

ENVELOPES IN BANACH SPACES

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ABSTRACT. We introduce the notion of isometric envelope of a subspace in a Banach space, establishing its connections with several key elements: a) we explore its relation to the mean ergodic projection on fixed points within a semigroup of contractions, b) we draw parallels with Korovkin sets from the 1970s, c) we investigate its impact on the extension properties of linear isometric embeddings. We use this concept to address the recent conjecture that the Gurarij space and the spaces L_p , $p \notin 2\mathbb{N} + 4$ are the only separable approximately ultrahomogeneous Banach spaces (a certain multidimensional transitivity of the action of the linear isometry group on the space considered in [FLAMT20]). The similar conjecture for Fraïssé Banach spaces (a strengthening of the approximately homogeneous property) is also considered. We characterize the Hilbert space as the only separable reflexive space in which any closed subspace coincides with its envelope; and we show that the Gurarij space satisfies the same property. We compute some envelopes in the case of Lebesgue spaces, showing that the reflexive L_p -spaces are the only reflexive rearrangement invariant spaces on $[0, 1]$ for which all 1-complemented subspaces are envelopes. We also identify the isometrically unique "full" quotient space of L_p by a Hilbertian subspace, for appropriate values of p , as well as the associated topological group embedding of the unitary group into the isometry group of L_p .

1. INTRODUCTION AND NOTATION

1.1. **The motivation.** An initial motivation for this work stems from the famous rotations problem of Mazur. Assume the linear isometry group of a separable Banach space X acts transitively on its unit sphere; must X be isometric to the Hilbert space? (see, for example, [CS97] or the recent survey [CSFRar]). Many examples of non-Hilbertian spaces which are almost transitive (AT) can be found, meaning that the orbits of the action of the isometry group on the sphere are dense. Striking examples of this are the Lebesgue spaces $L_p(0, 1)$ for $1 \leq p < \infty$. For $p \neq 2$, these spaces, although not transitive, are very close to being so: they admit exactly two orbits for the action of the isometry group, the orbit of functions with full support, resp. without full support. This suggests that the L_p spaces are transitive "with respect to the supports", that is, provided one sees each function f inside its "appropriate L_p -space" $L_p(\text{supp } f)$. The concept of an isometric envelope, defined in the present paper, initially aimed to imitate some aspects of the support of a function in an abstract setting that would not require the structure of a function space.

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The correct setting for defining the isometric envelope is multidimensional, so let us recall a few facts. A Banach space X is ultrahomogeneous when any isometric map between finite-dimensional subspaces extends to a (surjective) isometry on X . A multidimensional version of Mazur’s rotation problem asks: must any separable ultrahomogeneous Banach space be isometric to the Hilbert space ([FLAMT20])?

The separable Lebesgue spaces are very close to being ultrahomogeneous, although one has to exclude the even values of $p \geq 4$. For $p \notin 2\mathbb{N} + 4$, $L_p(0, 1)$ is approximately ultrahomogeneous (AUH), meaning that any isometric map between finite-dimensional subspaces may be approximated as accurately as desired by a (surjective) isometry (essentially due to Lusky [Lus78]). The Gurarij space is also AUH [KS13]. A conjecture formulated in [FLAMT20] and which is our main interest in this paper is the following: Is every separable AUH space isometric either to some $L_p(0, 1)$ or to the Gurarij space?

Isometric maps between subspaces of the AUH L_p ’s extend to isometries between respective appropriate L_p -subspaces ”enveloping” them, a consequence of Plotkin and Rudin’s equimeasurability theorem [Plo74, Rud76], see also [KK01] (the ”envelope” terminology is ours and does not appear in those works). Based on these examples, our aim is to define envelopes of subspaces in general Banach spaces, so that AUH spaces can be proved to be “ultrahomogeneous with respect to envelopes”, in a precise sense to be defined later. The notion of envelope is then expected to be useful to address the conjecture from [FLAMT20].

Another motivation for studying (AUH) spaces is that they are natural universal objects. Two separable (AUH) spaces are isometric as soon as they have the same finite-dimensional subspaces up to isometry [FLAMT20, Proposition 2.22]. For a strengthening of the (AUH) property, called the ”Fraïssé” property (see Definition 2.44), there is a correspondence between these spaces and classes of finite-dimensional spaces with a natural amalgamation property. Thus, the separable Fraïssé Banach space corresponding to one of such classes \mathcal{F} (called the Fraïssé limit of \mathcal{F}) is the ”generic” universal space for all those spaces having dense subspaces consisting of an increasing union of elements in \mathcal{F} . Examples of this situation are precisely the Gurarij space and the spaces $L_p[0, 1]$, for $p \notin 2\mathbb{N} + 4$, that are the generic universal spaces corresponding to the amalgamation classes of all separable spaces, and the class of L_p -spaces, respectively (see [FLAMT20]). They are the only known examples, and it is unknown whether there exist (AUH) spaces which are not Fraïssé. Moreover, the Kechris-Pestov-Todorćević correspondence for (AUH) Banach spaces characterizes a fixed point property, called extreme amenability, of their linear isometry group in terms of an approximate Ramsey property of the collection of their finite-dimensional subspaces. All the known (AUH) Banach spaces have this property (see [GM83] for ℓ_2 , [GP07] and [FLAMT20] for the other L_p ’s, and [BLALM21] for the Gurarij space). New examples of (AUH) Banach spaces could provide new extremely amenable groups; on the other hand, if all separable (AUH) Banach spaces are either isometric to the L_p ’s or the Gurarij, then their isometry groups appear as canonical extremely amenable groups of isometries of Banach spaces. Let us note finally that Fraïssé spaces must contain an isometric copy of the Hilbert space [FLAMT20, Proposition 2.13.], and therefore the study of the envelopes of Hilbertian subspaces of Fraïssé spaces will be particularly relevant.

Here is the definition of the isometric envelope $\text{Env}(Y)$ of a subspace Y of X (for full details see Section 2): it is the subspace of $x \in X$ such that $(T_i(x))_i$ converges to x whenever the net $(T_i)_i$ in $\text{Isom}(X)$ converges pointwise on Y to Id_Y . Under reflexivity, we may relate the isometric envelope to ergodic

decompositions relative to semigroups of contractions, in the so-called Jacobs-de Leeuw-Glicksberg theory (see [EFHN15, Chapters 8 and 16]), as well as to results on Korovkin sets (see [Wul68]). In particular, we prove the fundamental result that if X is reflexive and strictly convex, then $\text{Env}(Y)$ is always 1-complemented, and if X is reflexive and locally uniformly convex, then $\text{Env}(Y)$ is actually the smallest superspace of Y complemented by a projection in the "isometric hull" $\overline{\text{conv}}(\text{Isom}(X))$. As a byproduct, the Hilbert space is the only separable reflexive space on which all subspaces are equal to their envelopes. We conclude Section 2 with a fundamental extension property of the envelope, which we state here for (AUH) spaces X : any partial isometry between subspaces Y, Z of X extends uniquely to a partial isometry between their envelopes $\text{Env}(Y)$ and $\text{Env}(Z)$; if Y is separable, then this extension map is SOT-SOT continuous and $\text{Env}(Y)$ is maximum among superspaces of Y satisfying this property. Of particular interest are the "full" subspaces of X , i.e., those whose envelope is equal to X : any partial isometry between full subspaces extends uniquely to a surjective isometry on X , and the isometry group of a full subspace embeds naturally as a subgroup of $\text{Isom}(X)$. Thus, we achieve the extension properties that envelopes were designed to satisfy.

In Section 3 we investigate envelopes in L_p spaces, and prove that if Y is unital, then the envelope coincides with some classical notions such as the sublattice generated by Y , the minimal 1-complemented subspace containing Y , or the range of the conditional expectation associated to Y . In particular, all 1-complemented unital subspaces are envelopes in those spaces, and we prove that the L_p 's, $1 < p < \infty$ are the only r.i. reflexive spaces on $[0, 1]$ for which this happens. On the other hand, we prove that in the Gurarij space, every closed subspace is equal to its envelope.

After proving that, for $p \notin 2\mathbb{N} + 4$, the L_p 's contain a full copy of the Hilbert space, we apply the isometric extension results of Section 2 to study the associated exact sequence of Banach spaces and define objects such as the group embedding of $U(\ell_2)$ inside $\text{Isom}(L_p)$ or the associated "full quotient" of L_p by ℓ_2 . In the final section, we establish relations between the (AUH) or Fraïssé property and geometrical conditions from local theory such as type or cotype or complementations of Euclidean or Hilbertian subspaces. With these estimates, we obtain "local versions" of the known fact that L_p -spaces are Fraïssé if and only if $p \notin 2\mathbb{N} + 4$. With this, we reinforce the conjecture that all separable reflexive Fraïssé spaces must be L_p -spaces, as well as identify natural intermediary steps towards proving this fact. We also answer a question of G. Godefroy by proving that any Fraïssé Banach space with a C_∞ -norm must be isomorphic to a Hilbert space.

We shall define envelopes in arbitrary Banach spaces, but with a focus on the reflexive case. We aimed at identifying the minimal hypotheses regarding the norm for each of our results (strict convexity/local uniform convexity, strict convexity of the dual, etc.). However, it should be noted that (AUH) spaces are always uniformly convex and uniformly smooth as soon as they are reflexive. Therefore, the reader mainly interested in the concept of envelope inside a reflexive (AUH) space X can safely assume that the norm on X is always uniformly convex and uniformly smooth.

1.2. Notation. Given a Banach space $X = (X, \|\cdot\|)$ we shall write $\text{Sph}(X)$ to denote the unit sphere of X and B_X to denote its unit ball. We write $\text{GL}(X)$ and $\text{Isom}(X)$ to denote the group of surjective linear isomorphisms and surjective linear isometries, respectively, $\mathcal{L}_1(X)$ to denote the semigroup of contractions on X , and if X is a Banach lattice, $\mathcal{L}_1^+(X)$ to denote the semigroup of positive contractions.

For the definition of classical properties of norms (strict convexity, uniform convexity, etc...) and related notions we refer to [DGZ93] and/or [LT77]. For completeness we recall the slightly less classical notion of local uniform convexity: a Banach space $(X, \|\cdot\|)$ is locally uniformly convex if $\forall x_0 \in \text{Sph}(X)$ and $\forall \varepsilon > 0$, there exists $\delta > 0$ such that whenever $x \in B_X$,

$$\text{if } \left\| \frac{x + x_0}{2} \right\| > 1 - \delta, \text{ then } \|x - x_0\| < \varepsilon.$$

Locally uniformly convex norms are also called locally uniformly rotund and for these reasons this property is abbreviated as ‘‘LUR’’.

If T is an operator on X we denote by $\text{Fix}(T)$ the closed subspace of fixed points of T , i.e. $\text{Fix}(T) = \{x \in X : Tx = x\}$. If S is a semigroup of operators on X then $\text{Fix}(S)$ denotes the closed subspace of fixed points of S , i.e. $\text{Fix}(S) := \bigcap_{T \in S} \text{Fix}(T)$ is the largest subspace where all $T \in S$ act as the identity. On the other hand, if Y is a subspace of X , then $\text{Stab}_S(Y)$ denotes the semigroup of operators in S acting as the identity on Y .

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2. ENVELOPES

An (abstract) envelope is a set-valued map e to every subset A of a Banach space X its envelope, i.e. a closed subspace $e(A)$ of X satisfying the following properties

- (a) $A \subseteq e(A)$.
- (b) If $B \subseteq A$, then $e(B) \subseteq e(A)$.
- (c) $e(\overline{\text{span}}(A)) = e(A)$.
- (d) $e(e(A)) = e(A)$.

So, e is a standard (algebraic) closure operator with the additional property (c). Alternatively, any envelope map on X may be fully described by the class \mathcal{E} of closed subspaces of X which are envelopes, and by the condition that $e(A)$ is the smallest element of \mathcal{E} containing A as a subset. It is routine that a necessary and sufficient condition for a given class \mathcal{E} to define an envelope map in this manner is to be stable by (finite or infinite) intersections.

2.1. The minimal envelope. Perhaps the most geometrical example of envelope is the *minimal* envelope that assigns to a subset, if it exists, the smallest superspace of it that is 1-complemented. This set-mapping is not always well defined: there are examples of Banach spaces containing pairs of 1-complemented subspaces whose intersection is not 1-complemented (and therefore does not have a minimal envelope) - see W.B. Johnson’s comment in Mathoverflow [Joh12] for a simple proof using pushout. However in reflexive spaces with strictly convex norm we have a positive result, which was also indicated to us by W.B. Johnson.

Proposition 2.1. *Let X be a reflexive space with strictly convex norm. Then any intersection of a family $(Y_i)_{i \in I}$ of 1-complemented subspaces of X must be 1-complemented.*

PROOF. Pick a family $(p_i)_{i \in I}$ such that p_i is a contractive projection on Y_i for each i . Given any non-empty finite $F \subseteq I$, strict convexity implies that the average $T_F := (1/\#F) \sum_{i \in F} p_i$ of the projections p_i for $i \in F$ has $\bigcap_{i \in F} Y_i$ as its space $\text{Fix}(T_F)$ of fixed points. Consequently, the Yosida's mean ergodic theorem for reflexive spaces and power-bounded operators [Yos38] implies that the averages of powers of T_F $(1/n \sum_{i=0}^{n-1} T_F^i)_n$ converges SOT to a contractive projection p_F onto $\bigcap_{i \in F} Y_i$. By reflexivity we can pick a subnet of $(p_F)_{F \subseteq I, \text{ finite}}$ that converges WOT. The limit will be a contractive projection onto $\bigcap_{i \in I} Y_i$. \square

Remark 2.2. Let us note from the proof of Proposition 2.1 that if X is reflexive strictly convex and p_i is a contractive projection onto Y_i for each $i \in I$, then a contractive projection onto $\bigcap_{i \in I} Y_i$ may be chosen in the WOT-closure of the convex hull of $\{p_i, i \in I\}$.

Definition 2.3 (Minimal envelope of a subspace). *Let X be a Banach space, and A be a subset of X . We let $\text{Env}_{\min}(A)$ be the envelope defined as the smallest 1-complemented subspace of X containing A , when such subspace exists.*

Remark 2.4. When $\text{Env}_{\min}(A)$ exists, $\text{Env}_{\min}(TA) = T(\text{Env}_{\min}(A))$ for every $T \in \text{Isom}(X)$.

From Proposition 2.1, the minimal envelope is always well-defined in reflexive spaces with strictly convex norm. Under the additional hypothesis that X^* is strictly convex, then we also have that the contractive projection on a 1-complemented subspace is unique [CS70]. In such spaces, the duality mapping plays a central role in understanding the contractive projection onto the minimal envelope.

Definition 2.5. *For a reflexive space X such that X, X^* are both strictly convex, we let $J : \text{Sph}(X) \mapsto \text{Sph}(X^*)$ be the duality mapping, i.e. $\langle Jx, x \rangle = 1$ for every $x \in \text{Sph}(X)$.*

It is convenient to extend J to the whole of X by homogeneity. We recommend the survey [Ran01], and also [FR17, Section 4], for a better understanding of the relationship between J and the isometry groups. In particular, there it is observed that J is well defined, and that it is a bijection between $\text{Sph}(X)$ and $\text{Sph}(X^*)$ whose inverse is the corresponding duality mapping J_* between $\text{Sph}(X^*)$ and $\text{Sph}(X)$; in addition, when X and X^* are assumed to be locally uniformly rotund, then J is an homeomorphism; finally, $JT = T^{*-1}J$ whenever $T \in \text{Isom}(X)$.

The next relations between J and 1-complemented subspaces are due to Calvert [Cal75], and Cohen and Sullivan [CS70], see [Ran01, Thm 5.5 and Thm. 5.6.].

