finite dimensional subspace of $L_p(0,1)$ generated by simple functions. By *copy of* ℓ_p^n we mean some linearly isometric copy of a finite dimensional ℓ_p^n inside $L_p(0,1)$. It is a classical result by S. Banach that if $1 \leq p < +\infty, p \neq 2$, any isometric embedding between L_p-spaces is automatically d.p. ([FleJa, Theorem 3.2.5]). In particular, any isometric embedding of a copy of ℓ_p^m into a copy of ℓ_p^n is disjoint preserving, and any copy of ℓ_p^n is a disjointly generated subspace. It is also an easy observation that, for $1 \leq p < +\infty$, any finite dimensional subspace generated by simple functions in $L_p(0,1)$ is a subspace of some simple copy of some ℓ_p^n (with same support). S. Banach, [Ba] p. 178, stated the general formula of surjective isometries on $L_p(0,1)$, $1 \leq p < +\infty, p \neq 2$; any such isometry T is defined by

$$T(f)(t) = h(t)f(\phi(t)) \text{ for every } t \in [0,1],$$

where ϕ is a measurable non-singular transformation of $[0,1]$ onto itself, and h is a function satisfying $|h|^p = d(\lambda \circ \phi)/d\lambda$, where λ is the Lebesgue measure (see [FleJa, Chapter 3]). When $p = 2$ this formula defines a linear isometry but there are, obviously, other isometries on $L_2(0,1)$.

For $1 \leq p < +\infty$, we shall say that an isometry on $L_p(0,1)$ is *simple preserving* if it maps simple functions to simple functions. If A is a subset of $[0,1]$ of positive measure, and F is a subspace of $L_p(A)$, then the support of F is the union of the supports of all f, $f \in F$; in particular, for a vector in, or a subspace of X, *full support* in $L_p(A)$ means that the corresponding support is equal to the support of X.

Lemma 3.3. *Let $1 \leq p < +\infty$. Let A, B be subsets of $[0,1]$ of positive measure, and let $u \in L_p(A)$, $v \in L_p(B)$ be functions with full support and with equal norms. Then there exists a (disjoint preserving) isometry T from $L_p(A)$ onto $L_p(B)$ such that $T(u) = v$. Furthermore if u and v were simple functions then T is simple preserving, and if u and v were non-negative then T is a lattice isometry.*

Proof. It is enough to show this for $A = [0,1]$ and $u = \mathbb{1}_{[0,1]}$. Furthermore using a natural isometry between $L_p(B)$ and $L_p(0,1)$ we may assume $B = [0,1]$. Since $|v| > 0$, It is then clear that if $V(x) = \int_0^x |v(t)|^p dt$, then

$$T_v(f)(x) = f(V(x))v(x) \text{ for every } x \in [0,1]$$

defines a linear isometry of $L_p(0,1)$ sending u to v. The fact that this map is disjoint preserving (and respectively simple preserving, a lattice isometry) is obvious from the definition of T_v. □

It is well-known that when $1 \leq p < +\infty, p \neq 2$, any isometry of $L_p(0,1)$ sends full support vectors to full support vectors see [FleJa, Theorem 3.2.2]. This proves that $L_p(0,1)$ cannot be ultrahomogeneous in this case, nor even lattice ultrahomogeneous (so, as we commented before, the isometry group does not even act transitively on the unit sphere of $L_p(0,1)$ when $p \neq 2$). The next lemma will be useful to deal with perturbations in this context.

Lemma 3.4. *Let $1 \leq p < +\infty$. Let F be a finite dimensional subspace of $L_p(0,1)$ and let $\varepsilon > 0$. Then there exists a linear, disjoint preserving, isometric embedding γ of $L_p(0,1)$ into itself, whose image does not have full support, and such that $\|(\gamma - Id) \restriction F\| \leq \varepsilon$. If F was a sublattice then L may be chosen to be a lattice isometric embedding.*

Proof. Write $F = \langle f_j \rangle_{j<n}$, where the f_i's have norm 1. It is enough to find some isometric embedding γ whose image does not have full support and such that $\|(\gamma - \mathrm{Id})(f_j)\| \leq \varepsilon$ for all j and for some ε small enough. By a perturbation argument we may also assume that each f_j is continuous. Let $M = \max_{1 \leq j \leq n} \|f_j\|_\infty$, and let $\delta(s)$ be a common modulus of uniform continuity of all f_j's, i.e. such that $|x - y| \leq \delta(s)$ implies $|f_j(x) - f_j(y)| \leq s$ for every $j = 1, \dots, n$. Let $\lambda < 1$ be close enough to 1 so that $\lambda(1/\lambda-1)M+\delta(1/\lambda-1))^p+(1-\lambda)M^p<\varepsilon$. We define an isometric embedding γ whose image is $L_p(0,\lambda)\subset L_p(0,1)$ by

$$\gamma(f)(x):=\mathbb{1}_{[0,\lambda]}(x)f(\frac{x}{\lambda}).$$

Then for $j=1,\dots,n$ one has that

$$\|\gamma(f_j)-f_j\|^p=\int_0^\lambda|\frac{1}{\lambda}f_j(\frac{x}{\lambda})-f_j(x)|^pdx+\int_\lambda^1|f_j(x)|^pdx\le\lambda((\frac{1}{\lambda}-1)M+\delta(\frac{1}{\lambda}-1))^p+(1-\lambda)M^p<\varepsilon.\ \square$$

Proposition 3.5. *For every $1\le p<\infty$ the Banach lattice $L_p(0,1)$ is approximately lattice homogeneous, or in other words, the actions $\mathrm{Iso}_\diamond(L_p(0,1))\curvearrowright\mathrm{Emb}_\diamond(\ell_p^n,L_p(0,1))$ are almost transitive.*

Proof. We start with the following.

Claim 3.5.1. Fix $n\in\mathbb{N}$, and let $\mathcal{E}_F$ and $\mathcal{E}_N$ be the families of lattice embeddings of ℓ_p^n into $L_p(0,1)$ whose image has full support and does not have full support, respectively. Then the canonical actions $\mathrm{Iso}_\diamond(L_p(0,1))\curvearrowright\mathcal{E}_F$ and $\mathrm{Iso}_\diamond(L_p(0,1))\curvearrowright\mathcal{E}_N$ are transitive.

Proof of Claim: We prove only that $\mathrm{Iso}_\diamond(L_p(0,1))\curvearrowright\mathcal{E}_F$ is transitive; the other proof is similar, and we leave the details to the reader. Let F and G be two copies of some ℓ_p^n, with full support, and let t be an isometry from F onto G. Write $F=\langle f_j\rangle_{j<n}$, where the f_i's are normalized disjointly supported, and let $A_i:=\mathrm{supp}\,f_i$. Likewise we define $G=\langle tf_j\rangle_{j<n}$ and $B_i=\mathrm{supp}\,tf_i$. By Lemma 3.3, we may for each i define an isometry T_i from $L_p(A_i)$ onto $L_p(B_i)$ sending f_i to tf_i. This defines a global linear isometry T on $L_p(0,1)=\oplus_iL_p(A_i)=\oplus_iL_p(B_i)$ which extends t. $\square$

We use the above notation, where t is an isometric embedding of F onto G in $L_p([0,1])$, and assume for example that F does not have full support and G has. By Lemma 3.4, there exists an isometric embedding t' of F onto some G' without full support with $\|t-t'\|\le\epsilon$. By the Claim, t' extends to a global isometry on $L_p(0,1)$ and we are done. $\square$

Remark 3.6. Note that by Lemma 3.3 the isometry T considered in the proof of Claim 3.5.1 is simple preserving when the f_j's and tf_j are simple functions.

We finish with the following very strong amalgamation property.

Proposition 3.7. *Let $1\le p<\infty$. Then*
1) the class $\{\ell_p^n\}_{n\in\mathbb{N}}$ is an amalgamation class with modulus independent of the dimension.
2) $L_p(0,1)$ is the Fraïssé limit of $\{\ell_p^n\}_n$.

For the proof we use the following remarkable result by G. Schechtman (as observed by D. Alspach [Als])

Theorem 3.8 (Schechtman [Sch])**.** *For any $1\le p<\infty$ there is a modulus of stability $\varpi_p:]0,\infty[\to]0,\infty[$ such that*

$$\mathrm{Emb}_\delta(\ell_p^n,L_p(\mu))\subseteq(\mathrm{Emb}(\ell_p^n,L_p(\mu)))_{\varpi_p(\delta)}.$$

for every $n\in\mathbb{N}$, $\delta>0$ and finite measure μ. Consequently, for every $d,m\in\mathbb{N}$ there is $n\in\mathbb{N}$ such that

$$\iota_{m,n}\circ\mathrm{Emb}_\delta(\ell_p^d,\ell_p^m)\subseteq(\mathrm{Emb}(\ell_p^d,\ell_p^n)_{\varpi_p(\delta)},$$

where $\iota_{m,n}:\ell_p^m\to\ell_p^n$ is the canonical isometric embedding $(a_j)_{j<m}\in\mathbb{F}^m\mapsto(a_0,\dots,a_{m-1},0,\dots,0)\in\mathbb{F}^n$.

Proof of Proposition 3.7. *1)*:We may assume $p\neq2$ and let $\mathcal{F}$ be the collection of finite dimensional simple subspaces of $L_p(0,1)$ without full support.

Claim 3.8.1. $\mathcal{F}$ has the amalgamation property. Consequently, $\{\ell_p^n\}_n$ has the amalgamation property.

This, combined with Schechtman's result readily gives the proposition.

Proof of Claim: Let $E, F, G \in \mathcal{F}$, and let $\gamma \in \mathrm{Emb}(E, F)$, $\eta \in \mathrm{Emb}(E, G)$. By the Claim 3.5.1 (see the Remark 3.6) there is a simple preserving $T \in \mathrm{Iso}_\diamond(L_p(0,1))$ such that $T \circ \gamma = \eta$. Let $J \in \mathrm{Emb}(G, L_p(0,1)$ be simple preserving and such that $TF + JG$ does not have full support, and let $V \in \mathcal{F}$ be containing $TF + JG$, let $I : F \to V$ be the restriction $I := T \restriction F$. Then clearly $I \circ \gamma = J \circ \eta$.

2): From *1)* and the Proposition 3.5, we obtain that $L_p(0,1)$ is $\{\ell_p^n\}_n$-Fraïssé. Since $L_p(0,1) \in [\{\ell_p^n\}_n]$, we obtain from Theorem 2.25 that $L_p(0,1) = \lim\{\ell_p^n\}_n$. □

□

4. The Fraïssé property of $L_p(0,1)$: Approximate equimeasurability in L_p spaces

The main result of this section is the following

Theorem 4.1. *If $1 \le p < \infty$, p not even, then $L_p(0,1)$ is Fraïssé.*

Lusky proof of the approximate ultrahomogeneity of the spaces $L_p(0,1)$ for $p \notin 2\mathbb{N}$ is a consequence of the following result, known as the *equimeasurability principle*, proved independently by A. I. Plotkhin [Plo1] and W. Rudin [Ru].

Theorem 4.2 (Equimeasurability principle). *Suppose that $p \notin 2\mathbb{N}$, (Ω_0, μ_0), (Ω_1, μ_1) are finite measure spaces, and $f_0, \dots, f_{n-1} \in L_p(\Omega_0, \mu_0)$ and $g_0, \dots, g_{n-1} \in L_p(\Omega_1, \mu_1)$ are such that*

$$\int \left|1 + \sum_{j<n} a_j f_j(\omega)\right|^p d\mu_0(\omega) = \int \left|1 + \sum_{j<n} a_j g_j(\omega)\right|^p d\mu_1(\omega)$$

for every scalars $a_0, \dots, a_{n-1}$. Then, $F = (f_j)_j$ and $G = (g_j)_j$ are jointly equidistributed*, that is, for every Borel subset $A \subseteq \mathbb{F}^n$,*

$$F_*\mu_0(A) = \mu_0(\{\omega \in \Omega_0 \,:\, (f_j(\omega))_{j<n} \in A\}) = G_*\mu_1(A) = \mu_1(\{\omega \in \Omega_1 \,:\, (g_j(\omega))_{j<n} \in A\}).$$

In other words, if $\|\mathbb{1}_{\Omega_0} + \sum_j a_j f_j\|_{L_p(\Omega_0,\mu_0)} = \|\mathbb{1}_{\Omega_1} + \sum_j a_j g_j\|_{L_p(\Omega_1,\mu_1)}$ for every scalars $a_0, \dots, a_{n-1}$, then the *pushforward measures* $F_*\mu_0$ and $G_*\mu_1$ are equal. Our demonstration relies on an extension of the Equimeasurability principle. In order to state it we will use the well-known *Lévy-Prokhorov metric.* Let (X, d) be a separable metric space. Let $\mathcal{B}(X)$ be the collection of Borel subsets of X, and let $\mathcal{M}(X)$ be the collection of Borel measures on X.

Definition 4.3 (Lévy-Prokhorov metric). The *Lévy–Prokhorov metric* $d_{\mathcal{LP}} : \mathcal{M}(X)^2 \to [0, +\infty)$ is defined by setting the distance between two finite measures μ and ν to be

$$d_{\mathcal{LP}}(\mu, \nu) := \inf\{\varepsilon > 0 \mid \mu(A) \le \nu(A_\varepsilon) + \varepsilon \text{ and } \nu(A) \le \mu(A_\varepsilon) + \varepsilon \text{ for all } A \in \mathcal{B}(X)\}.$$

This metric defines the complete convergence on $\mathcal{M}(X)$ (see [Bi, Section 6]).

Theorem 4.4 (Approximate Equimeasurability principle). *Suppose that $p \notin 2\mathbb{N}$, (Ω_0, μ_0) is a finite measure space. Then for every $\varepsilon > 0$, $I \subseteq [0, p]$ finite and $f_0, \dots, f_{n-1} \in L_p(\Omega_0, \mu_0)$ there is $\delta > 0$ such that if (Ω_1, μ_1) is a finite measure space and $g_0, \dots, g_{n-1} \in L_p(\Omega_1, \mu_1)$ are such that*

$$\frac{1}{1+\delta}\left\|\mathbb{1}_{\Omega_1} + \sum_j a_j g_j\right\|_{L_p(\Omega_1,\mu_1)} \le \left\|\mathbb{1}_{\Omega_0} + \sum_j a_j f_j\right\|_{L_p(\Omega_0,\mu_0)} \le (1+\delta)\left\|\mathbb{1}_{\Omega_1} + \sum_j a_j g_j\right\|_{L_p(\Omega_1,\mu_1)}$$

for every scalars $a_0, \dots, a_{n-1}$*, then,*

$$\max_{\alpha \in I, j<n} d_{\mathcal{LP}}(|z_j|^\alpha d(F_* \mu_0), |z_j|^\alpha d(G_* \mu_1)) \le \varepsilon.$$

Here we are using the standard euclidean metric on $\mathbb{F}^n$, and the fact that $F_*\mu_0, G_*\mu_1 \in \mathcal{M}(\mathbb{F}^n)$. A well-known consequence of the equimeasurability principle is that if $\gamma : X \subseteq L_p(\mu_0) \to L_p(\mu_1)$ is an isometric embedding, and $u \in X$ has *full support* in X, then γu has full support in γX. Similarly, Theorem 4.4 will provide the corresponding approximate result. Given $\varepsilon > 0$, we say that $f \in X$ has ε-full support if $\|P_{u=0} \restriction X\| \le \varepsilon$. We have then the following.

Theorem 4.5 (Approximate full support)**.** *Suppose that* X *is a finite dimensional subspace of* $L_p(\mu)$ *with* $p \notin 2\mathbb{N}$*,* $u \in X$ *has full support in* X *and suppose that* $\varepsilon > 0$*. Then there is some* $\delta > 0$ *such that if* $\gamma \in \mathrm{Emb}_\delta(X, L_p(\mu))$ *then* γu *has* ε*-full support in* γX*.*

The study of the approximate equimeasurability will be done in the next Subsection 4.2, first by finding the topological correspondence determining the convergence of p-characteristics (Theorem 4.14), and finishing with its metric version in terms of the Lévy-Prohorov metric (Theorem 4.25) . Before, we present how to use Theorem 4.4 and Theorem 4.5 to prove the Fraïssé property of $L_p(0,1)$ of $p \notin 2\mathbb{N}$.

4.1. **The proof of Theorem 4.1.** We have already seen that $\mathrm{Age}(L_p(0,1))$ is always compact, so we just need the weak-Fraïssé property of $L_p(0,1)$ for $p \notin 2\mathbb{N}$. Let us explain the strategy of the proof. Suppose that $X \in \mathrm{Age}(L_p(0,1))$ and $\gamma : X \to L_p(0,1)$ is a δ-isometric embedding, $Y := \mathrm{Im}\gamma$. Suppose that $\mathbb{1}_{[0,1]} \in X$ and choose a basis $(f_j)_{j<n}$ of X. In Definition 4.6 we introduce appropriate partitions $\mathcal{R}$ of $\mathbb{F}^n$ whose pieces are products $\prod_j I_j$ where each I_j is an interval with small diameter or an unbounded one such that $\int_{I_j} |f_j|^p$ is small enough. By considering pullback partition $F^{-1}(\mathcal{R})$, $F : [0,1] \to \mathbb{F}^n$, $x \mapsto F(x) := (f_j(x))_j$, we can naturally almost embed X into the finite dimensional L_p-space $\langle \mathbb{1}_{F^{-1}R} \rangle_{R \in \mathcal{R}}$ (Proposition 4.7). By the approximate equimeasurability principle, Y is almost embedded into $\langle \mathbb{1}_{G^{-1}(R)} \rangle_{R \in \mathcal{R}}$ and the mapping $\mathbb{1}_{F^{-1}(R)} \mapsto \mathbb{1}_{G^{-1}(R)}$ linearly defines an almost isometry that almost extends γ, and that can be easily perturbed to become a surjective isometry I (Lemma 4.9). Now we use the extension result for isometric embeddings defined on ℓ_p^m (Proposition 3.5) to find the isometry almost extending γ. If $\mathbb{1}_{[0,1]} \notin X$, then we choose some normalized $u \in X$ of full support in X, and by the approximate full support principle we know that γu will have approximate full support of Y. We can now rotate both u and γu to $\mathbb{1}$, and use the case when X was unital. We start by introducing appropriate partitions.

Definition 4.6 (Appropriate partitions)**.** Let $F = (f_j)_{j<n}$ be a sequence of functions in $L_p(\Omega, \Sigma, \mu)$, $0 < \varepsilon \le 1$, and let $K > 0$ such that $\max_j \int_{|f_j| \ge K} |f_j|^p < \frac{\varepsilon^p}{3}$. We say that a finite partition $\mathcal{R}$ of $\mathbb{F}^n$ is (ε, K)- *appropriate* for F when each $P \in \mathcal{R}$ is of the form $P = \prod_{j<n} I_j$ where each I_j is either an interval of diameter strictly less than $\varepsilon/(3\|\mu\|)^{1/p}$, or else equal to $\mathbb{F} \setminus B(0,K)$.

Associated to such partition $\mathcal{R}$, we consider the finite partition $\mathcal{P} := F^{-1}(\mathcal{R}) = \{F^{-1}R : R \in \mathcal{R}\}$ of Ω, and the corresponding *conditional expectation* $\mathbb{E}(\cdot; \mathcal{P}) : L_p(\Omega, \Sigma, \mu) \to L_p(\Omega, \mathcal{P}, \mu)$,

$$\mathbb{E}(f; \mathcal{P}) = \sum_{\mu(F^{-1}(R))>0} \left(\frac{1}{\mu_0(F^{-1}(R))} \int_{F^{-1}(R)} f d\mu_0 \right) \mathbb{1}_{F^{-1}(R)}.$$

It is a consequence of Jensen's inequality that the conditional expectations is a norm one projection. Given an (ε, K)-appropriate partition $\mathcal{R}$ for $F := (f_j)_{j<n}$, let

$$\begin{aligned}\mathcal{R}^F_+ :=&\{R \in \mathcal{R} \,:\, F_*\mu(R) > 0\},\\ \mathcal{R}^F_0 :=&\mathcal{R}^F \setminus \mathcal{R}^F_+,\ R^F_0 := \bigcup \mathcal{R}^F_0 \text{ and for each } k < n,\\ \mathcal{R}^F_{+,k} :=&\{R \in \mathcal{R}^F_+ \,:\, \pi_k(R) \text{ is bounded}\},\ R^F_{+,k} := \bigcup \mathcal{R}^F_{+,k} \text{ and}\\ \mathcal{R}^F_{\infty,k} :=&\mathcal{R}^F_0 \setminus \mathcal{R}^F_{+,k}, \text{ and } R^F_{\infty,k} := \bigcup \mathcal{R}^F_{\infty,k}.\end{aligned}$$

Proposition 4.7. *Suppose that $\mathcal{R}$ is (ε, K)-appropriate for $F = (f_j)_{j<n}$. Then, for every $k < n$,*
(a) $\|\mathbb{E}(f_k; \mathcal{P}) \restriction F^{-1}(R^F_{\infty,k})\|^p_{L_p(\mu)} \le \|f_k \restriction F^{-1}(R_{\infty,k})^F\|^p_{L_p(\mu)} < \varepsilon^p/3$,
(b) $\|\mathbb{E}(f_k; \mathcal{P}) \restriction R^F_{+,k} - f_k \restriction R^F_{+,k}\|^p_{L_p(\mu)} \le \varepsilon^p/3$,
(c) $\|f_k - \mathbb{E}(f_k; \mathcal{P})\|_{L_p(\mu)} \le \varepsilon$.

Proof. The proof is standard. To simplify the notation we will avoid the superindex ^{F}. Fix $k < n$. The fact that $\|\mathbb{E}(f_k; \mathcal{P}) \restriction F^{-1}(R_{\infty,k})\|^p_{L_p(\mu)} \le \|f_k \restriction F^{-1}(R_{\infty,k})\|^p_{L_p(\mu)}$ follows from Jensen's inequality; Now observe that $R_{\infty,+} \subseteq \{|f_k| \ge K\}$, hence,

$$\|f_k \restriction F^{-1}(R_{\infty,k})\|^p_{L_p(\mu)} \|f_k \restriction \{|f_k| \ge K\}\|^p_{L_p(\mu)} < \frac{\varepsilon^p}{3}.$$

Given $R \in \mathrm{R}_{+,k}$, and given $\omega \in F^{-1}C$,

$$\left|\frac{1}{\mu(F^{-1}(R))}\int_{F^{-1}(R)} f_k d\mu - f_k(\omega)\right| \le \frac{1}{\mu(F^{-1}(R))}\int_{F^{-1}(R)} |f_k(\alpha) - f_k(\omega)|\, d\mu(\alpha) \le \operatorname{diam}(I_k) \le \frac{\varepsilon}{\sqrt[p]{3\|\mu\|}}.$$

Hence,

$$\int_{F^{-1}R} \left|\frac{1}{\mu(F^{-1}(R))}\int_{F^{-1}(R)} f_k d\mu - f_k(\alpha)\right|^p d\mu(\alpha) \le \frac{\varepsilon^p}{3\|\mu\|}\mu(F^{-1}(R)).$$

Putting all together,

$$\begin{aligned}\|\mathbb{E}(f_k; \mathcal{P}) \restriction R_{+,k} - f_k \restriction R_{+,k}\|^p_{L_p(\mu)} =& \sum_{R \in \mathcal{R}_{+,k}} \int_{F^{-1}R} \left|\frac{1}{\mu(F^{-1}(R))}\int_{F^{-1}(R)} f_k d\mu - f_k(\alpha)\right|^p d\mu(\alpha) \le\\ \le& \frac{\varepsilon^p}{3\|\mu\|} \sum_{\mu(F^{-1}(R))>0,\, \pi_k(R) \neq B(0,K)^c} \mu(F^{-1}(R)) \le \frac{\varepsilon^p}{3}.\end{aligned}$$

(c) follows easily from *(a)* and *(b)*. □

In the next, $z_j : \mathbb{F}^n \to \mathbb{F}$ is the j^{th}-projection mapping $(a_0, \dots, a_{n-1}) \mapsto a_j$, and given $f \in L_p(\Omega, \Sigma, \mu)$, let $f d\mu$ be the measure $(f d\mu)(A) := \int_A f d\mu$. The following is easy to prove.

