TIGHTNESS OF BANACH SPACES AND BAIRE CATEGORY

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ABSTRACT. We prove several dichotomies on linear embeddings between Banach spaces. Given an arbitrary Banach space X with a basis, we show that the relations of isomorphism and bi-embedding are meager or co-meager on the Polish set of block-subspaces of X. We relate this result with tightness and minimality of Banach spaces. Examples and open questions are provided.

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1. INTRODUCTION

W.T. Gowers' fundamental results in geometry of Banach spaces [11, 12] opened the way to a loose classification of Banach spaces up to subspaces, known as Gowers' program (see [8]). We focus in this note on a specific question: how many subspaces - up to linear isomorphism - does a non-hilbertian Banach space contain? More precisely, this note gathers several observations in the spirit of previous work by C. Rosendal and the first-named author ([4, 5, 6, 8]). which were not spelled out before. These remarks relate in particular the notion of tightness (from [8]) to Baire category arguments.

Our main results are dichotomies: Theorem 3.2 is an embedding dichotomy into a Banach space with a basis. Theorem 4.1 states that the relations of linear isomorphism and of bi-embeddability are meager or co-meager on the set $b(X)$ of block-subspaces of a space X with a basis. Several examples are given and commented open questions conclude the note.

We use the classical notation and terminology for Banach spaces, as may be found in [14]. Our reference for topology and descriptive set theory results is [13].

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Specific pieces of notation are needed for block-bases and subspaces, and for these we follow [4] and [7]. We differ, however, on the following: what is denoted $bb_d(X)$ there is denoted here $b(X)$.

We recall what this notation means. Let X is a Banach space equipped with a basis (e_n). Given a field Q of scalars, denote by $c_{00}(Q)$ the Q-vector space generated by the basis (e_n). We fix a countable field Q containing the set $\mathbb{Q}$ of rationals (or $\mathbb{Q}+i\mathbb{Q}$ in the complex case), and the norm of any vector in $c_{00}(Q)$ - such a Q is easily constructed by induction. Then let Q_0 be the set of normalized vectors of $c_{00}(Q)$.

We equip the countable set Q_0 with the discrete topology. The set $b(X)$ consists of all block-bases made of vectors in Q_0, while $b^{<\omega}(X)$ is the set of finite block-bases made of such vectors. The set $b(X)$ is a closed subset of Q_0^{ω}, and thus it is a Polish space. Although the set $b(X)$ depends not only on X, but also on the choice of the basis (e_n), there will be no ambiguity from this notation, since we shall always work with a fixed basis (e_n) of X.

We recall that the support of a vector $x = \sum_n x_n e_n$ of Q_0 is the set

$$\text{supp } x = \{n : x_n \neq 0\},$$

while the range of x is the interval of integers

$$\text{range } x = [\min(\text{supp } x), \max(\text{supp } x)].$$

2. Some topological lemmas

We recall in this section some well-known results on Baire dichotomies. Our first lemma is called the first topological 0-1 law in [13] (Th. 8.46). It appears in [9], Lemma 2, but was certainly known earlier.

Lemma 2.1. *Let P be a Polish space, and G be a group of homeomorphisms of P such that for all U, V non-empty open sets in P, there is $g \in G$ such that $g(U) \cap V \neq \emptyset$. Let $A \subset P$ with the Baire Property such that $g(A) = A$ for all $g \in G$. Then A is meager or comeager.*

Proof. Let $B = P \setminus A$. If A and B are both non-meager, then there exist two non-empty open sets U and V such that $U \cap B$ and $V \cap A$ are both meager. Let $g \in G$ be such that the open set $W := g(U) \cap V$ is non-empty. Since $g(U) \cap B = g(U \cap B)$, we have that $W \cap B$ and $W \cap A$ are both meager, and this is a contradiction. □

Example 2.2. *The relations E_0 and E_0'.*

We see the Cantor set 2^{ω} as the set of subsets of ω, the set $2^{<\omega}$ as the set of finite subsets of ω, and we define on 2^{ω} the following relations.

(1) uE_0v if there is $n \geq 0$ such that

$$u \cap [n, +\infty) = v \cap [n, +\infty),$$

(2) $uE_0'v$ if there is $n \geq 0$ such that

$$u \cap [n, +\infty) = v \cap [n, +\infty)$$

and

$$|u \cap [0, n-1]| = |v \cap [0, n-1]|.$$

Then the equivalence classes for E_0 or E'_0 are orbits of groups of homeomorphisms, namely, for E_0,

$$G_0 = \{(u\Delta.), u \in 2^\omega\},$$

and for E'_0, the group G'_0 of permutations of ω with finite support. Therefore any subset of 2^ω with the Baire property which is E_0-, or (merely) E'_0-saturated, is meager or comeager.

Our second lemma is a standard compactness argument (see [5], Lemma 7).

