# Some results about the Schroeder-Bernstein Property for separable Banach spaces

by

Valentin Ferenczi and Elói Medina Galego

**Abstract.** We construct a continuum of mutually non-isomorphic separable Banach spaces which are complemented in each other. Consequently, the Schroeder-Bernstein Index of any of these spaces is $2^{\aleph_0}$. Our construction is based on a Banach space introduced by W. T. Gowers and B. Maurey in 1997. We also use classical descriptive set theory methods, as in some work of V. Ferenczi and C. Rosendal, to improve some results of P. G. Casazza and of N. J. Kalton on the Schroeder-Bernstein Property for spaces with an unconditional finite-dimensional Schauder decomposition. [1]

## 1. Introduction

Let $X$ and $Y$ be Banach spaces. We write $X \overset{c}{\hookrightarrow} Y$ if $X$ is isomorphic to a complemented subspace of $Y$, $X \sim Y$ if $X$ is isomorphic to $Y$ and $X \not\sim Y$ when $X$ is not isomorphic to $Y$. We also write $X \overset{c}{\sim} Y$ if both $X \overset{c}{\hookrightarrow} Y$ and $Y \overset{c}{\hookrightarrow} X$ hold. If $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, then $X^n$ denotes the sum of $n$ copies of $X$. The first infinite cardinal number will be indicated by $\aleph_0$. We shall write our proofs in the case of real Banach spaces, clearly the results hold in the complex setting as well.

The Schroeder-Bernstein Problem for Banach spaces asks whether isomorphism and complemented biembeddability must coincide for any pair of Banach spaces. In other words, if $X$ and $Y$ are Banach spaces such that $X \overset{c}{\sim} Y$, does it follow that $X \sim Y$? It was answered by the negative by Gowers [11], using separable spaces. Later on, Gowers and Maurey [13] provided other counterexamples to the Problem: in particular, they built a separable Banach space $X_1$, which is isomorphic to its cube but not to its square. So $X_1 \overset{c}{\sim} X_1^2$ while $X_1 \not\sim X_1^2$.

The answer by the negative given by Gowers to the Schroeder-Bernstein Problem opens two directions of research which are the guidelines of this paper.

### 1.1. The Schroeder-Bernstein Property for Banach spaces with unconditional Schauder decomposition.

The first direction of research is to ask what additional conditions ensure a positive answer to the Schroeder-Bernstein Problem. More precisely, a Banach space $X$ is said to have the Schroeder-Bernstein Property (in short, the SBP), if whenever a Banach space $Y$ satisfies $Y \overset{c}{\sim} X$, it follows that $Y \sim X$.

[1] 2000 *Mathematics Subject Classification.* Primary 46B03, 46B20.

*Key words and phrases:* complemented subspaces, Schroeder-Bernstein property.

We wish to find sufficient conditions on a Banach space to have the SBP. For example, according to the well known Pelczynski's Decomposition Method, a Banach space of the form $l_p(X)$ for $1 \leq p < +\infty$ has the SBP. For more information about Banach spaces having the SBP, see the survey of P. Casazza [2], and for more examples of Banach spaces failing the SBP, see [8].

We recall that a Banach space $X$ is said to have a *Schauder decomposition* $X = \sum_{n \in \mathbb{N}} E_n$, where $(E_n)_{n \in \mathbb{N}}$ is a sequence of closed subspaces of $X$, if every $x \in X$ can be written in a unique way as $x = \sum_{n \in \mathbb{N}} x_n$, with $x_n \in E_n$ for all $n$. It is *unconditional* if there exists a constant $C$ such that for all $x = \sum_{n \in \mathbb{N}} x_n$, and every subset $I$ of $\mathbb{N}$, we have that $\| \sum_{n \in I} x_n \| \leq C \|x\|$. A *finite-dimensional Schauder decomposition (or FDD)* is a Schauder decomposition $X = \sum_{n \in \mathbb{N}} E_n$ for which $E_n$ is finite-dimensional for all $n$. We shall use the classical abbreviation *UFDD* for an unconditional finite-dimensional Schauder decomposition.