Proposition 4.8. *Suppose that $\mathcal{R}$ is (ε, K)-appropriate for F. There is $\delta > 0$ such that if $G = (g_j)_j$ in $L_p(\nu)$ is such that $\max_j d_{\mathcal{L}P}(|z_j|^p d(F_*\mu), |z_j|^p d(G_*\nu)) \le \delta$, then $\mathcal{R}$ is also (ε, K)-appropriate for G.* □

For the next, recall that a μ-measurable set is of *μ-continuity* when $\mu(\partial A) = 0$, where ∂A is the topological boundary.

Lemma 4.9. *For every $X \subseteq L_p(\Omega_0, \Sigma_0, \mu_0)$ finite dimensional containing $\mathbb{1}_{\Omega_0}$ and every $\varepsilon > 0$ there are $\delta > 0$, $\mathcal{E}(X; \varepsilon) \subseteq L_p(\Omega_0, \Sigma_0, \mu_0)$ and $\xi_{X,\varepsilon} : X \to \mathcal{E}(X; \varepsilon)$ such that*
(a) $\mathcal{E}(X; \varepsilon)$ is isometric to some ℓ^m_p;
(b) $\|\xi_{X,\varepsilon} - i_X\|_{\mu_0} \le \varepsilon$;
(c) for every unital δ-isometric embedding $\gamma : X \to L_p(\Omega_1, \Sigma_1, \mu_1)$ there is an isometric embedding $I : \mathcal{E}(X; \varepsilon) \to L_p(\Omega_1, \Sigma_1, \mu_1)$ such that $\|I \circ \xi_{X,\varepsilon} - \gamma\| \le \varepsilon$.

We will say that the pair $(\mathcal{E}(X;\varepsilon),\xi_{X,\varepsilon})$ *is an* ε-envelope of X.

Proof. Fix ε_0, and let $F:=(f_j)_{j<n}$ be an Auerbach basis of X, that is $\max_j |a_j|\le \|\sum_j a_j f_j\|_{L_0(\Omega_0,\mu_0)}$. Let $\mathcal{R}$ be an $(\varepsilon/(6n),K)$ appropriate partition for F, consisting of $F_*\mu_0$-continuity sets and let $\mathcal{P}:=F^{-1}(\mathcal{R})$. It follows from Proposition 4.7 that $\max_j\|f_j-\mathbb{E}(f_j;\mathcal{P})\|\le\varepsilon/6n$, and since F is an Auerbach basis, we obtain that $\|f-\mathbb{E}(f;\mathcal{P})\|\le\varepsilon/6$. Let then $\mathcal{E}(X;\varepsilon):=\langle \mathbb{1}_P\rangle_{P\in\mathcal{P}}$ and let $\xi_{X,\varepsilon}:X\to\widehat{X}^\varepsilon$ be the restriction to X of the conditional expectation $\mathbb{E}(\cdot;\mathcal{P})$. We use Theorem 4.4 of approximate equimeasurability to find $\delta>0$ such that if $T:X\to L_p(\Omega_1,\Sigma_1,\mu_1)$ is a δ-embedding such that $T(\mathbb{1}_{\Omega_0})=\mathbb{1}_{\Omega_1}$, and setting $G:=(g_j)_{j<n}$, $g_j:=T(f_j)$, then

(i) $(1+\varepsilon_0)^{-1}\le\mu_1(G^{-1}R)/\mu_0(F^{-1}R)\le 1+\varepsilon_0$ for $R\in\mathcal{R}_+^F$, where $\varepsilon_0:=\varepsilon/(4nK(3\|\mu_0\|)^{1/p})$;

(ii) $\max_j\int_{G^{-1}R_0^F}|g_j|^p d\mu_1\le\varepsilon_1=\varepsilon^p/(n^p6)$;

(iii) $\mathcal{R}$ is $(\varepsilon/2n,K)$-adequate;

(iv) G is a 2-biorthogonal sequence of TX, that is $\max_j|a_j|\le 2\|\sum_j a_jg_j\|_{L_0(\Omega_1,\mu_1)}$.

(i) is possible because each C is a $F_*\mu_0$-continuity set, and (ii) and (iii) are possible because we can force each $|z_j|^pd(F_*\mu_0)$ and $|z_j|^pd(G_*\mu_1)$ to be close enough with respect to the Lévy-Prokhorov metric. Suppose that $\gamma:X\to L_p(\Omega_1,\mu_1)$ is a unital δ-embedding. Let $G:=(g_j)_{j<n}$, $g_j=\gamma(f_j)$, $\mathcal{Q}:=G^{-1}\mathcal{R}$. Observe that by (iv) and (v) we have that $\|i_{\gamma X}-\mathbb{E}(\cdot;\mathcal{Q})\restriction\gamma X\|_{L_p(\Omega_1,\mu_1)}\le\varepsilon$. Let $I:\mathcal{E}(X;\varepsilon)\to L_p(\Omega_1,\mu_1)$ be linearly defined for each $R\in\mathcal{R}_+^G$ by

$$I(\mathbb{1}_{F^{-1}R}):=\frac{\mu_0(F^{-1}R)}{\mu_1(G^{-1}R)}\mathbb{1}_{G^{-1}R}.$$

Note that by (i) above, $\mathcal{R}_+^F\subseteq\mathcal{R}_+^G$; hence, for scalars $(a_R)_{R\in\mathcal{R}_+^F}$,

$$\|I(\sum_{R\in\mathcal{R}_+^F}a_R\mathbb{1}_{F^{-1}R})\|_{L_p(\mu_1)}=\|\sum_{R\in\mathcal{R}_+^F}a_R\frac{\mu_0(F^{-1}R)}{\mu_1(G^{-1}R)}\|_{L_p(\mu_1)}=\sum_{R\in\mathcal{R}_+^F}a_R\mu_0(F^{-1}R)=\|\sum_{R\in\mathcal{R}_+^F}a_R\mathbb{1}_{F^{-1}R}\|_{L_p(\mu_0)}.$$

So, I is an isometric embedding. The proof will be finished once we establish that

$$\|I\circ\xi_{X,\varepsilon}-\mathbb{E}(\cdot;\mathcal{Q})\circ\gamma\|\le\varepsilon.$$

Fix $k<n$. We have that

$$\begin{aligned}I(\xi_{X,\varepsilon}(f_k))=&I(\mathbb{E}(f_k;\mathcal{P}))=\sum_{R\in\mathcal{R}_+^F}\left(\frac{1}{\mu_1(G^{-1}(R))}\int_{F^{-1}(R)}f_kd\mu_0\right)\mathbb{1}_{G^{-1}(R)};\\ \mathbb{E}(g_k;\mathcal{Q})=&\sum_{R\in\mathcal{R}_+^F}\left(\frac{1}{\mu_1(G^{-1}(R))}\int_{G^{-1}(R)}g_kd\mu_1\right)\mathbb{1}_{G^{-1}(R)}.\end{aligned}$$

Fix $R\in\mathcal{R}_{+,k}^F$, and let $\alpha\in\pi_k(C)$. Then we have that

$$\begin{aligned}\left|\int_{F^{-1}(R)}f_kd\mu_0-\int_{G^{-1}(R)}g_kd\mu_1\right|\le&\int_{F^{-1}(R)}|f_k-\alpha|\,d\mu_0+\int_{G^{-1}(R)}\left|g_k-\frac{\mu_0(F^{-1}(R)}{\mu_1(G^{-1}(R)}\alpha\right|d\mu_1\le\\ \le&\frac{\varepsilon}{6n\sqrt[p]{3\|\mu_0\|}}\mu_0(F^{-1}(R))+\left(\frac{\varepsilon}{6n\sqrt[p]{3\|\mu_0\|}}+\varepsilon_0K\right)\mu_1(G^{-1}(R))\le\\ \le&\left(\frac{3\varepsilon}{4n\sqrt[p]{3\|\mu_0\|}}\right)\mu_1(G^{-1}(R)).\end{aligned}$$

Hence,

$$\|I(\xi_{X,\varepsilon}(f_k)) \restriction G^{-1}(R^F_{+,k}) - \mathbb{E}(g_k;\mathcal{Q}) \restriction G^{-1}(R^F_{+,k})\|^p_{L_p(\mu_1)} =$$
$$= \sum_{R\in\mathcal{R}^F_{+,k}} \int_{G^{-1}(R)} \left|\frac{1}{\mu_1(G^{-1}(R))}\left(\int_{F^{-1}(R)} f_k d\mu_0 - \int_{G^{-1}(R)} g_k d\mu_1\right)\right|^p \le$$
$$\le \sum_{R\in\mathcal{R}^F_{+,k}} \left(\frac{3\varepsilon}{4n\sqrt[p]{3\|\mu_0\|}}\right)^p \mu_1(G^{-1}(R)) \le \sum_{R\in\mathcal{R}^F_{+,k}} \left(\frac{3\varepsilon}{4n}\right)^p \frac{\mu_1(G^{-1}(R))}{3\|\mu_0\|}.$$

On the other hand, by (iii) above,

$$\|\mathbb{E}(g_k;\mathcal{Q}) \restriction G^{-1}(R^F_0)\|^p_{L_p(\mu_1)} \le \|g_k \restriction G^{-1}(R^F_0)\|^p_{L_p(\mu_1)} \le \|g_k \restriction G^{-1}(R^G_0)\|^p_{L_p(\mu_1)} \le \frac{\varepsilon^p}{6n^p}$$

Finally, since $\mathcal{R}$ is $(\varepsilon/6n, K)$-adequate for both F and G, we obtain from (a) in Proposition 4.7 that

$$\|\mathbb{E}(g_k;\mathcal{Q}) \restriction G^{-1}(R^F_{\infty,k})\|^p_{L_p(\mu_1)} \le \frac{\varepsilon^p}{3(6n)^p}.$$

Since

$$I(\xi_{X,\varepsilon}(f_k) \restriction G^{-1}(R^F_{\infty,k})) = I(\mathbb{E}(f_k;\mathcal{P}) \restriction F^{-1}(R^F_{\infty,k}))$$

and I is an isometry, we obtain, using again (a) in Proposition 4.7 that

$$\|I(\xi_{X,\varepsilon}(f_k) \restriction G^{-1}(R^F_{\infty,k}))\|^p_{L_p(\mu_1)} = \|\mathbb{E}(f_k;\mathcal{P}) \restriction F^{-1}(R^F_{\infty,k})\|^p_{L_p(\mu_0)} \le \frac{\varepsilon^p}{3(6n)^p}.$$

Putting all together,

$$\|I\circ\xi_{X,\varepsilon}(f_k) - \mathbb{E}(g_k;\mathcal{Q})\|^p = \|(I\circ\xi_{X,\varepsilon}(f_k) - \mathbb{E}(g_k;\mathcal{Q})) \restriction G^{-1}(R^F_{+,k})\|^p + \|\mathbb{E}(g_k;\mathcal{Q}) \restriction G^{-1}(R^F_0)\|^p_{L_p(\mu_1)} +$$
$$+ \|(I\circ\xi_{X,\varepsilon}(f_k) - \mathbb{E}(g_k;\mathcal{Q})) \restriction G^{-1}(R^F_{\infty,k})\|^p \le$$
$$\le \sum_{R\in\mathcal{R}_{+,k}} \left(\frac{3\varepsilon}{4n}\right)^p \frac{\mu_1(G^{-1}(R))}{3\|\mu_0\|} + \frac{\varepsilon^p}{3(6n)^p} + 2\frac{\varepsilon^p}{3(6n)^p} \le \frac{2}{3}\left(\frac{3\varepsilon}{4n}\right)^p + \frac{\varepsilon^p}{(6n)^p} \le \frac{\varepsilon^p}{n^p}.$$

Since $(f_k)_{k<n}$ is an Auerbach basis of X, we obtain that $\|I\circ\xi_{X,\varepsilon} - \mathbb{E}(\cdot;\mathcal{Q})\circ\gamma\|_{L_p(\Omega_0,\mu_0),L_p(\Omega_1,\mu_1)} \le \varepsilon$. ☐

We are ready for the proof of the weak Fraïssé property of $L_p(0,1)$ for $p \notin 2\mathbb{N}$:

Proof of Theorem 4.1. Suppose that $X \in \mathrm{Age}(L_p(0,1))$, and $0 < \varepsilon \le 1$. Let A be the support of X. By making a small perturbation if needed, we assume that $\lambda(A) < 1$. Let $u \in S_X$ be such that $\operatorname{supp} u = A$, and let $\theta_X \in \mathrm{Iso}(L_p(A))$ be such that $\theta_X(u) = \lambda(A)^{-1/p}\mathbb{1}_A$ (see for example Lemma 3.3). Let now $X_0 := \theta(X)$. We apply Lemma 4.9 to X_0 and ε to find the corresponding $0 < \delta_0 \le \varepsilon$, and then Theorem 4.5 to u X and $\delta_0/2 > 0$ to find the corresponding $0 < \delta_1 \le \delta_0$. Let $0 < \delta \le \delta_1$ be such that $(1+\delta)^2 \le (1+\delta_0)(1-\delta_0/2)$. We claim that such δ works: For suppose that $\gamma : X \to L_p(0,1)$ is a δ-embedding. It follows that $\gamma(u)$ has $\delta_0/2$-full support in $\gamma(X)$. Let $B := \operatorname{supp}\gamma(u)$, that without loss of generality we assume that $\lambda(B) < 1$, and let $\gamma_0 : X \to L_p(0,1)$ be

$$\gamma_0 := \|\gamma(u)\|^{-1} P_B \circ \gamma.$$

By the choice of δ, we have that $\gamma_0 \in \mathrm{Emb}_{\delta_0}(X, L_p(B))$ and $\|\gamma_0 - \gamma\| \le 3\varepsilon$. Let now $\Phi : L_p(B) \to L_p(A)$ be an isometry onto. Let $Y := \Phi(P_B(\gamma(X))) := \{\Phi(g \restriction B) : g \in \gamma(X)\}$. We consider now $\gamma_1 : X \to Y \subseteq L_p(A)$,

$$\gamma_1 := \Phi \circ \gamma_0,$$

and note that $\gamma_1 \in \mathrm{Emb}_{\delta_0}(X, L_p(A))$. Set $v := \gamma_1(u)$, that is normalized and has support A, and let $\theta_Y \in \mathrm{Iso}(L_p(A))$ be such that $\theta_Y(v) = \lambda(A)^{-1/p}\mathbb{1}_A$. We set now $\gamma_2 : X_0 \to \theta_Y(Y) = Y_0$,

$$\gamma_2 := \theta_Y \circ \gamma_1 \circ \theta_X^{-1} \upharpoonright X_0.$$

Observe that $\mathbb{1}_A \in X_0 \cap Y_0$, and that $\gamma_2(\mathbb{1}_A) = \mathbb{1}_A$, so by the choice of δ_0, there is some $\widehat{X} \equiv \ell_p^m$, $\xi : X_0 \to \widehat{X}$ and an isometric embedding $I : \widehat{X} \to L_p(A)$ such that $\|\xi - i_{X_0}\| \le \varepsilon$, and $\|I \circ \xi - \gamma_2\| \le \varepsilon$. By Proposition 3.5, there is an isometry $\theta \in \mathrm{Iso}(L_p(A))$ such that $\|\theta \upharpoonright \widehat{X} - I\| \le \varepsilon$. Then, $\|\theta - \gamma_2\| \le 4\varepsilon$, or, equivalently,

$$\|\theta_2 - \gamma_0\| \le 4\varepsilon,$$

where $\theta_2 := \Phi^{-1} \circ \theta_Y^{-1} \circ \theta \circ \theta_X : L_p(A) \to L_p(B)$ is an isometry onto. Let $J \in \mathrm{Iso}(L_p(0,1))$ be extending θ_2. Let us see that this is possible: since we have that $0 < \lambda(A), \lambda(B) < 1$, we have that $L_p(A^c)$ and $L_p(B^c)$ are isometric, so we fix a surjective isometry $\Theta : L_p(A^c) \to L_p(B^c)$, and then $J(f) := \theta_2(f \upharpoonright A) + \Theta(f \upharpoonright A^c)$ makes the job. Since we know that $\|\gamma - \gamma_0\| \le 3\varepsilon$, putting all together we obtain that

$$\|J \upharpoonright X - \gamma\| \le 7\varepsilon. \qquad \square$$

Corollary 4.10. *For $p \neq 4, 6, \dots$ the class $\mathrm{Age}(L_p(0,1))$ of finite dimensional subspaces of $L_p(0,1)$ is a Fraïssé class whose limit is $L_p(0,1)$.*

In a personal communication I. Ben Yaacov [BY1] mentioned to us that there might be connections of this result with the Ryll-Nardzewski-type theorem and the quantifier elimination in the context of continuous logic.

4.2. **The approximate equimeasurability principle.** As mentioned before, Plotkhin and Rudin independently proved that for $p \notin 2\mathbb{N}$ the transform $\widehat{\mu}^{(p)}(a) := \|1 + \langle a, z\rangle\|_{L_p(\mu)}$ determines the measure μ (for which $\mathbb{E}_\mu(|z|^p) < \infty$), much like the *Fourier-Stieltjes transform* $\widehat{\mu}(a) := \int e^{i\Re\langle a,u\rangle} d\mu(u)$ does for an arbitrary measure μ. In this case, there is also a continuity aspect of it, called *Lévy's continuity theorem*, stating that for finite measures $\widehat{\mu_n} \to \widehat{\mu}$ converges uniformly on compacta, then $\mu_n \to_n \mu$ completely. The goal now is to see that a similar statement holds for the transform $\widehat{\mu}^{(p)}$. In fact, the distance between the transforms $\widehat{\mu}^{(p)}$ and $\widehat{\nu}^{(p)}$ will also determine the distance between the finite measures $|z|^\alpha d\mu$ and $|z|^\alpha d\nu$ for every $0 \le \alpha \le p$. Our proof follows some ideas of the standard proof of Levy's continuity theorem (see for example [Cu, Theorem 2.6.8]) and the proof of the equimeasurability principle given by C. D. Hardin in [Har, Theorem 1.1a] (see also [FleJa, Theorem 3.3.2]).

We start by recalling some basic concepts in measure theory. We refer the reader to [Ha] or [Cu] for more details. Given a separable metric space (X, d), let $\mathcal{M}(X)$ be the collection of all finite measures on $\mathcal{B}(X)$, the class of Borel subsets of (X, d). Given $\mu \in \mathcal{M}(X)$, $A \in \mathcal{B}(X)$ is called a *μ-continuity set* if $\mu(\partial A) = 0$, where ∂A is the topological boundary of A. Recall that a sequence $(\mu_n)_n$ of finite measures *converges weakly* to $\mu \in \mathcal{M}(X)$ when $\mu_n(A) \to_n \mu(A)$ for every bounded continuity set $A \in \mathcal{B}(X)$. The sequence $(\mu_n)_n$ converges *completely* to μ if it converges weakly and $\|\mu_n\| \to_n \|\mu\|$. It is well-known that $(\mu_n)_n$ converges completely to μ exactly when $(\mu_n)_n$ converges weakly to μ and $(\mu_n)_n$ is *tight*, or, equivalently, when $\mu_n(A) \to_n \mu(A)$ for every continuity set $A \in \mathcal{B}(X)$. Recall that $(\mu_n)_n$ is tight if for every $\varepsilon > 0$ there is a compact set $K \subseteq X$ such that $\sup_n \mu_n(X \setminus K) \le \varepsilon$. Given a function $f : X \to \mathbb{R}$ that is integrable with respect to $\mu \in \mathcal{M}(X)$ one defines the (signed) measure $f d\mu$ by $f d\mu(A) := \int_A f d\mu$. The *Fourier-Stieltjes transform* of $\mu \in \mathcal{M}(\mathbb{F}^n)$ is the function $\widehat{\mu} : \mathbb{F}^n \to \mathbb{C}$,

$$\widehat{\mu}(a) = \int e^{i\Re(\langle a, \bar{b}\rangle)} d\mu(b),$$

where $\Re(\alpha)$ is the real part of α, and $\overline{(b_1,\dots,b_n)} = (\bar{b}_1,\dots,\bar{b}_n)$ is the sequence of conjugates of $b_1,\dots,b_n$. Notice that when $\mathbb{F} = \mathbb{C}$, the definition above coincides with the standard definition of the Fourier-Stieltjes transform for measures on $\mathbb{R}^{2n}$, via the canonical identification of $\mathbb{C}$ with $\mathbb{R}^2$. Recall also that given $\mu \in \mathcal{M}(\mathbb{F}^n)$ and given $f : \mathbb{F}^n \to \mathbb{R}$ that is integrable with respect to λ and μ, one defines the convolution $f * \mu : \mathbb{F}^n \to \mathbb{R}$,

$$(f * \mu)(b) := \int f(b-a)d\mu(a),$$

that corresponds to the convolution $(fd\lambda) * \mu$ of the measures $fd\lambda$ and μ. A basic property we will use is that $\widehat{f * \mu} = \widehat{f} \cdot \widehat{\mu}$.

Definition 4.11. Let $\mathcal{M}^{(p)}(\mathbb{F}^n)$ be the collection of all Borel measures on $\mathbb{F}^n$ such that $|z|^p d\mu \in \mathcal{M}_n(\mathbb{F}^n)$, that is, $\int |z|^p d\mu(z) < \infty$.

In the previous definition $|z|$ is the euclidean norm of the vector z. It follows that $\mathcal{M}^{(0)}(\mathbb{F}^n) = \mathcal{M}(\mathbb{F}^n)$ is the collection of finite Borel measures on $\mathbb{F}^n$, and that $\mu \in \mathcal{M}^{(p)}(\mathbb{F}^n)$ if and only if $\int |z_j|^p d\mu(z) < \infty$ for every $1 \le j \le n$, where each $z \in \mathbb{F}^n \mapsto z_j \in \mathbb{F}$ is the canonical j^{th}-projection. Recall that given a measurable function $T : (\Omega_0, \Sigma_0) \to (\Omega_1, \Sigma_1)$ and a measure μ on Σ_0 one defines the *pushforward measure* $T_*\mu$ on Σ_1 by $(T_*\mu)(A) := \mu(T^{-1}(A))$ for $A \in \Sigma_1$. In particular, each sequence $F = (f_1,\dots,f_n)$ of elements of a Lebesgue space $L_p(\Omega,\Sigma,\mu)$ defines the measure $F_*\mu$ on $\mathbb{F}^n$, where F is interpreted as the measurable function $F : \Omega \to \mathbb{F}^n$, $F(\omega) = (f_1(\omega),\dots,f_n(\omega))$. Notice that for $H(z_1,\dots,z_n) \in L_p(F_*\mu)$ one has that $\int H(z_1,\dots,z_n)d(F_*\mu)(z_1,\dots,z_n) = \int H(f_1(\omega),\dots,f_n(\omega))d\mu(\omega)$.

Definition 4.12 (p-characteristics)**.** Given $\mu \in \mathcal{M}^{(p)}(\mathbb{F}^n)$, we define the *p-characteristics* $\widehat{\mu}^{(p)} : \mathbb{F}^n \to \mathbb{R}$ of μ by

$$\widehat{\mu}^{(p)}(a) := \|1 + \langle a, z\rangle\|_{L_p(\mu)} = \left(\int |1 + \langle a, z\rangle|^p d\mu(z)\right)^{\frac{1}{p}} \text{ for every } a \in \mathbb{F}^n.$$

With this terminology, Plotkin and Rudin results states

Theorem 4.13 (Uniqueness of the p-characteristics)**.** *Suppose that $p \notin 2\mathbb{N}$. If $\mu, \nu \in \mathcal{M}^{(p)}(\mathbb{F}^n)$ are such that $\widehat{\mu}^{(p)} = \widehat{\nu}^{(p)}$, then $\mu = \nu$.*

This is the corresponding continuity statement for p-characteristics.