Theorem 2.6. *Let X be reflexive strictly convex and with strictly convex dual. A closed subspace Y of X is 1-complemented in X if and only if JY is a linear subspace of X^* . In this case there is a unique contractive projection of X onto Y which is associated to the decomposition $X = Y \oplus (JY)^\perp$, and if JY is closed then it is 1-complemented, associated to $X^* = JY \oplus (Y)^\perp$.*

Assume that the space X is reflexive and both X and X^* have strictly convex norms. Of special interest is the minimal (contractive) projection associated to $\text{Env}_{\min}(Y)$. Recall that a *minimal contractive projection* on a Banach space is a contractive projection which is minimal with respect to the usual order $p \leq q$ i.e. if and only if $pq = qp = p$. Under the current hypothesis on X , the intersection of

1-complemented subspaces is again 1-complemented, but it is not so obvious how to find (if it exists) a minimal contractive projection for a 1-complemented subspace. To this end we describe the *mean ergodic* projection associated to a semigroup of contractions, which may be seen as part of the *Jacobs-de Leeuw-Glicksberg* (or JdLG) theory, see [EFHN15, Chapters 8 and 16] for a reference. Recall that $\text{Fix}(S)$ denotes the subspace of points which are fixed by all elements of a semigroup S of operators on X . We denote by S^* the corresponding semigroup $\{s^* : s \in S\}$ of operators on X^* .

Part of the following description of the decomposition of the mean ergodic projection was obtained in [FR13] under the restriction that S is an isometry group and that X^* is strictly convex. We include a description through the convex hull of the semigroup. Note that the description of the second summand through duality differs from the classical JdLG description.

Proposition 2.7. *Assume X is reflexive strictly convex, and that S is a semigroup of contractions on X . Then*

- (1) $\text{Fix}(S)$ is 1-complemented by a projection in $\overline{\text{conv}(S)}^{\text{WOT}}$.

If furthermore X^* is assumed strictly convex, then

- (2) The contractive projection onto $\text{Fix}(S)$ is the unique minimal projection p_S in $\overline{\text{conv}(S)}^{\text{WOT}}$, and we have the formula

$$p_S(x) = \text{the unique element of } \text{Fix}(S) \cap \overline{\text{conv}}(Sx).$$

- (3) The associated decomposition of X is the S -invariant decomposition

$$X = \text{Fix}(S) \oplus J(\text{Fix}(S))^\perp = \text{Fix}(S) \oplus \text{Fix}(S^*)^\perp.$$

PROOF. We assume X is reflexive and strictly convex and we prove (1). For any $T \in S$, the mean ergodic Theorem states that the averages of powers of T converge to a contractive projection p_T onto $\text{Fix}(T)$; therefore p_T belongs to the SOT closure of $\text{conv}(S)$. By Remark 2.2, there exists a contractive projection onto $\bigcap_{T \in S} \text{Fix}(T) = \text{Fix}(S)$, belonging to the WOT-closure of the convex hull of the p_T 's, and therefore to the WOT-closure of $\text{conv}(S)$.

We now assume X^* is strictly convex and prove (2) and (3). From the fact that J is a bijection it follows immediately that $J(\text{Fix}(S)) = \text{Fix}(S^*)$ is a linear subspace. Therefore the 1-complementation of $\text{Fix}(S)$ by a unique projection (called q for now) satisfying decomposition (3) follow from previously mentioned [Ran01, Theorems 5.5 and 5.6.].

By the mean ergodic Theorem [EFHN15, Theorem 8.34], we know that $\overline{S}^{\text{WOT}}$ is mean ergodic, and therefore by [EFHN15, Theorem 8.33 b)] determines uniquely the so-called mean ergodic projection onto $\text{Fix}(S) = \text{Fix}(\overline{S}^{\text{WOT}})$ called p for now, which is characterized by $px \in \overline{\text{conv}}(Sx)$ and $pT = Tp = p$ for all $T \in S$. Strict convexity of X^* imply by [CS70] that $p = q$.

The uniqueness and existence of the minimal idempotent p_S in $\overline{\text{conv}(S)}^{\text{WOT}}$ is in [EFHN15, Theorem 16.20]. Since $pT = Tp = p$ for all $T \in S$ it is immediate that this also holds for all $T \in \overline{\text{conv}(S)}^{\text{WOT}}$ and therefore for p_S , i.e. $p \leq p_S$. To conclude that $p = p_S$ it is now enough to observe that $p \in \overline{\text{conv}(S)}^{\text{WOT}}$ by (1). \square

Remark 2.8. The condition that X^* strictly convex is necessary for the uniqueness in (2) and the decomposition in (3). Indeed if z is a point of X of norm 1 admitting two norming functionals ϕ_0 and ϕ_1 , then consider the two contractive projections onto the span of z defined by $p_i(x) = \phi_i(x)z$, and let $S = \{p_0, p_1\}$. Note that $\text{Fix}(S) = [z]$ and both p_0 and p_1 are projections onto it. Also for $i = 0, 1$, $\text{Fix}(p_i^*) = [\phi_i]$, therefore $\text{Fix}(S^*) = \{0\}$ and $\text{Fix}(S^*)^\perp = X$.

2.2. The algebraic envelope. Recall that given a semigroup S and a subset Y of X , the (point) S -stabilizer $\text{Stab}_S(Y)$ of Y is the collection of $s \in S$ that acts as the identity on Y . Inspired by the decomposition in Proposition 2.7 we define the following.

Definition 2.9 (S -algebraic envelope). *Given a subspace Y of X and a semigroup S of operators on X , the S -algebraic envelope of Y is defined as*

$$\text{Env}_S^{\text{alg}}(Y) = \text{Fix}(\text{Stab}_S(Y)).$$

In other words, the S -algebraic envelope of Y is the subspace of points which are fixed by all operators of S fixing each point of Y .

It is worth noting that when J is a bijection between the unit spheres of X and X^* , and S a semigroup of contractions, then $(\text{Stab}_S(Y))^* := \{s^* : s \in \text{Stab}_S(Y)\} = \text{Stab}_{S^*}(JY)$. The inclusion $\overline{\text{Stab}_S(Y)}^{\text{WOT}} \subseteq \text{Stab}_{S^*}^{\text{WOT}}(JY)$ holds, but the equality does not seem to in general; unless S is WOT-closed, in which case $\text{Stab}_S(Y)$ is also WOT-closed. The next relates the minimal and algebraic envelope.

Proposition 2.10. *Assume X is reflexive strictly convex, S is a semigroup of contractions on X , and Y is a closed subspace of X . Then $\text{Env}_S^{\text{alg}}(Y)$ is 1-complemented in X by a contractive projection in $\overline{\text{conv}(\text{Stab}_S(Y))}^{\text{WOT}}$. If furthermore X^* is strictly convex, then holds the $\text{Stab}_S(Y)$ -invariant decomposition*

$$X = \text{Env}_S^{\text{alg}}(Y) \oplus \text{Env}_{S^*}^{\text{alg}}(JY)^\perp,$$

and the projection upon the first summand is the unique minimal projection of $\overline{\text{conv}(\text{Stab}_S(Y))}^{\text{WOT}}$.

PROOF. This is a consequence of Proposition 2.7, of the fact that $\text{Env}_S^{\text{alg}}(Y) = \text{Fix}(\text{Stab}_S(Y))$ and of the fact that $\text{Env}_{S^*}^{\text{alg}}(JY) = \text{Fix}(\text{Stab}_{S^*}(JY)) = \text{Fix}((\text{Stab}_S(Y))^*) = J(\text{Fix}(\text{Stab}_S(Y))) = J(\text{Env}_S^{\text{alg}}(Y))$. \square

Remark 2.11. If $T \in S$ is an automorphism then $\text{Env}_S^{\text{alg}}(T(Y)) = T(\text{Env}_S^{\text{alg}}(Y))$: note that $\text{Stab}_S TY = T\text{Stab}_S(Y)T^{-1}$ and that $\text{Fix}(TST^{-1}) = T(\text{Fix}(S))$.

Even though the equality $\text{Fix}(S) = \text{Fix}(\text{conv}(S))$ always holds, the algebraic envelopes of a semigroup S and of its convex hull do not seem to coincide in general. Indeed we have $\text{conv}(\text{Stab}_S(Y)) \subseteq \text{Stab}_{\text{conv}(S)}(Y)$ but no reason to assume equality, unless under additional hypotheses:

Proposition 2.12. *Let X be a space with strictly convex norm, let S be a semigroup of contractions on X , and let Y be a subspace of X . Then*

$$\text{Env}_S^{\text{alg}}(Y) = \text{Env}_{\text{conv}(S)}^{\text{alg}}(Y).$$

PROOF. It is enough to prove that $\text{conv}(\text{Stab}_S(Y)) = \text{Stab}_{\text{conv}(S)}(Y)$, the direct inclusion of which always holds. Assume $T \in \text{conv}(S)$ acts as the identity on Y . Writing T as a convex combination $\sum_k \lambda_k s_k$, $s_k \in S$, we see that for any $y \in Y$, $y = \sum_k \lambda_k s_k(y)$. By strict convexity and the fact that the s_k 's are contractions, we deduce $s_k(y) = y$ for every k and $y \in Y$, i.e. $s_k \in \text{Stab}_S(Y)$. This means that T belongs to $\text{conv}(\text{Stab}_S(Y))$. We have proved that $\text{conv}(\text{Stab}_S(Y)) \supseteq \text{Stab}_{\text{conv}(S)}(Y)$. \square

Definition 2.13 (Algebraic envelope). *The algebraic envelope of a subspace Y of X , denoted by $\text{Env}^{\text{alg}}(Y)$, is the subspace $\text{Env}_{\text{Isom}(X)}^{\text{alg}}(Y)$.*

2.3. Korovkin and isometric envelopes. In some aspects the algebraic envelope seems to be “too rigid” and does not capture topological properties of a semigroup. We introduce a topological modification of the algebraic envelope.

Definition 2.14 (Korovkin envelope). *Let X be a Banach space and S be a bounded semigroup of operators. We define the (Korovkin) envelope $\text{Env}_S(A)$ of a subset A of X as the set of $x \in X$ such that whenever a net $(T_i)_{i \in I}$ in S converges pointwise on A to Id_A , then the net $(T_i(x))_{i \in I}$ converges to x .*

It is straightforward that $Y \mapsto \text{Env}_S(Y)$ is an envelope map. Also, we have the following inequalities.

$$\text{Env}_S(Y) \subseteq \text{Env}_{\overline{S}^{\text{SOT}}}^{\text{alg}}(Y) \subseteq \text{Env}_S^{\text{alg}}(Y).$$

To see this, suppose that x belong to $\text{Env}_S(Y)$, and assume that $T = \lim_i T_i$, with each $T_i \in \overline{S}^{\text{SOT}}$ and that $T \upharpoonright Y = \text{Id}_Y$. Then $(T_i \upharpoonright Y)_i$ tends SOT to Id_Y , and since $x \in \text{Env}_S(Y)$, by definition of the Korovkin envelope, we have that $(T_i x)_i$ converges in norm to x , and consequently $Tx = x$.

The name we chose for this envelope has the following explanation. We recall that we denote by $\mathcal{L}_1(X)$ the semigroup of contractions on X , and if X is a Banach lattice, by $\mathcal{L}_1^+(X)$ the semigroup of positive contractions. When S is equal to $\mathcal{L}_1(X)$ (resp. $\mathcal{L}_1^+(X)$), results on envelope maps have been obtained by several authors as extensions of *Korovkin theorems*, as described for example in [Ber74], [BL74]. Namely, if $\mathbf{T} = (T_i)_{i \in I}$ is a net of contractions, the *convergence set* $C_{\mathbf{T}}$ is defined as

$$C_{\mathbf{T}} := \{x \in X : (T_i(x))_{i \in I} \text{ converges to } x\},$$

and if A is a subset of X , the *shadow* $\Gamma_1(A)$ (resp. $\Gamma_1^+(A)$) of A is the intersection of all convergence sets (resp. of positive contractions) containing A (in the 1994 survey of Altomare and Campiti [AC94, Def 3.1.1. p 142.], the term *Korovkin closure* is also used). In other words, $\Gamma_1(A)$ coincides with the envelope $\text{Env}_{\mathcal{L}_1(X)}(A)$ and $\Gamma_1^+(A)$ with $\text{Env}_{\mathcal{L}_1^+(X)}(A)$.

We use the terminology *Korovkin set* (resp. *Korovkin⁺ set*) in X for a set A for which $\Gamma_1(A) = X$ (resp. $\Gamma_1^+(A) = X$). Korovkin Theorem [Kor60] states that $\{1, t, t^2\}$ is a Korovkin⁺ set in $C(0, 1)$. This was later improved to being a Korovkin set by Wulbert [Wul68] and independently, Saskin [Š69]. Bernau [Ber74] proves that $\{1, t\}$ is a Korovkin set in $L_p(0, 1)$, $1 < p < +\infty$, and Berens-Lorentz [BL74] for $p = 1$. In [Cal75], Calvert obtains that if X is a reflexive space with locally uniformly convex norm and dual norm, then convergence sets are ranges of linear contractive projections, and the envelope $\Gamma_1(A)$ is the smallest 1-complemented subspace containing A , i.e. $\Gamma_1(A) = \text{Env}_{\min}(A)$, [Cal75] Corollary 2. We shall revisit these results in the present section.

Definition 2.15 (Isometric envelope). *When G is the isometry group of $(X, \|\cdot\|)$ then $\text{Env}_G(Y)$ is called the (isometric) envelope of Y in X , and we denote it by $\text{Env}_{\|\cdot\|}(Y)$, or simply $\text{Env}(Y)$, when the norm on X is clear from the context.*

It will be relevant to give a name to those subspaces whose envelope coincides with the whole space:

Definition 2.16 (Full envelope, full subspace). *A subspace Y of X has full envelope, or it is a full subspace of X , when $\text{Env}(Y) = X$.*

When G is a bounded group of isomorphisms, we have the following reformulation of the definition of the envelope $\text{Env}_G(A)$. Because of item c) in the definition of envelope, from now on we only consider envelopes of closed subspaces Y of a Banach space X .

Proposition 2.17. *Let X be a Banach space and Y a closed subspace of X . If G is a bounded group of isomorphisms on X , then a point $x \in X$ belongs to $\text{Env}_G(Y)$ if and only if whenever a net $(T_i)_{i \in I}$ in G converges pointwise on Y , then $(T_i(x))_{i \in I}$ also converges.*

PROOF. Let G be a bounded group of isomorphisms on X . Assume that x satisfies the last condition, and let us see that $x \in \text{Env}_G(Y)$, so suppose that $(T_i)_{i \in I}$ is a net that converges pointwise on Y to Id_Y . We consider the set $J := I \times \{0, 1\}$ lexicographically ordered, and the net $(U_{(i,j)})_{(i,j) \in J}$, $U_{(i,0)} := \text{Id}$ and $U_{(i,1)} := T_i$ for all i . It follows that $(U_{(i,j)})_{(i,j)}$ pointwise converges in Y to Id_Y , so by hypothesis, $(U_{(i,j)}(x))_{(i,j)}$ also converges, and the limit must be x . This means in particular that $(U_i(x))_i$ converges to x .

Suppose now that x does not satisfy the last condition. We can find a net $(T_i)_{i \in I}$ in G converging pointwise on Y and some $\varepsilon > 0$ such that the subset $J \subseteq I$ of those i such that there are $j_i, k_i \geq i$ such that $\|T_{j_i}(x) - T_{k_i}(x)\| \geq \varepsilon$. This implies that, $\|x - T_{j_i}^{-1}(T_{k_i}(x))\| \geq \varepsilon / \sup_{T \in G} \|T\|$ for every $i \in J$. For each $i \in J$ let $U_i := T_{j_i}^{-1} \circ T_{k_i} \in G$. Then the net $(U_i)_{i \in J}$ in G converges pointwise on Y to the identity of Y but $(U_i(x))_{i \in J}$ does not converge to x , so $x \notin \text{Env}_G(Y)$. \square

So, $\text{Env}_G(Y)$ is the maximum superset Z of Y so that pointwise convergence on Y of elements of G (to a map $t : Y \rightarrow X$ which does not necessarily extend to an element of G) implies pointwise convergence of these elements on Z (necessarily to an extension of t).