Lemma 2.3. *Let A be a subset of 2^ω. The following assertions are equivalent:*

(1) *A is comeager,*
(2) *there is a sequence $I_0 < I_1 < I_2 < \cdots$ of successive subsets of ω, and $a_n \subset I_n$, such that for any $u \in 2^\omega$, if the set $\{n : u \cap I_n = a_n\}$ is infinite, then $u \in A$.*

Proof. For the reverse implication, just note that

$$O_n = \{u \in 2^\omega \mid \exists k \geq n, u \cap I_k = a_k\}$$

is a dense open set of 2^ω for any $n \geq 1$, and that

$$\cap_{n\geq 1} O_n \subset A.$$

For the direct implication, assuming A is comeager, we write

$$2^\omega \setminus A \subset \cup_{n\geq 0} F_n,$$

where each F_n is closed with empty interior. The "compactness" we use is actually the trivial fact that a set with two points is compact. An easy induction argument provides I_n and a_n such that

$$u \cap I_n = a_n \Rightarrow u \notin \cup_{i<n} F_i.$$

If $u \in F_k$, then $u \cap I_n \neq a_n$ for all $n > k$, and the conclusion follows. □

It results from the proof that we can assume without loss of generality that the I_k's constitute a partition of ω into intervals.

Corollary 2.4. *Let A be a subset of ω such that:*

$$u \in A, u \subset v \Rightarrow v \in A.$$

Then

(a) *A is meager if and only if there exist $I_0 < I_1 < I_2 < \cdots$ such that*

$$u \in A \Rightarrow \{n; u \cap I_n = \emptyset\} \text{ is finite.}$$

(b) *A is comeager if and only if there exist $I_0 < I_1 < I_2 < \cdots$ such that if u contains infinitely many I_n's, then $u \in A$.*

This corollary easily follows from Lemma 2.3. We note that (a) is shown in [10], where it is applied to filters and simply additive measures on ω.

There is a counterpart of Lemma 2.3 for the space $b(X)$ of block-bases of X [6], which we state below as Proposition 2.5. In this case, one uses compactness and not finiteness, so the general result involves ϵ-nets. Here we shall only be interested in isomorphic properties of block-subspaces, which are preserved by small enough perturbations of the vectors of the basis, and therefore use a simpler form of the characterization of comeager sets of $b(X)$.

If $\tilde{x}$ is a finite block-sequence in $b^{<\omega}(X)$, we say that $z \in b(X)$ *passes through* $\tilde{x}$ if z may be written as the concatenation

$$z = \tilde{y}^\frown \tilde{x}^\frown w$$

for some $\tilde{y} \in b^{<\omega}(X)$ and some $w \in b(X)$.

Proposition 2.5. *Let X be a Banach space with a basis (e_n). Let $A \subset b(X)$ be such that*

$$(y \in A \wedge \overline{\text{span}}(y) = \overline{\text{span}}(z)) \Rightarrow z \in A.$$

Then the following assertions are equivalent:

(1) *A is comeager in $b(X)$,*
(2) *there is a sequence $\tilde{x}_0 < \tilde{x}_1 < \tilde{x}_2 < \cdots$ of successive elements of $b^{<\omega}(X)$, such that for any $z \in b(X)$, if the set $\{n : z$ passes through $\tilde{x}_n\}$ is infinite, then $z \in A$.*

3. Application to embeddings of Banach spaces

Let X be a Banach space with a basis $(e_n)_n$. Following [8], we say that an (infinite-dimensional) space Y is *tight in X* if there is a sequence $I_0 < I_1 < I_2 < \cdots$ of successive intervals such that for all infinite subset $J \subset \mathbb{N}$,

$$Y \not\sqsubseteq \overline{\text{span}}[e_n, n \notin \cup_{j \in J} I_j],$$

where $\sqsubseteq$ means "embeds isomorphically into". We say that the space X is *tight* if all infinite-dimensional spaces Y are tight in X.

Of course these notions really depend on the choice of the basis $(e_n)_n$, so the notation is not exactly accurate, but this will not cause any problem since we shall consider only one choice of basis for X.

We recall that X is *minimal* if every infinite dimensional subspace of X contains an isomorphic copy of X. The main result of [8] asserts that every Banach space contains a tight subspace or a minimal subspace.

The notion of tightness can be linked with the Baire category statements of the previous section through the following results.

Proposition 3.1. *Let X be a Banach space with a basis $(e_n)_n$ and let Y be a Banach space. Then the following are equivalent:*

(a) *Y is tight in X,*
(b) *$E_Y := \{u \in 2^\omega : Y \sqsubseteq \overline{\text{span}}[e_n, n \in u]\}$ is meager in 2^ω,*
(c) *$\text{Emb}_Y := \{z \in b(X) : Y \sqsubseteq \overline{\text{span}}[z]\}$ is meager in $b(X)$.*

Proof. The implication $(a) \Rightarrow (b)$ is an immediate consequence of Corollary 2.4 (a), applied to $E_Y = \{u \in 2^\omega : Y \sqsubseteq \overline{\text{span}}[e_n, n \in u]\}$.