Theorem 4.14 (Continuity of the p-characteristics)**.** *Suppose that $p \notin 2\mathbb{N}$. The following are equivalent for a sequence $(\mu_k)_k$ and a measure μ all in $\mathcal{M}^{(p)}(\mathbb{F}^n)$:*

1) $(|z|^\alpha d\mu_k)_k$ converges completely to $|z|^\alpha d\mu$ for all $0 \le \alpha \le p$;
2) $(|z|^p d\mu_k)_k$ converges completely to $|z|^p d\mu$ and $\|\mu_k\| \to_k \|\mu\|$;
3) $(\mu_k)_k$ converges completely to μ and $(|z|^p d\mu_k)_k$ is tight;
4) $(\widehat{\mu_k}^{(p)})_k$ converges to $\widehat{\mu}^{(p)}$ uniformly in all compacta of $\mathbb{F}^n$.

The proof of each implication is done in several steps, being *4)* $\implies$ *3)* the more interesting one.

Proposition 4.15. *Let $f : (X, d) \to \mathbb{R}^+$ be a continuous function and suppose that $(\mu_k)_k$ and μ satisfy that $\int f d\mu_n, \int f d\mu < \infty$. If $(\mu_k)_k$ converges weakly to μ and $(f d\mu_k)_k$ is tight, then $(\mu_k)_k$ is tight and $(f d\mu_k)_k$ converges completely to $f d\mu$. Consequently 3) $\implies$ 1) of Theorem 4.14 holds for every p.*

Proof. By one of the several characterizations of weak convergence (Portmanteau Theorem, [Bi, Theorem 2.1]) we have to check that $\limsup_k \int_C f d\mu_k \le \int_C f d\mu$ for every closed subset $C \subseteq X$. Fix $\varepsilon > 0$. and let $K \subseteq X$ be a compact subset such that $\sup_k (f d\mu_k)(K^c) < \varepsilon/2$. Now the function $g := \mathbb{1}_{K \cap C} \cdot f$

is upper semicontinuous and bounded, since $K \cap C$ is closed. Since $\mu_k \to_k \mu$ weakly, it follows that $\limsup_k \int g d\mu_k \leq \int g d\mu$. Consequently,

$$\limsup_k \int_C f d\mu_k \leq \limsup_k \int g d\mu_k + \limsup_k \int_{K^c} f d\mu_k \leq \int g d\mu + \varepsilon \leq \int_C f d\mu + \varepsilon.$$

Since ε is arbitrary, we are done. $\square$

Proposition 4.16. *Suppose that $(|z|^p d\mu_k)_k$ converges completely to $|z|^p d\mu$ and suppose that $\|\mu_k\| \to_k \|\mu\|$. Then $(\mu_k)_k$ converges completely to μ. Consequently 2) $\implies$ 3) of Theorem 4.14 holds for every p.*

Proof. The fact that $(|z|^p d\mu_k)_k$ converges completely gives that it is tight, and since $p \geq 0$, we obtain that $(\mu_k)_k$ is tight. Since this implies that every subsequence of $(\mu_k)_k$ has a further completely converging subsequence, the next claim implies that $(\mu_k)_k$ converges completely to μ.

Claim 4.16.1. Every complete converging subsequence of $(\mu_k)_k$ converges to μ.

Proof of Claim: Suppose that $(\mu_k)_{k\in M}$ converges completely to ν. It follows from Proposition 4.15 that $|z|^p d\mu_k \to_{k\in M} |z|^p d\nu(z)$ completely, so $|z|^p d\nu = |z|^p \mu$. We are going to see that this implies that $\nu = \mu$: Fix $\delta > 0$. Let $0 < \gamma \leq \delta$ be such that $(1+\delta)(1 - p\gamma/\delta) \geq 1$, and let $\mathcal{P}$ be a countable partition of $K_\delta := \mathbb{F}^n \setminus B(0,\delta)$ of (Borel) subsets of diameter at most γ. A simple computation gives that if $A \subseteq K_\delta$ as diameter at most γ, then $\sup_{a\in A} |a|^p \leq (1+\delta) \inf_{a\in A} |a|^p$. This implies that given $P \in \mathcal{P}$ one has that

$$\inf_{a\in P} |a|^p \mu(P) \leq \int_P |z|^p d\mu(z) = \int_P |z|^p d\nu(z) \leq \sup_{a\in P} |a|^p \nu(P) \leq (1+\delta) \inf_{a\in A} |a|^p \nu(P),$$

and hence $\mu(P) \leq (1+\delta)\nu(P)$. It follows that given $E \subseteq K_\delta$ and setting $\mathcal{P}_E := \{P \in \mathcal{P} : P \cap E \neq \emptyset\}$ we have that $E \subseteq \bigcup_{P\in\mathcal{P}_E} P \subseteq (E)_\gamma$, and consequently,

$$\mu(E) \leq \mu(\bigcup_{P\in\mathcal{P}_E} P) = \sum_{P\in\mathcal{P}_E} \mu(P) \leq (1+\delta) \sum_{P\in\mathcal{P}_E} \nu(P) \leq (1+\delta)\nu((E)_\gamma) \leq (1+\delta)\nu((E)_\delta).$$

Since δ was arbitrary, we obtain that $\mu(E) \leq \nu(\bar{E})$ and $\nu(E) \leq \mu(\bar{E})$ for every Borel $E \subseteq \mathbb{F}^n \setminus \{0\}$. In particular $\mu(K_\delta) = \nu(K_\delta)$ for every $\delta > 0$, so $\mu(\mathbb{F}^n \setminus \{0\}) = \nu(\mathbb{F}^n \setminus \{0\})$. Using that $\|\mu\| = \|\nu\|$ we obtain that $\mu(\{0\}) = \nu(\{0\})$. Now fix a Borel continuity subset $E \subseteq \mathbb{F}^n$ for μ and ν, and let $(\delta_n)_n$ be a decreasing sequence of strictly positive real numbers with limit 0 such that $\mu(\partial B(0,\delta_n)) = \nu(\partial B(0,\delta_n)) = 0$. Since $\partial(E \cap K_{\delta_n}) \subseteq \partial E \cup \partial B(0,\delta_n)$, it follows that $E \cap K_{\delta_n}$ is a continuity set, so $\mu(E \cap K_{\delta_n}) = \nu(E \cap K_{\delta_n})$ for all n. Hence, $\mu(E \setminus \{0\}) = \lim_{n\to\infty} \mu(E \cap K_{\delta_n}) = \lim_{n\to\infty} \nu(E \cap K_{\delta_n}) = \nu(E \setminus \{0\})$ and consequently $\mu(E) = \nu(E)$. $\square$

$\square$

We need the following simple estimates.

Proposition 4.17. *Suppose that $\mu, \nu \in \mathcal{M}^{(p)}(\mathbb{F}^n)$. Then*
(a) $0 \leq \widehat{\mu}^{(p)}(a) \leq \|\mu\|^{1/p} + |a| \cdot \| |z|^p d\mu\|^{1/p}$ for every $a \in \mathbb{F}^n$;
(b) $\widehat{\mu}^{(p)}$ is uniformly continuous; in fact, $|\widehat{\mu}^{(p)}(a) - \widehat{\mu}^{(p)}(b)| \leq |a-b| \cdot \||z|^p d\mu\|^{1/p}$.

Proof. *(a)*: Using Cauchy–Schwarz, $\widehat{\mu}^{(p)}(a) = \|1 + \langle a, z\rangle\|_{L_p(\mu)} \leq \|1\|_{L_p(\mu)} + |a|\||z|\|_{L_p(\mu)} = \|\mu\|^{\frac{1}{p}} + |a|\||z|^p d\mu\|^{\frac{1}{p}}$. *(b)*: $|\widehat{\mu}^{(p)}(a) - \widehat{\mu}^{(p)}(b)| \leq \|\langle a-b, z\rangle\|_{L_1(\mu)} \leq |a-b|\||z|\|_{L_1(\mu)} = |a-b| \cdot \||z|^p d\mu\|^{1/p}$. $\square$

Proposition 4.18. *Suppose that $(|z|^\alpha d\mu_k)_k$ converges completely to $|z|^\alpha d\mu$ for $\alpha = 0, p$. Then $(\widehat{\mu_k}^{(p)})_k$ converges to $\widehat{\mu}^{(p)}$ uniformly in all compacta of $\mathbb{F}^n$. Consequently 1) $\implies$ 4) of Theorem 4.14 holds for every p.*

Proof. Fix $M > 0$ and $\varepsilon > 0$. We know that $(|z|^p\mu_k)_k$ is tight, so there is some $K \geq 1$ such that $\int_{|z|>K} |z|^p d\mu_k(z) \leq \varepsilon$. Let D be a finite ε-dense subset of $B(0,M)$, and let k_0 be such that

$$\left|\int_{|z|\leq K} |1+\langle a,z\rangle|^p d\mu_k(z) - \int_{|z|\leq K} |1+\langle a,z\rangle|^p d\mu(z)\right| \leq \varepsilon$$

for every $a \in D$ and every $k \geq k_0$. This is possible since $\mathbb{1}_{|z|\leq K}|1+\langle a,z\rangle|^p$ is bounded and $\mu_k \to \mu$ completely. Then for such k and a,

$$\begin{aligned}|\widehat{\mu_k}^{(p)}(a) - \widehat{\mu}^{(p)}(a)| \leq & \left|\int_{|z|\leq K} |1+\langle a,z\rangle|^p d\mu_k(z) - \int_{|z|\leq K} |1+\langle a,z\rangle|^p d\mu(z)\right| + \mu(B(0,K)^c) + \\ & + \mu_k(B(0,K)^c) + |a|^p((|z|^p d\mu)(B(0,K)^c) + (|z|^p d\mu_k)(B(0,K)^c)) \leq \varepsilon(3+2|a|^p) \leq \\ & \leq \varepsilon(3+2M^p).\end{aligned}$$

Using this and Proposition 4.17 *(b)*, for every $|a| \leq M$ and $k \geq k_0$,

$$|\widehat{\mu_k}^{(p)}(a) - \widehat{\mu}^{(p)}(a)| \leq \varepsilon(\||z|^p d\mu\| + \||z|^p d\mu_k\| + 3 + 2M^p).$$

This shows that $\widehat{\mu_k}^{(p)} \to \widehat{\mu}^{(p)}$ uniformly on $B(0,M)$. □

We have already seen that *1)*, *2)* and *3)* of Theorem 4.14 are equivalent (*1)* implies *2)* trivially) and also that *3)* $\implies$ *4)* in there. We finish by showing that *4)* $\implies$ *3)*. We start with the following interesting criteria for complete convergence extending the proof given by Hardin in [Har, Theorem 1.1a] of the equimeasurability principle for $n = 1$.

Lemma 4.19. *Suppose that $(\mu_k)_k$ and μ are measures in $\mathcal{M}(\mathbb{F})$ such that $\|\mu_k\| \to_k \|\mu\|$ and such that there is a continuous $0 \neq f \in L_1(\mathbb{F},\lambda)$ such that $\int f(a+bz)d\mu_k(z) \to_k \int f(a+bz)d\mu(z)$ for every $(a,b) \in \mathbb{F}\times\mathbb{F}$. Then $(\mu_k)_k$ converges completely to μ.*

Proof. We fix all data. We start with the following.

Claim 4.19.1. $(\mu_k)_k$ is tight.

Proof of Claim: Fix $\varepsilon > 0$. Let z_0 be such that $|f(z_0)| = \|f\|_\infty$, and let $\delta > 0$ be such that $(1-\varepsilon)|f(z)| \leq |f(z_0)|$ for $|z - z_0| \leq \delta$. Fix $K > 0$ such that $\mu(B(0,K)^c) \leq \varepsilon$ and let $g(z) := |f(z_0)|^{-1} f((\delta/K)z + z_0)$. Notice that $\int g(a+bz)d\mu_k \to_k \int g(a+bz)d\mu$ for every $(a,b) \in \mathbb{F}\times\mathbb{F}$, that $1 = g(0) = \max_z g(z) = \|g\|_\infty$, and that $1-\varepsilon \leq g(z) \leq 1$ for every $|z| \leq K$. Let $L \geq K$ be such that $\max_{|z|\geq L} |g(z)| \leq \varepsilon$. Let k_0 be such that $|\int g d\mu_k - \int g d\mu|, |\|\mu_k\| - \|\mu\|| \leq \varepsilon$ for every $k \geq k_0$. Then for such k,

$$\begin{aligned}\mu_k(B(0,L)) \geq & \int_{|z|\leq L} g(z)d\mu_k(z) \geq \int g(z)d\mu_k(z) - \varepsilon\|\mu_k\| \geq \int g(z)d\mu(z) - \varepsilon(1+\|\mu_k\|) \geq \\ \geq & \int_{|z|\leq K} g(z)d\mu(z) - (2+\|\mu_k\|)\varepsilon \geq (1-\varepsilon)\mu(B(0,K)) - (2+\|\mu_k\|)\varepsilon \geq \\ \geq & (1-\varepsilon)(\|\mu\| - \varepsilon) - (2+\|\mu_k\|)\varepsilon \geq (1-\varepsilon)(\|\mu_k\| - 2\varepsilon) - (2+\|\mu_k\|)\varepsilon = \\ = & \|\mu_k\| - (4+2\|\mu_k\|)\varepsilon.\end{aligned}$$

So, $\mu_k(B(0,L)^c) \leq (4+2\|\mu_k\|)\varepsilon$, and since $(\mu_k)_k$ is bounded, the previous inequality shows that $(\mu_k)_k$ is tight. □

Claim 4.19.2. Suppose that ν is such that $\|\mu\| = \|\nu\|$ and that $\int f(a+bz)d\mu(z) = \int f(a+bz)d\nu(z)$ for every $a, b \in \mathbb{F}$. Then $\mu = \nu$.

Proof of Claim: This is essentially proved in [Har]. For the sake of completeness we give a proof. Given $c \in \mathbb{F}$, let $f_c(z) := f(cz)$. For $a, c \in \mathbb{F}$ we have $(f_c * \mu)(a) = \int f_c(a-u)d\mu(u) = \int f(ca-cu)d\mu(u) = \int f(ca-cu)d\nu(u) = \int f_c(a-u)d\nu(u) = (f*\nu)(a)$, so,

$$\widehat{f}_c\widehat{\mu} = \widehat{f_c * \mu} = \widehat{f_c * \nu} = \widehat{f}_c\widehat{\nu} \tag{6}$$

We prove that $\widehat{\mu} = \widehat{\nu}$, that, by the uniqueness of Fourier-Stieltjes transform, implies that $\mu = \nu$. So, fix $a \in \mathbb{F}$. First of all, if $a = 0$, then $\widehat{\mu}(0) = \|\mu\| = \|\nu\| = \widehat{\nu}(0)$. So, suppose that $a \neq 0$; let $0 \neq a_0 \in \mathbb{F}$ be such that $\widehat{f}(a_0) \neq 0$. This is possible because $f \neq 0$, and by the continuity of $\widehat{f}$. Set $c := a/a_0$. It follows from (6) that

$$\frac{1}{c}\widehat{f}(a_0)\widehat{\mu}(a) = \widehat{f}_c(a)\widehat{\mu}(a) = \widehat{f}_c(a)\widehat{\nu}(a) = \frac{1}{c}\widehat{f}(a_0)\widehat{\nu}(a),$$

hence $\widehat{\mu}(a) = \widehat{\nu}(a)$. □

We are going to see now that every subsequence of $(\mu_k)_k$ has a further subsequence converging completely to μ, which proves that $(\mu_k)_k$ converges completely to μ: Fix a subsequence $(\mu_k)_{k\in M}$ of $(\mu_k)_k$; by Helly's first Theorem, there is a further subsequence $(\mu_k)_{k\in N}$ of $(\mu_k)_{k\in M}$ converging completely to ν. Since for each $a, b \in \mathbb{F}$ the function $f(a+bz)$ is continuous and bounded, we have that $\int f(a+bz)d\mu_k(z) \to_{k\in N} \int f(a+bz)d\mu(z)$ for every $a, b \in \mathbb{F}$. Hence, $\int f(a+bz)d\mu(z) = \int f(b+bz)d\nu(z)$ for every $a, b \in \mathbb{F}$, and this implies, by Claim 4.19.2, that $\nu = \mu$. □

We are going to use Lemma 4.19 to show the implication *4)* $\implies$ *3)*, so we have to find the appropriate function. This is the content of the next result.

Proposition 4.20. *For every $p \notin 2\mathbb{N}$ and every $m \geq 2\lfloor p \rfloor + 6$ there exists a sequence $(a_j)_{j\leq m}$ of real numbers such that the function $f(z) := \sum_{j=0}^m a_j |z+j|^p$ satisfies*

(a) $f \neq 0$;
(b) $f \in L_1(\lambda)$;
(c) $\lim_{z\to 0} \frac{f(z)}{|z|^p} = a_0$.

Proof. Recall that for $|z| > m$ and $0 \leq j \leq m$ one has that

$$|z+j|^p = (z+j)^{\frac{p}{2}}(\overline{z}+j)^{\frac{p}{2}} = \sum_{k,l=0}^{\infty} \binom{\frac{p}{2}}{k}\binom{\frac{p}{2}}{l} z^{\frac{p}{2}-k}\overline{z}^{\frac{p}{2}-l} j^{k+l};$$

while if $|z| < 1$,

$$|z+j|^p = (z+j)^{\frac{p}{2}}(\overline{z}+j)^{\frac{p}{2}} = \sum_{k,l=0}^{\infty} \binom{\frac{p}{2}}{k}\binom{\frac{p}{2}}{l} j^{p-(k+l)} z^k \overline{z}^l;$$

Let $(a_j)_{j\leq m}$ be a non-trivial solution of the system

$$\begin{cases} \sum_{j=0}^m a_j j^k = 0 & 0 \leq k \leq \lfloor p \rfloor + 2 \\ \sum_{j=1}^m a_j j^{p-l} = 0 & 0 \leq l \leq \lfloor p \rfloor + 1. \end{cases}$$

Let $f(z) := \sum_{j=0}^m a_j|z+j|^p$. It follows that for $|z|>m$,

$$\begin{aligned}|f(z)| &= \left|\sum_{j=0}^m a_j|z+j|^p\right| = \left|\sum_{j=0}^m a_j \sum_{k,l=0}^\infty \binom{\frac{p}{2}}{k}\binom{\frac{p}{2}}{l} z^{\frac{p}{2}-k}\overline{z}^{\frac{p}{2}-l} j^r\right| = \\ &= \left|\sum_{r\geq 0}\left(\sum_{j=0}^m a_j j^r\right)\sum_{k+l=r}\binom{\frac{p}{2}}{k}\binom{\frac{p}{2}}{l} z^{\frac{p}{2}-k}\overline{z}^{\frac{p}{2}-l}\right| = \\ &= \left|\sum_{r=\lfloor p\rfloor+3}^\infty\left(\sum_{j=0}^m a_j j^r\right)\sum_{k+l=r}\binom{\frac{p}{2}}{k}\binom{\frac{p}{2}}{l} z^{\frac{p}{2}-k}\overline{z}^{\frac{p}{2}-l}\right| \leq \sum_{r=\lfloor p\rfloor+3}^\infty \left|\sum_{j=0}^m a_j j^r\right| \sum_{k+l=r}\left|\binom{\frac{p}{2}}{k}\binom{\frac{p}{2}}{l}\right| |z|^{p-r} = \\ &= O(|z|^{\lfloor p\rfloor - p+2}).\end{aligned}$$

Since $\lfloor p\rfloor - p + 2 > 1$, the previous inequality shows that $f\in L_1(\lambda)$. On the other hand, for $|z|<1$ and similarly as before,

$$\begin{aligned}|f(z)-a_0|z|^p| &= \left|\sum_{j=1}^m a_j|z+j|^p\right| = \left|\sum_{j=1}^m a_j\sum_{k,l=0}^\infty\binom{\frac{p}{2}}{k}\binom{\frac{p}{2}}{l} j^{p-k-l} z^k\overline{z}^l\right| = \\ &= \left|\sum_{r\geq 0}\left(\sum_{j=1}^m a_j j^{p-r}\right)\sum_{k+l=r}\binom{\frac{p}{2}}{k}\binom{\frac{p}{2}}{l} z^k\overline{z}^l\right| = \\ &= \left|\sum_{r=\lfloor p\rfloor+2}^\infty\left(\sum_{j=1}^m a_j j^{p-r}\right)\sum_{k+l=r}\binom{\frac{p}{2}}{k}\binom{\frac{p}{2}}{l} z^k\overline{z}^l\right| \leq \sum_{r=\lfloor p\rfloor+2}^\infty\left|\sum_{j=1}^m a_j j^{p-r}\right|\sum_{k+l=r}\left|\binom{\frac{p}{2}}{k}\binom{\frac{p}{2}}{l}\right| |z|^r = \\ &= O(|z|^{\lfloor p\rfloor+1}) \text{ as } z\to 0.\end{aligned}$$

So, $\lim_{z\to 0}|f(z)-a_0|z|^p|/|z|^p \leq \lim_{z\to 0} O(|z|^{\lfloor p\rfloor+1})/|z|^p = 0$. □

Lemma 4.21. *Suppose that $(\widehat{\mu_k})_k$ converges to $\widehat{\mu}$ uniformly on compacta of $\mathbb{F}^n$. Then $(|z|^p d\mu_k(z))_k$ is tight and $(\mu_k)_k$ converges completely to μ. Consequently, 4) $\implies$ 3) of Theorem 4.14 holds.*

Proof. We start with the following:

Claim 4.21.1. $(|z|^p d\mu_k(z))_k$ is tight.

Proof of Claim: For each $1\leq j\leq n$, let $\mu_k^{(j)} := (\pi_j)_*\mu_k \in \mathcal{M}_p(\mathbb{F})$, where $\pi_j:\mathbb{F}^n\to\mathbb{F}$ is the canonical projection $\pi_j(z_1,\dots,z_n)=z_j$. Observe that for each $K>0$,

$$\int_{\|z\|_\infty>K}\|z\|_\infty^p d\mu_k(z) \leq \sum_{j=1}^n\int_{K<\|z\|_\infty=|z_i|}|z_i|^p d\mu_k(z)\leq \sum_{j=1}^n\int_{K<|t|}|t|^p d\mu_k^{(j)}(t), \tag{7}$$

so, it suffices to show that each $(|t|^p d\mu_k^{(j)}(t))_k$ is a tight sequence for each $1\leq j\leq n$. We fix one of such j, and to simplify the notation we set $\nu := (\pi_j)_*\mu$ and $\nu_k := \mu_k^{(j)}$ for every k. Note that $\widehat{\nu_k}^{(p)} \to_k \widehat{\nu}$, so in particular $\|\nu_k\|\to_k\|\nu\|$. Let f be a function as in Proposition 4.20 for $m:=2\lfloor p\rfloor+6$. Then for each $a,b\in\mathbb{F}$ one has that $\int f(a+bz)d\nu_k \to_k \int f(a+bz)d\nu$, so it follows from Lemma 4.19 that $\nu_k\to_k\nu$ completely. Since $\widehat{\nu_k}^{(p)}\to\widehat{\nu}$, in particular one has that $\||t|^p d\nu_k(t)\|\to_k\||t|^p d\nu(t)\|$. Set $F(z) := f(z)/|z|^p$. For each $a,b\in\mathbb{F}$ we have that $F(a+bz)|z|^p$ is bounded and continuous, so it follows that $\int F(a+bt)|t|^p d\nu_k(t)\to_k\int F(a+bt)|t|^p d\nu(t)$. Again using Lemma 4.19 we obtain that $|t|^p d\nu_k(t)\to_k |t|^p d\nu(t)$ completely, so in particular $(|t|^p d\nu_k(t))_k$ is tight, as desired. □

Let us prove now that $\mu_k\to\mu$ completely. Since $(\mu_k)_k$ is tight, by Helly's first Theorem, it suffices to show that each completely convergent subsequence of $(\mu_k)_k$ converges completely to μ; so, fix one such completely convergent subsequences $\mu_{k_l} \to_l \nu$; Since $(|z|^p d\mu_{k_l}(z))_l$ is in addition tight, it follows from the implication *3)* $\implies$ *4)* of Theorem 4.14 that $\widehat{\mu_{k_l}}^{(p)} \to_l \widehat{\nu}^{(p)}$; in particular $\widehat{\nu}^{(p)} = \widehat{\mu}^{(p)}$, and by the uniqueness theorem of the p-characteristics, we have that $\nu = \mu$, as desired. □

4.2.1. *Inversion formulas for $\mathbb{F} = \mathbb{R}$ and $p \in 2\mathbb{N}+1$.* When working on real numbers and p an odd integer, there is a more direct and elementary proof of 3) $\implies$ 2) of Theorem 4.14.