Note that when $S \subseteq S'$ are bounded semigroups then $\text{Env}_{S'}(Y) \subseteq \text{Env}_S(Y)$, and also that whenever $T \in S$ is an automorphism, then $\text{Env}_S(TY) = T(\text{Env}_S(Y))$.

Example 2.18. *If $X = H$ is a Hilbert space and Y is a closed subspace of it then $\text{Env}(Y) = Y$. Indeed we can decompose H as $Y \oplus Y^\perp$ and define $T_n = \text{Id} \oplus -\text{Id}$. If x is not in Y , then we easily use $(T_n)_n$ to deduce that x is not in $\text{Env}(Y)$.*

A group of isomorphisms G on a space X is said to be trivial if all elements of G are multiples of the identity map.

Example 2.19. *If the isometry group $\text{Isom}(X)$ is trivial, then any subspace Y of X of dimension at least 1 has envelope equal to X : if a net $(T_i)_{i \in I}$ converges pointwise on Y , then each $T_i = \lambda_i \text{Id}_X$ with $(\lambda_i)_{i \in I}$ a converging net; thus, $(T_i)_{i \in I}$ converges pointwise on the whole of X .*

As in the case of the algebraic envelope, in general the Korovkin envelope for a semigroup S does not seem to need to coincide with the envelope for its WOT-closure or for its convex hull. There is however such an identification under mild topological conditions on the norm, which are related to the topological nature of the Korovkin envelope.

Proposition 2.20. *Let X be a space with a LUR norm and let S be a semigroup of contractions on X . Let Y be a subspace of X . Then*

$$\text{Env}_S(Y) = \text{Env}_{\overline{\text{conv}(S)}^{\text{WOT}}}(Y).$$

PROOF. Only the direct inclusion is not trivial. Let x be normalized in $\text{Env}_S(Y)$ and fix $\varepsilon > 0$. By definition of $\text{Env}_S(Y)$, there exist normalized y_1, \dots, y_n in Y and $\delta > 0$ such that for any $g \in S$,

$$\text{if } \|g(y_i) - y_i\| \leq \delta \text{ for every } i = 1, \dots, n, \text{ then } \|gx - x\| \leq \varepsilon.$$

We use the LUR property of the norm to choose for each $i = 1, \dots, n$ some $\delta_i > 0$ such that if $\|z\| \leq 1$ is such that $\|z + y_i\| \geq 2 - \delta_i$ then $\|z - y_i\| \leq \delta_i$. Let $\beta > 0$ be small enough so that $\sqrt{2\beta} \leq \min_i \delta_i$, $\beta \leq \varepsilon$ and $2n\sqrt{2\beta} \leq \varepsilon$. We claim that whenever $T \in \overline{\text{conv}(S)}^{\text{WOT}}$ satisfies that $\max_i \|Ty_i - y_i\| \leq \beta$, then it follows that $\|Tx - x\| \leq 3\varepsilon$. This implies that whenever $(T_i)_{i \in I}$ is a net in $\overline{\text{conv}(S)}^{\text{WOT}}$ converging pointwise to the identity on Y , then $(T_i(x))_{i \in I}$ converges to x , and this means that x belongs to $\text{Env}_{\overline{\text{conv}(S)}^{\text{WOT}}}(Y)$. This therefore proves the required direct inclusion.

We now give the proof of the claim. For $i = 1, \dots, n$ let ϕ_i be a normalized functional norming y_i , and let ϕ_0 be a normalized functional norming $Tx - x$. Denote $y_0 := x$ and consider a convex combination $\sum_{k \in K} \lambda_k g_k$ with each $g_k \in S$, such that $|\langle \phi_i, (T - \sum_{k \in K} \lambda_k g_k)y_i \rangle| \leq \beta$ for $i = 0, \dots, n$. Note that for every $i = 1, \dots, n$,

$$1 - 2\beta \leq \phi_i(T(y_i)) - \beta \leq \sum_{k \in K} \lambda_k \phi_i(g_k(y_i)). \quad (1)$$

For each $i = 1, \dots, n$, let A_i be the set of indices $k \in K$ such that $\phi_i(g_k(y_i)) \geq 1 - \sqrt{2\beta}$ and let $B_i = K \setminus A_i$ be the complement of A_i . Note that for $k \in A_i$, we have that $\|g_k(y_i) - y_i\| \leq \delta$. Also from the above equation (1) we have

$$1 - 2\beta \leq \sum_{k \in A_i} \lambda_k + (1 - \sqrt{2\beta}) \sum_{k \in B_i} \lambda_k,$$

from which one immediately deduces

$$\sum_{k \in B_i} \lambda_k \leq \sqrt{2\beta}.$$

Finally let $A = \bigcap_{i=1}^n A_i$ and $B = \bigcup_{i=1}^n B_i$. Whenever $k \in A$, we have $\|g_k y_i - y_i\| \leq \delta$ for all $i = 1, \dots, n$, and therefore $\|g_k x - x\| \leq \varepsilon$, by the choice of δ . Using this estimate, we compute

$$\|Tx - x\| = \phi_0(Tx - x) \leq \beta + \sum_{k \in K} \lambda_k \|g_k x - x\| \leq \beta + \left(\sum_{k \in A} \lambda_k \right) \varepsilon + 2 \sum_{i=1}^n \sum_{k \in B_i} \lambda_k \leq 2\varepsilon + 2n\sqrt{\beta} \leq 3\varepsilon,$$

and this concludes the proof of the claim. \square

Remark 2.21. An interesting consequence of Proposition 2.20 is that if Y is a subspace of an LUR space X and S a semigroup of contractions, then $\text{Env}_S(Y)$ is contained in the range of any contractive projection $p \in \overline{\text{conv}(S)}^{\text{WOT}}$ acting as the identity on Y . Indeed from $p \upharpoonright Y = \text{Id}_Y$ it follows that $px = x$ for any $x \in \text{Env}_{\overline{\text{conv}(S)}^{\text{WOT}}}(Y) = \text{Env}_S(Y)$.

We now show that in many cases the Korovkin envelope may be seen as an algebraic envelope. This is important to obtain a JdLG decomposition associated to this envelope. We shall use the following facts. If X is reflexive (actually the point of continuity property is enough) and if G is a bounded group of automorphisms on X , then the weak and norm topologies coincide on each G -orbit of a non-zero point x_0 of X [Meg01, Theorem 2.5], applied for G equipped with the topology of weak convergence in the point x_0 . Also it is a classical and easy fact that if the norm on X is locally uniformly rotund (LUR), then weak convergence of a net on the unit ball of X to a point of the sphere, implies strong convergence to this point. See [AFGR19, Remark 2.4.] for more details on both these facts.

Proposition 2.22. *Let X be a reflexive space. Assume that either S is a bounded group of isomorphisms on X , or that X is LUR and S is a semigroup of contractions. Then*

$$\text{Env}_S(Y) = \text{Env}_{\overline{S}}^{\text{alg}}(Y).$$

In particular, $\text{Env}_S(Y)$ and $\text{Env}_{\overline{S}}^{\text{alg}}(Y)$ coincide if S is WOT-closed.

PROOF. In the first case, we may assume that S is a group of isometries for some renorming of X . Therefore in either case, for any net $(T_i)_i$ in S , and normalized vector $x \in X$, weak convergence of $(T_i x)_i$ to x implies norm convergence. Assume some x does not belong to $\text{Env}_S(Y)$ and let $(T_i)_i$ converge pointwise to Id_Y on Y such that $(T_i x)_i$ does not converge to x in norm and therefore does not converge weakly to x . By reflexivity, the space of contractions with the WOT is compact, so there is convergent subnet for $(T_i)_i$. W.l.o.g., we may assume that $(T_i)_i$ converges WOT to some T but that Tx is not equal to x , and note that T is the identity on Y . This proves that x does not belong to $\text{Env}_{\overline{S}}^{\text{alg}}(Y)$.

Likewise, let $x \in \text{Env}_S(Y)$ and $T \upharpoonright Y = \text{Id}_Y$ for $T \in \overline{S}^{\text{WOT}}$, then let $(T_i)_i$ a net in S converging WOT to T , so that $(T_i)_i$ converges WOT to Id on Y . Then the convergence of $(T_i)|_Y$ to Id_Y is SOT and so $(T_i x)_i$ converges to x , so $Tx = x$, and consequently $x \in \text{Env}_{\overline{S}}^{\text{alg}}(Y)$. \square

Finally, the next proposition will allow us to compare the Korovkin and the minimal envelopes. We have the following decomposition for the Korovkin envelope that should be compared with Proposition 2.10.

Proposition 2.23. *Let X be reflexive strictly convex, S a semigroup of contractions on X , and let Y be a closed subspace of X . Assume additionally that either (i) S is a group of isometries on X , or that (ii) X is LUR. Then*

- (1) $\text{Env}_S(Y)$ is a 1-complemented subspace of X ; in case (ii) $\text{Env}_S(Y)$ is the smallest superspace of Y complemented by a projection in $\overline{\text{conv}(S)}^{\text{WOT}}$.

(2) If furthermore X^* is strictly convex, then holds the $\text{Stab}_{\overline{S}^{\text{WOT}}}(Y)$ -invariant decomposition

$$X = \text{Env}_S(Y) \oplus \text{Env}_S(JY)^\perp, \quad (2)$$

and in case (ii) the projection on the first summand is the unique minimal projection of the semigroup $\overline{\text{conv}(S)}^{\text{WOT}}$ acting as the identity on Y .

PROOF. The 1-complementation in (1) and the decomposition in (2) is a consequence of Proposition 2.10 and the relation between the algebraic and the Korovkin envelope in Proposition 2.22. Regarding the projection in case (1) (ii), note that as a first step the set it belongs to is $\overline{\text{conv}(\text{Stab}_{\overline{S}^{\text{WOT}}}(Y))}^{\text{WOT}}$. Under the LUR property and by Proposition 2.20, this set is the same as $\overline{\text{conv}(\text{Stab}_{\overline{\text{conv}(S)}^{\text{WOT}}}(Y))}^{\text{WOT}}$. But it is also the same as $\text{Stab}_{\overline{\text{conv}(S)}^{\text{WOT}}}(Y)$ since this set is convex and WOT-closed. This proves that $\text{Env}_S(Y)$ is complemented by a projection in $\overline{\text{conv}(S)}^{\text{WOT}}$, and that it is the smallest among those containing Y is Remark 2.21. The statement about the projection in (2)(ii) follows from the one in case (1)(ii) and strict convexity of the dual. \square

As an immediate consequence of Proposition 2.23. we spell out the following.

Proposition 2.24. *Let X be reflexive strictly convex and let Y be a closed subspace of X . Then*

(1) *the isometric envelope $\text{Env}(Y)$ is 1-complemented.*

If furthermore X is LUR then

(2) *$\text{Env}(Y)$ is the smallest superspace of Y complemented by a projection in $\overline{\text{conv}(\text{Isom}(X))}^{\text{WOT}}$; if X^* is strictly convex, then this projection is the minimal projection in $\overline{\text{conv}(\text{Isom}(X))}^{\text{WOT}}$ acting as the identity on Y .*

(3) *The Korovkin envelope $\text{Env}_{\mathcal{L}_1(X)}(Y)$ coincides with the minimal envelope $\text{Env}_{\min}(Y)$.*

Remark 2.25. Assertion (3) is an improvement of Calvert's result [Cal75] on the Korovkin envelope, since there X^* needs to be LUR and we impose no requirement on it.

Remark 2.26. Under the assumption that X is reflexive strictly convex, we have

$$\text{Env}_{\min}(Y) \subseteq \text{Env}(Y) \subseteq \text{Env}^{\text{alg}}(Y).$$

We do not know of a general characterization of situations where $\text{Env}(Y)$ and $\text{Env}^{\text{alg}}(Y)$ must coincide. However when $X = L_p$, $p \neq 2$, and Y is a unital subspace, then we shall see in Theorem 3.12 that $\text{Env}_{\min}(Y) = \text{Env}(Y) = L_p([0, 1], \Sigma_Y)$, where Σ_Y is the minimal subalgebra of the Borel algebra $\mathcal{B}([0, 1])$ making all functions in Y measurable. And in this case $\text{Env}(Y) = \text{Env}^{\text{alg}}(Y)$ exactly when Σ_Y is a *fixed-point subalgebra* (see Remark 3.6).

Remark 2.27. In [BM77] Beauzamy and Maurey consider a non-linear procedure associating to a subset M the set $M \subseteq \min M$ of its minimal points in a metric sense. When X is reflexive strictly convex and with strictly convex dual, it follows from their results that the minimal envelope of a subspace may be obtained through iteration of their "min" procedure. In this way they recover the results of Bernau [Ber74] and Calvert [Cal75] mentioned earlier.

Remark 2.28. We spell out, for X reflexive LUR: $Y = \text{Env}(Y)$ if and only if Y is complemented by a contractive projection in $\overline{\text{conv}(\text{Isom}(X))}^{\text{WOT}}$. This characterizes the subspaces which are isometric envelopes as those which are ranges of projections in the “isometric hull” $\overline{\text{conv}(\text{Isom}(X))}^{\text{WOT}}$, and this fully describes the isometric envelope map.

Proposition 2.29. *Assume X is separable reflexive, G is a subgroup of $\text{Isom}(X)$, and Y is a subspace of X . Then $\text{Env}_G(Y)$ is 1-complemented in X by a projection in $\overline{\text{conv}(G)}^{\text{WOT}}$.*

PROOF. By Lancien [Lan93] we may renorm X with a norm $\|\cdot\|$ which is $1 + 1/n$ -equivalent to the original norm and so that X is LUR and $G \subseteq \text{Isom}(X, \|\cdot\|)$. It follows from Proposition 2.23 (1)(ii) that $\text{Env}_G(Y)$ is 1-complemented with respect to $\|\cdot\|$ and therefore $(1 + 1/n)^2$ -complemented in the original norm, by a projection in $\overline{\text{conv}(G)}^{\text{WOT}}$. A WOT-cluster point of the sequence of associated projections P_n is a norm 1-projection onto $\text{Env}_G(Y)$ belonging to $\overline{\text{conv}(G)}^{\text{WOT}}$. \square

Corollary 2.30. *The Hilbert space is the only separable reflexive space for which each closed subspace Y (2-dimensional subspace is enough) is equal to its envelope $\text{Env}(Y)$.*

PROOF. Indeed it is a known result by Kakutani [Kak39] that X is Hilbertian if and only if all subspaces are 1-complemented (in fact it is enough to require that all 2-dimensional subspaces are 1-complemented); in the complex case this is due to Bohnenblust [Boh42]. \square

2.4. Strong isometric embeddings. Closely related to the isometric envelope are the embeddings who are limits or restriction of (global) isometries.

Definition 2.31. *Given a Banach space X , and a subspace $Y \subseteq X$, one defines the collection $\text{Emb}_{\text{ext}}(Y, X)$ of extendable isometric embeddings as*

$$\text{Emb}_{\text{ext}}(Y, X) := \text{Isom}(X) \upharpoonright Y \subseteq \text{Emb}(Y, X),$$

and the collection $\text{Emb}_s(Y, X)$ of strong isometric embeddings as

$$\text{Emb}_s(Y, X) := \overline{\text{Isom}(X) \upharpoonright Y}^{\text{SOT}} \subseteq \text{Emb}(Y, X).$$

Note that $\text{Emb}_s(Y, X)$ is by definition SOT-closed in $\text{Emb}(Y, X)$, hence a motivation for the name.