To prove $(b) \Rightarrow (c)$, assume that E_Y is meager. Let $\phi : b(X) \to 2^\omega$ be defined by

$$\phi((z_n)_n) = \cup_n \text{supp}\, z_n.$$

It is clear that $\overline{\text{span}}\, z \subset \overline{\text{span}}\{e_i, i \in \phi(z)\}$, and therefore

$$\text{Emb}_Y \subset \phi^{-1}(E_Y).$$

The map ϕ is continuous, and for any basic open set $U = N_{z_0,\ldots,z_n}$ of $b(X)$,

$$\phi(U) = N_{\cup_{i \le n} \text{supp}\, z_i} \setminus 2^{<\omega},$$

and therefore $\overline{\phi(U)} = N_{\cup_{i \le n} \text{supp}\, z_i}$ is open. This easily implies that

$$A \text{ meager in } 2^\omega \Rightarrow \phi^{-1}(A) \text{ meager in } b(X).$$

If now (b) E_Y is meager, then $\mathrm{Emb}_Y \subset \phi^{-1}(E_Y)$ is meager, and (c) holds.

To prove $(c) \Rightarrow (a)$, assume $\mathrm{Emb}_Y = \{z \in b(X) : Y \sqsubseteq \overline{\mathrm{span}}\ z\}$ is meager, and for all j, let $\tilde{x}_j \in b^{<\omega}(X)$ be such that if $z \in b(X)$, then

$$z \text{ passes through } \tilde{x}_j \text{ for infinitely many } j \Rightarrow Y \not\sqsubseteq \overline{\mathrm{span}}\ z.$$

Let $I_j = \mathrm{range}\ \tilde{x}_j$ for each j. Let J be an infinite subset of $\mathbb{N}$, and consider

$$W = \overline{\mathrm{span}}[e_n, n \notin \cup_{j \in J} I_j].$$

If z is the concatenation (in the appropriate order) of the e_n's for $n \notin \cup_{j\in J} I_j$ and of the $\tilde{x}_j$ for $j \in J$, then z passes through $\tilde{x}_j$ for all $j \in J$ and therefore Y does not embed into $\overline{\mathrm{span}}\ z$. Since $W \subset \overline{\mathrm{span}}\ z$, Y does not embed into W. Since J was arbitrary, we have proven that Y is tight in X. □

Theorem 3.2. *Let X be a Banach space with a basis (e_n), and let Y be a Banach space. Then exactly one of the two following assertions holds:*

(a) *there exists $I_0 < I_1 < I_2 < \cdots$ such that for any $J \subset \omega$ infinite,*

$$Y \not\sqsubseteq \overline{\mathrm{span}}[e_n, n \notin \cup_{j\in J} I_j],$$

(b) *there exists $J_0 < J_1 < J_2 < \cdots$ such that for any $I \subset \omega$ infinite,*

$$Y \sqsubseteq \overline{\mathrm{span}}[e_n, n \notin \cup_{i\in I} J_i].$$

Proof. Recall that

$$E_Y = \{u \in 2^\omega : Y \sqsubseteq \overline{\mathrm{span}}[e_n, n \in u]\}.$$

It is easy to check that E_Y is an analytic subset of 2^ω and thus it is has the Baire property. Obviously, $u \in E_Y$ and $u \subset v$ implies that $v \in E_Y$.

if $uE_0'v$ (see Example 2.2), then the closed linear spans of $[e_n, n \in u]$ and $[e_n, n \in v]$ are isomorphic. Hence E_Y is E_0'-saturated and thus by Lemma 1.1 it is meager or comeager. The result now follows from Corollary 2.4.

Note that for checking that (a) and (b) are mutually exclusive, it suffices to apply them with two infinite sets I and J such that $\left(\cup_{j\in J} I_j\right) \cap \left(\cup_{i\in I} J_i\right) = \emptyset$. □

Example 3.3. *Tightness and minimality.*

A space X is tight when (a) holds for any Y, or equivalently, for any block-subspace $Y = \overline{\mathrm{span}}\ y$ generated by some $y \in b(X)$.

On the other hand, (b) holds for any minimal subspace Y of X: indeed if (a) holded, and if we picked a subspace Z of Y embedding isomorphically into $\overline{\mathrm{span}}[e_n; n \in \cup_{j\in K} I_j]$ for some $K \subset \omega$ coinfinite, then we would deduce from (a) that Y does not embed into Z, contradicting minimality.

In particular we see that a tight space does not contain any minimal subspace. Also, since every subspace of a minimal space is minimal, it follows that if X is minimal, then (b) holds for every subspace Y of X,

Example 3.4. *Tightness with constants.*

If there are successive subsets I_j of $\mathbb{N}$ such that for each j,

$$Y \not\sqsubseteq_j \overline{\mathrm{span}}[e_n, n \notin I_j],$$

where $\sqsubseteq_j$ means "embeds with constant j", then we may use the I_j's to prove that Y is tight in X; we say in that case that Y is tight in X *with constants*. When all infinite-dimensional spaces are tight in X with constants, then X is said to be *tight* *with constants*. This notion was defined and studied in [8]; Tsirelson's space T is the typical space satisfying tightness with constants.