Two important open problems about the *SBP* are to know whether every primary Banach space or every Banach space with an unconditional basis has the *SBP*. Recall that a Banach space $X$ is said to be *primary* if whenever $X = Y \oplus Z$, then $Y \sim X$ or $Z \sim X$. The following definitions introduced by N. J. Kalton [15] are the starting point of our research in this direction.

A Banach space $X$ has the *SBP restricted to spaces with UFDD*, if $X$ has an UFDD and whenever a Banach space $Y$ with UFDD satisfies $Y \overset{c}{\sim} X$, it follows that $Y \sim X$. A Banach space $X$ is said to be *countably primary* if there is a countable set $(S_n)_{n \in \omega}$ of Banach spaces such that whenever $X = A \oplus B$, then there exists $n \in \omega$ such that $A \sim S_n$ or $B \sim S_n$.

Kalton obtained various results about countably primary Banach spaces with an unconditional basis or an unconditional Schauder decomposition. In particular, a countably primary Banach space with an unconditional basis has the SBP restricted to spaces with UFDD, see Theorem 2.8 in [15].

In [6], most results of Kalton concerning Banach spaces with an unconditional basis were improved. The method was simplified, using classical results of descriptive set theory. Uniformity of the constants of isomorphism was obtained. The results were also extended to $\kappa$-primary for any $\kappa < 2^\omega$ (with the obvious definition).

In Section 2, after noting that a primary Banach space with a UFDD must have the SBP (Proposition 2.1), we show how to extend the methods of [6] to the case of spaces with an unconditional Schauder decomposition. In consequence, we improve in a similar way the results of Kalton about spaces with an unconditional Schauder decomposition, and our method is also more direct. One application related to SBP is that for $\kappa < 2^\omega$, a $\kappa$-primary Banach space with an unconditional basis has the SBP restricted to spaces with UFDD (Theorem 2.10). In fact, it is more generally true of Banach spaces with an unconditional basis which are not *perfectly decomposable* (roughly speaking, a Banach space $X$ is perfectly decomposable if there are perfectly many mutually non-isomorphic ways of decomposing $X$). We shall define precisely this topological notion in

Section 2.

The new uniformity result we get allows us to improve on some work of Casazza concerning primary spaces: we prove that an $l_p$-sum of finite-dimensional spaces, which is $\kappa$-primary for some $\kappa < 2^\omega$ (or more generally, which is not perfectly decomposable) must have the SBP (Corollary 2.12).

### 1.2. The Schroeder-Bernstein Index for Banach spaces.

The second direction of research about the SBP is the following. Given the existence of non trivial families of Banach spaces which are mutually non isomorphic yet complementably biembeddable in each other, we wish to know what are the possible structures for these families, for example in terms of cardinality. This question was formalized by the definition of the Schroeder-Bernstein index $SBi(X)$ of a Banach space $X$ in [9]. Here, we shall use a modified Schroeder-Bernstein Index $SBI(X)$ given by Definition 1.1 below. This definition is simpler and more natural than the one of $SBi(X)$. Both indexes are cardinal numbers, and denoting by $\alpha^+$ the successor cardinal of any cardinal $\alpha$, it is direct to check that they are related by $SBi(X) = SBI(X)^+$.

**Definition 1.1.** Let $X$ be a Banach space. Let $CB(X)$ be the set of subspaces $Y$ of $X$ such that $Y \overset{c}{\sim} X$. Let $\overline{CB}(X) = CB(X)/\sim$ be the set of isomorphism classes of elements of $CB(X)$. Then

$$SBI(X) := |\overline{CB}(X)|.$$

More simply said, $SBI(X)$ is the number of $\sim$-classes on the $\overset{c}{\sim}$-class of $X$.

Observe that for any Banach space $X$, $SBI(X) = 1$ if and only if $X$ has the SBP. Also, $SBI(X) \leq 2^{dens(X)}$, where $dens(X)$ denotes the density character of $X$. Indeed there is a basis of open sets for the topology of $X$, of cardinality $dens(X)$. So there are no more than $2^{dens(X)}$ open subsets of $X$, and in particular no more than $2^{dens(X)}$ (isomorphism classes of) closed subspaces of $X$.

Our goal is to see what are the possible values of the Schroeder-Bernstein Index for a Banach space.