When X is a Hilbert space, then $\text{Emb}_{\text{ext}}(F, X) = \text{Emb}(F, X)$ for all finite dimensional subspaces F of X . Spaces with these properties are called *ultrahomogeneous*. The Hilbert space is the only known separable example; non separable examples appear in [ACSC⁺16] and [FLAMT20]. When $\text{Emb}_s(F, X) = \text{Emb}(F, X)$ for every finite dimensional F , the space is called approximately ultrahomogeneous (see [FLAMT20]). Separable examples are the Hilbert spaces, all $L_p[0, 1]$ for $p \notin 4 + 2\mathbb{N}$, or the Gurarij space.

We shall sometimes use the word *partial isometry* on a space X to mean a surjective isometry t between subspaces Y and Z of X . A *strong partial isometry* is a partial isometry $t : Y \rightarrow Z$ such that $i_{ZX} \circ t$ belongs to $\text{Emb}_s(Y, X)$, where i_{ZX} is the inclusion map $Z \rightarrow X$.

Lemma 2.32. *The inverse of a strong partial isometry is a strong partial isometry. When defined, the composition of two strong partial isometries is a strong partial isometry.*

PROOF. Let $\gamma : Y \rightarrow Z$ be a strong partial isometry. Given $z_1, \dots, z_n \in Z$ of norm 1, let $y_j := \gamma^{-1}z_j$ for each $1 \leq j \leq n$. Let $g \in \text{Isom}(X)$ be such that $\|g(y_j) - \gamma(y_j)\| \leq \varepsilon$ for every $1 \leq j \leq n$. Then for each $1 \leq j \leq n$ one has that $\|\gamma^{-1}(z_j) - g^{-1}(z_j)\| = \|y_j - g^{-1}(\gamma(y_j))\| = \|g^{-1}gy_j - g^{-1}\gamma(y_j)\| = \|gy_j - \gamma y_j\| \leq \varepsilon$. Similarly one proves the second part of the statement. \square

We may therefore define an equivalence relation between subspaces of X as follows: Y and Z are *strongly isometric* if there exists a strong partial isometry between Y and Z . The next proposition substantiates the idea that isometric envelopes are well preserved under strong partial isometries.

Proposition 2.33. *Let X be a Banach space and Y a closed subspace of X . Then*

- (1) *Every $t \in \text{Emb}_s(Y, X)$ extends uniquely to some $\tilde{t} \in \text{Emb}_s(\text{Env}(Y), X)$.*
- (2) *The image $\tilde{t}(\text{Env}(Y))$ is equal to $\text{Env}(tY)$.*

In particular if Y and Z are strongly isometric subspaces of X , then so are $\text{Env}(Y)$ and $\text{Env}(Z)$. Furthermore,

- (3) *If Y is separable, the map $t \mapsto \tilde{t}$ is SOT-SOT continuous from $\text{Emb}_s(Y, X)$ to $\text{Emb}_s(\text{Env}(Y), X)$.*

PROOF. (1): An extension to $\text{Env}(Y)$ is defined by $\tilde{t}(x) = \lim_{i \in I} T_i(x)$, if $t = \lim_i T_i|_Y$ with $T_i \in \text{Isom}(X)$. If t' is a strong isometric embedding on $\text{Env}(Y)$ defined as pointwise limit of some $(U_i)_i$, extending t , then note that since the sequence $(T_i^{-1}U_i)_i$ is pointwise convergent to Id_Y on Y it is also pointwise convergent on $\text{Env}(Y)$, implying that \tilde{t} and t' coincide on $\text{Env}(Y)$.

(2): We claim that $\tilde{t}(\text{Env}(Y)) = \text{Env}(tY)$. We first prove that $\tilde{t}(\text{Env}(Y)) \subseteq \text{Env}(tY)$, so fix $x \in \tilde{t}(\text{Env}(Y))$. Let $(T_i)_{i \in I}$ be a net of isometries of X that defines \tilde{t} . Suppose that $(U_j)_{j \in J}$ is an arbitrary sequence on $\text{Isom}(X)$ such that $(U_j t y)_j$ converges for all $y \in Y$. Without loss of generality we may assume that $I = J$: One can use the product ordering $I \times J$ and define $T_{i,j} := T_i$ and $U_{i,j} := U_j$. It follows that $(U_i \circ T_i y)_i$ converges for all $y \in Y$, so, by hypothesis, $(U_i \circ T_i x)_i$ converges, or equivalently $(U_i \tilde{t}(x))_i$ converges. This means that $\tilde{t}(x) \in \text{Env}(tY)$. Now let $u : tY \rightarrow Y$ be the inverse of t . We know that $u \in \text{Emb}_s(tY, X)$, hence, $\tilde{u} \text{Env}(tY) \subseteq \text{Env}(utY) = \text{Env}(Y)$. This means that if $x \in \text{Env}(tY)$, then $\tilde{u}(x) \in \text{Env}(Y)$, and so $x = \tilde{t}(\tilde{u}(x)) \in \tilde{t}(\text{Env}(Y))$.

(3): By separability of Y , $\text{Emb}(Y, X)$ is a metric space. Assume $(t_n)_n$ converges SOT to t and let $x \in \text{Env}(Y)$. Let V_n be an element of $\text{Isom}(X)$ such that $d(V_n|_Y, t_n) \leq 1/n$, where d is the SOT-convergence metric on $\text{Emb}(Y, X)$, and so that $\|V_n x - \tilde{t}_n x\| \leq 1/n$ (by definition of \tilde{t}_n). Then note that V_n converges SOT to t on Y and therefore $(V_n x)_n$ converges to $\tilde{t}x$ (by definition of \tilde{t}). So, $\tilde{t}_n x$ converges to $\tilde{t}x$ for all $x \in \text{Env}(Y)$. \square

Corollary 2.34. *The following are equivalent for $Y \subseteq X$:*

- (1) *Y is strongly isometric to a full subspace of X .*
- (2) *$\text{Env}(Y)$ is strongly isometric to X .*

PROOF. Suppose that $t : Y \rightarrow X$ is a strong isometry such that tY is a full subspace. It follows from Proposition 2.33 (2) that $\tilde{t}(\text{Env}(Y)) = \text{Env}(tY) = X$, hence $\tilde{t} : \text{Env}(Y) \rightarrow X$ is a (surjective) strong isometry. Conversely, assume that $T : \text{Env}(Y) \rightarrow X$ is a surjective strong isometry. Then $t := T|_Y$

$Y \in \text{Emb}_s(Y, X)$, $\tilde{t} = T$, and it follows from Proposition 2.33 (2) that $X = T(\text{Env}(Y)) = \text{Env}(T(Y))$, so TY is a full subspace of X . \square

The envelope of Y admits an important characterization when Y is assumed separable.

Proposition 2.35. *Assume Y is a separable subspace of X . Then $\text{Env}(Y)$ is the maximum among subspaces Z of X with the property that*

- (1) Z contains Y .
- (2) Any embedding t in $\text{Emb}_s(Y, X)$ extends uniquely to an embedding \tilde{t} in $\text{Emb}_s(Z, X)$.
- (3) The mapping $t \mapsto \tilde{t}$ is SOT-SOT continuous.

PROOF. The space $\text{Env}(Y)$ does satisfy (1), (2) and (3). Let now Z be such a subspace and $z \in Z$. If $(T_n)_n$ is a sequence of isometries on X converging SOT on Y , let $t_n = T_n|_Y \in \text{Emb}_{\text{ext}}(Y, X) \subseteq \text{Emb}_s(Y, X)$ and $t = \lim_n t_n$. By definition of Z , $(T_n|_Z)_n = (\tilde{t}_n)_n$ tends SOT to \tilde{t} ; therefore $(T_n z)_n$ converges. This proves that $z \in \text{Env}(Y)$ and therefore that $Z \subseteq \text{Env}(Y)$. \square

Propositions 2.33 and 2.35 lead us to see the class of subspaces of X which are isometric envelopes as a “strongly isometric skeleton” of X , in the sense that it is a family of subspaces of X “containing all the information” on strong isometric embeddings inside X .

Finally a relevant case of strong isometric embedding defined on a subspace Y are those whose image is equal to Y , and may therefore be seen as surjective isometries on Y . In general, not all surjective isometries on Y are of this form. However in a class of spaces that we will consider in the next subsection 2.5 this is the case and we will see that it implies that $Y \subseteq X$ is a “ g -embedding”.

Definition 2.36 (Strong isometry). *Let X be a Banach space and Y a closed subspace of X . We let $\text{Isom}_s(Y)$ be the set of strong isometries on Y , i.e., of strong isometric embeddings of Y whose image is Y .*

In other words, elements of $\text{Isom}_s(Y)$ are isometries on Y which, as maps with value in X , belong to $\text{Emb}_s(Y, X)$. In what follows we always consider the SOT.

Lemma 2.37. *Let X be a Banach space and Y a closed subspace of X . Then $\text{Isom}_s(Y)$ is a closed subgroup of $\text{Isom}(Y)$.*

PROOF. It is clearly closed. If t, u are two such elements, with $t = \lim_i T_i|_Y$ and $u = \lim_i U_i|_Y$, then $(T_i U_i|_Y)_i = (T_i(U_i|_Y - u) + T_i u)_i$ also tends SOT to tu . Likewise $t^{-1} = \lim_i (T_i^{-1})|_Y$. \square

Corollary 2.38. *If Y is a closed subspace of X , then $t \in \text{Isom}_s(Y) \mapsto \tilde{t} \in \text{Isom}_s(\text{Env}(Y))$ is an embedding of topological groups.*

PROOF. The group embedding property is clear. Let us denote this embedding by e . Then let $t_i \in \text{Isom}_s(Y)$ converge to $t \in \text{Isom}_s(Y)$, and let $T_i = e(t_i)$ defined on $\text{Env}(Y)$ extending t_i . Note that T_i converges pointwise to t on Y , so by Proposition 2.17, $i_{\text{Env}(Y), X} T_i$ converges pointwise on $\text{Env}(Y)$

to some $T \in \text{Emb}_s(\text{Env}(Y))$ such that $T|_Y = t$. By the uniqueness result (1) in Proposition 2.33, T_i converges pointwise to $e(t)$. \square

2.5. Envelopes in AUH or Fraïssé spaces. Recall that a space X is called ultrahomogeneous (UH) when $\text{Emb}_{\text{ext}}(Y, X) = \text{Emb}(Y, X)$ for every finite dimensional subspace Y of X ; and approximately ultrahomogeneous (AUH) when $\text{Emb}_s(Y, X) = \text{Emb}(Y, X)$ for such subspaces Y of X .

We obtain the following results regarding isometric embeddings in (AuH) spaces, easy to prove directly from the corresponding results for strong isometric embeddings in general spaces. The next is Proposition 2.33 reformulated for (AuH) spaces:

Proposition 2.39. *Assume X is (AuH), let Y be a subspace of X , let $t \in \text{Emb}(Y, X)$ be an isometric embedding of Y into X . Then*

- (1) *t extends uniquely to an isometric embedding $\tilde{t} \in \text{Emb}(\text{Env}(Y), X)$; and $\tilde{t}(\text{Env}(Y)) = \text{Env}(t(Y))$.*

Assume that Y is separable. Then,

- (2) *the map $t \mapsto \tilde{t}$ is SOT-SOT continuous.*
- (3) *$\text{Env}(Y)$ is the maximum among spaces Y' such that any isometric embedding $t : Y \rightarrow X$ extends uniquely to an isometric embedding $\tilde{t} : Y' \rightarrow X$, and such that $t \mapsto \tilde{t}$ is SOT-SOT continuous.* \square

Definition 2.40 (linear g -embedding). *We say that a subspace Y of X is a linear g -embedding when there is an embedding of topological groups $e : \text{Isom}(Y) \rightarrow \text{Isom}(X)$ such that $e(h)|_Y = h$ for every $h \in \text{Isom}(Y)$.*

This is the linear version of the notion of g -embedding of a subspace M of a metric space L introduced in [Usp08, Definition 3.1] (see also [Pes06, Definition 5.2.6]), that demands the existence of an embedding of topological groups $e : \mathcal{I}so(M) \rightarrow \mathcal{I}so(L)$ such that $e(h)|_M = h$ for every $h \in \mathcal{I}so(M)$, where $\mathcal{I}so(N)$ is the group of isometries of N . Recall that the classical Mazur-Ulam Theorem states that the group $\mathcal{I}so(Y)$ of surjective, not necessarily linear, isometries on Y is the group of the affine ones on Y . From this it follows easily that if $e : \text{Isom}(Y) \rightarrow \text{Isom}(X)$ is an embedding of topological groups, then $\bar{e} : \mathcal{I}so(Y) \rightarrow \mathcal{I}so(X)$ defined by $\bar{e}(\gamma) = e(\gamma - \gamma(0)) + \gamma(0)$ is also an embedding as topological groups, and consequently every linear g -embedding is a g -embedding. We have the following direct consequence of Corollary 2.38.

Proposition 2.41. *Assume X is (AuH) and let Y be a subspace of X . Then*

- (1) *any surjective isometry t on Y extends uniquely to a surjective isometry \tilde{t} on $\text{Env}(Y)$.*
- (2) *The map $t \mapsto \tilde{t}$ defines a topological (group) embedding of $\text{Isom}(Y)$ into $\text{Isom}(\text{Env}(Y))$.*
- (3) *this is the unique such map for which \tilde{t} is an extension of t for each $t \in \text{Isom}(Y)$.*

Consequently $Y \subseteq \text{Env}(Y)$ is a linear g -embedding. \square

Corollary 2.42. *Suppose that X is (AuH), let $Y \subseteq X$ be a subspace, and let $\gamma \in \text{Emb}(Y, \text{Env}(Y))$. Then γ is such that $\text{Env}(\gamma Y) = \text{Env}(Y)$ if and only if $\gamma \in \text{Emb}_{\text{ext}}(Y, \text{Env}(Y))$, that is, if γ is a restriction of a surjective isometry of $\text{Env}(Y)$. In particular, any partial isometry between full subspaces of X extends to an isometry of X .*

PROOF. For suppose that $g \in \text{Isom}(\text{Env}(Y))$, and set $\gamma := g \upharpoonright Y$. Then, by uniqueness, $\bar{\gamma} = g$ and, by Proposition 2.41, $\text{Env}(\gamma Y) = \bar{\gamma} \text{Env}(Y) = g \text{Env}(Y) = \text{Env}(Y)$. Suppose now that $\gamma \in \text{Emb}(Y, \text{Env}(Y))$ is such that $\text{Env}(\gamma Y) = \text{Env}(Y)$. We claim that $\bar{\gamma} \in \text{Isom}(\text{Env}(Y))$: First of all, $\bar{\gamma}(\text{Env}(Y)) = \text{Env}(\gamma Y) \subseteq \text{Env}(Y)$, so $\bar{\gamma} \in \text{Emb}(\text{Env}(Y), \text{Env}(Y))$. Let $\eta \in \text{Emb}(\gamma Y, \text{Env}(Y))$ be such that $\eta(\gamma Y) = Y$ and $\eta \circ \gamma = \text{Id}_Y$. Then, similarly we get that $\bar{\eta} \in \text{Emb}(\text{Env}(\gamma Y), \text{Env}(\gamma Y)) = \text{Emb}(\text{Env}(Y), \text{Env}(Y))$. Since $\bar{\eta} \circ \bar{\gamma} \upharpoonright Y = \eta \circ \gamma = \text{Id}_Y$, it follows by uniqueness of the extensions that $\bar{\eta} \circ \bar{\gamma} = \text{Id}_{\text{Env}(Y)}$, and consequently $\bar{\eta}$ is surjective and $\bar{\gamma}$ as well. \square

Observe that a subspace Y of X and gY for $g \in \text{Isom}(X)$ are placed “similarly” inside X because both $Y \equiv gY$ but also the quotient map $\hat{g} : X/Y \rightarrow X/(gY)$, $x + Y \mapsto g(x) + gY$ is a surjective isometry compatible with $g \upharpoonright Y : Y \rightarrow gY$. In this way, any of the $\text{Isom}(X)$ -orbits of the action by composition in $\text{Emb}(Y, X)$ is called an *isometric position* of Y inside X , and the previous Corollary says that when X is (AuH) the isometric position defined by the inclusion $i : Y \rightarrow \text{Env}(Y)$ consists of the isometric embeddings $\gamma : Y \rightarrow \text{Env}(Y)$ such that $\text{Env}(\gamma Y) = \text{Env}(Y)$. From a homological point of view, the exact sequences $0 \rightarrow Y \rightarrow E(Y) \rightarrow E(Y)/Y \rightarrow 0$ and $0 \rightarrow \gamma Y \rightarrow E(\gamma Y) \rightarrow E(\gamma Y)/\gamma Y \rightarrow 0$ are isometrically equivalent (in the sense that the extension of γ to a map between the envelopes making the diagram commute is an isometric map), and the associated quotient $E(Y)/Y$ is isometrically unique (in the sense that $E(\gamma Y)/\gamma Y$ is independent of γ).