Defining for $j \in \mathbb{N}$,

$$E_Y(j) := \{u \in 2^\omega : Y \sqsubseteq_j \overline{\text{span}}[e_n, n \in u]\},$$

we have of course

$$E_Y = \cup_{j \in \mathbb{N}} E_Y(j).$$

The next proposition, a counterpart of Proposition 3.1 in the case of j-embeddings, shows that Y being tight in X with constants is equivalent to saying that all $E_Y(j)$'s are nowhere dense in 2^ω. In other words we have in that case a very natural description of E_Y as a countable union of nowhere dense sets. A similar result holds for Emb_Y.

Proposition 3.5. *Let X be a Banach space with a basis $(e_n)_n$ and let Y be a Banach space. Then the following are equivalent:*

(a) *Y is tight in X with constants,*
(b) *$E_Y(j) := \{u \in 2^\omega : Y \sqsubseteq_j \overline{\text{span}}[e_n, n \in u]\}$ is nowhere dense in 2^ω for all $j \in \mathbb{N}$,*
(c) *$\text{Emb}_Y(j) := \{z \in b(X) : Y \sqsubseteq_j \overline{\text{span}}[z]\}$ is nowhere dense in $b(X)$ for all $j \in \mathbb{N}$.*

Proof. $(a) \Rightarrow (b)$: let (I_j) be successive such that for each j, $Y \not\sqsubseteq_j [e_n, n \notin I_j]$. This means that

$$E_Y(j) \subset \{u \in 2^\omega \mid u \cap I_j \neq \emptyset\},$$

and since $E_Y(j) \subseteq E_Y(k)$ whenever $j \leq k$, that

$$E_Y(j) \subset \bigcap_{k \geq j} \{u \in 2^\omega \mid u \cap I_k \neq \emptyset\}.$$

The set on the right hand side of this inclusion is closed with empty interior, so $E_Y(j)$ is nowhere dense for all j.

To prove $(b) \Rightarrow (c)$, let $\phi : b(X) \to 2^\omega$ be defined as in Proposition 3.1 by $\phi((z_n)_n) = \cup_n \text{supp } z_n$, therefore for $j \in \mathbb{N}$,

$$\text{Emb}_Y(j) \subset \phi^{-1}(E_Y(j)).$$

Since the map ϕ is continuous, and for any basic open set U of $b(X)$, $\overline{\phi(U)}$ is open, it follows that

$$E_Y(j) \text{ nowhere dense } \Rightarrow \phi^{-1}(E_Y(j)) \text{ nowhere dense } \Rightarrow \text{Emb}_Y(j) \text{ nowhere dense.}$$

To prove $(c) \Rightarrow (a)$, assume $\text{Emb}_Y(j) = \{z \in b(X) : Y \sqsubseteq \overline{\text{span}}\ z\}$ is nowhere dense for all $j \in \mathbb{N}$. We may use induction to find successive $\tilde{x}_j \in b^\omega(X)$ so that if I_j denotes range $\tilde{x}_j$ and $n_j := \max I_j$, then

$$N_{(e_0, e_1, \ldots, e_{n_{j-1}})^\frown \tilde{x}_j} \cap \text{Emb}_Y(j) = \emptyset$$

for all j. We may assume the I_j form a partition of $\mathbb{N}$, and this implies that Y does not embed with constant j into $\overline{\text{span}}[e_i, i \notin I_j]$. Therefore (a) is proved. □

In the next section, we will display an embedding dichotomy similar to Theorem 3.2 within the set $b(X)$ of block-subspaces of a given Banach space with a basis.

4. TOPOLOGICAL $0-1$-LAWS FOR THE CLASSICAL RELATIONS ON $b(X)$

Let X be a Banach space with a basis (e_n), and let $b(X)$ be the Polish space of its block sequences. We denote by $\sqsubseteq$ (resp. $\simeq$) the relation of embeddability (resp. isomorphism) between subspaces of X. We consider the following subsets of $b(X)^2$:

$$\mathrm{Is} = \left\{(y,z) \in b(X)^2 : \overline{\mathrm{span}}\ y \simeq \overline{\mathrm{span}}\ z\right\},$$

$$\mathrm{Be} = \left\{(y,z) \in b(X)^2 : \overline{\mathrm{span}}\ y \sqsubseteq \overline{\mathrm{span}}\ z \text{ and } \overline{\mathrm{span}}\ z \sqsubseteq \overline{\mathrm{span}}\ y\right\},$$

$$\mathrm{Emb} = \left\{(y,z) \in b(X)^2 : \overline{\mathrm{span}}\ y \sqsubseteq \overline{\mathrm{span}}\ z\right\}.$$