Lemma 4.22. *For $a \neq 0$ we have*

$$\mu(]-\infty, a]) = \frac{1}{2} + \frac{1}{2\cdot(p!)} \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^p} \sum_{j=0}^{p} (-1)^{j+1}\binom{p}{j} |a+j\varepsilon|^p \left(\widehat{\mu}^{(p)}\left(\frac{-1}{a+j\varepsilon}\right)\right)^p$$

Observe that the previous formula shows that a sequence $(\mu_k)_k$ in $\mathcal{M}^{(p)}(\mathbb{R})$ converges completely to $\mu \in \mathcal{M}^{(p)}(\mathbb{R})$ when $(\widehat{\mu_k}^{(p)})_k$ converges to $\widehat{\mu}^{(p)}$ uniformly on compacta. Given $a \in \mathbb{R}$ and $\varepsilon > 0$, let

$$G_p(x,a,\varepsilon) := \frac{1}{2} + \frac{1}{2(p!)\varepsilon^p} \sum_{j=0}^{p} (-1)^{j+1}\binom{p}{j} |x-(a+j\varepsilon)|^p.$$

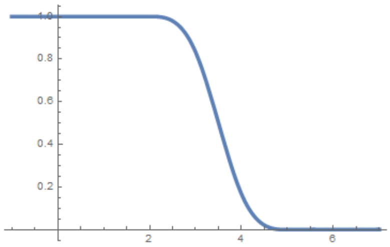

FIGURE 2. $G_3(x,2,1)$

Proposition 4.23. $G_p(x,a,\varepsilon) = 1$ *if* $x \leq a$, $G_p(x,a,\varepsilon)(x) = 0$ *if* $x \geq a+\varepsilon p$ *and* $0 \leq G_p(\cdot,a,\varepsilon) \leq 1$*; consequently,*

$$\mu(]-\infty,a]) \leq \int G_p(x,a,\varepsilon) d\mu(x) \leq \mu(]-\infty,a+\varepsilon p]); \tag{8}$$

PROOF. *(a)*: Suppose that $x \leq a$; then $|x-(a+j\varepsilon)|^p = (a-x+j\varepsilon)^p$, so

$$G_p(x,a,\varepsilon) = \frac{1}{2} + \frac{1}{2}\frac{1}{p!\varepsilon^p} \sum_{k=0}^{p} \binom{p}{k} (a-x)^{p-k}(-\varepsilon)^k \left(\sum_{j=0}^{p} (-1)^{j+1}\binom{p}{j} j^k\right) = 1,$$

because $\sum_{j=0}^p (-1)^{j+1}\binom{p}{j} j^k = 0$ for $0 \leq k < p$ and $\sum_{j=0}^p (-1)^{j+1}\binom{p}{j} j^p = p!$, because p is odd. The estimate for $x \geq a+\varepsilon p$ is similar. Then note that $G_p(x,a,\varepsilon) + G_p(2a+p\varepsilon-x,a,\varepsilon) = 1$ so it is enough to check the inequality $0 \leq G_p(x,a,\varepsilon) \leq 1$ for $a \leq x \leq a+\varepsilon p/2$. For $1 \leq r < p/2$ we consider the $2r$-derivative of $G_p(x,a,\varepsilon)$ with respect to x,

$$G_p^{(2r)}(x,a,\varepsilon) = \frac{1}{2}\frac{1}{(p-2r)!\varepsilon^p} \sum_{j=0}^{p} (-1)^{j+1}\binom{p}{j} |x-(a+j\varepsilon)|^{p-2r}.$$

and note that it assumes the value 0 in $x = a$ and in $x = a+\varepsilon p/2$, and that when $p = 2r+1$ then it is affine on each interval $[a+j\varepsilon, a+(j+1)\varepsilon]$, $0 \leq j \leq p-1$ and therefore admits at most $\frac{p+1}{2}$ zeros on $[a, a+\varepsilon p/2]$ (an easy computation shows that actually exactly $\frac{p+1}{2}$ zeros are obtained). By standard analysis we also note that if for $r \geq 1$, $G_p^{(2r)}(x,a,\varepsilon)$ admits at most m zeros (including a and $a+\varepsilon p/2$) in $[a, a+\varepsilon p/2]$, then $G_p^{(2r-1)}(x,a,\varepsilon)$ admits at most $m-1$ zeros (including a), and then again $G_p^{(2r-2)}(x,a,\varepsilon)$ admits at most $m-1$ zeros in the same interval. From these two facts we deduce that $G_p'(x,a,\varepsilon)$ admits at most one zero in $[a, a+\varepsilon p/2]$, which must be a, and therefore $G_p(x,a,\varepsilon)$ is monotonous (actually decreasing) there and in particular takes values between $\frac{1}{2}$ and 1. This concludes the proof. □

Now Lemma 4.22 follows from this Proposition, simply using that for $c \neq 0$ one has that

$$\int |x+c|^p d\mu(x) = |c|^p \left(\widehat{\mu}^{(p)}\left(\frac{1}{c}\right)\right)^p.$$

4.2.2. *Metrics on $\mathcal{M}(\mathbb{F}^n)$.* We give now a quantitative version of the Continuity Theorem 4.14.

Definition 4.24. Given $\mu, \nu \in \mathcal{M}^{(p)}(\mathbb{F}^n)$, we define

$$\partial_p(\mu,\nu) := \inf\{K \geq 1 \, : \, \frac{1}{K}\widehat{\nu}^{(p)}(a) \leq \widehat{\mu}^{(p)}(a) \leq K\widehat{\nu}^{(p)}(a) \text{ for all } a \in \mathbb{F}^n\}.$$

Such K always exists because the basic sequence $(\mathbb{1}_{\mathbb{F}^n}, z_0, \dots, z_{n-1}\rangle)$ in $L_p(\mu)$ and in $\subseteq L_p(\nu)$ must be equivalent. The function ∂_p is a multiplicative pseudometric, that because of the uniqueness of the p-characteristics, it is a multiplicative metric. It is easy to see that ∂_p defines the compact convergence on $\mathcal{M}^{(p)}(\mathbb{F}^n)$.

We give now a quantitative version of the continuity theorem of the p-characteristics. Given $\theta : \mathbb{N} \to \mathbb{N}$, let

$$\mathcal{T}_\theta := \{\mu \in \mathcal{M}(\mathbb{F}^n) \, : \, \|\mu\| \leq \theta(0) \text{ and } \mu(\mathbb{F}^n \setminus B(0,\theta(m+1))) \leq \frac{1}{2^m} \text{ for every } m \in \mathbb{N}\}.$$

Notice that $\bigcup_{\theta \in^{\mathbb{N}}\mathbb{N}} \mathcal{T}_\theta = \mathcal{M}(\mathbb{F}^n)$. As a consequence of the continuity theorem, we obtain the following.

Corollary 4.25 (Quantitative Continuity Theorem)**.** *Let $\theta : \mathbb{N} \to \mathbb{N}$ and let $\varepsilon > 0$.*

(a) Given $0 \leq \alpha \leq p$ there is some $\delta > 0$ such that if $|z|^p d\mu \in \mathcal{M}^{(p)}(\mathbb{F}^n) \cap \mathcal{T}_\theta$, $\|\mu\| \leq \theta(0)$ and $\nu \in \mathcal{M}^{(p)}(\mathbb{F}^n)$ is such that $\partial_p(\mu,\nu) < 1+\delta$, then $d_{\mathcal{LP}}(|z|^\alpha d\mu, |z|^\alpha d\nu) < \varepsilon$.

(b) There is some $\delta > 0$ such that if $\mu, \nu \in \mathcal{M}^{(p)}(\mathbb{F}^n)$ are such that $\|\mu\|, \|\nu\| \leq \theta(0)$, $|z|^p d\mu, |z|^p d\nu \in \mathcal{T}_\theta$ and $d_{\mathcal{LP}}(\mu,\nu) < \delta$, then $\partial_p(\mu,\nu) < 1+\varepsilon$.

Proof. *(a)*: Suppose otherwise; we can find sequences $(|z|^p d\mu_k)_k$ in $\mathcal{M}^{(p)}(\mathbb{F}^n) \cap \mathcal{T}_\theta$ and with $\sup_k \|\mu_k\| \leq \theta(0)$, and $(\nu_k)_k$ in $\mathcal{M}^{(p)}(\mathbb{F}^n)$ such that $\lim_k \partial_p(\mu_k,\nu_k) = 1$ and $\inf_k d_{\mathcal{LP}}(|z|^\alpha d\mu_k, |z|^\alpha d\nu_k) \geq \varepsilon$. Since $(\mu_k)_k$ is bounded in norm and tight, there is a completely convergent subsequence to μ. Without loss of generality, we assume that $(\mu_k)_k$ converges completely to μ, Since $(|z|^p d\mu_k)_k$ is tight, it follows from the implications 2. $\implies$ 3. and 2. $\implies$ 1. in Theorem 4.14 that $\lim_k \partial_p(\mu_k,\mu) = 1$ and that $\lim_k d_{\mathcal{LP}}(|z|^\alpha d\mu_k, |z|^\alpha d\mu) = 0$. Resuming, $\lim_k \partial_p(\mu,\nu_k) = 1$ and $\inf_k d_{\mathcal{LP}}(|z|^\alpha d\mu, |z|^\alpha d\nu_k) \geq \varepsilon$, contradicting the implication 3. $\implies$ 1. in Theorem 4.14. The proof of *(b)* is similar. We leave the details to the reader. □

4.2.3. *Approximate full support.* We finish the section by proving the approximate full support principle in Theorem 4.5. We assume that $L_p(\mu)$ is separable. Recall that for a μ-measurable subset A, P_A denotes be the Boolean projection on $L_p(\mu)$ defined by $P_A(f) := f \cdot \mathbb{1}_A$, and that given a subspace X of $L_p(\mu)$, and let $\varepsilon > 0$, we say that f has *ε-full support in X* if $\|P_{\{f=0\}} \upharpoonright X\| \leq \varepsilon$. We recall that Theorem 4.5 states that if $u \in X$ has full support in X then for every $\varepsilon > 0$ there is some $\delta > 0$ such that if $\gamma \in \mathrm{Emb}_\delta(X, L_p(\mu))$ then γu has ε-full support in γX. We will follow the same strategy than in [Har, Section 3]. We need the following preliminary result.

Lemma 4.26. *Let $D \subseteq L_p(\Omega)$ be countable. For every $0 < \varepsilon < 1$ there is $a_1, a_2 \in [0,1]$ such that for every $f_1, f_2 \in D$ one has that $a_1 f_1 + a_2 f_2$ has full support in $\langle f_1, f_2\rangle$ and $1 - a_1, a_2 \leq \varepsilon$.*

Proof. Fix $g_1, g_2 \in L_p(\mu)$. Let $S := \operatorname{supp} g_1 \cup \operatorname{supp} g_2$ and let $g : S \times [0,1] \to \mathbb{F}$, $g(\omega, t) := (1-\varepsilon)tg_1(\omega) + \varepsilon(1-t)g_2(\omega)$. We consider the product measure $\mu \times \lambda$ on $S \times [0,1]$, where λ is the Lebesgue measure on $[0,1]$. Let $A \subseteq S \times [0,1]$ be the set $A := \{(\omega, t) \in S \times [0,1] : g(\omega, t) = 0\}$. Notice that given $\omega \in S$ we have that $\lambda(A_\omega) = 0$, where $A_\omega := \{t \in [0,1] : (\omega, t) \in A\}$: We have that $t \in A_\omega$ if and only if $t((1-\varepsilon)g_1(\omega) - \varepsilon g_2(\omega)) = -\varepsilon g_2(\omega)$, and since $\omega \in S$, we obtain that $(1-\varepsilon)g_1(\omega) - \varepsilon g_2(\omega) \neq 0$, so $A_\omega = \{-\varepsilon g_2(\omega)((1-\varepsilon)g_1(\omega) - \varepsilon g_2(\omega))^{-1}\}$ is a singleton. We define also, given $t \in [0,1]$, $A^t = \{\omega \in S : (\omega, t) \in A\}$. Now using Fubini Theorem,

$$0 = \int_{\omega \in S} \lambda(A_\omega) d\mu(\omega) = (\mu \times \lambda)(A) = \int_{t \in [0,1]} \mu(A^t) d\lambda(t).$$

So $H_{g_1,g_2} := \{t \in [0,1] : \mu(A^t) = 0\}$ has Lebesgue Measure 1 in $[0,1]$. Notice that for $t \in H_{g_1,g_2}$ we have that $\mu(S \setminus \operatorname{supp} g(\cdot, t)) = 0$, so $g(\cdot, t)$ has full support in $\langle g_1, g_2 \rangle$. Notice also that Now $\bigcap_{\{g_1,g_2\} \subseteq D} H_{g_1,g_2}$ has measure 1, and $a_1 = (1-t)\varepsilon, a_2 = t(1-\varepsilon)$ for t in that intersection will work. $\square$

Lemma 4.27. *Let X be a finite dimensional subspace of $L_p(\mu)$ and suppose that $u \in X$ has full support in X and it is normalized. Suppose that $T : X \to L_p(\mu)$ is an isomorphic embedding such that Tu has full support in TX. Given $f \in X$ of norm one, let $\nu := (f/u)_*(|u|^p d\mu)$ and $\eta := (Tf/Tu)_*(|Tu|^p d\mu)$. Then $\nu, \eta \in \mathcal{M}^{(p)}(\mathbb{F})$ and $\partial_p(\nu, \eta) \leq \max\{\|T\|, \|T^{-1}\|\}$.*

Proof. This follows from the simple observation that $\widehat{\nu}^{(p)}(a) = \|u + f\|_{L_p(\mu)}$. $\square$

Proof of Theorem 4.5. Fix $0 < \varepsilon \leq 1$, and a finite dimensional subspace $X \subseteq L_p(\mu)$. Let $f_0 \in X$ with full support, that without loss of generality we assume that f_0 is normalized. Let $(f_1, \dots, f_n)$ be a normalized basis of X, and let D be a countable dense subset of the centered ball $B(0,2)$ of $L_p(\mu)$ that contains $f_0, f_1, \dots, f_n$, and let $a_1, a_2 \in [0,1]$ be the result of applying Lemma 4.26 to D and ε_0 such that $(4\varepsilon_0)^p + \varepsilon_0^p \|\mu\| + 3\varepsilon_0 \leq \varepsilon$. For each $1 \leq j \leq n$, let $\varphi_j := a_0 f_0 + a_1 f_j$; notice that $\mu(\operatorname{supp} f_0 \setminus \operatorname{supp} \varphi_j) = 0$, so $f_0/\varphi_j \in L_p(|\varphi_j|^p d\mu)$. For each $j = 1, \dots, n$, let $\mu_j := (f_0/\varphi_j)_*(|\varphi_j|^p d\mu)$. Observe that

$$\mu_j(\{0\}) = \int_{\frac{f_0}{\varphi_j} = 0} |\varphi_j|^p d\mu = \int_{f_0 = 0} |\varphi_j|^p d\mu = 0,$$

because f_0 has full support in X. Let $0 < \varepsilon_1 \leq \varepsilon_0$ be such that $\mu_j(B(0, 2\varepsilon_1)) < \varepsilon_0$ for every $j = 1, \dots, n$, and let $\varepsilon_2 \leq \varepsilon_1$ be such that for $j = 1, \dots, n$, if $\nu \in \mathcal{M}^{(p)}(\mathbb{F})$ is such that $\partial_p(\mu_j, \nu) \leq 1 + \varepsilon_2$, then $d_{\mathcal{LP}}(\mu_j, \nu) \leq \varepsilon_1$. We claim that $\delta := \varepsilon_2/2$ works. For suppose that $T : X \to L_p(\mu)$ is such that $\|T\|, \|T^{-1}\| \leq 1 + \delta$. We will show that $\int_{Tf_0 = 0} |Tf_j|^p \leq \varepsilon$: Fix $1 \leq j \leq n$. We assume that f_j and f_0 are linearly independent, since otherwise $\operatorname{supp} Tf_0 = \operatorname{supp} Tf_j$. Let $\gamma_0, \gamma_j \in D$ and $0 < \varepsilon_3 \leq \varepsilon_2$ be such that

(a) $\varepsilon_3 < \sqrt{\varepsilon_2} \varepsilon_1^{(2p+1)/p}$;
(b) $\int_{|Tf_j| > \frac{1}{(\varepsilon_3)^{1/2}}} |Tf_j|^p d\mu < \varepsilon_0$;
(c) $\|Tf_0 - \gamma_0\|, \|Tf_j - \gamma_j\|, \|T(\varphi_j) - \psi_j\| \leq \varepsilon_3$ where $\psi_j = a_0\gamma_0 + a_1\gamma_j$;
(d) the linear mapping U defined by $U(f_0) := \gamma_0$ and $U(f_j) = \gamma_j$ satisfies that $\|U\|, \|U^{-1}\| \leq 1 + \varepsilon_2$

Since $\psi_j := a_0\gamma_0 + a_1\gamma_j$, has full support in $\langle \gamma_0, \gamma_j \rangle$, it follows that $\nu_j := (\gamma_0/\psi_j)_*(|\psi_j|^p d\mu) \in \mathcal{M}^{(p)}(\mathbb{F})$ and $\partial_p(\mu_j, \nu_j) \leq 1 + \varepsilon_2$, by (d) above. Hence, by Lemma 4.27, $d_{\mathcal{LP}}(\mu_j, \nu_j) \leq \varepsilon_1$. It follows that

$$\int_{|\frac{\gamma_0}{\psi_j}| \leq \varepsilon_1} |\psi_j|^p d\mu = \nu_j(B(0, \varepsilon_1)) \leq \nu_j(B(0, 2\varepsilon_1)) + \varepsilon_1 \leq 2\varepsilon_0. \tag{9}$$

Now observe that

$$\int_{|\gamma_0| \leq \varepsilon_1^2} |\psi_j|^p d\mu \leq \int_{|\gamma_0| \leq \varepsilon_1^2 \& |\psi_j| \leq \varepsilon_1} |\psi_j|^p d\mu + \int_{|\gamma_0| \leq \varepsilon_1^2 \& |\psi_j| \geq \varepsilon_1} |\psi_j|^p d\mu \leq \varepsilon_1^p \|\mu\| + 2\varepsilon_0. \tag{10}$$

Since $0 \le a_0 \le \varepsilon_0$ and $1-\varepsilon_0 \le a_1 \le 1$, it follows that $\|Tf_j - \psi_j\| \le \|Tf_j - \gamma_j\| + \|\psi_j - \gamma_j\| \le \varepsilon_3 + \varepsilon_0(\|\gamma_0\| + \|\gamma_j\|) \le \varepsilon_3 + 3\varepsilon_0$. Hence,

$$\int_{|\gamma_0|\le\varepsilon_1^2} |Tf_j|^p d\mu \le (\varepsilon_3 + 3\varepsilon_0)^p + \varepsilon_1^p\|\mu\| + 2\varepsilon_0. \tag{11}$$

Now,

$$\int_{Tf_0=0} |Tf_j|^p d\mu \le (\varepsilon_3 + 3\varepsilon_0)^p + \varepsilon_1^p\|\mu\| + 2\varepsilon_0 + \int_{Tf_0=0\&|\gamma_0|>\varepsilon_1^2} |Tf_j|^p d\mu. \tag{12}$$

Since

$$\varepsilon_0^{2p}\mu(Tf_0 = 0\&|\gamma_0| > \varepsilon_1^2) \le \int_{Tf_0=0\&|\gamma_0|>\varepsilon_1^2} |\gamma_0|^p d\mu \le \|Tf_0 - \gamma_0\|^p \le \varepsilon_3^p \tag{13}$$

it follows that

$$\begin{aligned}\int_{Tf_0=0\&|\gamma_0|>\varepsilon_1^2} |Tf_j|^p d\mu \le& \int_{Tf_0=0\&|\gamma_0|>\varepsilon_1^2\&|Tf_j|\le 1/\sqrt{\varepsilon_2}} |Tf_j|^p d\mu + \int_{Tf_0=0\&|\gamma_0|>\varepsilon_1^2\&|Tf_j|>1/\sqrt{\varepsilon_2}} |Tf_j|^p d\mu \\ \le& \left(\frac{\varepsilon_3}{\sqrt{\varepsilon_2}\varepsilon_1^2}\right)^p + \varepsilon_1 \le 2\varepsilon_1 \end{aligned}\tag{14}$$

Combining (12) and (14) we obtain that

$$\int_{Tf_0=0} |Tf_j|^p d\mu \le (\varepsilon_3 + 3\varepsilon_0)^p + \varepsilon_1^p\|\mu\| + 2\varepsilon_0 + 2\varepsilon_1 \le \varepsilon. \qquad \square$$

5. Approximate Ramsey properties of L_p spaces

The *approximate Ramsey property (ARP)* is an extension of the near amalgamation property that is known to characterize the extreme amenability of the isometry group of (AuH) Banach spaces. This is a particular instance of the Kechris-Pestov-Todorcevic (KPT) correspondence (see Proposition 5.10) for Banach spaces. We will give a proof of the (ARP) of the class $\{\ell_p^n\}_n$, and we will relate it with some approximate Ramsey principles of certain regular partitions. Our proof uses a discrete form of the method of concentration of measure applied to these partitions.

5.1. **Approximate Ramsey properties of classes of finite dimensional spaces.** We start by recalling some combinatorial useful concepts and terminology. Let (A, d_A) be a metric space. Given $r \in \mathbb{N}$, an r-coloring of a set A is simply a mapping $c : A \to r = \{0, 1, \dots, r-1\}$. A *monochromatic* set of an r-coloring c of A is a subset B of A on which c is constant. We say that $B \subseteq A$ is ε-monochromatic, $\varepsilon \ge 0$, if there is some $\hat{r} \in r$ such that $B \subseteq (c^{-1}\{\hat{r}\})_\varepsilon$. A *continuous coloring* of A is a 1-Lipschitz mapping $c : A \to [0, 1]$.

Definition 5.1. Let $\mathcal{F}$ be a family of finite dimensional normed spaces.

(a) $\mathcal{F}$ has the *Approximate Ramsey Property* (ARP) when for every X and Y in $\mathcal{F}$ and every $\varepsilon > 0$ there exists $Z \in \mathcal{F}$ such that every continuous coloring c of $\mathrm{Emb}(X, Z)$ ε-stabilizes on $\gamma \circ \mathrm{Emb}(X, Y)$ for some $\gamma \in \mathrm{Emb}(Y, Z)$, that is, such that

$$\mathrm{osc}(c \restriction \gamma \circ \mathrm{Emb}(X, Y)) = \sup_{\psi,\eta\in\mathrm{Emb}(X,Y)} |c(\gamma \circ \psi) - c(\gamma \circ \eta)| < \varepsilon;$$

(b) $\mathcal{F}$ has the *Approximate Ramsey Property*$^+$ (ARP$^+$) when for every X and $\varepsilon > 0$ there is $\delta > 0$ such that for any Y in $\mathcal{F}$ there exists $Z \in \mathcal{F}$ such that every continuous coloring c of $\mathrm{Emb}_\delta(X, Z)$ ε-stabilizes on $\gamma \circ \mathrm{Emb}_\delta(X, Y)$ for some $\gamma \in \mathrm{Emb}(Y, Z)$;

(c) $\mathcal{F}$ has the *Steady Approximate Ramsey Property*$^+$ (SARP$^+$) with modulus of stability $\varpi:\mathbb{N}\times[0,\infty[\to[0,\infty[$ when for every $X\in\mathcal{F}_k$, every $Y\in\mathcal{F}$, $\varepsilon>0$ and $\delta\geq 0$ there exists $Z\in\mathcal{F}$ such that every continuous coloring c of $\mathrm{Emb}_{\varpi(k,\delta)}(X,Z)$ $(\varpi(k,\delta)+\varepsilon)$-stabilizes on $\gamma\circ\mathrm{Emb}_\delta(X,Y)$ for some $\gamma\in\mathrm{Emb}(Y,Z)$.