Definition 2.43 (Position). *Given a (AuH) space X , the position of a subspace Y inside its envelope $\text{Env}(Y)$ is the $\text{Isom}(\text{Env}(Y))$ -orbit of the inclusion $i : Y \rightarrow \text{Env}(Y)$.*

The full position of a subspace Y of X is (if exists) the position of some (any) full isometric copy of Y . We call full quotient of X by Y the isometrically unique associated quotient X/Y .

This definition was inspired by a notion of *isomorphic position* related to the so-called automorphic space problem, see [CP10].

We finish this subsection with some additional observations on (AuH) with a stronger extension property. Recall that $\text{Emb}_\delta(E, X)$ is the collection of all δ -embeddings $\gamma : E \rightarrow X$, i.e. linear mappings such that $(1 + \delta)^{-1}\|e\| \leq \|\gamma e\| \leq (1 + \delta)\|e\|$ for every $e \in E$.

Definition 2.44 (Fraïssé space). *A Banach space X is Fraïssé when for every $\varepsilon > 0$ and every $k \in \mathbb{N}$, there is a $\delta > 0$ such that for every k -dimensional subspace E of X and every $t \in \text{Emb}_\delta(E, X)$, there exists $T \in \text{Isom}(X)$ such that $\|T \upharpoonright E - t\| \leq \varepsilon$.*

The spaces L_p for $p \neq 4, 6, \dots$ and the Gurarij space are Fraïssé [FLAMT20], and the following is the main conjecture on such spaces.

Conjecture 2.45. *The spaces L_p for $p \neq 4, 6, \dots$ and the Gurarij space \mathbb{G} are the only separable Fraïssé or even (AuH) spaces.*

It was proved in [FLAMT20, Proposition 2.13.] that a Fraïssé space must contain an isometric copy of the Hilbert space. It remains open whether any (AuH) space is Fraïssé, or simply whether it contains an isometric copy of the Hilbert. In any case, Hilbertian subspaces seem to be particularly relevant to the study of envelopes, even though we do not know whether every separable Fraïssé space contains a full copy of the Hilbert space. In Section 3 we shall give a positive answer in the case of the Fraïssé L_p -spaces. For more general Fraïssé spaces we have the following:

Proposition 2.46. *Consider X an (AuH) Banach space and Y a full Hilbertian subspace of X . Then Y must be a minimal full subspace of X and a maximal Hilbertian subspace of X .*

PROOF. Let Y be some full Hilbertian subspace of X . Note that no proper subspace Z of Y has full envelope: Indeed the identity embedding $i_{Z,X}$ of Z into X admits several extensions as an isometric embedding of Y into X and therefore several extensions as an isometric embedding of X into X ; while the unique isometric embedding of $\text{Env}(Z)$ extending $i_{Z,X}$ is $i_{\text{Env}(Z),X}$ by Proposition 2.39 (1). On the other hand all superspaces of Y have full envelope. Therefore Y is minimal with full envelope.

This also means that Y is maximal Hilbertian inside X . Indeed a non-trivial Hilbertian extension of Y would have full envelope and would not be a minimal full subspace, contradicting the first assertion of the proposition. \square

Remark 2.47. It does not seem possible to extend the result of uniqueness of full position inside the space L_1 to the context of “almost-isometric” position: It is proved in [GKL00, Theorem V.1 and Remark (2) on page 284] that there is a subspace X of L_1 such that (a) every isometric embedding of X into L_1 extends uniquely to a surjective isometry on L_1 , but (b) for every $\delta > 0$ there is a δ -embedding $\gamma_\delta : X \rightarrow L_1$ such that $\inf_{\delta > 0} d_{\text{BM}}(L_1/X, L_1/\gamma_\delta(X)) > 1$, where d_{BM} is the Banach-Mazur (multiplicative) distance.

3. ENVELOPES IN REARRANGEMENT INVARIANT SPACES, L_p -SPACES AND THE GURARIJ SPACE

In this section we identify the envelope of subspaces of L_p -spaces, $1 \leq p < +\infty$. We start with facts valid for general reflexive spaces, and then for r.i. spaces on $[0, 1]$ (also called symmetric by Peller [Pel81]). We start with the following result that allows to compute the Korovkin envelope.

Proposition 3.1. *Assume X is reflexive LUR. Let S be a semigroup of contractions on X . Let $Y \subseteq X$ be the closure of a directed sequence $(Y_i)_{i \in I}$ of subspaces of X . Then*

$$\text{Env}_S(Y) = \overline{\bigcup_{i \in I} \text{Env}_S(Y_i)}.$$

PROOF. Only the direct inclusion is not trivial. For each $i \in I$ let p_i be a contractive projection $p_i \in \overline{\text{conv}(S)}^{\text{WOT}}$ onto $\text{Env}_S(Y_i)$ provided by Proposition 2.23. Let p be a WOT-cluster point of $\{p_i\}_{i \in I}$; note that $p \in \overline{\text{conv}(S)}^{\text{WOT}}$ and that p is a contractive projection onto $\overline{\bigcup_{i \in I} \text{Env}_S(Y_i)}$. In particular, $p \upharpoonright Y = \text{Id}_Y$. Therefore by Remark 2.21, the range of p contains $\text{Env}_S(Y)$. \square

Remark 3.2. It does not seem that the previous proposition holds for the algebraic envelope. It is clear that $\text{Env}_S^{\text{alg}}(Y) \supseteq \overline{\bigcup_{i \in I} \text{Env}_S^{\text{alg}}(Y_i)}$. As for the direct inclusion, assume that $d(x, \bigcup_{i \in I} \text{Env}_S^{\text{alg}}(Y_i)) > \epsilon$. We use the following claim

Claim 3.2.1. Assume X is reflexive strictly convex. Let S be a semigroup of contractions on X and let $Y \subseteq X$. If $d(x, \text{Env}_S^{\text{alg}}(Y)) > \epsilon$, then there exists $s \in S$ such that $s \upharpoonright Y = \text{Id}_Y$ and $d(x, sx) > \epsilon$.

To prove this, by Proposition 2.10, let p be a contractive projection on $\text{Env}_S^{\text{alg}}(Y)$ belonging to the set $\overline{\text{conv}(\text{Stab}_S(Y))}^{\text{WOT}}$. Note that for such x one has that $\|x - px\| > \epsilon$; therefore $\phi(x - px) > \epsilon$ for some norm 1 functional ϕ . Since $p \in \overline{\text{conv}(\text{Stab}_S(Y))}^{\text{WOT}}$, there is some $T \in \text{conv}(\text{Stab}_S(Y))$ such that $\phi(x - Tx) > \epsilon$ and therefore some $s \in \text{Stab}_S(Y)$ such that $\phi(x - sx) > \epsilon$.

Once this is established, we can get $(s_i)_{i \in I}$ with s_i acting as the identity on Y and $d(x, s_i x) > \epsilon$. Under appropriate hypotheses one would obtain a WOT-cluster point T of $(s_i)_i$ with $T \upharpoonright Y = \text{Id}_Y$ and $d(x, Tx) > \epsilon$. If T were an element of S , this would prove that x is not in $\text{Env}_S^{\text{alg}}(Y)$; but T only belongs to $\overline{S}^{\text{WOT}}$ a priori. This perhaps explains why the result in the desired equality was proved for the Korovkin envelope and not the algebraic one. In the case of $X = L_p$, $p \neq 2$, we know the following. Suppose that $Y = L_p([0, 1], \Sigma)$, and $(\Sigma_n)_n$ is an increasing sequence of finite subalgebras whose union is dense in Σ . Then $\text{Env}^{\text{alg}}(Y) = \overline{\bigcup_n \text{Env}^{\text{alg}}(L_p([0, 1], \Sigma_n))}$ only holds when Σ is a fixed-point subalgebra (see Remark 3.6), because $\text{Env}^{\text{alg}}(L_p([0, 1], \Sigma_n)) = L_p([0, 1], \Sigma_n)$ (Lemma 3.5) and consequently the required equality means that $\text{Env}^{\text{alg}}(Y) = Y$.

Using Lancien's theorem about the existence of LUR renormings preserving the isometry group in separable reflexive Banach spaces [Lan93], we deduce the following.

Corollary 3.3. *Assume X is separable reflexive. Let $Y \subseteq X$ be the closure of a directed sequence $(Y_i)_{i \in I}$ of subspaces of X . Then*

$$\text{Env}(Y) = \overline{\bigcup_{i \in I} \text{Env}(Y_i)}.$$

3.1. Envelopes in rearrangement invariant spaces.

In this subsection we consider rearrangement invariant (r.i.) spaces on $[0, 1]$ as defined in [LT79, Definition 2.a.1.]. It is well-known that reflexive r.i. spaces on $[0, 1]$ are separable (see for example [LT79, pp 118-119]).

Definition 3.4. *For X a r.i. space on $I = [0, 1]$, and every σ -subalgebra Σ of measurable subsets of I , let X_Σ be the subspace of X consisting of the Σ -measurable functions in X .*

Note that X_Σ is a unital subspace, which is 1-complemented by [Ran01, Theorem 5.25].

Lemma 3.5. *If X is a r.i. space on $[0, 1]$, and Y is a finite dimensional unital sublattice, then $\text{Env}^{\text{alg}}(Y) = Y$.*

PROOF. The sublattice Y is generated by the characteristic functions of n disjoint measurable subsets (A_i) partitioning $[0, 1]$. Let f be such that $T(f) = f$ for all $T \in \text{Stab}_{\text{Isom}(X)}(Y)$. We claim that $f \upharpoonright A_i$ is constant for each i , and consequently, $f \in Y$. We prove the claim by contradiction. We can find $c \in \mathbb{R}$, and B, C of positive (equal) measure in some A_i such that $f \geq c$ on B and $f < c$ on C . Let σ be a measure isomorphism such that $\sigma(B) = C$, and $\sigma(A_j) = A_j$ for all $j = 1, \dots, n$. Observe that the isometry T_σ determined by σ belongs to $\text{Stab}_{\text{Isom}(X)}(Y)$, but $T_\sigma(f) \neq f$, because $T_\sigma(f) \upharpoonright C$ is a

function with value $\geq c$ while $f|_C$ has always value $< c$. This is impossible since we are assuming that $f \in \text{Env}^{\text{alg}}(Y)$. \square

Remark 3.6. Recall that a σ -subalgebra \mathcal{B} of a measure algebra (\mathcal{A}, μ) is called a *fixed-point subalgebra* when there is a set Γ of measure-preserving isomorphisms of \mathcal{A} such that $\mathcal{B} = \{a \in \mathcal{A} : \pi a = a \text{ for every } \pi \in \Gamma\}$ (see [Fre04, Section 333]). This subalgebras are characterized in [Fre04, Theorem 333R], where in particular is shown that fixed-point subalgebras can be determined by a single measure-preserving isomorphism of (\mathcal{A}, μ) . It is straightforward to see that A unital sublattice Y of $L_p[0, 1]$, $p \neq 2$, satisfies that $\text{Env}^{\text{alg}}(Y) = Y$ exactly when $Y = L_p([0, 1], \Sigma)$ for some fixed-point subalgebra of the Borel algebra $\mathcal{B}([0, 1])$. Moreover, we will see in Theorem 3.12 that if $Y \subseteq L_p$, $p \neq 2$, is a unital subspace, then $\text{Env}_{\min}(Y) = \text{Env}(Y) = L_p([0, 1], \Sigma_Y)$, and $\text{Env}^{\text{alg}}(Y) = L_p([0, 1], \Sigma_Y)$ iff Σ_Y is a fixed-point subalgebra.

Proposition 3.7. *If X is a reflexive r.i. space on $I = [0, 1]$, then for any σ -subalgebra Σ one has that $\text{Env}(X_\Sigma) = X_\Sigma$.*

PROOF. Each σ -algebra Σ is separable, hence we can find an increasing sequence $(\Delta_n)_n$ of finite subalgebras of Σ whose union is dense (see [Hal50]), and consequently $\bigcup_n X_{\Delta_n}$ is dense in X_Σ , because X is a r.i. space. Since each X_{Δ_n} is a finite dimensional unital sublattice, it follows from the previous lemma that $\text{Env}^{\text{alg}}(X_{\Delta_n}) = X_{\Delta_n} = \text{Env}(X_{\Delta_n})$. We know that X is separable, so by Corollary 3.3 we deduce that $\text{Env}(X_\Sigma) = \overline{\bigcup_n \text{Env}(X_{\Delta_n})} = \overline{\bigcup_n X_{\Delta_n}} = X_\Sigma$. \square

Note that holds: $X_{\cap_i \Sigma_i} = \bigcap_i X_{\Sigma_i}$. Therefore we may define the following envelope.

Definition 3.8 (Conditional envelope). *Suppose that X is a r.i. space on $I = [0, 1]$. For $Y \subseteq X$, let Σ_Y be the smallest σ -algebra making all functions of Y measurable. The conditional envelope $\text{Env}_c(Y)$ is the space X_{Σ_Y} of Σ_Y -measurable functions of X .*

The name conditional we chose is due to the fact that $\text{Env}_c(Y) = L_p([0, 1], \Sigma_Y)$ is the range of the conditional expectation projection \mathcal{E}^{Σ_Y} .

When $X = L_p[0, 1]$, X_Σ is the Lebesgue space $L_p([0, 1], \Sigma)$, hence $\text{Env}_c(Y) = L_p([0, 1], \Sigma_Y)$. We also have the following classical result.

Proposition 3.9. *Suppose that $1 \leq p < \infty$, $p \neq 2$, and Y is a unital subspace of $L_p[0, 1]$. Then,*

$$\text{Env}_{\min}(Y) = \text{Env}_c(Y) = \text{Lat}(Y),$$

where $\text{Lat}(Y)$ is the sublattice generated by Y .

PROOF. We first prove that $\text{Env}_c(Y) = \text{Env}_{\min}(Y)$. We know that the conditional expectation operator \mathcal{E}^{Σ_Y} is a contractive projection whose range is $L_p([0, 1], \Sigma_Y) = \text{Env}_c(Y)$. On the other hand, classical results by Douglas [Dou65] (for $p = 1$) and by Ando [And66] (for the rest of p 's) states that a contractive projection that fixes $\mathbb{1}_{[0,1]}$ must be a conditional expectation defined by some σ -subalgebra. Hence any contractive projection q whose range $R(q)$ contains Y satisfies $R(q) = L_p([0, 1], \Sigma)$ for some

σ -subalgebra Σ . Since $Y \subseteq R(q)$, it follows that $\Sigma_Y \subseteq \Sigma$, so $\text{Env}_c(Y) = L_p([0, 1], \Sigma_Y) \subseteq L_p([0, 1], \Sigma) = R(q)$. This proves that $\text{Env}_c(Y) = \text{Env}_{\min}(Y)$ (including that the latter exists when $p = 1$).

We now prove that $\text{Env}_c(Y) = \text{Lat}(Y)$. We start with the following

Claim 3.9.1. Suppose that Z is a unital sublattice. Then $\Sigma_Z = \{A : \mathbb{1}_A \in Z\}$ and $Z = L_p([0, 1], \Sigma_Z)$.