Obviously

$$\mathrm{Is} \subseteq \mathrm{Be} \subseteq \mathrm{Emb}.$$

The main result of this section is:

Theorem 4.1. *The relations* Is*,* Be*, and* Emb *are meager or comeager in the Polish space* $b(X)^2$*.*

Proof. Pick $\tilde{x}$ and $\tilde{y}$ in $b^{<\omega}(X)$ with same length, and denote by $T_{\tilde{x},\tilde{y}}$ the homeomorphism T of $b(X)$ defined by

$$T(\tilde{x}^\frown z) = \tilde{y}^\frown z,$$

$$T(\tilde{y}^\frown z) = \tilde{x}^\frown z,$$

$$T(z) = z \text{ if } z \notin N(\tilde{x}) \cup N(\tilde{y}).$$

In other words, $T_{\tilde{x},\tilde{y}}$ substitutes $\tilde{x}$ to $\tilde{y}$ (and conversely) in $N(\tilde{x}) \cup N(\tilde{y})$ and does nothing else.

Let G be the group of homeomorphisms of $b(X)^2$ generated by the maps $(T_{\tilde{x},\tilde{y}}, T_{\tilde{u},\tilde{v}})$, where $(\tilde{x},\tilde{y})$ and $(\tilde{u},\tilde{v})$ are arbitrary pairs of elements of $b^{<\omega}(X)$ with same length. It is easily seen that the G-orbit of any point $(x,y) \in b(X)^2$ is dense. Moreover, one clearly has

$$\overline{\mathrm{span}}\ T_{\tilde{x},\tilde{y}}(u) \simeq \overline{\mathrm{span}}\ u$$

for all $u \in b(X)$, and it follows that the sets Is, Be and Emb are G-invariant. They are clearly analytic, hence have the Baire Property, and Lemma 2.1 concludes the proof. □

Remark 4.2. *Continuum of non-isomorphic subspaces:*

The Kuratowski-Mycielski theorem (see (19.1) in [13]) asserts that if a relation R is meager in the perfect Polish space $b(X)^2$, then there is an homeomorphic copy K of the Cantor set in $b(X)$ such that $\neg(xRy)$ for all $x \neq y$ in K. Hence if Is is meager, the space X contains a continuum of non-isomorphic subspaces.

If $j \in \mathbb{N}$, we denote $\mathrm{Emb}(j) = \left\{(y,z) \in b(x)^2 \,\middle|\, \overline{\mathrm{span}}\ y \sqsubseteq_j \overline{\mathrm{span}}\ z\right\}$, and observe that

$$\mathrm{Emb} = \cup_{j \in \mathbb{N}} \mathrm{Emb}(j).$$

The next observation makes a link with the previous section.

Proposition 4.3. *Let X be a space with a basis (e_n). The following hold:*

(a) *If X is tight, then the relation* Emb *is meager in* $b(X)^2$*.*
(b) *If X is tight with constants, then the relation* Emb(j) *is nowhere dense in* $b(X)^2$ *for all* $j \in \mathbb{N}$*.*

Proof. (a) If X is tight, then by Proposition 3.1 (c), the set

$$\mathrm{Emb}_y = \{z \in b(X) : (y,z) \in \mathrm{Emb}\}$$

is meager in $b(X)$ for any $y \in b(X)$. The Kuratowski-Ulam theorem ([13] Th. 8.41) then shows that Emb is meager since all its fibers are.

(b) If X is tight by constants, then in particular, by [8] Proposition 4.1, we may find a successive sequence (I_j) of intervals such that for all j,

$$\overline{\mathrm{span}}[e_i, i \in I_j] \not\sqsubseteq_j \overline{\mathrm{span}}[e_i, i \notin I_j].$$

Fix $k \in \mathbb{N}$, and given $\tilde{x}, \tilde{y}$ in $b^{<\omega}(X)$, pick $j \geq k$ so that I_j is supported after $\tilde{x}$ and $\tilde{y}$. Then it follows that

$$\left(N(\tilde{x}^\frown (e_i)_{i \in I_j}) \times N(\tilde{y}^\frown e_{1+\max I_j})\right) \cap \mathrm{Emb}(j) = \emptyset.$$

Since $\mathrm{Emb}(k) \subset \mathrm{Emb}(j)$ we deduce that $N(\tilde{x}) \times N(\tilde{y})$ contains an open set which is disjoint from $\mathrm{Emb}(k)$. Since $\tilde{x}, \tilde{y}$ were arbitrary, this means that $\mathrm{Emb}(k)$ is nowhere dense. □

Observe that from Proposition 3.1 (b), we may deduce equivalently to (a) that if X is tight, then the set

$$\{(y,u) \in b(X) \times 2^\omega : \overline{\mathrm{span}}\ y \sqsubseteq \overline{\mathrm{span}}[e_n, n \in u]\}$$

is meager in $b(X) \times 2^\omega$.