It is defined in [BaLALuMbo2] that when ϖ does not depend on the dimension, $\mathcal{F}$ is said to have the *stable* approximate Ramsey *Property*$^+$.

Up to now, the following classes are known to have approximate Ramsey properties.

Example 5.2. *The class* $\{\ell_2^n\}_n$ *of finite dimensional euclidean spaces has the* (SARP$^+$)*. First of all, M. Gromov and V. Milman* [GrMi] *proved that the unitary group* $\mathrm{Iso}(\ell_2)$ *with its strong operator topology is a Lévy group, so it is extremely amenable. Since* ℓ_2 *is* (uH)*, this last fact is equivalent to saying that* $\mathrm{Age}(\ell_2)$ *has the* (ARP) *(see Theorem 5.10 below). Moreover,* ℓ_2 *is Fraïssé, hence* $\mathrm{Age}(\ell_2)$ *is an amalgamation class. This implies that* $\mathrm{Age}(\ell_2)\equiv\{\ell_2^n\}_n$ *has the* (SARP$^+$) *(see Proposition 5.9).*

Example 5.3. *For* $1\leq p<\infty$*,* $p\notin 2\mathbb{N}$*, the class* $\mathrm{Age}(L_p(0,1))$ *has the* (SARP$^+$)*: The* (ARP) *of* $\mathrm{Age}(L_p(0,1))$ *is a consequence of the fact that* $\mathrm{Iso}(L_p(0,1))$ *is extremely amenable, proved by T. Giordano and V. Pestov* [GiPe]*, and that those* L_p *spaces are* (AuH)*. Moreover, we proved that these spaces* $L_p(0,1)$ *are Fraïssé, so,* $\mathrm{Age}(L_p(0,1))$ *is an amalgamation class.*

Example 5.4. *For all* $1\leq p\neq 2<\infty$ *the class* $\{\ell_p^n\}_n$ *has the* (SARP$^+$)*: We give a direct proof in Section 5 of the* (ARP) *of* $\{\ell_p^n\}_n$*. This, and the fact that* $\{\ell_p^n\}_n$ *has the amalgamation, proved in Proposition 3.7 gives the desired* (SARP$^+$) *of* $\{\ell_p^n\}_n$*, and that also gives another proof of the extreme amenability of the isometry group* $\mathrm{Iso}(L_p(0,1))$.

Example 5.5. *The classes* $\{\ell_\infty^n\}_n$*, the finite dimensional* polyhedral spaces*, and all finite dimensional normed spaces have the* (SARP$^+$) *(proved by D. Bartošová, M. Lupini, B. Mbombo and the second author of this paper, in* [BaLALuMbo2] *(see also* [BaLALuMbo1]*).*

The (ARP) has the following reinterpretation in terms of finite colorings.

Proposition 5.6. [BaLALuMbo2] *For a class* $\mathcal{F}$ *of finite dimensional spaces the following are equivalent:*

1) $\mathcal{F}$ *has the approximate Ramsey property;*

2) $\mathcal{F}$ *has the discrete approximate Ramsey property, that is, for every* X *and* Y *in* $\mathcal{F}$*, every* $r\in\mathbb{N}$ *and every* $\varepsilon>0$ *there exists* $Z\in\mathcal{F}$ *such that every* r*-coloring of* $\mathrm{Emb}(X,Z)$ *has a* ε*-monochromatic set of the form* $\gamma\circ\mathrm{Emb}(X,Y)$ *for some* $\gamma\in\mathrm{Emb}(Y,Z)$.

Similar equivalences are true for the (ARP$^+$) and the (SARP$^+$).

Proof. For the sake of completeness we sketch the proof. *2)* $\implies$ *1)*: Given any continuous coloring $c:\mathrm{Emb}(X,Z)\to[0,1]$ and given $\varepsilon>0$, one can induce the discretization of c, $\widehat{c}:\mathrm{Emb}(X,Z)\to r$, where r is chosen such that there is a partition of $[0,1]$ into r-many disjoint intervals of diameter less than ε. Since this assignment does not depend on X or Z, we can use *2)* to deduce *1)*. The proof of *1)* $\implies$ *2)* is done by induction on the number of colors r: Given an $r+1$-coloring $c:\mathrm{Emb}(X,Z)\to r+1=\{0,1,\dots,r\}$, we can define the induced continuous coloring $\widehat{c}(\gamma):=(1/2)d(\gamma,c^{-1}(r))$, and then use the inductive hypothesis for r and *1)*. □

Similarly to the case of discrete structures, approximate Ramsey properties extend the corresponding amalgamation properties.

Proposition 5.7. *Suppose that $\mathcal{F}$ has the* (JEP)*. If $\mathcal{F}$ has the* (ARP), (ARP$^+$), (SARP$^+$)*, then $\mathcal{F}$ has the* (NAP)*, is a weak amalgamation class, is an amalgamation class, respectively.*

Proof. We only prove that (ARP$^+$) implies weak amalgamation; the other implications are proved in a similar way. Suppose that $X \in \mathcal{F}$ and $\varepsilon > 0$. We claim that $0 < \delta \le 1$ witnessing the (ARP$^+$) for the initial parameters X and $\mathcal{F}$ works. For suppose that $\gamma \in \mathrm{Emb}_\delta(X,Y)$, $\eta \in \mathrm{Emb}_\delta(X,Z)$. Let $V \in \mathcal{F}$, and $f \in \mathrm{Emb}(Y,V)$, and $g \in \mathrm{Emb}(Z,V)$. By the (ARP$^+$), we can find $W \in \mathcal{F}$ such that for the particular coloring $c : \mathrm{Emb}_\delta(X,W) \to [0,1]$, $c(h) := (1/4)d(h, \mathrm{Emb}(V,W)\circ f \circ \gamma)$ we can find $\varrho \in \mathrm{Emb}(V,W)$ for which $\mathrm{Osc}(c \restriction \varrho \circ \mathrm{Emb}_\delta(X,V)) \le \varepsilon$. Observe that $c(\varrho\circ f\circ\gamma) = 0$, so there is $\nu \in \mathrm{Emb}(V,W)$ such that $\|\nu\circ g\circ\eta - \varrho\circ f\circ\gamma\| \le \varepsilon$, as desired. □

Problem 5.8. *Does there exist a Fraïssé class of finite dimensional spaces not having the* (ARP)*?*

Proposition 5.9. *Let $\mathcal{F}$ be a class of finite dimensional normed spaces. Then,*

1) $\mathcal{F}$ has the (ARP$^+$) *if and only if $\mathcal{F}$ has the* (ARP) *and weak amalgamation.*

2) $\mathcal{F}$ has the (SARP$^+$) *if and only if $\mathcal{F}$ has the* (ARP) *and it is an amalgamation class.*

Proof. We use the following, that has to be compared with Lemma 2.31 and that is proved similarly.

Claim 5.9.1. $\mathcal{F}$ has weak amalgamation if and only if for every $\varepsilon > 0$ and $X \in \mathcal{F}$ there is $\delta > 0$ such that for every $Y, Z \in \mathcal{F}$ there is some $V \in \mathcal{F}$ and some $I \in \mathrm{Emb}(Y,V)$ such that for every $\gamma \in \mathrm{Emb}_\delta(X,Y)$ and $\eta \in \mathrm{Emb}_\delta(X,Z)$ there is $J \in \mathrm{Emb}(Z,V)$ such that $\|I\circ\gamma - J\circ\eta\| \le \varepsilon$. □

Now suppose that $\mathcal{F}$ has both the (ARP) and weak amalgamation. Fix $\varepsilon > 0$ and $X \in \mathcal{F}$. We use first the claim to find the corresponding δ for $\varepsilon/3$. Now given $Y \in \mathcal{F}$ we use the property of δ to find $V \in \mathcal{F}$ and $I \in \mathrm{Emb}(Y,Z)$ such that, in particular, $I \circ \mathrm{Emb}_\delta(X,Y) \subseteq (\mathrm{Emb}(X,V))_{\varepsilon/3}$. Now we use the (ARP) of $\mathcal{F}$ applied to X, V and ε to find Z, that we claim that it works for our purposes: For suppose that $c : \mathrm{Emb}_\delta(X,Z) \to [0,1]$ is a continuous coloring. By the (ARP) of $\mathcal{F}$, there is $J \in \mathrm{Emb}(V,Z)$ such that $\mathrm{Osc}(c \restriction J \circ \mathrm{Emb}(X,V)) \le \varepsilon/3$. Let us see that $\mathrm{Osc}(c \restriction J\circ I\circ \mathrm{Emb}_\delta(X,Y)) \le \varepsilon$: For suppose that $\gamma, \eta \in \mathrm{Emb}_\delta(X,Y)$. There are $\iota, \xi \in \mathrm{Emb}(X,V)$ such that $\|\iota - I\circ\gamma\|, \|\xi - I\circ\eta\| \le \varepsilon/2$. Hence $|c(J\circ I\circ\gamma) - c(J\circ I\circ\eta)| \le |c(J\circ I\circ\gamma) - c(J\circ\iota)| + |c(J\circ\iota) - c(J\circ\xi)| + |c(J\circ\xi) - c(J\circ I\circ\eta)| \le \varepsilon$. *2)* is proved similarly. □

The following connects the approximate Ramsey property of $\mathrm{Age}(E)$ and the extreme amenability of $\mathrm{Iso}(E)$. It is a slight extension of the correspondence given in [BaLALuMbo2], and a particular case of the metric KPT correspondence for metric structures (see [MeTsa, Theorem 3.10]).

Theorem 5.10 (KPT correspondence)**.** *Suppose that E is* (AuH)*. The following are equivalent:*

1) The group $\mathrm{Iso}(E)$ with its strong operator topology is extremely amenable; that is, every continuous action of $\mathrm{Iso}(E)$ on a compact space has a fixed point.

2) $\mathrm{Age}(E)$ has the (ARP)*.*

Suppose that $\mathcal{G} \preceq \mathrm{Age}(E)$ is an amalgamation class such that $E \in [\mathcal{G}]$ (See Definition 2.16). Then the previous are equivalent to

3) $\mathcal{G}$ has the (ARP) *(equiv* (SARP$^+$)*).*

Before giving the proof, two interesting consequences.

Corollary 5.11. *Suppose that $\mathcal{G}$ is an hereditary family with the* (SARP$^+$)*.*

1) There is a unique separable Fraïssé Banach space E such that $\mathrm{Age}(E) \equiv \overline{\mathcal{G}}^{\mathrm{BM}}$ *and such that its isometry group is extremely amenable. Moreover* $E = \mathrm{Flim}\,\overline{\mathcal{G}}^{\mathrm{BM}}$.

2) $\overline{\mathcal{G}}^{\mathrm{BM}}$ *is* (SARP$^+$).

PROOF. *1)* and *2)*: Set $\mathcal{H} := \overline{\mathcal{G}}^{\mathrm{BM}}$. We know by Proposition 5.9 *2)* that $\mathcal{G}$ is an amalgamation class, hence, by Proposition 2.23 *2)*, also $\mathcal{H}$ is an amalgamation class. Hence, $\mathcal{H}$ is a Fraïssé class. Let E be its Fraïssé limit $\mathrm{Flim}\,\mathcal{H}$. The family $\mathcal{G}$ fulfills the conditions in the last part of Theorem 5.10, so $\mathrm{Iso}(E)$ is extremely amenable and $\mathcal{H}$ has the (ARP).

□

Proof of Theorem 5.10. We prove that *1)* implies *2)*: We need to introduce some concepts. Given two metric spaces (A, d_A) and (B, d_B), let $\mathrm{Lip}(A, B)$ be the collection of 1-Lipschitz mappings from A to B. When A is compact, we endow it with the uniform metric $d(c, d) := \sup_{a\in A} d_B(c(a), d(a))$. Observe that when B is also compact, $(\mathrm{Lip}(A, B), d)$ is also compact. For each $W \in \mathrm{Age}(E)$, let $\langle W\rangle := \{X \in \mathrm{Age}(E) : W \subseteq X\}$. Note that $\{\langle W\rangle\}_{W\in\mathrm{Age}(E)}$ has the finite intersection property. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathrm{Age}(E)$ containing all $\langle W\rangle$. Define the ultraproduct

$$\mathrm{Lip}_{\mathcal{U}}(\mathrm{Emb}(X, E), [0, 1]) := \left(\prod_{X\subseteq Y\in\mathrm{Age}(E)} \mathrm{Lip}(\mathrm{Emb}(X, Y), [0, 1])\right) / \sim_{\mathcal{U}},$$

where $(c_Y)_Y \sim_{\mathcal{U}} (d_Y)_Y$ if and only if for every $(\gamma_j)_{j<n}$ in $\mathrm{Emb}(X, E)$, and every $\varepsilon > 0$ one has that $\{Y \in \langle\sum_{j<n} \mathrm{Im}\gamma_j\rangle \,:\, |\max_{j<n} |c_Y(\gamma_j) - d_Y(\gamma_j)| \le \varepsilon\} \in \mathcal{U}$. We consider the canonical action of $\mathrm{Iso}(E)$ in $\mathrm{Lip}(\mathrm{Emb}(X, E), [0, 1])$, $(g \cdot c)(\gamma) := c(g \circ \gamma)$, and the corresponding (algebraic) action $\mathrm{Iso}(E) \curvearrowright \mathrm{Lip}_{\mathcal{U}}(\mathrm{Emb}(X, E), [0, 1])$, $g \cdot [(c_Y)_Y]_{\mathcal{U}} = [(d_Y)_Y]_{\mathcal{U}}$, where each $d_Y(\gamma) := c_{g(Y)}(g \circ \gamma)$. Finally, let $\Phi : \mathrm{Lip}(\mathrm{Emb}(X, E), [0, 1]) \to \mathrm{Lip}_{\mathcal{U}}(\mathrm{Emb}(X, E), [0, 1])$, $\Phi(c) = (c_Y)_Y$, where $c_Y(\gamma) := c(\gamma)$.

Claim 5.11.1. Φ is a $\mathrm{Iso}(E)$-bijection.

Proof of Claim: Suppose that $\Phi(c) = [(c_Y)_Y]_{\mathcal{U}}$ and $\Phi(g \cdot c) = [(d_Y)_Y]_{\mathcal{U}}$. Then for each Y and $\gamma \in \mathrm{Emb}(X, Y)$, $c_Y(\gamma) = c(\gamma)$ and $d_Y(\gamma) = (g \cdot c)(\gamma) = c(g \circ \gamma)$, so $g \cdot [(c_Y)_Y]_{\mathcal{U}} = [(d_Y)_Y]_{\mathcal{U}}$. It is easy to see that Φ is 1-1. We prove that Φ is onto: Fix $[(c_Y)_Y]_{\mathcal{U}}$. We are going to find c such that $\Phi(c) = [(c_Y)_Y]_{\mathcal{U}}$. Fix $\gamma \in \mathrm{Emb}(X, E)$. Since $\langle\mathrm{Im}\gamma\rangle \in \mathcal{U}$ and since $(c_Z(\gamma))_{W\subseteq Z}$ is a bounded sequence, the $\mathcal{U}$-limit $c(\gamma) := \lim_{Y\to\mathcal{U}} c_Y(\gamma)$ exists. It is ease to see that $c \in \mathrm{Lip}(\mathrm{Emb}(X, E), [0, 1])$ and that $\Phi(c) = [(c_Y)_Y]_{\mathcal{U}}$. □

Now suppose that $\mathrm{Iso}(E)$ is extremely amenable, and let us prove the (ARP) of $\mathrm{Age}(E)$: Fix $X, Y \in \mathrm{Age}(E)$ and $\varepsilon > 0$, and let $c : \mathrm{Emb}(X, E) \to [0, 1]$ be 1-Lipschitz. Let $d \in \overline{\mathrm{Iso}(E)c}$ be such that $g \cdot d = d$ for every $g \in \mathrm{Iso}(E)$, i.e., $d(\gamma) = d(g \circ \gamma)$ for every $\gamma \in \mathrm{Emb}(X, E)$. Since we are assuming that E is (AuH), it follows from this that d is a constant function. Now, since $\mathrm{Emb}(X, Y)$ is compact, we can find $g \in \mathrm{Iso}(E)$ such that $\sup_{\gamma\in\mathrm{Emb}(X,Y)} |g \cdot c(\gamma), d(\gamma)| \le \varepsilon/2$. Let us see that $\mathrm{Osc}(c \upharpoonright g \circ \mathrm{Emb}(X, Y)) \le \varepsilon$: For suppose that $\gamma, \eta \in \mathrm{Emb}(X, Y)$; Then, $|c(g \circ \gamma) - c(g \circ \eta)| \le |c(g \circ \gamma) - d(\gamma)| + |c(g \circ \eta) - d(\eta)| \le \varepsilon$. Since Φ is a $\mathrm{Iso}(E)$-bijection, given $(c_Z)_Z \in \prod_{Z\in\mathrm{Age}(E)} \mathrm{Lip}(\mathrm{Emb}(X, Z), [0, 1])$ one has that

$$\{Z \in \mathrm{Age}(E) \,:\, \text{there is } \gamma \in \mathrm{Emb}(Y, Z) \text{ such that } \mathrm{Osc}(c_Z \upharpoonright \gamma \circ \mathrm{Emb}(X, Y)) \le \varepsilon\} \in \mathcal{U},$$

and consequently,

$$\{Z \in \mathrm{Age}(E) \,:\, \forall c \in \mathrm{Lip}(\mathrm{Emb}(X, Z), [0, 1])\ \exists\gamma \in \mathrm{Emb}(Y, Z) \text{ with } \mathrm{Osc}(c \upharpoonright \mathrm{Emb}(X, Y)) \le \varepsilon\} \in \mathcal{U}$$

We prove that *2)* implies *1)*. We use the following known characterization of existence of a fixed point.

Claim 5.11.2. Let G be a topological group, $G \curvearrowright K$, and suppose that $p \in K$ has dense orbit. The following are equivalent.

a) There is a fixed point for the action $G \curvearrowright K$.
b) For every entourage U in K and every finite set $F \subseteq G$ there is some $g \in G$ such that $F \cdot (g \cdot p)$ is U-small, that is, for every $f_0, f_1 \in F$ one has that $(f_0 \cdot (g \cdot p), f_1 \cdot (g \cdot p)) \in U$.

Proof of Claim: We assume that all entourages considered are symmetric. For suppose that $q \in K$ is a fixed point; Fix $F \subseteq G$ finite and an entourage U; let V be an entourage such that $V \circ V \subseteq U$. Using that $g\cdot : K \to K$ is uniformly continuous, we find an entourage W such that $gW \subseteq V$ for every $g \in F$. Let $h \in G$ be such that $(h \cdot p, q) \in W$. It follows that $(gh \cdot p, q) = (gh \cdot p, gq) \in V$ for all $g \in F$; hence $(gh \cdot p, g'h \cdot p) \in U$. Suppose now that b) holds, and for every finite set F and entourage U choose $g_{F,U} \in G$ such that $(F \cup \{e\}) \cdot (g_{F,U} \cdot p)$ is U-small, hence $\{f \cdot (g_{F,U} \cdot p)\}_{f \in F} \subseteq U[g_{F,U} \cdot p]$ for every F and U. Then any accumulation point q of $\{g_{F,U} \cdot p\}_{F,U}$ is a fixed point. □

Now suppose that $\mathrm{Age}(E)$ has the (ARP). Again, we assume that all entourages considered are symmetric. Fix $\mathrm{Iso}(E) \curvearrowright K$, $p \in K$, an entourage U and a finite set $F \subseteq \mathrm{Iso}(E)$. Set $H := F^{-1}$, and let V be an entourage such that $V \circ V \circ V \circ V \subseteq U$. For each $X \in \mathrm{Age}(E)$, we define the pseudometric d_X on $\mathrm{Iso}(E)$ by $d_X(g,h) := \|g \upharpoonright X - h \upharpoonright X\|$. The family $\mathcal{D} = \{d_X\}_{X \in \mathrm{Age}(E)}$ defines the left-uniformity of $\mathrm{Iso}(E)$, hence its SOT. Now the mapping $\mathrm{Iso}(E) \to K$, $g \mapsto g^{-1}p$ is uniformly continuous with respect to the family $\mathcal{D}$, so there is some $X \in \mathrm{Age}(E)$ and some $\delta > 0$ such that $d_X(g,h) \le \delta$ implies that $(g^{-1} \cdot p, h^{-1} \cdot p) \in V$. Let $Y := \sum_{g \in H} gX$. Finally, let $\{x_j\}_{j<r} \subseteq K$ be such that $K = \bigcup_{j<r} V[x_j]$, where $V[x] := \{y \in K \,:\, (x,y) \in V\}$. We apply the (ARP) of $\mathrm{Age}(E)$ to X, Y, $\delta/3$ and r to find the corresponding Z. We define the coloring $c : \mathrm{Emb}(X,Z) \to r$ for $\gamma \in \mathrm{Emb}(X,Z)$ by choosing $g \in \mathrm{Iso}(E)$ such that $\|g \upharpoonright X - \gamma\| \le \delta/3$, and then by declaring $c(\gamma) = j$ if j is (the first) such that $g^{-1}p \in V[x_j]$. By the Ramsey property of Z, we can find $\varrho \in \mathrm{Emb}(Y,Z)$ and $j < r$ such that, in particular, for every $\eta \in \mathrm{Emb}(X,Y)$ there is some $g_\eta \in \mathrm{Iso}(E)$ such that $(g_\eta)^{-1} \cdot p \in V[x_j]$ and $\|\varrho \circ \eta - g_\eta\| \le 2\delta/3$. Choose $h \in \mathrm{Iso}(E)$ such that $\|h \upharpoonright Y - \varrho\| \le \delta/3$. Then, for every $f \in H$, setting $\eta := f \upharpoonright X_n$, then $d_X(h \circ f, g_\eta) \le \delta$, and $g_\eta^{-1} \cdot p \in V[x_j]$. Consequently, $(f_0 \circ h^{-1} \cdot p, f_1 \circ h^{-1} \cdot p) \in U$ for every $f_0, f_1 \in F$, as desired.

Suppose that $\mathcal{G} \preceq \mathrm{Age}(E)$ is an amalgamation class such that $E \in [\mathcal{G}]$, that is, such that the collection of subspaces of elements of $\mathcal{G}_E$ is Λ_E-dense in $\mathrm{Age}(E)$. Then the family of pseudometrics $\{d_X\}_{X \in \mathcal{G}_E}$ defines the SOT of $\mathrm{Iso}(E)$. The strategy of the proof used to see that *2)* implies *1)*, with the natural modifications, works here. We leave the details to the reader.

Suppose now that $\mathrm{Age}(E)$ has the (ARP). Fix $X, Y \in \mathcal{G}$, $r \in \mathbb{N}$, and $\varepsilon > 0$. Let $0 < \varepsilon_0 < \varepsilon/3$ and let $Z \in \mathrm{Age}(E)$ be such that every r-coloring of $\mathrm{Emb}(X,Z)$ has an ε_0-monochromatic set of the form $\gamma \circ \mathrm{Emb}(X,Y)$ for some $\gamma \in \mathrm{Emb}(Y,Z)$. Since $E \in [\mathcal{G}]$ and $\mathcal{G}$ is an amalgamation class, we can find $0 < \delta < \varepsilon/(3\varepsilon_0) - 1$, some $V \in \mathcal{G}$ and some $\theta \in \mathrm{Emb}_\delta(Z,V)$ such that for every $\gamma \in \mathrm{Emb}(X,Z)$ and $\eta \in \mathrm{Emb}(Y,Z)$ there are isometric embeddings $i \in \mathrm{Emb}(X,V)$ and $j \in \mathrm{Emb}(Y,V)$ such that $\|\theta \circ \gamma - i\|, \|\theta \circ \eta - j\| \le \varepsilon_0$. We claim that V works for our purposes: For suppose that $c : \mathrm{Emb}(X,V) \to r$. We induce the coloring $\widehat{c} : \mathrm{Emb}(X,Z) \to r$ by choosing for each $\gamma \in \mathrm{Emb}(X,Z)$ some $i \in \mathrm{Emb}(X,V)$ such that $\|i - \theta \circ \gamma\| \le \varepsilon_0$ and declare $\widehat{c}(\gamma) := c(i)$. Let $\eta \in \mathrm{Emb}(Y,Z)$ and $\bar{r} < r$ be such that $\eta \circ \mathrm{Emb}(X,Y) \subseteq (\widehat{c}^{-1}(\bar{r}))_{\varepsilon_0}$. Then one can show that $j \circ \mathrm{Emb}(X,Y) \subseteq (c^{-1}(\bar{r}))_\varepsilon$, where $j \in \mathrm{Emb}(Y,V)$ is such that $\|j - \theta \circ \eta\| \le \varepsilon_0$. □

Observe that the previous requirement on $\mathcal{G}$ is satisfied when $E = \mathrm{Flim}\,\mathcal{G}$. Observe also that in the proof we are not assuming that E is necessarily separable, and since for Fraïssé spaces we have that $\mathrm{Age}(E) \equiv \mathrm{Age}(E_\mathcal{U})$ for any ultrafilter $\mathcal{U}$, we obtain the following.