Proof of Claim: Set $\Delta := \{A : \mathbb{1}_A \in Z\}$. Obviously, $\Delta \subseteq \Sigma_Z$. Observe that in general for an arbitrary subspace V one has that Σ_V is the σ -subalgebra generated by the sets $\{f > c\}$ for $c > 0$ and $f \in V$, and when V is assumed to be a lattice then those $f \in V$ can be assumed to be positive. Fix now a positive $f \in Z$, $c > 0$, and set $g := (f - c\mathbb{1}_{[0,1]})^+ \in Z$. Then,

$$\mathbb{1}_{\{f > c\}} = \mathbb{1}_{\{g > 0\}} = \left(\sup_{n \in \mathbb{N}} n \cdot g \right) \wedge \mathbb{1}_{[0,1]} \in Z,$$

and consequently $\{f > c\} \in \Delta$. By the previous remark, $\Sigma_Z \subseteq \Delta$. Finally, trivially we have that $Z \subseteq L_p([0, 1], \Sigma_Z)$; as for the reverse inclusion, observe that it follows from the equality $\Sigma_Z = \{A : \mathbb{1}_A \in Z\}$ that every simple function in $L_p([0, 1], \Sigma_Z)$ belongs to Z , hence $L_p([0, 1], \Sigma_Z) \subseteq Z$. \square

We have by minimality that $\text{Lat}(Y) \subseteq L_p([0, 1], \Sigma_Y) = \text{Env}_c(Y)$. Since $\text{Lat}(Y)$ is 1-complemented, it follows that $\text{Env}_c(Y) = \text{Env}_{\min}(Y) \subseteq \text{Lat}(Y)$ and we are done. \square

Lemma 3.10. *If X is a reflexive r.i. space on $[0, 1]$, then for any unital subspace Y one has that*

$$\text{Env}_{\min}(Y) \subseteq \text{Env}(Y) \subseteq \text{Env}_c(Y).$$

PROOF. From Proposition 2.24 and Proposition 2.29 we know that $\text{Env}(Y)$ is 1-complemented, hence $\text{Env}_{\min}(Y) \subseteq \text{Env}(Y)$. Also, it follows from Proposition 3.7 that $\text{Env}(\text{Env}_c(Y)) = \text{Env}_c(Y)$ and consequently, $\text{Env}(Y) \subseteq \text{Env}_c(Y)$. \square

Lemma 3.11. *If Y is a unital subspace of L_1 , then $\text{Lat}(Y) \subseteq \text{Env}(Y)$.*

PROOF. By the equimeasurability formula, any isometric map t from Y into L_1 extends uniquely to an isometric map T on $\text{Lat}(Y)$ by the formula

$$T(|f|) := \frac{t(\mathbb{1})}{|t(\mathbb{1})|} |t(f)|.$$

So, fix $f \in Y$, and assume $(t_n)_n$ tends SOT to t on Y . We claim that $(t_n(|f|))_n$ converges to $T(|f|)$, that by the characterization of $\text{Env}(Y)$ will mean that $|f| \in \text{Env}(Y)$. Let us prove this claim. By composing by an isometry acting by change of signs, it is enough to prove the claim when $t(\mathbb{1})$ is non-negative, and by composing with an isometric embedding of $L_1(\text{supp}(t(\mathbb{1})))$ onto L_1 we may assume that $t(\mathbb{1})$ is positive, and therefore that $T(|f|) = |t(f)|$. We compute:

$$T_n(|f|) - T(|f|) = \frac{t_n(\mathbb{1})}{|t_n(\mathbb{1})|} |t_n(f)| - |t(f)| = \frac{t_n(\mathbb{1})}{|t_n(\mathbb{1})|} (|t_n(f)| - |t(f)|) + \left(\frac{t_n(\mathbb{1})}{|t_n(\mathbb{1})|} - \mathbb{1} \right) |t(f)| \quad (3)$$

The norm of the first part of the sum is at most $\int_0^1 |t_n(f)| - |t(f)| \leq \int_0^1 |(t_n - t)(f)|$, and this tends to 0. Since $(t_n(\mathbb{1}))_n$ tends to $t(\mathbb{1})$ in the L_1 -norm, the measure of the sets $A_n := \{s \in [0, 1] : t_n(\mathbb{1})(s) \leq 0\}$ tend to 0. The norm of the second summand in (3) is controlled by $2 \int_{A_n} |t(f)|$ and therefore tends to 0 as well. \square

Theorem 3.12. *If $1 \leq p < +\infty, p \neq 2$ and Y is a unital subspace of L_p then*

$$\text{Env}(Y) = \text{Lat}(Y) = \text{Env}_c(Y) = \text{Env}_{\min}(Y).$$

PROOF. For $p > 1$, we know from Proposition 3.9 and Lemma 3.10 that $\text{Lat}(Y) = \text{Env}_{\min}(Y) \subseteq \text{Env}(Y) \subseteq \text{Env}_c(Y) = \text{Lat}(Y)$.

The proof for $p = 1$ uses the Mazur maps. For some fixed $1 < p \neq 2$ we let $\phi : S_{L_1} \rightarrow S_{L_p}$ denote the Mazur map, extend it by homogeneity to a map between L_1 and L_p . Observe that $T \in \text{Isom}(L_1)$ if and only if $\phi T \phi^{-1} \in \text{Isom}(L_p)$. Let us also note that if Y is a unital lattice of L_1 then $\phi(Y) = L_p \cap Y$ which is a unital sublattice of L_p . As we have seen in Claim 3.9.1, there is a 1-1 correspondence between the class of unital sublattices of L_p and the class of sub- σ -algebras of Borel subsets of $[0, 1]$,

$$Y \mapsto \Sigma_Y = \{A \in \mathcal{B} : 1_A \in Y\}$$

and

$$\Sigma \mapsto Y_\Sigma = L_p([0, 1], \Sigma).$$

To avoid misinterpretations, we use the terminology $\text{Env}_{(L_p)}(Z)$ to denote the isometric envelope of some $Z \subseteq L_p[0, 1]$ in $L_p[0, 1]$.

Claim 3.12.1. $\phi(\text{Env}_{(L_1)}(Y) \subseteq \text{Env}_{L_p}(\phi(Y))$.

Admitting the claim, we deduce that if $Z \subseteq L_1[0, 1]$ is a unital sublattice, then so is $\phi(Z)$, hence $\phi(\text{Env}_{(L_1)}(Z)) \subseteq \text{Env}_{(L_p)}(\phi(Z)) = \phi(Z)$, and consequently $\text{Env}_{(L_1)}(Z) = Z$, because ϕ is 1-1. This implies that $\text{Env}(Y) \subseteq \text{Env}(\text{Lat}(Y)) = \text{Lat}(Y)$, and the equality $\text{Env}(Y) = \text{Lat}(Y)$ follows from Lemma 3.11. The rest of the equalities are in Theorem 3.9.

To prove the claim, assume $(T_i)_i$ is a net of isometries of L_p so that $(T_i(g))_i$ tends to g for all $g \in \phi(Y)$. This is equivalent to that $(\phi U_i(y))_i$ tends to $\phi(y)$ for all y in Y where each $U_i := \phi^{-1} T_i \phi$ is the associated isometry on L_1 to T_i . Since the Mazur map is an homeomorphism (see [Maz29]), this is equivalent to that $(U_i)_i$ converges to Id pointwise on Y , which, since $f \in \text{Env}_{(L_1)}(Y)$, implies that $(U_i(f))_i$ tends to f . Therefore $(T_i \phi(f))_i$ converges to ϕf , because $T_i \phi(f) = \phi U_i(f)$. \square

Remark 3.13. It follows easily that the previous proposition is true for the Lebesgue spaces $L_p(A)$ for every Borel subset A of the unit interval. Hence, if $Y \subseteq L_p[0, 1]$, $p \neq 2$, is a subspace such that $1_A \in Y$ where A is the support of Y , then $\text{Env}(Y) = \text{Lat}(Y) = \text{Env}_c(Y) = \text{Env}_{\min}(Y)$. This is so, because all these envelopes on the space $L_p(A)$ are equal to their versions in the full space $L_p[0, 1]$.

Remark 3.14. In the case $1 < p < +\infty$ this result can also be deduced from Proposition 2.24(b) and the description by Peller of the WOT closure \mathcal{M} of the isometry group of L_p [Pel81], from which it follows that any contractive projection on L_p belongs to \mathcal{M} . There does not seem to be such a direct proof in the case $p = 1$, since our WOT characterization of the envelope is, apparently, only valid in the reflexive case.

The envelopes of non unital subspaces of L_p can also be computed.

Proposition 3.15. *Suppose that $1 \leq p < +\infty$, and let Y be a subspace of L_p . Then we have that*

$$(1) \text{ Env}(Y) = \text{Env}_{\min}(Y).$$

(2) *If p is not even, and $g \in Y$ is of full support in Y then*

$$\text{Env}(Y) = g \cdot \text{Env}(Y/g) = g \cdot \text{Lat}(Y/g) = g \cdot \text{Env}_c(Y/g).$$

PROOF. Pick g of full support in Y . To prove (1) first assume Y has full support. Let T be an isometry on L_p sending g to 1, so that TY is unital. Then (1) follows from the equality $\text{Env}(TY) = \text{Env}_{\min}(TY)$ (Proposition 3.12), and from Remarks 2.4 and before Example 2.18 about preservation of the minimal and isometric envelopes by isometries. When Y does not have full support the proof is similar using Remark 3.13.

The last two equalities in (2) for *every* $p \neq 2$ also follow from Remark 3.13. We have to show that $\text{Env}(Y) = g \text{Env}(Y/g)$ for non-even p 's. It is well-known that for every $1 \leq p < \infty$, the envelope $\text{Env}_c(Y) = L_p([0, 1], \Sigma_Y)$ is the closure of the subspace of functions $B(f_1, \dots, f_n)$ where $f_1, \dots, f_n \in Y$ and $B : \mathbb{R}^n \rightarrow \mathbb{R}$ is Borel. Suppose now that $p \notin 2\mathbb{N}$, fix $Y \subseteq L_p[0, 1]$ and $g \in Y$ with full support. Let t be an isometry between a unital subspace Z of L_p and Y sending $\mathbb{1}$ to g . Since $L_p[0, 1]$ is an approximately ultrahomogeneous space, it follows that there is a unique extension T of t from $\text{Env}(Z)$ onto $\text{Env}(Y)$. In fact, T is defined by

$$T(B(f_1, \dots, f_n)) = g \cdot B(tf_1/g, \dots, tf_n/g)$$

for every $f_1, \dots, f_n \in Z$ and every Borel function $B : \mathbb{R}^n \rightarrow \mathbb{R}$ (see [KK01, Theorem 4]). Hence,

$$\begin{aligned} \text{Env}(Y) &= T(\text{Env}(Z)) = \{T(B(f_1, \dots, f_n)) : f_1, \dots, f_n \in Z, B \text{ Borel}\} = \\ &= g \cdot \{B(g_1/g, \dots, g_n/g) : g_1, \dots, g_n \in Y, B \text{ Borel}\} = g \cdot L_p(\text{supp } g, \Sigma_{Y/g}) = \\ &= g \cdot L_p([0, 1], \Sigma_{Y/g}) = g \cdot \text{Env}_c(Y/g) = g \cdot \text{Env}(Y/g). \end{aligned}$$

□

Finally we may observe the following regarding separable r.i. spaces on $[0, 1]$ different from the L_p 's. Recall that a contractive map on a r.i. space on $[0, 1]$ is absolute if it defines a contraction on L_1 and on L_∞ .

Theorem 3.16. *The L_p 's, $1 < p < +\infty$ are the unique reflexive r.i. spaces on $[0, 1]$ for which all 1-complemented subspaces (1-dimensional subspaces is enough) are envelopes.*

PROOF. Let X be a reflexive r.i. space on $[0, 1]$. In particular, X is separable. The assertion that all 1-dimensional subspaces are envelopes implies, by Proposition 2.29, that any 1-dimensional subspace is the range of a contractive projection belonging to the WOT-closure of the convex hull of the isometry group. On the other hand, it is a classical result (see [Zai77, Theorem 2]) that if X is different from one of the L_p 's, then it follows that every isometry of X is absolute. Since absolute contractions are closed under convex combinations and WOT-limits (using for example [Pel81] Theorem 8), it follows that under our hypothesis every 1-dimensional subspace of X is 1-complemented by an absolute contractive projection, and we are going to see (probably a well-known result) in the next claim that this is impossible.

Claim 3.16.1. Every reflexive r.i. space on $[0, 1]$ admits a 1-dimensional subspace which is not the range of an absolute projection. More precisely: if u and v are disjointly supported elements of L_∞ and $\|u\|_\infty > \|v\|_\infty > 0$, then the span of $u + v$ is not the range of an absolute projection.

Proof of Claim: We may assume without loss of generality that $x_0 := u + v$ has norm 1 in X . Let P be a contractive projection with range equal to the span of x_0 . It must satisfy the formula $P(x) = \langle \phi_0, x \rangle x_0$ where $\phi_0 \in X^*$ and $\phi_0(x_0) = \|\phi_0\| = 1$. If P is also a contraction in L_∞ , let us define $\gamma : L_\infty \rightarrow \mathbb{R}$ by $\gamma(x) := \|x\|_\infty - \langle \phi_0, x \rangle \|x_0\|_\infty$, and let us observe that

$$\gamma(x) = \|x\|_\infty \pm \|P(x)\|_\infty \geq 0.$$

We see now that ϕ_0 and v are disjointly supported: Let $h := \mathbb{1}_A$ where $A \subseteq \text{supp } v$. Note that for $|t|$ small enough one has that $\|x_0 + th\|_\infty = \|x_0\|_\infty$, hence

$$\gamma(x_0 + th) = \|x_0 + th\|_\infty - (1 + t\langle \phi_0, h \rangle) \|x_0\|_\infty = -t \|x_0\|_\infty \langle \phi_0, h \rangle.$$

Since γ is non-negative we deduce that $\phi_0(h) = 0$, and this implies that $\text{supp } (\phi_0) \cap \text{supp } (v) = \emptyset$ and therefore $\phi_0(u) = \phi_0(x_0) = 1$. It follows that $P(u) = x_0$.

Therefore $\|P(u)\|_1 = \|x_0\|_1 = \|u\|_1 + \|v\|_1 > \|u\|_1$, and P cannot also be a contraction in L_1 . □

□

We do not know whether the minimal and isometric envelopes coincide for an arbitrary reflexive Fraïssé or even (AuH) Banach space.

3.2. Full copies of L_q -spaces inside L_p . We study the envelopes of L_q -spaces inside L_p , obtaining the following.

Theorem 3.17. *Assume that $1 \leq p < +\infty$ is not even. Then*

- (1) *the envelope of any Hilbertian subspace of L_p of dimension at least 2 is isometric to L_p .*
- (2) *If q satisfies $1 \leq p \leq q \leq 2$, then the envelopes of L_q and ℓ_q inside L_p are isometric to L_p .*

Consequently,

- (3) *L_p admits full copies of ℓ_2^n , $n \geq 2$, ℓ_2 , and of L_q and ℓ_q provided that $1 \leq p \leq q \leq 2$.*

PROOF. In fact, we have the following formally stronger statement: Assume $1 \leq p < +\infty$, p not even and let Y be a subspace of L_p such that one of the two conditions hold:

- i) the group $\text{Isom}(Y)$ acts almost transitively on S_Y and Y does not embed isometrically into ℓ_p ;
- ii) Y has a 1-symmetric Schauder basis (e_n) such that $\inf_n \frac{\|\sum_{i=1}^n e_i\|_Y}{n^{1/p}} = 0$

Then the envelope of Y is isometric to L_p . In particular some isometric copy of Y has full envelope.