It follows from Proposition 4.3 and the Kuratowski-Mycielski theorem that every tight space contains a continuum of subspaces which do not embed into each other. This also follows from ([8] Th.7.3).

In fact, this argument goes beyond the case of tight spaces, since we have:

Example 4.4. *A space with an unconditional basis which is not tight, although* Emb *is meager.*

Proof. Let G_u be Gowers' "tight by support" space, that is, such that all disjointly supported subspaces on its canonical basis (u_n) are totally incomparable [11]. Let (f_n) be the canonical basis of ℓ_2. We consider $X = G_u \oplus \ell_2$, equipped with the basis $(u_0, f_0, u_1, f_1, \ldots)$.

By the remarks of Example 3.3, (b) of Theorem 3.2 holds for $Y = \ell_2$. Therefore (a) does not hold for this choice of Y and therefore X is not tight.

To prove that Emb is meager, it is enough by the Kuratowski-Ulam theorem to prove that for y in a comeager subset of $b(X)$, the set Emb_y is meager in $b(X)$, or equivalently by Proposition 3.1, that the set

$$E_Y = \{u \in 2^\omega : Y \sqsubseteq \overline{\mathrm{span}}[e_n, n \in u]\}$$

is meager (where Y denotes $\overline{\mathrm{span}}\ y$). Let therefore Ω be the comeager set of all $y \in b(X)$ which pass through infinitely many u_n's. We claim that for $y \in \Omega$, the set E_Y is meager in 2^ω.

We may and do assume that y is a subsequence of (u_n). If E_Y is not meager, then it is comeager, and therefore by Corollary 2.4, there exist $I_0 < I_1 < I_2 < \cdots$ such that if u contains infinitely many I_n's, then $u \in E_Y$. In other words, if u contains infinitely many I_n's, then

$$Y \sqsubseteq \overline{\mathrm{span}}[e_n, n \in u].$$

Passing to a subsequence of y whose supports on (e_i) are disjoint from infinitely many I_n's, and letting u be the union of these I_n's, we may therefore assume that

$$(\text{supp } y) \cap u = \emptyset.$$

This implies that Y embeds into a direct sum $\ell_2 \oplus Z$, where Z is a subspace of G_u which is disjointly supported from Y. On the other hand, since G_u is tight by support, Y is totally incomparable with ℓ_2 and with Z, therefore ([14] Prop. 2.c.5) every operator from W to $\ell_2 \oplus Z$ is strictly singular, which is a contradiction. □

Hence the largest relation Emb can be meager for spaces which are not tight. On the other hand, the relation Be - and thus the relation Emb - is trivial for minimal spaces, and hence it is of course comeager. In the next section we shall see that the converse is false, even for the smallest relation Is. We shall also show that Is may be meager for minimal spaces.

5. Some more examples

We start by giving two non-minimal examples of spaces for which Is is comeager. The first is an easily defined infinite ℓ_2-sum which is not minimal. The second is more involved and does not even contain a minimal subspace.

Example 5.1. *An ℓ_2-sum with an unconditional basis which is not minimal, but such that* Is *is comeager.*

Proof. We fix $p \neq 2$ and let $X = \left(\sum_n \oplus \ell_p^n\right)_2$. The space X is not minimal, since it contains ℓ_2 but does not embed into ℓ_2. On the other hand it can be shown - using e.g. the arguments from [14], Prop. 1.g.4 - that if $z \in b(X)$, then $\overline{\text{span}}\, z$ is isomorphic to ℓ_2 or to X. If $b(X) = A \cup B$ is the partition of $b(X)$ into the two corresponding Is-classes, we deduce that A or B is non-meager. Hence Is is non-meager and therefore comeager by Theorem 4.1 - equivalently, A or B is comeager in $b(X)$. □

The comeager class in Example 5.1 is actually the class of X. This follows for instance from the next observation.

Remark 5.2. *Let X be a space with an unconditional basis (e_n). If*

$$\left\{z \in b(X) \,\middle|\, \overline{\text{span}}\, z \simeq \ell_2\right\}$$

is comeager, then X is isomorphic to ℓ_2.

Proof. Assuming $A = \{z \in b(X) : \overline{\text{span}}\, z \simeq \ell_2\}$ is comeager in $b(X)$, let $\tilde{x}_n \in b^{<\omega}(X)$ be successive such that if z passes through infinitely many $\tilde{x}_n$'s, then $\overline{\text{span}}\, z$ is isomorphic to ℓ_2. W may assume that the intervals $I_n = \text{range } \tilde{x}_n$ form a partition of ω. Then the concatenation of the $\tilde{x}_{2n}$'s and of the e_i for $i \in \cup_n I_{2n+1}$ is in A, from which it follows that

$$\overline{\text{span}}[e_i, i \in \cup_n I_{2n+1}]$$

embeds into ℓ_2 and therefore is isomorphic to ℓ_2. The same holds for

$$\overline{\text{span}}[e_i, i \in \cup_n I_{2n}].$$

By unconditionality of (e_n), it follows that X is isomorphic to ℓ_2. □

For the next two examples we shall make use of several properties of Tsirelson's space T, its dual or its 2-convexification $T^{(2)}$; all may be found in [3]. We shall also use the result from [8] stating that T and $T^{(2)}$ are tight.