Clearly the counterexample $X_G$ of Gowers in [11] satisfies $SBI(X_G) > 1$. In 1997, Gowers and Maurey ([13], page 559) constructed, for each $p \in \mathbb{N}$, $p \geq 2$, a Banach space $X_1(\mathcal{S}_p)$ (this notation will be explicited in Section 4), which is isomorphic to its subspaces of codimension $n$ if and only if $p$ divides $n$. Consequently $SBI(X_1(\mathcal{S}_p)) \geq p$, consider the family of spaces $(X_1(\mathcal{S}_p) \oplus \mathbb{R}^n)_{0 \leq n \leq p-1}$. In fact, using the properties of $X_1(\mathcal{S}_p)$ mentioned in the remark in [13] after Theorem 19, it is not difficult to prove that $SBI(X_1(\mathcal{S}_p)) = p$.

Recently the second author [9] found a Banach space $X_E$ such that $X_E^2$ is isomorphic to a complemented subspace of $X_E$, but $X_E^m$ is not isomorphic to $X_E^n$, for every $m \neq n$. Hence $(X_E^n)_{n \in \mathbb{N}}$ is an infinite sequence of mutually non-isomorphic Banach spaces which are complemented in each other. Thus $SBI(X_E) \geq \aleph_0$.

The main aim of Section 3, and the main result of this paper, is to provide a family of cardinality the continuum of mutually non-isomorphic separable Banach spaces which are complemented in each other (Theorem 3.9). In particular, if $X$ is any member of this family then $SBI(X) = 2^{\aleph_0}$. The construction of such a family is inspired by the construction by Gowers and Maurey of the Banach space $X_1$ isomorphic to its cube but not to its square ([13], Section 4.4).

In Section 4, we note some open problems about the SBP as well as some side consequences of some of our techniques. For example, we show that failing the SBP is not a three-space property (Proposition 4.3).

Finally, we end with an appendix which contains the proof of two technical lemmas needed in Section 3.

## 2. On the SBP restricted to Banach spaces with unconditional finite-dimensional Schauder decomposition.

We start by noting a direct and interesting consequence of Kalton's results.

**Proposition 2.1.** *Let $X$ be a primary Banach space with a UFDD. Then $X$ has the SBP.*

**Proof.** Being primary, $X$ must be isomorphic to its hyperplanes. By Theorem 2.3 from [15], for some $N \in \mathbb{N}$, $X \sim X \oplus \sum_{n \geq N} E_n$, where the $E_n$'s are the summands of the UFDD of $X$. From this and from the fact that each $E_n$ is finite-dimensional, we deduce that $X$ must be isomorphic to its square.

But a primary space isomorphic to its square is easily seen to have the SBP. Indeed, assume $Y \overset{c}{\sim} X$. There exists $Z$ such that $Y \sim X \oplus Z$, so $Y \sim X \oplus Z \sim X \oplus X \oplus Z \sim X \oplus Y$. On the other hand $Y$ embeds complementably in $X$, so by the primariness of $X$, either $Y \sim X$, and we are done, or $X \sim X \oplus Y \sim Y$. ■

We now turn to our generalization of Kalton's result about the SBP restricted to spaces with UFDD, for countably primary Banach spaces. We shall use classical results and definitions from descriptive set theory, and our reference for these will be the book of Kechris [16].

Let $X$ be a separable Banach space with a Schauder decomposition $\sum_{n=1}^{+\infty} E_n$. We start as in [15] or [6] by assigning to each element $\alpha$ of $2^\omega$ a subspace $X(\alpha)$ of $X$ in the obvious way:

$$X(\alpha) = \sum_{\alpha(n)=1} E_n.$$

The relation of isomorphism between spaces of the form $X(\alpha)$, $\alpha \in 2^\omega$, induces a relation on $2^\omega$ that we shall denote by $\simeq$, and it is not difficult to check that it is analytic.