Corollary 5.12. *Suppose that E is a Fraïssé space such that* $\mathrm{Age}(E)$ *has the* (ARP) *then* $\mathrm{Iso}(E_{\mathcal{U}})$ *is extremely amenable for every ultrafilter $\mathcal{U}$ on $\mathbb{N}$. In particular,* $\mathrm{Iso}(\mathbb{G}_{\mathcal{U}})$ *and* $\mathrm{Iso}(L_p(0,1)_{\mathcal{U}})$ *are extremely amenable non separable groups.* □

5.1.1. *Multidimensional Borsuk-Ulam.* The approximate Ramsey property of the family $\{\ell_p^n\}_n$ has a natural reinterpretation as a version of a multidimensional *Borsuk-Ulam Theorem.* Recall that the equivalent reformulation by Lusternik and Shnirel'man [LuSch] (see also [Ma, Theorem 2.1.1]) of the Borsuk-Ulam theorem states that if the sphere $S_{\ell_2^{n+1}}$ is covered by $n+1$ open sets, then one of these sets contains a pair $(x,-x)$ of antipodal points. Given $0<p<\infty$, let $\mathbf{n}_{\boldsymbol{p}}(d,m,r,\varepsilon)$ be the minimal number n such that for every coloring $c:\mathrm{Emb}(\ell_p^d,\ell_p^n)\to r$ there is $\gamma\in\mathrm{Emb}(\ell_p^m,\ell_p^n)$ and $i<r$ such that $\gamma\circ\mathrm{Emb}(\ell_p^d,\ell_p^m)\subseteq(c^{-1}\{i\})_\varepsilon$. The (ARP) of $\{\ell_p^n\}_n$ is exactly the statement saying that $\mathbf{n}_{\boldsymbol{p}}(d,m,r,\varepsilon)$ exists.

Recall that given $\delta>0$ and a subset A of a metric space X, one defines $A_{-\delta}:=X\setminus((X\setminus A)^c)_\delta$. It is easy to see that $(A_{-\delta})_\delta\subseteq A$. We will say that an open covering $\mathcal{U}$ of a metric space X is called *ε-fat* when $\{V_{-\varepsilon}\}_{U\in\mathcal{U}}$ is also a covering on X.

The following is a sort of Lebesgue's Number Lemma.

Proposition 5.13. *Suppose that X is a compact metric space. Then every open covering of X is an ε-open covering for some $\varepsilon>0$.*

Proof. Suppose that for some open covering $\mathcal{U}$ of X such $\varepsilon>0$ does not exists. For each $n\in\mathbb{N}$ we can find a point x_n of X not in $\bigcup_{U\in\mathcal{U}}U_{-2^{-n}}$. Since X is compact, there is a subsequence $(x_{n_k})_k$ converging to some $x\in X$. Choose $U\in\mathcal{U}$ such that $x\in U$, and also $\delta>0$ such that the ball $B(x,2\delta)\subseteq U$. Observe that $B(x,\delta)\subseteq(B(x,2\delta))_{-\delta}\subseteq U_{-\delta}$. Now let $k\in\mathbb{N}$ be such that $\delta n_k\ge 1$ and such that $d(x_{n_k},x)<\delta$. It follows then that $x_{n_k}\in U_{-\delta}\subseteq U_{-2^{-n_k}}$, a contradiction. □

Definition 5.14. Given $0<p\le\infty$, integers d,m, and r and $\varepsilon>0$, let $\mathbf{n}_{\mathbf{BU},\boldsymbol{p}}(d,m,r,\varepsilon)$ be the minimal integer n such that for every ε-fat open covering $\mathcal{U}$ of $\mathrm{Emb}(\ell_p^d,\ell_p^n)$ with at most r many pieces there exists $\gamma\in\mathrm{Emb}(\ell_p^d,\ell_p^m)$ and some $U\in\mathcal{U}$ such that

$$\gamma\circ\mathrm{Emb}(\ell_p^d,\ell_p^m)\subseteq U.$$

Notice that by assigning to each $x\in S_{\ell_p^n}$ the embedding $1\mapsto x$ we can identify topologically $S_{\ell_p^n}$ and $\mathrm{Emb}(\ell_p^1,\ell_p^n)$. Since $\mathrm{Emb}(\ell_p^1,\ell_p^1)=\{\pm\mathrm{Id}\}$, it follows that $\mathbf{n}_{\mathbf{BU},\boldsymbol{p}}(1,1,r,\varepsilon)$ is the minimal integer such that for every open covering of $S_{\ell_p^n}$ of cardinality r there exists $U\in\mathcal{U}$ containing some pair of antipodal vectors. Hence, by Borsuk-Ulam, $\mathbf{n}_{\mathbf{BU},\boldsymbol{p}}(1,1,r,\varepsilon)\le r$ for every $\varepsilon>0$. In this way, we have Borsuk-Ulam Theorem is the following statement.

Theorem 5.15 (Lusternik and Shnirel'man)**.** $\mathbf{n}_{\mathbf{BU},\boldsymbol{p}}(1,1,r,\varepsilon)\le r$ *for every $\varepsilon>0$.*

We have the following relation, easy to prove.

Proposition 5.16. *For every $0<p\le\infty$ every d,m, r and every $0<\varepsilon<\delta$ one has that*

$$\mathbf{n}_{\boldsymbol{p}}(m,d,r,\delta)\le\mathbf{n}_{\mathbf{BU},\boldsymbol{p}}(d,m,r,\varepsilon)\le\mathbf{n}_{\boldsymbol{p}}(m,d,r,\varepsilon).$$

Problem 5.17. *Is always $\mathbf{n}_{\mathbf{BU},\boldsymbol{p}}(d,m,r,\varepsilon)$ independent of ε?*

5.2. **The** (ARP) **of the family** $\{\ell_p^n\}_n$**.** We give a direct proof of the Approximate Ramsey property of the family $\{\ell_p^n\}_n$ for $1\le p\ne 2<\infty$ and then of the (ARP$^+$) of $\mathrm{Age}(L_p(0,1))$ for $p\notin 2\mathbb{N}$.

Theorem 5.18. *For $1\le p\ne 2<\infty$ the family $\{\ell_p^n\}_n$ has the* (ARP)*.*

Its proof is done by relating this (ARP) with the (ARP) of approximate equipartitions, that is shown to be true by an instance of the phenomenon of concentration of measure. First of all, we try to reduce Theorem 5.18 to the case of $p = 1$. Notice that it follows from the Banach-Lamperti Theorem on isometries of $L_p(0,1)$, $p \neq 2, \infty$, that all isometry groups for those p's are topologically isomorphic, as was observed and used in [GiPe]. There is some similar fact concerning embeddings using the *Mazur mapping*.

Definition 5.19. Given $0 < p, q < \infty$, and given n, let $\mathrm{M}_{p,q,n} : \ell_p^n \to \ell_q^n$, be the *Mazur map* defined for $x \in \ell_p^n$ by

$$\mathrm{M}_{p,q,n}(x) := \sum_{\xi<n} \mathrm{sign}(x(\xi))|x(\xi)|^{\frac{p}{q}} u_\xi. \tag{15}$$

The following facts are known and easy to prove:

(a) $\mathrm{M}_{p,q,n}$ preserves the support and the signs of the coordinates; in fact, if x and y are disjointly supported then $\mathrm{M}_{p,q,n}(\lambda x + \mu y) = \mathrm{sign}(\lambda)|\lambda|^{p/q}\mathrm{M}_{p,q,n}(x) + \mathrm{sign}(\lambda)|\lambda|^{p/q}\mathrm{M}_{p,q,n}(y)$.

(b) $\mathrm{M}_{q,p,n} \circ \mathrm{M}_{p,q,n} = \mathrm{Id}_{\ell_p^n}$.

(c) $\mathrm{M}_{p,q,n}$ is a uniform homeomorphism between the corresponding unit spheres (note that $\|\mathrm{M}_{p,q,n}(x)\|_q^q = \|x\|_p^p$). If $\tau_{p,q}$ is the modulus of uniform continuity of $\mathrm{M}_{p,q}$, then

$$\tau_{p,q}(t) \le \begin{cases} \frac{p}{q}t & \text{if } p \ge q \\ c_{p,q}t^{\frac{p}{q}} & \text{if } p < q. \end{cases}$$

For $1 \le p, q < \infty$, $p, q \neq 2$, the Mazur mapping naturally extends to $\mathrm{M}_{p,q,n}^d : \mathrm{Emb}(\ell_p^d, \ell_p^n) \to \mathrm{Emb}(\ell_q^d, \ell_q^n)$ defined for $\gamma \in \mathrm{Emb}(\ell_p^d, \ell_p^n)$ by

$$\mathrm{M}_{p,q,n}(\gamma) := \mathrm{M}_{p,q,n} \circ \gamma \circ \mathrm{M}_{q,p,d}.$$

$\mathrm{M}_{p,q,n}(\gamma)$ is a linear isometric embedding because γ sends disjointed supported to disjointed supported, and in fact,

$$\mathrm{M}_{p,q,n}^d(\gamma)(\sum_{i<d} a_i u_i) = \sum_{i<d} a_i \mathrm{M}_{p,q,n}(\gamma(u_i)).$$

Proposition 5.20. $\mathrm{M}_{p,q,n}^d$ *is an uniform homeomorphism with modulus of continuity* $\tau_{p,q}$ *with inverse* $\mathrm{M}_{q,p,n}^d$.

Proof.

$$\begin{aligned}\|\mathrm{M}_{p,q,n}^d(\gamma) - \mathrm{M}_{p,q,n}^d(\eta)\|_{q,q} &= \max_{x \in S_{\ell_q^d}} \|\mathrm{M}_{p,q,n}(\gamma(\mathrm{M}_{q,p,d}(x))) - \mathrm{M}_{p,q,n}(\eta(\mathrm{M}_{q,p,d}(x)))\|_q = \\ &= \max_{y \in S_{\ell_p^d}} \|\mathrm{M}_{p,q,n}(\gamma(y)) - \mathrm{M}_{p,q,n}(\eta(y))\|_q \le \max_{y \in S_{\ell_p^d}} \tau_{p,q}(\|\gamma(y) - \eta(y)\|_p) = \\ &= \omega_{p,q}(\|\gamma - \eta\|_{p,p}),\end{aligned}$$

because $\tau_{p,q}$ is increasing. □

Given $p, d, m \in \mathbb{N}$ and $\varepsilon > 0$, the integer $\mathbf{n}_{\boldsymbol{p}}(d, m, r, \varepsilon)$ is the minimal integer n witnessing the (ARP) of $\{\ell_p^k\}_k$ for the initial parameters d, m and ε. We obtain the following

Proposition 5.21. $\mathbf{n}_{\boldsymbol{p}}(d, m, r, \varepsilon) = \mathbf{n}_{\boldsymbol{q}}(d, m, r, \tau_{p,q}(\varepsilon))$ *for every* $0 < p, q < \infty$, $p, q \neq 2$.

Proof. Fix p, q as above, and fix all the parameters. Let $n := \mathbf{n}_{\mathbf{p}}(d, m, r, \varepsilon)$, and let $c : \mathrm{Emb}(\ell_q^d, \ell_q^n) \to r$. Let $\widehat{c} : \mathrm{Emb}(\ell_p^d, \ell_p^n) \to r$ be the induced coloring $\widehat{c} = c \circ \mathrm{M}_{p,q,n}^d$. Then, let $\gamma \in \mathrm{Emb}(\ell_p^m, \ell_p^n)$, and $i < r$ be such that $\gamma \circ \mathrm{Emb}(\ell_p^d, \ell_p^m) \subseteq (\widehat{c}^{-1}\{i\})_\varepsilon$. Let $\widehat{\gamma} = \mathrm{M}_{p,q,n}^m(\gamma)$. We claim that $\widehat{\gamma}$ and $i < r$ do the job. Fix $\sigma \in \mathrm{Emb}(\ell_q^d, \ell_q^m)$. Then $\bar{\sigma} := \mathrm{M}_{q,p,m}^d(\sigma) \in \mathrm{Emb}(\ell_p^d, \ell_p^m)$, so there is some $\psi \in \mathrm{Emb}(\ell_p^d, \ell_p^n)$

with $c(\mathrm{M}^d_{p,q,n}(\psi)) = i$ and $\|\gamma\circ\bar\sigma - \psi\|_{p,p} \le \varepsilon$. Hence $\|\mathrm{M}^d_{p,q,n}(\gamma\circ\bar\sigma) - \mathrm{M}^d_{p,q,n}(\psi)\|_{q,q} \le \tau_{p,q}(\varepsilon)$, and since $\mathrm{M}^d_{p,q,n}(\gamma\circ\bar\sigma) = \mathrm{M}^d_{p,q,n}(\gamma\circ \mathrm{M}^d_{q,p,m}(\sigma)) = \mathrm{M}^m_{p,q,n}(\gamma)\circ\sigma$, we are done. □

So, in order to have the (ARP) of $\{\ell_p^n\}_n$, $p\neq 2,\infty$, it suffices to prove the (ARP) of $\{\ell_1^n\}_n$, and this is what we do next. The proof we give is a byproduct of an extension of the fact that $\mathbf{n}_1(1,m,r,\varepsilon)$ exists, proved by J. Matoušek and V. Rödl [MaRo], and the existence of the Ramsey number corresponding to unital embeddings. The proof of the existence of $\mathbf{n}_1(1,m,r,\varepsilon)$ was done, as we mentioned before, by Matoušek and Rödl using combinatorial methods (spread vectors), and, independently, by E. Odell, H. P. Rosenthal and Th. Schlumprecht in [OdRoSchl], by using tools of Banach space theory, such as different type of bases.

We introduce some notation. In ℓ_1^n, we denote by $\mathbb{1}$ to the sequence $(1/n)\sum_{j<n}^n u_j = (1/n,\dots,1/n)\in \mathbb{F}^n$. We will denote by $\mathrm{Emb}((\ell_1^d,\mathbb{1}),(\ell_1^n,\mathbb{1}))$ the collection of unital isometric embeddings. We have the following consequence of the equimeasurability principle of Plotkin and Rudin mentioned above in §4 (Theorem 4.2).

Proposition 5.22. *An isometric embedding $\gamma:\ell_1^d\to\ell_1^n$ is unital if and only if $d|n$ and $\gamma(u_j) = (d/n)\sum_{k\in s_j} u_k$ for $j<d$ such that $\{s_j\}_{j<d}$ is a d-*equipartition *of n, that is, $\#s_j = \#s_l = n/d$ for every $j,l<d$.* □

Definition 5.23. Given $d|n$ let $\mathcal{EQ}(n,d)$ be the set of *equipartitions* of n with d many pieces. Given in addition $d|m|n$, and $\mathcal{R}\in\mathcal{EQ}(n,m)$ let $\langle\mathcal{R}\rangle_d^{\mathrm{eq}}$ be the collection of d-equipartitions of n coarser than $\mathcal{R}$.

We identify it with the set of all *rigid surjections* that is, onto mappings $F:n\to d$ such that $\min F^{-1}(i) < \min F^{-1}(j)$ for every $i<j<d$.

It follows that d-dimensional unital subspaces of ℓ_1^n that are isometric to ℓ_1^d are the of the form $\langle (d/n)\sum_{k\in s_j} u_k\rangle_{j<d}$ for d-equipartitions $\{s_j\}_{j<d}$ of n. This means that the following are equivalent:

(i) The class $\{(\ell_1^n,\mathbb{1})\}_n$ has the *Structural Ramsey Property*, that is, for every d,m and every r there is n such that every r-coloring of the collection $\binom{(\ell_1^n,\mathbb{1})}{(\ell_1^d,\mathbb{1})}$ of unital subspaces of ℓ_1^n isometric to ℓ_1^d has a monochromatic set of the form $\binom{X}{(\ell_1^d,\mathbb{1})}$ for some unital $X\in\binom{(\ell_1^n,\mathbb{1})}{(\ell_1^m,\mathbb{1})}$.

(ii) The collection of equipartitions have the Structural Ramsey Property, that is, for every d,m and r there is n such that every r-coloring of $\mathcal{EQ}(n,d)$ has a monochromatic set of the form $\langle\mathcal{R}\rangle_d^{\mathrm{eq}}$ for some $R\in\mathcal{EQ}(n,m)$.

It is interesting to compare the previous equivalence between Ramsey properties with what happens in the case of $p=\infty$: Observe that $\binom{(\ell_1^n,\mathbb{1})}{(\ell_1^d,\mathbb{1})}$ is exactly the collection of unital *sublattices* of ℓ_1^n of dimension d. Now we define $\binom{(\ell_\infty^n,\mathbb{1})}{(\ell_\infty^d,\mathbb{1})}$ in the same way as the collection of d-dimensional unital (i.e. containing the unit $\sum_{j<n} u_j$ of ℓ_∞^n) sublattices of ℓ_∞^n. Then we have the following similar equivalence.

(iii) The class $\{(\ell_\infty^n,\mathbb{1})\}_n$ has the *Structural* Ramsey Property, that is, for every d,m and every r there is n such that every r-coloring of the collection $\binom{(\ell_\infty^n,\mathbb{1})}{(\ell_\infty^d,\mathbb{1})}$ of unital subspaces of ℓ_∞^n isometric to ℓ_∞^d has a monochromatic set of the form $\binom{X}{(\ell_\infty^d,\mathbb{1})}$ for some unital $X\in\binom{(\ell_\infty^n,\mathbb{1})}{(\ell_\infty^m,\mathbb{1})}$.

(iv) The collection of partitions have the Structural Ramsey Property, that is, for every d,m and r there is n such that every r-coloring of the collection $\mathcal{E}(n,d)$ of partitions of n with d many pieces has a monochromatic set of the form $\langle\mathcal{P}\rangle$, the collection of d-partitions coarser than $\mathcal{P}$, for some $\mathcal{P}\in\mathcal{E}(n,m)$.

This latter statement is the well-known *Dual Ramsey Theorem* of Graham and Rothschild [GrRo], that was recently used in [BaLALuMbo2] to prove the (ARP) of $\{\ell_\infty^n\}_n$. We have the following open problem posed by A. S. Kechris, M. Sokić and S. Todorcevic in [KeSoTo].

Problem 5.24. *Does the collection of equipartitions have the structural Ramsey property?*

We now present a positive answer for the approximate version of this problem, and that will be used to show the (ARP) of $\{\ell_1^n\}_n$. First, endow $\mathcal{EQ}(n,d)$ with the normalized *Hamming* metric as follows: We identify each equipartition $\mathcal{P}$ with the corresponding rigid surjection $F_{\mathcal{P}}$, and then we define

$$d_{\mathrm{H}}(\mathcal{P},\mathcal{G}) := d_{\mathrm{H}}(F_{\mathcal{P}},F_{\mathcal{G}}) := \frac{1}{n}\#[F_{\mathcal{P}} \neq F_{\mathcal{Q}}] = \frac{1}{n}\#\{j<n \,:\, F_{\mathcal{P}}(j) \neq F_{\mathcal{Q}}(j)\}$$

We will prove that

Theorem 5.25 (Approximate Structural Ramsey Property of Equipartitions)**.** *For every $d|m$, every $r \in \mathbb{N}$ and every $\varepsilon > 0$ there is n divided by m such that for every r-coloring of $\mathcal{EQ}(n,d)$ has an ε-monochromatic set of the form $\langle \mathcal{R}\rangle_d^{\mathrm{eq}}$ for some $\mathcal{R} \in \mathcal{EQ}(n,m)$.*

To prove this, we introduce the notion of *approximate equipartitions and equisurjections* and using a discrete case of the method of concentration of measure we prove the approximate Ramsey result in Theorem 5.27 below, that easily implies Theorem 5.25.

Definition 5.26. Given two finite sets S and T, let ${}^{T}S$ be the set of mappings from T to S, and let $\mathrm{Epi}(T,S)$ be the subset of the surjective ones. Given $\delta \geq 0$, let $\mathrm{Equi}_\delta(T,S)$ be the collection of all δ-*equisurjections* $F : T \to S$; that is, those T such that

$$\frac{\#T}{\#S}(1-\delta) \leq \#F^{-1}(s) \leq \frac{\#T}{\#S}(1+\delta) \text{ for all } s \in S. \tag{16}$$

So δ-equisurjections are "up-to δ" *equisurjections*, that is, surjections $F : T \to S$ such that $\#F^{-1}(s)$ is always the same. Notice that when $\delta < 1$, we have

$$\frac{1-\delta}{1+\delta} \leq \frac{\#F^{-1}(s)}{\#F^{-1}(t)} \leq \frac{1+\delta}{1-\delta} \tag{17}$$

for every $F \in \mathrm{Equi}_\varepsilon(T,S)$ and $s,t \in S$. The set $\mathrm{Equi}_0(T,S)$ will be denoted by $\mathrm{Equi}(T,S)$ and its elements equisurjections instead of 0-equisurjections.

Finally, observe also that $\mathrm{Equi}_{\delta_0}(S,R) \circ \mathrm{Equi}_{\delta_1}(T,S) \subseteq \mathrm{Equi}_\delta(T,R)$ if δ is such that $(1-\delta) \leq (1-\delta_0)(1-\delta_1) \leq (1+\delta_0)(1+\delta_1) \leq (1+\delta)$. We consider also ${}^{T}S$ as a metric space endowed with the normalized *Hamming* distance

$$d_{\mathrm{H}}(F,G) := \frac{1}{\#T}\#(\{t \in T \,:\, F(t) \neq G(t)\}). \tag{18}$$

We will prove the following slight generalization of the (ARP) of equisurjections.

Theorem 5.27 (Approximate Ramsey property for δ-equisurjections)**.** *Let $d|m$, $r \in \mathbb{N}$, $\delta \geq 0$ and $\varepsilon > 0$. There is a multiple n of m such that every r-coloring of* $\mathrm{Equi}_\delta(n,d)$ *has an $(\delta+\varepsilon)$-monochromatic set of the form* $\mathrm{Equi}_\delta(m,d) \circ R$ *for some* $R \in \mathrm{Equi}(n,m)$.

It follows for example from the approximate equimeasurability principle that if $\gamma : \ell_1^d \to \ell_1^n$ is a unital approximate isometric embedding, then $\gamma(u_j)$ and $\gamma(u_k)$ are almost disjointly supported for $j \neq k < d$, so these unital quasi isometric embeddings can be approximated by those linear mappings of the form $u_j \mapsto (1/\#s_j)\mathbb{1}_{s_j}$ where $(s_j)_{j<d}$ is an approximate equisurjection of n. As a consequence, the (ARP) of approximate equipartitions is exactly a reformulation of the (ARP) of unital approximate isometric embeddings between ℓ_1^n's. The proof of Theorem 5.27 will be given later on Subsection 5.2.1, but before we come back to it, we introduce a combinatorial tool that will be used. We recall the notion of *spread vector* introduced by Matoušek and Rödl to prove the existence of $\mathbf{n}_{\boldsymbol{p}}(1,m,r,\varepsilon)$.

Definition 5.28. Given a vector $\mathbf{a} = (a_j)_{j<n} \in \mathbb{R}^n$, we say that $v \in c_{00}$ is a *spread* of $\mathbf{a}$ if $v = \sum_{j<n} a_j u_{m_j}$ for some increasing sequence $(m_j)_{j<n}$ of integers. In this case, we write that $v = \mathrm{Spread}(\mathbf{a}, \{m_j\}_{j<n})$.