Recall that two bases (e_n) and (f_n) are equivalent when the map defined by $T(e_n) = f_n$ for all n extends to a linear isomorphism of the closed linear spans of (e_n) and (f_n). A basis is subsymmetric if it is unconditional and equivalent to all its subsequences, and symmetric when it is equivalent to all its permutations.

Lemma 5.3. *Let X be a space with a subsymmetric basis (e_n). Then* Is *is comeager.*

Proof. Assume (e_n) is subsymmetric. If $x = \sum_{i\in\mathrm{supp}(x)} x_i e_i$ and $y = \sum_{j\in\mathrm{supp}(y)} y_j e_j$ are finitely supported, we say that they have same distribution if there is an order preserving bijection σ between supp(x) and supp(y) such that $y_{\sigma(i)} = x_i$ for all i. Note that for vectors of Q_0, there are only countably many possible distributions, which we denote by $\{d_k, k \geq 1\}$. Let

$$A = \left\{(z_n)_n \in b(X) \,\middle|\, \forall k \geq 1,\ z_n \text{ has distribution } d_k \text{ for infinitely many } n'\text{s} \right\}.$$

We claim that A is comeager and contained in a Is-class in $b(X)$. Then it follows immediately that Is is comeager.

To prove the second part of the claim, note that if y, z belong to A, then one easily constructs by induction a subsequence $(z_{n_i})_i$ of z such that each z_{n_i} has the same distribution as y_i. By subsymmetry of the basis, it follows that the subsequence $(z_{n_i})_i$ is equivalent to y. Likewise y is equivalent to a subsequence of z. Since both are unconditional, it follows by the Schroeder-Bernstein property for unconditional sequences (first proved by Mityagin [15]) that z is equivalent to a permutation of y and therefore that $\overline{\mathrm{span}}\ y \simeq \overline{\mathrm{span}}\ z$. So A is contained in a single Is-class.

Finally to prove the first part of the claim, let $(\tilde{x}_n)_n$ be successive elements of $b^{<\omega}(X)$, such that each $\tilde{x}_n$ is a sequence of n vectors of respective distributions $d_1, d_2, \ldots, d_n$. Let C be the comeager set of all z in $b(X)$ which pass through infinitely many of the $\tilde{x}_n$'s. It is clear that any $z \in C$ contains infinitely many terms of distribution d_k for each $k \geq 1$. Therefore $C \subset A$ and A is comeager. □

Example 5.4. *A space without minimal subspaces, although* Is *is comeager.*

Proof. Let $X = S(T^{(2)})$, the symmetrization of the 2-complexification of Tsirelson's space. The canonical basis of X is symmetric, so Is(X) is comeager by Lemma 5.3. On the other hand, by [3] Notes and Remarks 7) a) p.118, every subspace Y of X contains an isomorphic copy of a subspace of $T^{(2)}$. Since $T^{(2)}$ is tight, it contains no minimal subspace, which implies that Y cannot be minimal. □

We note at this point that the spaces for which Is is comeager are those for which the existence of a continuum of non-isomorphic subspaces remains to be shown - and is still open in some simple cases, such as ℓ_p for $2 < p < +\infty$.

Conversely to Example 5.4, the relation Is may be meager even for minimal spaces:

Example 5.5. *A space which is minimal although* Is *is meager.*

Proof. We shall consider the space T^*, which is minimal by [3], and prove that Is is meager on $b(T^*)^2$. First we denote by (e_n) the canonical basis of T and by $\simeq$ the relation on 2^ω induced by isomorphism on T, i.e.

$$u \simeq v \Leftrightarrow \overline{\mathrm{span}}\,[e_i, i \in u] \simeq \overline{\mathrm{span}}\,[e_i, i \in v].$$

We observe that any $\simeq$-class on 2^ω is meager. Indeed if $u_0 \in 2^\omega$ and if Y_0 denotes $\overline{\text{span}}[e_n, n \in u_0]$, then

$$\{u \in 2^\omega : u_0 \simeq u\} \subseteq \{u \in 2^\omega : Y_0 \sqsubseteq \overline{\text{span}}[e_n, n \in u]\} = E_{Y_0},$$

which is meager because T is tight.