Its proof goes as follows. Composing with an isometric embedding of Y into L_p sending some vector of full support in Y to 1, we may assume Y is unital, hence $\text{Env}(Y) = L_p([0, 1], \Sigma_Y)$. The proof is finished once we argue that Σ_Y does not have atoms. Suppose otherwise. Then $\text{Env}(Y) = L_p(A_1, \Sigma_1) \oplus_p$

$L_p(A_2, \Sigma_2)$ where $\Sigma_1, \Sigma_2 \subseteq \Sigma_Y$ are atomless and atomic σ -algebras in A_1 and A_2 , respectively, $A_1 \sqcup A_2 = [0, 1]$, and Σ_2 is non-empty.

We denote by i the inclusion map of Y into $L_p(A_1, \Sigma_1) \oplus_p L_p(A_2, \Sigma_2)$, and choosing $i_1 = 0$ if necessary, we write $i = (i_1, i_2)$ with respect to the decomposition $L_p(A_1, \Sigma_1) \oplus_p L_p(A_2, \Sigma_2)$, that is $i_j(f) = f|_{A_j}$ for every $f \in Y$. Note that by the Banach-Lamperti formula for isometries of L_p -spaces, all isometries on this space can be written as $T = (T_1, T_2)$ where each T_i is an isometry on $L_p(A_i, \Sigma_i)$ for $i = 1, 2$. Observe that $i_2 \neq 0$, because otherwise $A_2 = \emptyset$.

Claim 3.17.1. The norm of i_2y is constant on $\text{Isom}(Y)$ -orbits.

Proof of Claim: For suppose that $t \in \text{Isom}(Y)$ and $y \in Y$. Since p is not even, we know that L_p is (AuH), and it follows from Proposition 2.41 that there is a unique extension of t to $T \in \text{Isom}(\text{Env}(Y))$. Then $(i_1ty, i_2ty) = ity = Tiy = (T_1i_1y, T_2i_2y)$, and consequently $\|i_2ty\| = \|T_2i_2y\| = \|i_2y\|$. \square

Suppose that (i) holds. Then it follows from the previous claim that the norm of i_2 has constant value $k > 0$ on S_Y (because $i_2 \neq 0$). Hence, i_2/k is an isometric embedding of Y into $L_p(A_2, \Sigma_2)$, and since Σ_2 is atomic, $L_p(A_2, \Sigma_2)$ is isometric to $\ell_p(I)$, where I is the (countable) set of atoms of Σ_2 , contradicting the hypothesis.

Suppose that (ii) holds. Since $(e_n)_n$ is a 1-symmetric basis of Y , the isometries of Y act transitively on the basis, and by the previous claim this implies that the norm of i_2e_n for $n \in \mathbb{N}$ is constant, with value c . As before, we have that $c > 0$. By w^* -compactness there exists $u \in \ell_p^{**}$ and N an infinite subset of \mathbb{N} such that $(i_2e_n)_{n \in N} \rightarrow^{w^*} u$. We claim that $u = 0$. To see this, note that for any normalized ϕ in ℓ_p^* , and any $n \in \mathbb{N}$ we can find a finite $F \subseteq N$ of cardinality n such that $\phi(\sum_{j \in F} i_2e_j) \geq \frac{1}{2}n\phi(u)$. It follows that $n\phi(u) \leq 2\|i_2(\sum_{j \in F} e_j)\| \leq 2\|i_2\| \|\sum_{j=1}^n e_j\|_Y$. The estimate in ii) then implies that $\phi(u) = 0$ and therefore the claim that $u = 0$. Now since $(i_2e_n)_{n \in N}$ tends weakly to 0 in ℓ_p , we can, by a gliding hump argument and passing to a subset of N , assume that the sequence $(i_2e_n)_{n \in N}$ is almost successive in the sense that $\|\sum_{j \in F} i_2e_j\| \geq \frac{1}{2}(\sum_{j \in F} \|i_2e_j\|^p)^{1/p}$ whenever $F \subseteq N$ is finite. Therefore $c|F|^{1/p} \leq 2\|i_2\| \|\sum_{j \in F} e_j\|_Y$, contradicting ii) when $|F|$ is large enough. \square

3.3. On g -embeddings of ℓ_2 into L_p and the spaces L_p/ℓ_2 . Knowing that ℓ_2 admits a full copy in L_p , we have an associated exact sequence

$$0 \rightarrow \ell_2 \rightarrow L_p \rightarrow L_p/\ell_2 \rightarrow 0,$$

to which we may apply Corollary 2.42 and the commentary thereafter.

Definition 3.18 (The full quotient space L_p/ℓ_2). *Let $1 \leq p < \infty$, $p \notin 2\mathbb{N}$. The quotient L_p/ℓ_2 of L_p by any full copy of ℓ_2 is isometrically unique. We call this space the full quotient of L_p by ℓ_2 .*

Proposition 3.19. *For each $1 \leq p < \infty$, $p \notin 2\mathbb{N} + 4$, there is a linear g -embedding (see definition just before Definition 2.41) of $U(\ell_2)$ into $\text{Isom}(L_p)$. It is defined by extension of unitaries on any fixed full copy of ℓ_2 inside L_p , to isometries on L_p .*

PROOF. This is a consequence of Theorem 3.17 and of Proposition 2.41. \square

Many questions remain open about concrete representations of full copies of ℓ_2 inside L_p , about the exact sequence $0 \rightarrow \ell_2 \rightarrow L_p \rightarrow L_p/\ell_2 \rightarrow 0$ associated to the linear g -embedding of ℓ_2 inside L_p , for appropriate values of p , about the associated full quotient of L_p by ℓ_2 , or about the topological embedding of $U(H)$ into $\text{Isom}(L_p)$ from a concrete point of view. We shall not address them in this paper but we do make a few initial comments below.

If $1 < p < +\infty$, then the exact sequence above splits, by complementation of ℓ_2 inside L_p . This means that L_p/ℓ_2 is isomorphic to L_p (since it is isomorphic to a complemented subspace of L_p , and since L_p is primary). On the other hand when $p = 1$, then

$$0 \rightarrow \ell_2 \rightarrow L_1 \rightarrow L_1/\ell_2 \rightarrow 0$$

does not split, and the isomorphic type of the full quotient L_1/ℓ_2 is unknown.

Many other homological questions arise: what description can be made of the induced action of $U(\ell_2)$ on L_p/ℓ_2 (which is actually an action by bounded isomorphisms on L_p when $p > 1$)? what can be said of the quasilinear map Ω associated to the exact sequence and of the commutator estimates of Ω with the action of the unitary group on L_p ? How may the dual exact sequence be described? For details we refer to [CG97], for the general theory of exact sequences of Banach spaces, and to [CF21], for a theory of their compatibility with group actions.

Another direction which seems worthwhile to explore is the relation between the isometric envelopes of the Hilbert space in an L_p -space and Gaussian Hilbert spaces. See in particular the extension Theorem 4.12 in [Jan97]. Note however that the theory of Gaussian Hilbert spaces seems to be limited to the context of L_p -spaces and not to generalize to the (AuH) or Fraïssé situation.

3.4. Envelopes in the Gurarij space. One may wonder whether there is a canonical definition of the envelope of a separable space Y , not depending on the choice of the space where Y embeds. Since the Gurarij space \mathbb{G} is the only (AuH) separable and universal space, it would be natural to define the canonical envelope of Y to be its envelope inside \mathbb{G} . However we shall now see that any subspace of \mathbb{G} is equal to its own envelope, so this concept is not as relevant as one would hope. On the other hand, the result leads to interesting questions.

Theorem 3.20. *$\text{Env}(X) = X$ for every subspace $X \subset \mathbb{G}$.*

The proof relies on two steps, starting with an easy lemma.

Lemma 3.21. *Let E be a Banach space. Assume that for any finite dimensional subspace F of E , for any $x \in E \setminus F$, there exists $\delta = \delta(x, F) > 0$ such that for any $\varepsilon > 0$, there exists $T \in \text{Iso}(E)$ such that $\|Tf - f\| \leq \varepsilon$ for all $f \in F$ and $\|Tx - x\| \geq \delta - \varepsilon$. Then $\text{Env}(X) = X$ for every separable subspace $X \subset E$.*

PROOF. Suppose that $X \subset E$ is arbitrary. Let $\{F_n\}_n$ be an increasing sequence of finite dimensional subspaces of X whose union is dense in X . Suppose that $x \notin X$. We use the hypothesis to find for each k some $T_k \in \text{Isom}(\mathbb{G})$ such that $\|T_k \upharpoonright F_k - \text{Id}_{F_k, \mathbb{G}}\| \leq 2^{-k}$ and $\|T_k(x) - x\| \geq \delta - 2^{-k}$. Consequently, $T_k \upharpoonright X \rightarrow_k \text{Id}_X$ pointwise but $T_k(x) \not\rightarrow_k x$, and therefore, $x \notin \text{Env}(X)$. \square

Proposition 3.22. *The Gurarij space satisfies the hypothesis of Lemma 3.21 with $\delta(x, F) = 2d(x, F)$.*

PROOF. We prove slightly more than what is required. Fixing $F \subset \mathbb{G}$, x, y in \mathbb{G} , let G (resp. G') be a finite dimensional subspace of \mathbb{G} containing x (resp. y) and F . We consider the push-out of G with G' , $H := (G \oplus G')/S$, where $S := \{(f, -f) \in G \times G' : f \in F\}$. Let $x_0 := (x, 0) + S$, $x_1 := (0, y) + S$. It is easy to see that $\|x_0 - x_1\|_H \geq \delta := d(x, F) + d(y, F)$. Similarly, let $G_0 := (G \times \{0\}) + S$, $G_1 := (\{0\} \times G') + S$. Using that \mathbb{G} is universal for separable spaces and approximate ultrahomogeneous we can find an isometry $\theta_H : H \rightarrow \mathbb{G}$, such that

a) the mapping $\theta : G \rightarrow \mathbb{G}$ linearly defined by $\theta(z) = \theta_H((z, 0) + S)$ satisfies that $\|\theta(z) - z\| \leq \varepsilon\|z\|$ for every $z \in G$.

b) the mapping $\theta' : G' \rightarrow \mathbb{G}$ linearly defined by $\theta'(z) = \theta_H((0, z) + S)$ satisfies that $\|\theta(x) - \theta'(y)\| = \|\theta_H(x_0 - x_1)\| \geq \delta$.

To finalize the proof, we now restrict to the case where y is chosen equal to a given x sitting outside of F . We let $G = G'$ be equal to the subspace generated by F and x . Observe that $(1 + 2\varepsilon)^{-1}\|y\| \leq \|\theta(y)\| \leq (1 + \varepsilon)\|y\|$ (since $\varepsilon \leq 1/2$). Setting $\bar{H} := \text{Im}\theta_H(H)$ and $\bar{G}_0 := \theta_H(G_0)$, $\bar{G}_1 := \theta_H(G_1)$, let $\bar{\eta} : \bar{G}_0 \rightarrow \bar{G}_1$ be the surjective isometry defined by $\bar{\eta}(\theta_H((g, 0) + S)) := \theta_H((0, g) + S)$. Then, the composition $\bar{\eta} \circ \theta : G \rightarrow \bar{H}$ is a $(1 + 2\varepsilon)$ -isometry. Since \mathbb{G} is a stable Fraïssé space with modulus $2t$ ([FLAMT20] Example 2.5) there is some isometry $T \in \text{Isom}(\mathbb{G})$ such that $\|T - \bar{\eta} \circ \theta\| \leq 4\varepsilon$. So, given $f \in F$ we have that

$$\|T(f) - f\| \leq \|T(f) - \bar{\eta}(\theta(f))\| + \|f - \bar{\eta}(\theta(f))\| \leq 4\varepsilon\|f\| + \|f - \theta(f)\| \leq 5\varepsilon\|f\|,$$

so T approximates the identity on F , while

$$\|T(x) - x\| \geq \|\bar{\eta}(\theta(x)) - \theta(x)\| - \|T(x) - \bar{\eta}(\theta(x))\| - \|\theta(x) - x\| \geq \delta - 2\varepsilon\|x\|,$$

with $\delta = 2d(x, F)$. □

In the proof we have observed that \mathbb{G} satisfies the following property, for every F, G, G' finite dimensional subspace of \mathbb{G} with $F \subseteq G, G'$, for every $x \in G$ and $y \in G'$, and for every $\varepsilon > 0$: there are a finite dimensional subspace V of \mathbb{G} , $I \in \text{Emb}(G, V)$ and $J \in \text{Emb}(G', V)$ such that $\|I \circ i_{F,G} - J \circ i_{F,G'}\| \leq \varepsilon$ and

$$\|I(x) - J(y)\| \geq (1 - \varepsilon)d(x, \gamma(F)) + d(y, \eta(F)).$$

This is a property which recalls the free amalgamation property in model theory for relational structures (see [Fra00]).

A similar property is easily seen to be satisfied by the Hilbert space, and in particular the Hilbert space satisfies Lemma 4.2 with $\delta = \sqrt{2}d(x, F)$, which is another way of proving that every of its subspaces is equal to its own envelope.

Problem 3.23. *Show that the Hilbert space and the Gurarij space are the only separable Fraïssé spaces X for which each subspace is equal to its envelope.*

4. COMPLEMENTATIONS IN (AuH) SPACES: NUMERICAL ESTIMATES

The aim of this section is to establish relations between local properties of a space and the AUH or Fraïssé property. Recall that a Fraïssé Banach space X always contains isometric copies of the Hilbert space. We shall see that the constant of complementation of such copies in X determines for which

values of p the space X is possibly isometric to L_p . We shall also relate these values to the type and cotype of the space.

In consequence we shall obtain “local versions” of the fact that L_p -spaces are Fraïssé only if $p \notin 2\mathbb{N} + 4$, and reinforce the conjecture that those spaces, together with the Gurarij space, are the only separable Fraïssé spaces.

4.1. Complementations in (AuH) spaces. Our first result relates to constants of complementation in (AuH)-spaces. This extends some results known for L_p -spaces, but our point is to obtain these only through properties of the isometry group.

Proposition 4.1. *Let X be (AuH) and 1-complemented in its bidual. Then any isometric embedding of a subspace Y of X into X extends to a contraction on X .*

PROOF. Let t be an embedding of a subspace Y of X . For any F finite-dimensional subspace of Y and $\varepsilon > 0$, there is a map $T_{F,\varepsilon} \in \text{Isom}(X)$ such that $\|(t - T_{F,\varepsilon})\upharpoonright F\| \leq \varepsilon$. Using a non-trivial ultrafilter refining the natural filter on the set of (F, ε) , we define a map $T : X \rightarrow X^{**}$ by $Tx = w^* - \lim_U T_{F,\varepsilon}x$. If P is a norm 1 projection from X^{**} onto X then PT defines a contraction on X extending t . \square

Proposition 4.2. *Let X be (AuH), and let Y be C -complemented in X for some $C \geq 1$. Then any isometric copy of Y inside X is also C -complemented, as soon as one of the following conditions holds:*

- (1) X is 1-complemented in its bidual, or
- (2) Y is finite dimensional, or
- (3) Y is 1-complemented in its bidual and can be written as the closure of an increasing union of a net $(F_i)_i$ of finite dimensional subspaces such that F_i is 1-complemented in Y for every i .