Theorem 5.29 (The spreading vector Theorem)**.** [MaRo] *For every $\varepsilon > 0$ and m there exists k and a normalized vector $\mathbf{a} \in S_{\ell_1^k}$ with the following property: for each $j < m$, let*

$$x_j = x_j^{(\varepsilon,m)} := \sum_{l<k} a_l u_{k^2 j + k(l+1)}.$$

Then for every $x = \sum_j b_j x_j$ of norm 1 there is $s \subseteq k^2 m$ of cardinality k such that $\|x - \mathrm{spread}(\mathbf{a}, s)\|_1 < \varepsilon$ and such that $s \subseteq \bigcup_{b_j \neq 0} [k^2 j + k/2, k^2(j+1) + k/2[$. Consequently, for every isometric embedding $\gamma : \ell_1^m \to \langle x_j \rangle_{j<m}$ there is an isometric embedding $\eta : \ell_1^d \to \ell_1^{k^2 m}$ such that $\|\gamma - \eta\|_{1,1} \leq \varepsilon$ and such that each $\eta(u_j) = \mathrm{spread}(\mathbf{a}, s_j)$ for some s_j. □

This statement and the classical Ramsey theorem easily proves the existence of $\mathbf{n}_1(1, m, r, \varepsilon)$.

Proof of Theorem 5.18. By Proposition 5.21, it suffices to prove the case $p = 1$. We use the Spreading Theorem 5.29 applied to m and $\varepsilon/2$ to find the corresponding $\mathbf{a} = (a_l)_{l<k} \in S_{\ell_1^k}$. Fix $d, m, r \in \mathbb{N}$ and $\varepsilon > 0$. We use the Approximate Ramsey property for equipartitions in Theorem 5.27 (see the comment after the statement) applied to $d = k^2 m$, $k^2 m$, number of colors r, $\delta = 0$, and admitted error $\varepsilon/2$ to find the corresponding n divided by $k^2 m$, i.e., n has the property that for every r-coloring of $\mathrm{Emb}((\ell_1^{k^2 m}, \mathbb{1}), (\ell_1^n, \mathbb{1}))$ has an $\varepsilon/2$-monochromatic set of the form $\gamma \circ \mathrm{Iso}(\ell_1^{k^2 m})$ for some unital isometric embedding $\gamma \in \mathrm{Emb}((\ell_1^{k^2 m}, \mathbb{1}), (\ell_1^n, \mathbb{1}))$. Let us see that this n works. We fix $c : \mathrm{Emb}(\ell_1^d, \ell_1^n) \to r$. Let $\tau : \ell_1^m \to \ell_1^{k^2 m}$ be the linear mapping defined by $\tau(u_j) := x_j^{(\varepsilon/2,m)}$ for every $j < m$, and let $\iota : \ell_1^d \to \ell_1^m$ be the canonical isometric embedding $\iota(u_i) := u_i$ for every $i < d$. Now let $\widehat{c} : \mathrm{Emb}((\ell_1^{k^2 m}, \mathbb{1}), (\ell_1^n, \mathbb{1})) \to r$ be defined by $\widehat{c}(\gamma) := c(\gamma \circ \tau \circ \iota)$. By the (ARP), there is some unital $\varrho : \ell_1^{k^2 m} \to \ell_1^n$ and some $s < r$ such that $\tau \circ \mathrm{Iso}(\ell_1^{k^2 m}) \subseteq (\widehat{c}^{-1}(s))_{\varepsilon/2}$. Finally, we claim that $(\varrho \circ \tau) \circ \mathrm{Emb}(\ell_1^d, \ell_1^m) \subseteq (c^{-1}(s))_\varepsilon$: For suppose that $\gamma \in \mathrm{Emb}(\ell_1^d, \ell_1^m)$. By the property of $(x_j^{(\varepsilon,m)})_{j<m}$ in the spreading vector Theorem, there are pairwise disjoint subsets $(s_j)_{j<d}$ of $k^2 m$, each of cardinality k, such that $\|\tau(\gamma(u_j)) - \mathrm{spread}(\mathbf{a}, s_j)\|_1 < \varepsilon/2$ for every $j < d$. Let now $\theta \in \mathrm{Iso}(\ell_1^{k^2 m})$ be such that $\theta(x_j^{(\varepsilon,m)}) = \mathrm{spread}(\mathbf{a}, s_j)$ for every $j < d$. Note that is possible because $x_j^{(\varepsilon,m)} = \mathrm{spread}(\mathbf{a}, \{k^2 j + k(l+1)\}_{l<k})$ and $(s_j)_{j<d}$ is a pairwise disjoint sequence. It follows that

$$\|\theta \circ \tau \circ \iota - \tau \circ \gamma\|_{1,1} = \max_{j<d} \|\theta(x_j^{(\varepsilon,m)}) - \tau(\gamma(u_j))\|_1 \leq \max_{j<d} \|\mathrm{spread}(\mathbf{a}, s_j) - \tau(\gamma(u_j))\|_1 \leq \frac{\varepsilon}{2},$$

hence $\|\varrho \circ \theta \circ \tau \circ \iota - \varrho \circ \tau \circ \gamma\|_{1,1} \leq \varepsilon/2$. Let $\psi : \ell_1^{k^2 m} \to \ell_1^n$ be unital such that $\widehat{c}(\psi) = s$ and such that $\|\psi - \varrho \circ \theta\|_{1,1} \leq \varepsilon/2$. It follows that $c(\psi \circ \tau \circ \iota) = s$ and

$$\|\psi \circ \tau \circ \iota - \varrho \circ \tau \circ \gamma\| \leq \|\psi \circ \tau \circ \iota - \varrho \circ \theta \circ \tau \circ \iota\| + \|\varrho \circ \theta \circ \tau \circ \iota - \varrho \circ \tau \circ \gamma\| \leq \varepsilon.$$ □

5.2.1. *The proof of the* (ARP) *for approximate equisurjections.* The intention here is to give a proof of the (ARP) of approximate equisurjections in Theorem 5.27. This statement is a consequence of the concentration of measure phenomenon that approximate equisurjections have. Let us recall some basic fact and definitions on this.

Definition 5.30. Recall that an *mm*-space is a triple (X, d, μ) where (X, d) is a metric space and μ is a (probability) measure on X. Recall that the *extended concentration function* $\alpha_X(\delta, \varepsilon)$ for $\varepsilon, \delta > 0$ is defined by

$$\alpha_X(\delta, \varepsilon) = 1 - \inf\{\mu((A)_\varepsilon \,:\, \mu(A) \geq \delta\}.$$

The *concentration function* $\alpha_X(\varepsilon)$ is $\alpha_X(1/2,\varepsilon)$. A sequence $(X_n)_n$ of mm spaces is called *Lévy* when

$$\alpha_{X_n}(\varepsilon)\to_n 0 \text{ for every } \varepsilon>0.$$

The sequence $(X_n)_n$ is called *normal Lévy* when there are $c_1,c_2>0$ such that

$$\alpha_{X_n}(\varepsilon)\le c_1\exp(-c_2\varepsilon^2 n). \tag{19}$$

We say that $(X_n)_n$ is *asymptotically normal Lévy* when there are $c_1,c_2>0$ such that for every $\varepsilon>0$ there exists n_ε such that for every $n\ge n_\varepsilon$ the inequality in (19) holds.

Proposition 5.31. *Suppose that $((X_n,d_n,\mu_n))_n$ is a normal Lévy sequence, and suppose that $A_n\subseteq X_n$ for every n is such that $\inf_n\mu_n(A_n)>0$. Then $(A_n,d_n,\mu_n(\cdot|A_n))_n$ is an asymptotic normal Lévy sequence.*

The proof is based on the following simple fact.

Proposition 5.32. $\alpha_X(\delta,\varrho+\varepsilon)\le\alpha_X(\varepsilon)$ *for every δ such that $\alpha_X(\varrho)<\delta$.*

Proof. Fix A such that $\mu(A)\ge\delta$. We see first that $\mu(A_\varrho)\ge 1/2$: Otherwise, $\mu(X\setminus A_\varrho)\ge 1/2$, hence $\mu(A)\le 1-\mu((X\setminus A_\varrho)_\varrho)\le\alpha_X(\varrho)<\delta$, and this is impossible. So it follows that $\mu(A_{\varrho+\varepsilon})\ge 1-\alpha_X(\varepsilon)$. □

A standard way to estimate the concentration functions is by studying lengths of filtrations. Recall that given a measure space (Ω,Σ), a *filtration* is an $\subseteq$-increasing sequence $(\Sigma_n)_n$ of σ-subalgebras of Σ. For what we are interested in, we assume that the filtration is finite, starting and finishing with the trivial subalgebras $\{\emptyset,\Omega\}$ and Σ. A finite metric space (Ω,d) is of *length l* (see for example [MiSch]) if there are numbers $a_0,\dots,a_n$ and a filtration $(\mathcal{F}_k)_{k=0}^n$ such that $l=(\sum_i a_i^2)^{1/2}$ and such that for every k and every $A,B\in\mathcal{F}_k$ such that $A,B\subseteq C\in\mathcal{F}_{k-1}$ there is a bijection $\theta:A\to B$ such that $\max_{a\in A}d(a,\theta(a))\le a_k$. It follows then (see [Le, Theorem 4.2]) that if (Ω,d) is of length l, and if μ_C denotes the normalized counting measure on Ω, then,

$$\alpha_{(\Omega,d,\mu_C)}(\varepsilon)\le\exp(-\varepsilon^2/(8l^2)) \tag{20}$$

It is well known that the *mm space* $\mathcal{X}_n:=({}^nS,d_{\mathrm{H}},\mu_{\mathrm{C}})$, where d_{H} is the normalized Hamming distance and μ_{C} is the normalized counting measure, is of *length* $1/\sqrt{n}$, so it follows from the inequality (20) that

$$\alpha_{\mathcal{X}_n}(\varepsilon)\le\exp(-\frac{\varepsilon^2 n}{8}). \tag{21}$$

The following is a well-known result that follows from the weak law of large numbers (see below). We give an alternative proof using concentration of measure.

Proposition 5.33. *Suppose that $\delta>0$ and $\#S\ge 2$. Then there exists some n_δ such that for every $n\ge n_\delta$ one has that*

$$\mu(\mathrm{Equi}_\delta(n,S))\ge 1-\exp(-\frac{\delta^2}{9(\#S(\#S-1))^2}n). \tag{22}$$

Proof. For each $s\in S$, let $A_s:=\{F\in{}^nS\,:\,\#F^{-1}(s)\ge n/\#S\}$. Notice that $\mu_{\mathrm{C}}(A_s)\ge 1/\#S$: Observe that $\mu_{\mathrm{C}}(A_s)=\mu_{\mathrm{C}}(A_t)$ because the transposition π sending s to t transforms A_s into A_t in a measure preserving way. Since ${}^nS=\bigcup_{s\in S}A_s$, we obtain the desired bound. Now we see that

$$\bigcap_{s\in S}(A_s)_\delta\subseteq\mathrm{Equi}_{\delta\#S(\#S-1)}(n,S):$$

Given $F\in(A_s)_\delta$, let $G\in A_s$ be such that $d_{\mathrm{C}}(F,G)\le\delta$. It follows that

$$\#F^{-1}(s)\ge\#G^{-1}(s)-d_{\mathrm{C}}(F,G)\ge\frac{n}{\#S}(1-\delta\#S). \tag{23}$$

So, if $F \in \bigcap_{s\in S}(A_s)_\delta$, then, using (23), one has that

$$\#F^{-1}(s) = n - \sum_{t\neq s} \#F^{-1}(t) \leq n - \frac{n}{\#S}(1-\delta\#S)(\#S-1) = \frac{n}{\#S}(1+\#S(\#S-1)). \tag{24}$$

Since by hypothesis $\#S \geq 2$, it follows from (23) and (24) that $F \in \mathrm{Equi}_{\delta\#S(\#S-1)}(n,S)$. Let $\gamma > 0$ be such that $8/9 < (1-\gamma)^2$, and let n_δ be such that for all $n \geq n_\delta$ one has that

$$\exp(-\frac{\delta^2(1-\gamma)^2}{8}n) \leq \frac{\exp(-\frac{\delta^2}{9}n)}{\#S} \text{ and } \exp(-\frac{(\delta\gamma)^2}{8}n) < \frac{1}{\#S}.$$

Fix $n \geq n_\delta$. Then, $\alpha_{\mathcal{X}_n}(\delta\gamma) < \frac{1}{\#S}$. It follows from this, (21) and Proposition 5.32 that $\alpha_{\mathcal{X}_n}(\frac{1}{\#S}, \delta) \leq \alpha_{\mathcal{X}_n}(\delta(1-\gamma)) \leq \frac{\exp(-\frac{\delta^2}{9}n)}{\#S}$. Hence, for every $s \in S$, $\mu_{\mathrm{C}}((A_s)_\delta) \geq 1 - \frac{\exp(-\frac{\delta^2}{9}n)}{\#S}$, so, we have that

$$\mu_{\mathrm{C}}(\mathrm{Equi}_{\delta\#S(\#S-1)}(n,S)) \geq \mu_{\mathrm{C}}(\bigcap_{s\in S}(A_s)_\delta) \geq 1 - \exp(-\frac{\delta^2}{9}n).$$

Since $\delta > 0$ is arbitrary, we obtain (22). □

As a consequence,

Corollary 5.34. *Given $\delta > 0$, the sequence $(\mathrm{Equi}_\delta(n,S), d_{\mathrm{H}}, \mu_{\mathrm{C}})_n$ is an asymptotically normal Lévy sequence.* □

Proposition 5.35. *Let T, S be two finite sets, $\delta, \delta' \geq 0$. For every $\phi \in \mathrm{Equi}_\delta(T,S)$ and $\psi \in \mathrm{Equi}_{\delta'}(T,S)$ there is some permutation π of T such that $d_{\mathrm{H}}(\psi\circ\pi, \phi) \leq (\delta+\delta')/2$. In particular, if in addition $\#S|\#T$, then $\mathrm{Equi}_\delta(T,S) \subseteq (\mathrm{Equi}(T,S))_{\delta/2}$.*

Proof. For each $s \in S$, let $A_s := \phi^{-1}(s)$ and $B_s := \psi^{-1}(s)$. Let also $S_0 := \{s \in S : \#A_s \geq \#B_s\}$ and $S_1 := S \setminus S_0$. By symmetry, without loss of generality we assume that $2\#S_0 \leq \#S$. We define $\pi : T \to T$ as follows: For every $s \in S_0$, let $g_s : B_s \to A_s$ be an injection, and similarly for $s \in S_1$, let $f_s : A_s \to B_s$ be also an injection. Let $\pi : T \to T$ be any bijection such that $\pi \upharpoonright A_s = f_s$ for $s \in S_1$, and such that $\pi \upharpoonright g_s(B_s) = g_s^{-1}$ for every $s \in S_0$. It follows that for every $s \in S_1$ and every $t \in A_s$ one has that $\phi(t) = s = \psi(\pi(t))$, while for $s \in S_0$ and $t \in g_s(B_s)$, $t = g_s(\bar{t})$ one has that $\phi(t) = \phi(g_s(\bar{t})) = s = \psi(\bar{t}) = \psi(\pi(t))$. This means that

$$\begin{aligned} d_{\mathrm{H}}(\phi, \psi\circ\pi) \leq & \frac{1}{\#T}\sum_{s\in S_0} \#(A_s \setminus g_s(B_s)) = \frac{1}{\#T}\sum_{s\in S_0}(\#(A_s) - \#(B_s)) \leq \\ \leq & \#S_0((1+\delta)\frac{1}{\#S} - (1-\delta')\frac{1}{\#S}) \leq \frac{1}{2}(\delta+\delta'). \end{aligned}$$ □

Proposition 5.36. *Let $\phi_0, \phi_1 \in \mathrm{Equi}_{\delta_0}(T,S)$ and $\psi_0, \psi_1 \in \mathrm{Equi}_{\delta_1}(S,R)$. Then $d_{\mathrm{H}}(\psi_0\circ\phi_0, \psi_1\circ\phi_0) \leq (1+\delta_0)d_{\mathrm{H}}(\psi_0,\psi_1)$ and $d_{\mathrm{H}}(\psi_0\circ\phi_0, \psi_0\circ\phi_1) \leq d_{\mathrm{H}}(\phi_0,\phi_1)$.*

Proof. We have that $d_{\mathrm{H}}(\psi_0\circ\phi_0, \psi_1\circ\phi_0) = (\#T)^{-1}\sum_{\psi_0(s)\neq\psi_1(s)} \#(\phi_0^{-1}(s)) \leq (\#T)^{-1} \cdot d_{\mathrm{H}}(\psi_0,\psi_1) \cdot (\#S)(1+\delta_0)\#T(\#S)^{-1}$. The other inequality is easy to check. □

The following particular case of Theorem 5.27 will be used later to show the general case. In the next, $\mathcal{S}_X$ denotes the group of permutations of a set X.

Lemma 5.37. *For every finite set X, $\delta, \varepsilon > 0$ and $r \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that every r-coloring of $\mathrm{Equi}_\delta(n,X)$ has an ε-monochromatic set of the form $\mathcal{S}_X \circ F$ for some $F \in \mathrm{Equi}_\delta(n,X)$.*

PROOF. The group of permutations $\mathcal{S}_X$ of the set X acts by bijections on $\mathrm{Equi}_\delta(n,X)$, hence by μ_{C}-preserving transformations. This implies that if $A \subseteq \mathrm{Equi}_\delta(n,X)$ is such that $\mu_{\mathrm{C}}(A) > 1-(\#X!)$, then $\bigcap_{\pi\in\mathcal{S}_X} \pi\cdot A \neq \emptyset$. Let $n\in\mathbb{N}$ be such that $\alpha_{X_n}(1/r,\varepsilon) < 1/\#X!$, where $X_n := (\mathrm{Equi}_\delta(n,X), d_{\mathrm{H}}, \mu_{\mathrm{C}})$. Then n works: Given a coloring $c : \mathrm{Equi}_\delta(n,X) \to r$, let $j<r$ be such that $A = c^{-1}(j)$ has μ_{C}-measure $\geq 1/r$. Hence, $\mu_{\mathrm{C}}((A)_\varepsilon) > 1-1/m!$. Let $F \in \bigcap_{\pi\in S_X} \pi((A)_\varepsilon)$. Then, $\pi\cdot F \in (c^{-1}(j))_\varepsilon$ for every $\pi \in \mathcal{S}_X$. □

PROOF OF THEOREM 5.27. Fix all parameters. We choose $n\in\mathbb{N}$ given by Lemma 5.37 when applied to m, δ,ε and r, and such that $m|n$. We claim that such n works. For suppose that $c : \mathrm{Equi}_\delta(n,d) \to r$. Fix $\sigma \in \mathrm{Equi}(m,d)$, and define $\widehat{c} : \mathrm{Equi}_\delta(n,m) \to r$, by $\widehat{c}(\varrho) := c(\sigma\circ\varrho)$. By the choice of n, there is some $F \in \mathrm{Equi}_\delta(n,m)$ and $j<r$ such that $S_m \circ F \subseteq (\widehat{c}^{-1}(j))_\varepsilon$. We apply Proposition 5.35 to F to find $F_0 \in \mathrm{Equi}(n,m)$ such that $d_{\mathrm{H}}(F,F_0) \leq \delta/2$. We claim that $\mathrm{Equi}_\delta(m,d)\circ F_0 \subseteq (\widehat{c}^{-1}(j))_\varepsilon$. For suppose that $\pi \in \mathrm{Emb}_\delta(m,d)$. We apply Proposition 5.35 to $\pi\in\mathrm{Emb}_\delta(m,d)$ and $\sigma\in\mathrm{Equi}(m,d)$ to find a permutation $\theta\in\mathcal{S}_m$ such that $d_{\mathrm{H}}(\sigma\circ\theta,\pi) \leq \delta/2$. Since $S_m\circ F \subseteq (\widehat{c}^{-1}(j))_\varepsilon$, we can find $G \in \mathrm{Equi}_\delta(n,m)$ such that $c(\sigma\circ G) = \widehat{c}(G) = j$ and such that $d_{\mathrm{H}}(G,\theta\circ F) \leq \varepsilon$. We use Proposition 5.36 to conclude that

$$\begin{aligned} d_{\mathrm{H}}(\pi\circ F_0, \sigma\circ G) \leq & d_{\mathrm{H}}(\pi\circ F_0, \sigma\circ\theta\circ F_0) + d_{\mathrm{H}}(\sigma\circ\theta\circ F_0, \sigma\circ G) \leq d_{\mathrm{H}}(\pi,\sigma\circ\theta) + d_{\mathrm{H}}(\theta\circ F_0, G) \leq \\ \leq & d_{\mathrm{H}}(\pi,\sigma\circ\theta) + d_{\mathrm{H}}(F_0,F) + d_{\mathrm{H}}(\theta\circ F, G) \leq \delta+\varepsilon. \end{aligned}$$ □

6. Gurarij M-space

We finish the paper by presenting a Fraïssé space in the category of M-spaces. Recall that a Banach lattice X is called an M-space when $\|\sum_{j<n} x_j\| = \max_{j<n}\|x_j\|$ for every sequence $(x_j)_{i<n}$ of pairwise disjoint elements of X. In particular, for finite dimensional Banach lattices X this is equivalent to saying that X is lattice isometric to $\ell_\infty^{\dim X}$. We are going to prove that the class of finite dimensional M-spaces is a Fraïssé class, and we will find the corresponding Fraïssé M-space $\mathbb{G}_\diamond$. For this we revisit some results of F. Cabello-Sanchez [CaSa] on the existence of an almost transitive renorming of $C[0,1]$. To extend his results to the setting of extensions of partial isometries between finite dimensional subspaces, we shall extend the notions of ultrahomogeneity which are specific to normed lattices that were introduced in Section 3. In this context, suppose that A and B are Banach lattices and suppose that $\delta>0$. Let $\gamma : A\to B$ be a 1-1 linear mapping.

(a) γ is *δ-disjoint preserving* if $\||\gamma(a_j)|\wedge|\gamma(a_k)|\| \leq \delta, j\neq k$,
(b) γ is *δ-positive* if $d(\gamma(a),B_+) \leq \delta$ for every positive $a\in A_+$, where A_+, B_+ denote the corresponding set of positive elements.
(c) γ is a *δ-isometric disjoint-preserving embedding* when γ is δ-disjoint preserving and δ-isometric embedding.
(d) γ is a *δ-isometric lattice embedding* when it is δ-disjoint preserving, δ-positive and δ-isometric embedding.

Some of these definition already appear in [OiTr] Let $\mathrm{Emb}_\delta^\diamond(A,B)$ and $\mathrm{Emb}_\delta^\perp(A,B)$ be the collection of δ-isometric lattice embeddings and δ-isometric disjoint preserving embeddings, respectively. Let $\mathcal{E}$ be a Banach lattice. We introduced in Definition 3.1 the notion of approximately lattice ultrahomogeneous ($\diamond$ −AuH). Let $\mathrm{Age}^\diamond(\mathcal{E})$ be the collection of finite dimensional sublattices of $\mathcal{E}$. Similarly, let $\mathrm{Iso}_\diamond(\mathcal{E})$ and $\mathrm{Iso}_\perp(\mathcal{E})$ be the group of lattice and of disjoint-preserving isometries on $\mathcal{E}$, respectively.

Definition 6.1. We say that $\mathcal{E}$ is a *Fraïssé Banach lattice* when there is a modulus of stability $\varpi :]0,\infty[\times\mathbb{N} \to]0,\infty]$ such that for every $X\in\mathrm{Age}_k^\diamond(\mathcal{E})$, every $\delta\geq 0$ and $\varepsilon>0$ the canonical action $\mathrm{Iso}_\diamond(\mathcal{E}) \curvearrowright \mathrm{Emb}_\delta^\diamond(X,\mathcal{E})$ is $(\varpi(k,\delta)+\varepsilon)$-transitive for every $k\in\mathbb{N}$ and every $X\in\mathrm{Age}_k^\diamond(\mathcal{E})$.