On the other hand, since the basis of T is unconditional and T is reflexive, we note that $\simeq$ is also the relation on 2^ω induced by isomorphism on T^*, i.e.

$$u \simeq v \Leftrightarrow \overline{\text{span}}\ [e_n^*, n \in u] \simeq \overline{\text{span}}\ [e_n^*, n \in v],$$

where (e_n^*) is the canonical basis of T^*. So we may relate $(2^\omega, \simeq)$ to $(b(T^*), \text{Is})$ as follows. Let $\phi : b(T^*) \to 2^\omega$ be defined by

$$\phi((z_n)_n) = \cup_n \min(\text{supp } z_n).$$

By the properties of T^* we have that the sequences (z_n) and $(e^*_{\min(\text{supp } z_n)})$ are equivalent and in particular span isomorphic subspaces of T^*. In other words the spaces $\overline{\text{span}}\ [e_i^*, i \in \phi(z)]$ and $\overline{\text{span}}\ z$ are isomorphic for each $z \in b(T^*)$, and therefore

$$(z, y) \in \text{Is} \Leftrightarrow \overline{\text{span}}\ z \simeq \overline{\text{span}}\ y \Leftrightarrow \phi(z) \simeq \phi(y)$$

for $z, y \in b(T^*)$. Now if A is any Is-class on $b(T^*)$, then $\phi(A)$ is contained in a single $\simeq$-class on 2^ω, and therefore is meager. The map ϕ is continuous, and for any basic open set $N_{z_0,\dots,z_n}$ of $b(X)$, $\overline{\phi(N_{z_0,\dots,z_n})}$ is a basic open set of 2^ω. It follows easily that $A = \phi^{-1}(\phi(A))$ is meager. So all Is-classes are meager in $b(T^*)$, and Kuratowski-Ulam theorem implies that Is is meager in $b(T^*)^2$. □

Finally if we note

$$\text{Emb}^* = \left\{(y, z) \in b(X)^2 : \ \overline{\text{span}}\ z \sqsubseteq \overline{\text{span}}\ y\right\},$$

then of course

$$\text{Be} = \text{Emb} \cap \text{Emb}^*.$$

Since Emb* is homeomorphic to Emb, it follows that Be is comeager if and only if Emb is comeager. The above example shows however that Is can be meager while Be is comeager - and even equal to $b(X)^2$.

6. Some open questions

This work is motivated by the crucial problem of estimating the complexity of the linear isomorphism relation $\simeq$ on the set $SB(X)$ of subspaces of a Banach space X. Gowers and Komorowski - Tomczak-Jaegermann' solution to Banach homogeneous space problem [12] asserts that if $X \not\simeq \ell_2$, then $SB(X)$ contains at least two classes, but it is not known if, for example, it necessarily contains infinitely many classes.

Following [6], we say that a separable Banach space X is *ergodic* if E_0 Borel reduces to $\simeq$ on $SB(X)$, i.e. if there is

$$f : 2^\omega \to SB(X)$$

a Borel map (when $SB(X)$ is equipped with the natural Effros Borel structure, see [2]), such that

$$uE_0v \Leftrightarrow f(u) \simeq f(v).$$

It is shown in [8] Th. 7.3 that every tight space has a strong E_0-antichain and thus is in particular ergodic. it is interesting to notice that spaces which are "close to ℓ_2" but not ℓ_2 are ergodic: indeed [1] weak Hilbert spaces and asymptotically hilbertian spaces non-isomorphic to ℓ_2 are ergodic. We recall that by Kuratowski

- Mycielski, any space X such that Is is meager in $b(X)^2$ contains a continuum of non-isomorphic subspaces. Actually a similar argument shows that if Is is meager then E_0 reduces to Is, and thefore X is ergodic (Proposition 7 of [6]).

The main conjecture, already stated in [6], is therefore:

Problem 6.1. *Let X be a separable Banach space which is not isomorphic to ℓ_2. Is X ergodic?*

A slightly weaker form of this conjecture would be to show that any $X \not\simeq \ell_2$ contains a continuum of non-isomorphic subspaces. This is not known for ℓ_p, $2 < p < \infty$, although it is known that ℓ_p contains uncountably many non-isomorphic subspaces. And since $b(\ell_p)$ consists of a single isomorphism class, one has to deal with the whole set $SB(\ell_p)$ of closed linear subspaces of ℓ_p.

By a theorem of Silver [16], every Borel - or more generally coanalytic - equivalence relation on a Polish space with uncountably many classes actually has a continuum of classes. To show this for $\simeq$ on $SB(\ell_p)$, it would therefore be sufficient to answer positively the following question:

Problem 6.2. *Is the isomorphism relation $\simeq$ a Borel subset of $SB(\ell_p)^2$ ($1 \leq p < +\infty, p \neq 2$)?*

Note that it is analytic in $SB(X)^2$ for any Banach space X, and it is known to be non-Borel if e.g. $X = \mathscr{C}(\Delta)$ ([2]). We conjecture - unfortunately - a negative answer to Problem 6.2.

Finally, in Example 5.5, it is known that T^* it is not "block-minimal", meaning that it is not true that it embeds as a block-subspace of all its block-subspaces. So arguably the minimality of T^* does not have much to do with the structure of the relation of isomorphism between block-subspaces of T^*. In this direction, the following remains open:

Problem 6.3. *Find a space X which embeds as a block-subspace of all its block-subspaces, but such that* Is *is meager.*

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