PROOF. Let Y be C -complemented by a projection p and t be an isometric embedding of Y into X . (1): Using Proposition 4.1, tY is complemented by tpT' , if T' is a contraction on X extending t^{-1} . (2): for any $\varepsilon > 0$ we can find, by classical perturbation arguments, a surjective $1 + \varepsilon$ -isometry T_ε on X extending t , and then $p_\varepsilon := T_\varepsilon p T_\varepsilon^{-1}$ is a projection onto tY of norm at most $C(1 + \varepsilon)^2$. A compactness argument provides a projection onto tY of norm at most C . (3): Applying (2) we find a projection p_i onto tF_i of norm at most C for each i . Using a non-trivial ultrafilter refining the net, we define a map $p : X \rightarrow (tY)^{**}$ by $px = w^* - \lim_U p_i x$. If P is a norm 1 projection from $(tY)^{**}$ onto tY then Pp defines a projection onto tY of norm at most C . \square

From the (AuH) properties of L_p -spaces, we recover a known result in the theory of L_p -spaces ([Ran99]), i.e. the above holds inside $L_p(0, 1)$, $1 \leq p < +\infty$, $p \notin 2\mathbb{N} + 4$. It also proved in [Ran99] that the result does not hold for $p \in 2\mathbb{N} + 4$, since in this case there is a complemented subspace admitting an isometric uncomplemented copy. One may actually prove the following:

Remark 4.3. No (AuH) space can contain a complemented isometric copy of any L_p with $p \in 2\mathbb{N} + 4$. This follows from the existence of a constant K and a sequence of pairwise isometric finite dimensional subspaces F_n and G_n of L_p such that F_n is K -complementend and G_n is not n -complemented in L_p , for each $n \in \mathbb{N}$. It is easy to see that this fact, together with item (2) of Proposition 4.2, is incompatible

with complementation of a copy of L_p inside X . The existence of K , (F_n) and (G_n) is based on the unconditionality of the subspaces used in [Ran99] and is detailed in the proof of Proposition 2.10 of [FLAMT20].

As for the Hilbert space H , a classical result states that $U(H)$ is WOT-dense in $\mathcal{L}_1(H)$ (see for example [Pel81]). This and Proposition 4.1 gives us the following.

Corollary 4.4. *Assume X is (AuH) 1-complemented in its bidual. Then every operator defined on a hilbertian subspace of X extends to an operator of same norm on X .*

PROOF. For suppose that $Y \subseteq X$ is hilbertian and $t : Y \rightarrow Y$ be a contraction of norm 1. We choose a net $(t_i)_i$ of isometries of Y converging WOT to T . By Proposition 4.1, we can extend each t_i to norm one operators $T_i : X \rightarrow X$. Let $U : X \rightarrow X^{**}$ be WOT-limit of $(T_i)_i$, and let $P : X^{**} \rightarrow X$ be a contractive projection. Then PU has norm 1 and extends t . \square

Note that this holds in particular for for all the separable Fraïssé L_p -spaces.

A similar result seems possible for subspaces Y for which the unital algebra $\mathcal{L}(Y)$ is *unitary*, that is, when the convex hull of its isometries is norm dense in $\mathcal{L}_1(Y)$. It is classical that $\mathcal{L}(H)$ is unitary in the complex case and the real case is due to Navarro-Pascual and Navarro [NPN12] who actually prove that the infinite convex hull is equal to $\mathcal{L}_1(H)$.

Definition 4.5. *For X a Banach space, $1 \leq p \leq \infty$, and $n \in \mathbb{N}$, let us denote $c_p^n(X) \in [1, +\infty]$ the infimum of constants of complementation of an isometric copy of ℓ_p^n inside X , and $c_p(X)$ the infimum of constants of complementation of an isometric copy of L_p inside X . Let us also denote by $\text{Fr}(X)$ the set of p 's such that ℓ_p is finitely representable in X .*

Proposition 4.6. *Assume X is Fraïssé. The following holds for every $p \in \text{Fr}(X)$, $1 \leq p < +\infty$:*

- (1) $c_p^n(X)$ is attained.
- (2) $c_p(X)$ is attained when it is finite.
- (3) We have $c_p(X) = \lim_n c_p^n(X)$.
- (4) If there is $C \geq 1$ such that for any n , and for any $\varepsilon > 0$, X contains a C -complemented $1 + \varepsilon$ -copy of ℓ_p^n , then $c_p(X) \leq C$.

Furthermore

- (5) for fixed n , the map $p \mapsto c_p^n(X)$ is continuous on $\text{Fr}(X)$

PROOF. Recall that L_p embeds into X as soon as $p \in \text{Fr}(X)$ (see [FLAMT20]). Here, except for (1), we are using that the space X is assumed to be Fraïssé and not only (AuH). (1): Since X is (AuH), any isometric map between two copies of ℓ_p^n may be approximated by a global isometry, therefore extends to a $1 + \varepsilon$ -isometry for ε arbitrarily small, and we deduce that any isometric copy of ℓ_p^n is complemented by a projection of norm at most $c_p^n(X) + \delta$, for arbitrary $\delta > 0$. We may then define a

projection as some weak limit of almost optimal projections along an ultrafilter to see that all copies of ℓ_p^n are complemented by a projection of norm $c_p^n(X)$.

Note that the sequence $c_p^n(X)$ is non-decreasing, and assume it is bounded by C . Writing L_p as the closure of the increasing union of ℓ_p^n 's, and again defining a projection onto L_p as some weak limit of projections on the ℓ_p^n 's (using that L_p is 1-complemented in its bidual), we deduce that both (2) and (3) hold.

(4): Let $\varepsilon > 0$, $\eta > 0$, and consider a C -complemented $1 + \eta$ -isometric copy F of ℓ_p^n . If η had been chosen small enough, the Fraïssé property allows to extend a partial $1 + \eta$ -isometry t between F and an isometric copy G of ℓ_p^n to a global isomorphism T on X such that $\|T\|\|T^{-1}\| < 1 + \varepsilon$, guaranteeing that G is $C(1 + \varepsilon)$ -complemented. Then we deduce that $c_p^n(X) \leq C$, and, by c), that $c_p(X) \leq C$.

(5): Let $p \in \text{Fr}(X)$, $\varepsilon > 0$, $\eta > 0$. For q close enough to p in $\text{Fr}(X)$, there is a partial $1 + \eta$ -isometry t between copies of ℓ_p^n and ℓ_q^n . If η had been chosen small enough, the Fraïssé property allows to extend t to a global isomorphism T on X such that $\|T\|\|T^{-1}\| < 1 + \varepsilon$, guaranteeing that $c_q^n(X) \leq (1 + \varepsilon)c_p^n(X)$. \square

In the case (4) we shall say that ℓ_p is uniformly complementably finitely representable in X ; we noted that this implies that X contains a complemented copy of L_p .

4.2. The hilbertian case. We now concentrate on the case of Hilbertian/euclidean copies. The exact value of the (best) constant of complementation of isometric copies of ℓ_2^n or ℓ_2 inside L_p , $1 \leq p \leq +\infty$ is known (see Gordon-Lewis-Retherford [GLR72, Theorem 6]). Let us note:

$$c_2(L_p) = \frac{2}{\sqrt{\pi}} \left(\Gamma\left(\frac{p+1}{2}\right) \right)^{1/p} \left(\Gamma\left(\frac{p'+1}{2}\right) \right)^{1/p'}, \quad 1 < p < +\infty \quad (4)$$

and

$$c_2(L_\infty) = c_2(L_1) = +\infty$$

where for $p \in [1, +\infty]$ we use the notation $p' \in [1, +\infty]$ for the conjugate of p .

Recall that $p(X) := \sup\{p : X \text{ has type } p\}$ and that $q(X) := \inf\{p : X \text{ has cotype } q\}$. The space X is said to be near Hilbert when $p(X) = q(X) = 2$. It is well-known that X does not necessarily have type $p(X)$ or cotype $q(X)$, however the Maurey-Pisier Theorem [MP76] states that both $\ell_{p(X)}$ and $\ell_{q(X)}$ are finitely representable in X . Regarding duality, it is immediate that if X has type $p > 1$ then X^* has cotype p' . A profound result by Pisier [Pis82] states conversely that if X has non-trivial type and cotype q , then X^* has type q' . Another result by Pisier [Pis82] states that a Banach space X has non-trivial type exactly when X is *locally π -euclidean*, that is, when for some $c \geq 1$, for every n and $\varepsilon > 0$, there exists N such that any N -dimensional subspace of X admits an n -dimensional subspace which is $1 + \varepsilon$ -isometric to an euclidean space and which is c -complemented in X (see also Pełczyński-Rosenthal [PR75, Definition and Observation on p284], noting that the definitions of locally π -euclidean in [Pis82] and [PR75] are equivalent by Dvoretzky theorem).

Lemma 4.7. *The following hold:*

- (1) $c_2(L_p) = c_2(L_{p'})$ for $1 \leq p \leq +\infty$.
- (2) If $1 \leq p < \infty$ is in $\text{Fr}(X)$ for some Fraïssé space X , then $c_2(L_p) \leq c_2(X)$.

(3) The map $p \mapsto c_2(L_p)$ is decreasing on $[1, 2]$ and increasing on $[2, \infty]$.

PROOF. Assertion (1) is a direct consequence of the formula (4). (2): Since X is Fraïssé and $p \in \text{Fr}(X)$, it follows that X contains an isometric copy of L_p . For a given n , if ℓ_2^n is C -complemented in X , then any copy of ℓ_2^n inside L_p is also C -complemented in X (because X is Fraïssé) and consequently also in L_p . This implies that $c_2^n(L_p) \leq c_2^n(X)$, and by Proposition 4.6 (3) we have that $c_2(L_p) \leq c_2(X)$. (3): since L_q embeds isometrically into L_p for $1 \leq p \leq q \leq 2$, this and assertion (2) implies that the map is decreasing on $[1, 2]$, and then, by assertion (1), increasing on $[2, +\infty]$ by (2). \square

The next relates $p(X)$, $q(X)$, and complemented copies of L_p -spaces.

Proposition 4.8. *Let X be a Fraïssé Banach space.*

- (1) *If $q(X) = +\infty$ then X contains an isometric copy of the Gurarij space, and if X separable then X is isometric to the Gurarij space.*
- (2) *If $p(X) = 1$ and $q(X) < +\infty$, then X contains isometric copies of L_1 , ℓ_2 is uncomplemented in X , and $c_2^n(X) \geq n \cdot \Gamma(n/2) / (\sqrt{\pi} \cdot \Gamma(1/2 + n/2))$.*
- (3) *If $p(X) > 1$ then ℓ_2 is complemented in X . There exists a unique $1 < p \leq 2$ such that $c_2(X) = c_2(L_p)$, and we have $p(X) \geq p$ and $q(X) \leq p'$.*
- (4) *If $p(X) > 1$ and X has type $p(X)$ then X contains a complemented isometric copy of $L_{p(X)}$.*
- (5) *If $p(X) > 1$ and X has cotype $q(X)$ then X contains a complemented isometric copy of $L_{q(X)}$ and $q(X) \notin 2\mathbb{N} + 4$.*

PROOF. Assertion (1) was already observed in [FLAMT20]. It is a consequence of the Maurey-Pisier Theorem in [MP76] that ℓ_∞ is finitely representable in any space with non non-trivial cotype, implying that a space satisfying (1) must be isometrically universal for separable spaces, and the fact that the Gurarij space is the unique separably universal Fraïssé space.

Assertion (2) is a consequence of Maurey-Pisier Theorem and the Fraïssé property - Lemma 4.7 (2). Since ℓ_2 is uncomplemented in L_1 it has to be uncomplemented in X and this also provides a lower bound for $c_2^n(X)$ by $c_2^n(L_1) = n \cdot \Gamma(n/2) / (\sqrt{\pi} \cdot \Gamma(1/2 + n/2))$.

Regarding (3), since X has non-trivial type, it is locally π -euclidean. Therefore X contains uniformly complemented $1 + \varepsilon$ -copies of euclidean spaces of arbitrary size and therefore a complemented copy of the Hilbert space. The formula (4) implies that $p \mapsto c_2(L_p)$ is continuous, and by Lemma 4.7 (3), is also decreasing on $[1, 2]$. Since $c_2(L_1) = \infty$ and $c_2(L_2) = 1$, it follows that there must be a unique $1 < p \leq 2$ such that $c_2(X) = c_2(L_p)$. By the Maurey-Pisier Theorem on the finite representability of $L_{p(X)}$ in X , and by using that X is Fraïssé, X contains an isometric copy of $L_{p(X)}$. It follows that $c_2(L_{p(X)}) \leq c_2(X) = c_2(L_p)$, and therefore $p(X) \geq p$. The same reasoning holds for cotype.

(4): Following [MS86, Proposition 13.16], if we suppose that $p(X) > 1$ and X has type $p(X)$, then X contains for some $C \geq 1$ and for any $\varepsilon > 0$, $1 + \varepsilon$ -isomorphic copies of ℓ_p^n C -complemented, with $p = p(X)$, and therefore a C -complemented copy of $L_{p(X)}$ by Proposition 4.6 (4).

(5): the Pisier duality result [Pis82] holds to deduce that $p(X^*) = q(X)'$ and X^* has type $p(X^*)$. Therefore as in (4) above X^* contains $1 + \varepsilon$ -isomorphic copies of $\ell_{p(X^*)}^n$'s uniformly complemented, and by duality and local reflexivity, X contains $1 + \varepsilon$ -isomorphic copies of $\ell_{q(X)}^n$'s uniformly complemented. The rest of the reasoning is as in (5) to deduce that X admits a complemented copy of $L_{q(X)}$. The fact that $q(X) \notin 2\mathbb{N} + 4$ then follows from Remark 4.3. \square

We have the following striking corollary of Proposition 4.8:

Corollary 4.9. *Any Fraïssé space X satisfies exactly one of the two following properties: either*

- (1) *X contains a complemented isometric copy of ℓ_2 , or*
- (2) *X contains an isometric copy of L_1 .* \square

Applying Proposition 4.8 to the case of 1-complemented copies of L_p spaces we also get:

Proposition 4.10. *Let X be a Fraïssé Banach space. Then*

- (1) *there are at most 2 values of p (which must be conjugate) such that X contains a 1-complemented copy of L_p .*
- (2) *if X contains a 1-complemented copy of L_p and $p \leq 2$ (resp. $p \geq 2$), then $p(X) = p$ (resp. $q(X) = p'$). In particular if X contains a 1-complemented copy of L_2 then X is near Hilbert.*

PROOF. Suppose that X contains a 1-complemented copy of L_p . Then, by Lemma 4.7 (1), (2), $c_2(L_{p'}) = c_2(L_p) \leq c_2(X)$. On the other, the 1-complementation of a copy of L_p implies that $c_2(X) \leq c_2(L_p)$. This determines the set $\{p, p'\}$. (2): if $p \leq 2$ then it follows that $p(X) \geq p$, but also since X contains a copy of L_p , that $p(X) \leq p$. Likewise if $p \geq 2$ then it follows that $q(X) = p'$. \square

We do not know if any Fraïssé space contains a 1-complemented copy of some L_p and if it contains at most such 1-complemented L_p . There are other properties of Fraïssé L_p spaces which may be seen as natural steps towards the conjecture that all separable Fraïssé spaces are either L_p -spaces or the Gurarij space. We just list two of them here: must any Fraïssé space X attain its type and cotype? if X is Fraïssé and reflexive, must $\text{Fr}(X)$ be either an interval of the form $[p, 2]$ or a singleton $\{q\}$ for $2 \leq q < +\infty$? The reader will easily find other natural conjectures in this vein.

We conclude with a solution to a weak version of the Mazur rotation problem posed by G. Godefroy (see [CSFRar, Problem 5.20, and the paragraph before it]).

Theorem 4.11. *A Fraïssé space admitting C_∞ bump functions must be isomorphic to a Hilbert space.*

PROOF. Assume X is as above and not isomorphic to a Hilbert space. Since a space with a Fréchet-differentiable bump function is necessarily Asplund, X cannot contain a copy of L_1 , whereby by Proposition 4.8 (2), X has non trivial type. According to a Theorem of Deville, see [DGZ93, Chapter V.4] for an exposition, this implies that $q(X)$ is an even number larger than 2 and X has cotype $q(X)$, and this contradicts Proposition 4.8 (5). \square

A Banach space X is Lipschitz transitive [CS21] when there exists $C \geq 1$ such that for any two normalized vectors x, y of X , there is a surjective isometry T on X such that $Tx = y$ and $\|T - Id\| \leq C\|y - x\|$. According to [CS21, Lemma 2.6], any Lipschitz transitive norm on a separable space is C_∞ . We immediately deduce:

Proposition 4.12. *A separable Fraïssé space which is Lipschitz transitive must be isomorphic to Hilbert space.* \square

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