It follows from Proposition 3.5 and Schechtman's Theorem 3.8 that $L_p(0,1)$ is a Fraïssé Banach lattice for every $1 \le p < \infty$. Now we present a new one. In [CaSa], F. Cabello-Sanchez defined $\mathfrak{X} = (\Pi_{n=1}^{+\infty} L_{p_n}(0,1))_{\mathcal{U}}$, where $p_n \to +\infty$, and proved that the non-separable Banach lattice $\mathfrak{X}$ is an M-space that is transitive, meaning that the isometry group acts transitively on the unit sphere of $\mathfrak{X}$. We prove with similar methods that his results may be improved as follows.

Theorem 6.2. *The lattice $\mathfrak{X}$ is a Fraïssé Banach lattice with modulus $\varpi(\delta,k) \le 3\cdot k\cdot\delta$. In addition, for every $X \in \mathrm{Age}_k^{\diamond}(\mathfrak{X})$, every $\delta \ge 0$ and $\varepsilon > 0$ the canonical action $\mathrm{Iso}_\perp(\mathfrak{X}) \curvearrowright \mathrm{Emb}_\delta^{\diamond}(X,\mathfrak{X})$ is $(3\cdot k\cdot\delta+\varepsilon)$-transitive for every $k \in \mathbb{N}$ and every $X \in \mathrm{Age}_k^{\diamond}(\mathfrak{X})$.*

The proof readily follows from the following two lemmas.

Lemma 6.3. *The lattice $\mathfrak{X}$ is lattice ultrahomogeneous and disjointly ultrahomogeneous, that is, for every $X \in \mathrm{Age}^{\diamond}(\mathfrak{X})$ the action $\mathrm{Iso}_\perp(\mathfrak{X}) \curvearrowright \mathrm{Emb}^{\perp}(X,\mathfrak{X})$ is transitive.*

Proof. Let F, G be finite dimensional sublattices of $\mathfrak{X}$ and t be an isometry from F onto G. Write $F = \langle f_j\rangle_{j<d}$ and $G = \langle tf_j\rangle_{j<d}$, where $(f_j)_{j<d}$ is a normalized, positive and pairwise disjoint sequence. The vectors $f_0,\dots,f_{d-1}$ may be represented as $f_j = (f_n^j)_n$, where for each n the $(f_j^n)_{j<d}$'s is a normalized, positive and pairwise disjoint sequence in $L_{p_n}(0,1)$ (see in [CaSa, the proof of Lemma 3.2]), and the same holds for each $tf_j = (g_n^j)_n$. If we call t_n the isometric map sending f_n^j to g_n^j we know by Lemma 3.5 on the approximate lattice ultrahomogeneity of L_p's that there exists a lattice isometry T_n on L_{p_n} such that $\|T_n \restriction F_n - t_n\| \le 2^{-n}$. We note that $T = (T_n)_n$ is a lattice isometry on $\mathfrak{X}$ and that $T(f_j) = (T_n(f_j^n))_n = (t_n(f_j^n))_n = (g_n^j)_n = t(f_j)$, for each $j < d$. Disjoint ultrahomogeneity follows from a similar proof and the fact that each $L_{p_n}(0,1)$ is approximately disjointly ultrahomogeneous (see Proposition 3.2). □

It seems to remain open whether $\mathfrak{X}$ is actually ultrahomogeneous as a Banach space.

Remark 6.4. It follows that $\mathfrak{X}$ is also of lattice disposition, that is, for any $F \subset G$, where F and G are finite sublattices of $\mathfrak{X}$, and for any lattice isometric embedding t of F into M, there exists a lattice isometric embedding T of G into $\mathfrak{X}$ such that $T \restriction F = t$.

The other lemma which is needed is a form of amalgamation property, in the lattice setting, of the class of finite dimensional M-spaces:

Lemma 6.5. *Suppose that A and B are two finite dimensional M-spaces and suppose that $\delta \ge 0$. Then $\mathrm{Emb}_\delta^{\diamond}(A,B) \subseteq (\mathrm{Emb}^{\diamond}(A,B))_{3\delta \dim A}$ and $\mathrm{Emb}_\delta^{\perp}(A,B) \subseteq (\mathrm{Emb}^{\perp}(A,B))_{3\delta \dim A}$.*

Proof. We write $A = \langle a_j\rangle_{j<m}$, $B = \langle b_j\rangle_{j<n}$ both normalized pairwise disjointly supported, and suppose that $\gamma \in \mathrm{Emb}_\delta^{\diamond}(A,B)$. Let $g : A \to B$ linearly defined by $g(a_j) := \sum_{|b_k^*(\gamma(a_j))|>\delta} b_k^*(\gamma(a_j))b_k$, where b_k^* is the sequence of functionals dual to b_k. The fact that γ is an δ-isometric lattice embedding implies that $(g(a_j))_{j<m}$ is pairwise disjointly supported, positive and $\|g(a_j) - \gamma(a_j)\| \le \delta$ for all $j < m$. Hence, $|\|g(a_j)\| - 1| \le 2\delta$ for all $j < m$. Let $\xi : A \to B$ linearly defined by $\xi(a_j) := g(a_j)/\|g(a_j)\|$. It is clear that ξ is a lattice isometric embedding. also, $\|\xi(a_j) - \gamma(a_j)\| \le 3\delta$; hence, $\|\gamma - \xi\| \le 3\delta \dim A$. □

This lemma should be compared with the fact that without lattice constraints one has $\mathrm{Emb}_\delta(\ell_\infty^d, \ell_\infty^n) \subseteq \mathrm{Emb}(\ell_\infty^d, \ell_\infty^n)_\delta$.

Proposition 6.6. *The M-space $\mathfrak{X}$ admits a separable sublattice $\mathbb{G}_\diamond$ that is a Fraïssé Banach lattice with modulus of stability $\varpi(k,\delta) \le 3\cdot\delta\cdot k$, and such that for every finite dimensional sublattice X of $\mathbb{G}_\diamond$, $\delta \ge 0$ and $\varepsilon > 0$ the canonical action $\mathrm{Iso}_\perp(\mathbb{G}_\diamond) \curvearrowright \mathrm{Emb}_\delta^{\perp}(X,\mathbb{G}_\diamond)$ is $3\cdot\delta\cdot\dim X+\varepsilon$-transitive.*

PROOF. We will find a separable sublattice $\mathbb{G}_\diamond$ of $\mathfrak{X}$ and some countable dense set $D \subseteq \bigcup_n \mathrm{Emb}^\perp(\ell_\infty^n, \mathbb{G}_\diamond)$ such that

a) for every $\gamma, \eta \in D \cap \mathrm{Emb}^\perp(\ell_\infty^n, \mathbb{G}_\diamond)$ there is some $T \in \mathrm{Iso}_\perp(\mathbb{G}_\diamond)$ such that $\eta = T \circ \gamma$.
b) $D \cap (\bigcup_n \mathrm{Emb}^\diamond(\ell_\infty^n, \mathbb{G}_\diamond))$ is dense in $\bigcup_n \mathrm{Emb}^\diamond(\ell_\infty^n, \mathbb{G}_\diamond)$.
c) for every $\gamma, \eta \in D \cap \mathrm{Emb}^\diamond(\ell_\infty^n, \mathbb{G}_\diamond)$ there is some $T \in \mathrm{Iso}_\diamond(\mathbb{G}_\diamond)$ such that $\eta = T \circ \gamma$.

This, together with Lemma 6.5 gives the Fraïssé property of $\mathbb{G}_\diamond$ with the corresponding modulus. Let Y_0 be a separable sublattice of $\mathfrak{X}$ and let $D_0 \subseteq \bigcup_n \mathrm{Emb}^\diamond(\ell_\infty^n, Y_0)$ be countable and dense in $\bigcup_n \mathrm{Emb}^\perp(\ell_\infty^n, Y_0)$ such that $C_0 \cap (\bigcup_n \mathrm{Emb}^\diamond(\ell_\infty^n, Y_0))$ is dense in $\bigcup_n \mathrm{Emb}^\diamond(\ell_\infty^n, Y_0)$.

By the lattice and disjoint ultrahomogeneity of $\mathfrak{X}$, given $\gamma, \eta \in D_0$, $\gamma, \eta \in \mathrm{Emb}^\perp(\ell_\infty^n, Y_0)$ it is possible to select a global disjoint isometry $T_{\gamma,\eta}$ on $\mathfrak{X}$ such that $\eta = T_{\gamma,\eta}\gamma$, in a way that if γ, η are positive, then $T_{\gamma,\eta}$ is also positive. Let G_0 be the countable subgroup of $\mathrm{Iso}(\mathfrak{X})$ generated by the disjoint isometries $\{T_{\gamma,\eta} : \gamma, \eta \in D_0 \cap (\bigcup_{n\in\mathbb{N}} \mathrm{Emb}^\perp(\ell_\infty^n, Y_0)\}$, and let Y_1 be the separable sublattice generated by the spaces TY_0 with $T \in G_0$. In this way, we can find $\subseteq$-increasing sequences $(Y_k)_k$, $(D_k)_k$ and $(G_k)_k$ where each Y_k is a separable Banach sublattices of $\mathfrak{X}$, D_k is a countable dense subset of $\bigcup_n \mathrm{Emb}^\perp(\ell_\infty^n, Y_k)$ such that D_k is also dense in $\bigcup_n \mathrm{Emb}^\diamond(\ell_\infty^n, Y_k)$, and G_k is a countable subgroup of $\mathrm{Iso}_\perp(\mathfrak{X})$ such that for every $\gamma, \eta \in D_k \cap \mathrm{Emb}^\perp(\ell_\infty^n, Y_k)$ there is $T \in G_k$ such that $\eta = T \circ \gamma$, that is positive if γ, η are so, and such that Y_{k+1} is the sublattice generated by $\bigcup_{T\in G_k} TY_k$. Let $Y := \bigcup_k Y_k$, and let $\gamma, \eta \in D_k \cap \mathrm{Emb}^\perp(\ell_\infty^n, Y_k)$. Choose $T \in G_k$ such that $\eta = T \circ \gamma$, and such that it is positive if γ, η are so. Since $TY_l \subseteq Y_{l+1}$ for all $l \geq k$, we have that $T \upharpoonright Y : Y \to Y$ is an isometric embedding; since G_k is a subgroup, similarly we have that $T^{-1} : Y \to Y$, so T is a surjective isometry on Y. Then the closure $\mathbb{G}^\diamond$ of Y is the desired sublattice. □

REMARK 6.7. $\mathbb{G}^\diamond$ is of steady approximate lattice and disjoint preserving disposition with modulus $\leq 3{\cdot}k{\cdot}\delta$; that is, for every $X \subseteq Y$ both in $\mathrm{Age}^\perp(\mathbb{G}_\diamond)$, $\delta \geq 0$, $\varepsilon > 0$, and $\gamma \in \mathrm{Emb}_\delta^\perp(X, \mathbb{G}_\diamond)$ there is some $\eta \in \mathrm{Emb}^\diamond(Y, \mathbb{G}_\diamond)$ such that $\|\eta \upharpoonright X - \gamma\| \leq 3 \cdot \dim X \cdot \delta + \varepsilon$, and η is can be chosen to be positive when X, Y are lattices and γ is positive as well.

Of course since the above construction depends on choices of subspaces and embeddings, the lattice $\mathbb{G}^\diamond$ is not unique, but lattice isometrically $\mathbb{G}^\diamond$ is unique. Concerning its proof, is worth noticing that the proof for the uniqueness principle for (AuH) Banach spaces (Theorem 2.21) does not seem to work directly; the reason is that, in general, given two finite dimensional sublattices A and B of a Banach lattice X, the lattice generated by A and B may not be finite dimensional. On the other hand, the approach in Theorem 2.18, with the obvious modifications, shows the next uniqueness statement. We leave the details to the reader.

Theorem 6.8. *Suppose that X and Y are two separable Fraïssé Banach lattices. The following are equivalent.*

1) $\mathrm{Age}^\diamond(X) \equiv \mathrm{Age}^\diamond(Y)$.
2) X and Y are lattice isometric. □

Corollary 6.9. *There is a renorming of $C[0,1]$ that is a Fraïssé Banach lattice.*

PROOF. According to [CaSa, Theorem 3.4], every separable almost transitive infinite dimensional M-space is isomorphic to $C[0,1]$ (a consequence of Miljutin's Theorem). □

It is worth noting that $\mathbb{G}_\diamond$, although isomorphic to $C[0,1]$, cannot be isometric to a $C(K)$ space itself. Indeed separable $C(K)$ spaces are not almost transitive unless in the trivial case:

REMARK 6.10. If K is a metrizable compact space with $|K| \geq 2$, then the group of isometries on $C(K)$ is never almost transitive.

PROOF. Fix a compact metric space $(K, \mathbb{1}_K)$. First note that isometries on $C(K)$ act by multiplication of a unimodular continuous function with $f \mapsto f(\sigma)$ for some homeomorphism of σ, so the orbit of $\mathbb{1}_K$ is the set of unimodular functions. So if $a \in K$, the function $x \mapsto f(x) = d_K(x,a)/\|d(x,a)\|_\infty$ is a norm 1 function which is at distance 1 for the orbit of $\mathbb{1}_K$. □

We finish this part by exposing new extremely amenable groups.

Theorem 6.11. *The groups* $\mathrm{Iso}_\perp(\mathbb{G}_\diamond)$ *and* $\mathrm{Iso}_\diamond(\mathbb{G}_\diamond)$ *are extremely amenable.*

We have the following correspondence.

Theorem 6.12 (KPT correspondence for Banach lattices)**.** *Let* $\mathcal{E}$ *be a separable* (AuH) *Banach lattice. Then the following are equivalent*

1) $\mathrm{Iso}_\diamond(\mathcal{E})$ *is extremely amenable;*
2) $\mathrm{Age}^\diamond(\mathcal{E})$ *has the* (ARP).

Its proof is exactly the obvious modification of that for Banach spaces in Theorem 5.10.

Theorem 6.13. *The class* $\mathrm{Age}_\diamond(\mathfrak{X})$ *of finite dimensional sublattices with lattice or disjoint preserving isometric embeddings has the* (ARP).

In its proof we will use the Dual Ramsey theorem by Graham and Rothschild. Recall that given two finite linear orderings $\mathbf{R} = (R, <_R)$ and $\mathbf{S} = (S, <_S)$, a mapping $\sigma : S \to R$ is a *rigid surjection* when σ is a surjection such that for every $r_0 <_R r_1$ one has that $\min_{<_S} \sigma^{-1}(r_0) <_S \min_{<_S} \sigma^{-1}(r_1)$. Let $\mathrm{Epi}(\mathbf{S}, \mathbf{R})$ be the collection of rigid surjections from $\mathbf{S}$ onto $\mathbf{R}$.

Theorem 6.14 (Dual Ramsey Theorem [GrRo])**.** *For every finite linearly ordered sets* $\mathbf{R}$ *and* $\mathbf{S}$*, and every* r *there is some linearly ordered set* $\mathbf{T}$ *such that every* r*-coloring of* $\mathrm{Epi}(\mathbf{T}, \mathbf{R})$ *has a monochromatic set of the form* $\mathrm{Epi}(\mathbf{S}, \mathbf{R}) \circ \sigma$ *for some* $\sigma \in \mathrm{Epi}(\mathbf{T}, \mathbf{S})$.

Note that a linear mapping $\gamma : \ell_\infty^d \to \ell_\infty^n$ is a disjoint preserving embedding if and only if its dual operator $\gamma^* : \ell_1^n \to \ell_1^d$ satisfies that

$$\{u_k\}_{k<d} \subseteq \{\pm\gamma^*(u_j)\}_{j<n} \subseteq [-1,1] \cdot \{u_k\}_{k<d}.$$

In other words, if A denotes the matrix representing γ in the unit bases of $\mathbb{R}^d$ and $\mathbb{R}^n$, then the row vectors of A must be of the form $a_k u_k$ for some $|a_k| \leq 1$, and for each $k < d$ there must be a row vector with $a_k = \pm 1$. Let $\mathrm{Quo}^\perp(\ell_1^n, \ell_1^d)$ be the collection of such surjections σ. Similarly, γ is a lattice embedding when

$$\{u_k\}_{k<d} \subseteq \{\gamma^*(u_j)\}_{j<n} \subseteq [0,1] \cdot \{u_k\}_{k<d}.$$

Let $\mathrm{Quo}^\diamond(\ell_1^n, \ell_1^d)$ be the collection of such surjections σ. Given a 1-1 mapping $f : d \to m$ and $\theta := (\theta_k)_{k<d} \in \{-1,1\}^d$, let $\gamma_{f,\theta} \in \mathrm{Emb}^\perp(\ell_\infty^d, \ell_\infty^m)$ be linearly defined by $\eta_{f,\theta}(u_k) := \theta_k u_{f(k)}$. Observe that for every $\sigma \in \mathrm{Quo}^\perp(\ell_\infty^d, \ell_\infty^m)$ there are f and θ such that $\sigma \circ \eta_{f,\theta} = \mathrm{Id}_{\ell_\infty^d}$, and if in addition σ is positive, $\theta = (1)$ and consequently $\gamma_{f,(1)} \in \mathrm{Emb}^\diamond(\ell_\infty^d, \ell_\infty^m)$. We choose one of those embedding and we denote its parameters by $(f_\sigma, \theta_\sigma)$.

Proof of Theorem 6.13. We prove the (ARP) of finite dimensional M-spaces with respect to disjoint preserving in its dual form; that is, for every $d, m, r \in \mathbb{N}$ and every $\varepsilon > 0$ there is some n such that every r-coloring of $\mathrm{Quo}^{\perp}(\ell_1^n, \ell_1^d)$ has an ε-monochromatic set of the form $\mathrm{Quo}^{\perp}(\ell_1^m, \ell_1^d) \circ \sigma$ for some $\sigma \in \mathrm{Quo}^{\perp}(\ell_1^n, \ell_1^m)$; the corresponding (ARP) for lattice embeddings is proved similarly, and we leave the details to the reader.

Fix $d, m, r \in \mathbb{N}$ and $\varepsilon > 0$. Let $e := \lceil 1/\varepsilon \rceil$, and let $\Delta := \{s(l/e)u_k \ : \ s \in \{-1, 1\}, 0 \le l \le t, k < d\}$ ordered by $\prec$ such that if $l < l'$, then $s(l/e)u_k \prec s'(l'/e)u_{k'}$ Let also $\Lambda := \Delta \times E$, where $E = \{(f, \theta) : f : d \to m$ is 1-1 and $\theta \in \{-1, 1\}^d\}$. We linearly order E arbitrarily and then Λ by the corresponding lexicographic ordering. Let n, ordered canonically, be the result of applying the dual Ramsey Theorem for the parameters Δ, Λ and number of colors r. We claim that n works. For suppose that $c : \mathrm{Quo}^{\perp}(\ell_1^n, \ell_1^d) \to r$. Let $\Phi : \mathrm{Epi}(n, \Delta) \to \mathrm{Quo}^{\perp}(\ell_1^n, \ell_1^d)$ be linearly defined for $f \in \mathrm{Epi}(n, \Delta)$ and $j < n$ by $\Phi(f)(u_j) := s \cdot a \cdot u_k$ if $f(j) = (a, s, k) \in A \times \{-1, 1\} \times d$. Let $\widehat{c} := c \circ \Phi$. By the "Ramsey property" of the number n, there is some $g \in \mathrm{Epi}(n, \Lambda)$ such that $\mathrm{Epi}(\Lambda, \Delta) \circ g$ is $\widehat{c}$-monochromatic with constant value $\bar{r} < r$. Let $\sigma \in \mathrm{Quo}^{\perp}(\ell_1^n, \ell_1^m)$ be linearly defined for $j < n$ by $\sigma(u_j) := s \cdot a \cdot \gamma_{f,\theta}(u_k)$, if $g(j) = ((a, s, k), (f, \theta)) \in \Lambda$. We see that $\mathrm{Quo}(\ell_1^m, \ell_1^d) \circ \sigma \subseteq (c^{-1}(\bar{r}))_{\varepsilon}$ by establishing the following.

Claim 6.14.1. For every $\tau \in \mathrm{Quo}(\ell_1^m, \ell_1^d)$ there is some rigid surjection $h \in \mathrm{Epi}(\Lambda, \Delta)$ such that $\|\tau \circ \sigma - \Phi(h \circ g)\| \le \varepsilon$.

Proof of Claim: Fix such τ, and $\lambda := (s(l/e)u_k, (f, \theta)) \in \Lambda$. If $\tau(l \cdot u_{f(k)}) = 0$, then we declare $h(\lambda) = 0$. Suppose that $\tau(l \cdot u_{f(k)}) \neq 0$. There are two cases to consider:

a) $(f, \theta) = (f_\tau, \theta_\tau)$. We let $h(\lambda) = s(l/e)u_k$.
b) $(f, \theta) \neq (f_\tau, \theta_\tau)$. Write $\tau(\gamma_{f,\theta}(s(l/e)u_k)) = bu_i \neq 0$, for some $b \in [-1, 1]$ and $i < d$. Let $0 \le l \le e$ and $c \in \{-1, 1\}$ be such that $|b - cl/e| \le \varepsilon$ and be such that $j/e < |b|$, and we set $h(\lambda) := c(l/e)u_i$

We show that h is a rigid surjection by proving that $\min_\Lambda h^{-1}(0) = (0, \min E)$, and $\min_\Lambda h^{-1}(s(l/e)u_k) = (s(l/e)u_k, (f_\tau, \theta_\tau))$. The fact that $\min_\Lambda h^{-1}(0) = (0, \min E)$ is trivial; as for the other equality, set $\lambda := (s(l/e)u_k, (f_\tau, \theta_\tau))$. First of all, $l \neq 0$, and $\tau(s(l/e)u_{f_\tau(k)}) = a\tau(\gamma_{f_\tau,\theta_\tau}(u_k)) = au_k$, where $a = t\theta_k(l/e)$ with θ_k being the k^{th}-coordinate of θ_τ. So, in particular, $\tau(lu_{f_\tau(k)}) \neq 0$, and consequently, by definition, $h(s(l/e)u_k, (f_\tau, \theta_\tau)) = s(l/e)u_k$. On the other hand, if $h(\lambda') = s(l/e)u_k$ with $\lambda \neq \lambda' = (s'(l'/e)u_{k'}, (f, \theta))$, then necessarily $s'(l'/e)u_{k'} \neq s(l/e)u_k$, and consequently we are in case b) of the definition of h, so $l/e < \|\tau(\gamma_{f',\theta'}(s'(l'/e)u_{k'}))\| \le \|s'(l'/e)u_{k'}\| = l'/e$, because τ is a contraction and $\gamma_{f',\theta'}$ is an embedding. Hence, by the ordering on Δ, $s(l/e)u_k \prec s'(l'/e)u_{k'}$ and consequently, $\lambda < \lambda'$.

Finally, let us see that $\|\tau \circ \sigma - \Phi(h \circ g)\| \le \varepsilon$, that is, $\|\tau(\sigma(u_j)) - \Phi(h \circ g)(u_j)\| \le \varepsilon$ for every $j < n$, because we consider operators from ℓ_1^n onto ℓ_1^d. So, fix $j < n$, and set $\lambda = (s(l/e)u_k, (f, \theta)) = g(j)$. By definition, $\sigma(u_j) = \gamma_{f,\theta}(s(l/e)u_k)$, while $\Phi(h \circ g)(u_j) = h(\lambda)$. Suppose first that $\tau(\sigma(u_j)) = 0$; then by definition of h, $h(\lambda) = 0$, hence $\tau(\sigma(u_j)) = \Phi(h \circ g)(u_j) = 0$. Suppose now that $\tau(\sigma(u_j)) \neq 0$. Now suppose that $(f, \theta) = (f_\tau, \theta_\tau)$. It follows that $\tau(\sigma(u_j)) = \tau \circ \gamma_{f,\theta}(s(l/e)u_k) = s(l/e)u_k$, while $\Phi(h \circ g)(u_j) = h(\lambda) = s(l/e)u_k = \tau(\sigma(u_j))$. Finally, suppose that $(f, \theta) \neq (f_\tau, \theta_\tau)$. By the choice of $h(\lambda)$, $\|\tau(\sigma(u_j)) - h(\lambda)\| \le \varepsilon$. □

□

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