We first give some definitions. For a cardinal $\kappa \leq 2^\omega$, we say that a Schauder decomposition is *$\kappa$-primary* if there is a set $\mathcal{S}$ of Banach spaces, of cardinality $\kappa$, such that for every subset $I$ of $\mathbb{N}$, there exists $S \in \mathcal{S}$ such that $\sum_{n\in I} E_n \sim S$ or $\sum_{n\in\mathbb{N}\setminus I} E_n \sim S$. A Banach space $X$ is *$\kappa$-primary* if if there is a set $\mathcal{S}$ of Banach spaces, of cardinality $\kappa$, such that whenever $X \sim A \oplus B$, there exists $S \in \mathcal{S}$ such that $A \sim S$ or $B \sim S$. Evidently an unconditional Schauder decomposition of a $\kappa$-primary Banach space is $\kappa$-primary.

We shall say that a Schauder decomposition is *perfect* if there is a perfect subset $P$ of $2^\omega$ such that, for all $\alpha, \beta \in P$ with $\alpha \neq \beta$,

$$\sum_{\alpha(n)=1} E_n \not\sim \sum_{\beta(n)=1} E_n,$$

$$\sum_{\alpha(n)=0} E_n \not\sim \sum_{\beta(n)=1} E_n,$$

$$\sum_{\alpha(n)=0} E_n \not\sim \sum_{\beta(n)=0} E_n.$$

In particular, note that a Schauder decomposition which is $\kappa$-primary for some $\kappa < 2^\omega$, is not perfect.

For our last definition, we need to recall that the set of separable Banach spaces, seen as subspaces of an isometrically universal separable Banach space such as $C([0,1])$, or more generally, the set of subspaces of a given separable Banach space $X$, may be equipped naturally with a Borel structure called the Effros-Borel structure (see e.g. [6]). In this setting we may talk about Borel or analytic sets of separable Banach spaces, or of subspaces of a given separable Banach space $X$; note that any uncountable Borel set of Banach spaces is necessarily of cardinality $2^\omega$.

For a Banach space $X$, we call *decomposition of* $X$ a pair $(A_0, A_1)$ of subspaces of $X$ such that $X = A_0 \oplus A_1$. We say that a separable Banach space $X$ is *perfectly decomposable* if there is a Borel set $\{(A^0_\alpha, A^1_\alpha), \alpha \in 2^\omega\}$ of decompositions of $X$, such that for $\alpha \neq \beta$ and any $(\epsilon, \gamma)$ in $\{0,1\}^2$, $A^\epsilon_\alpha \not\sim A^\gamma_\beta$. So a Banach space which is $\kappa$-primary for some $\kappa < 2^\omega$, is not perfectly decomposable.

Finally and evidently, if a separable Banach space has a perfect unconditional Schauder decomposition then it is perfectly decomposable.

We need to recall two theorems from descriptive set theory (Theorems 19.1 and 8.41 from [16]).

**Theorem 2.2 (Kuratowski-Mycielski)** *Let $E$ be a perfect Polish space, and $R$ be a relation on $E$ which is meager in $X^2$. Then there exists a homeomorphic copy $C$ of the Cantor space such that $\forall x, y \in C$ with $x \neq y$, we have $x \neg R y$.*

**Theorem 2.3 (Kuratowski-Ulam)** *Let $E$ be a Polish space and $D$ be a subset of $E^2$ having the Baire property. Then $D$ is nonmeager if and only if*

$$\exists^* x \exists^* y : (x, y) \in D.$$

Here $\exists^* x P(x)$ signifies the existence of a nonmeager set of $x$ such that $P(x)$.

We are now ready to prove a proposition in the spirit of [6].

**Proposition 2.4.** *Let $X$ be a separable Banach space with a Schauder decomposition $X = \sum_{n=1}^{+\infty} E_n$ which is not perfect. Then there exists an $\simeq$-class which is non-meager in $2^\omega$.*

**Proof.** For $\alpha \in 2^\omega$, we shall denote by $\mathcal{C}\alpha$ the element $(1-\alpha(n))_{n\in\omega}$ of $2^\omega$. As in [6] we define the relations $\simeq_1$ and $\simeq_2$ on $2^\omega$ by

$$\alpha \simeq_1 \beta \Leftrightarrow \alpha \simeq \mathcal{C}\beta,$$

$$\alpha \simeq_2 \beta \Leftrightarrow \mathcal{C}\alpha \simeq \mathcal{C}\beta.$$

The first case in our proof is to assume that $\simeq$, $\simeq_1$ and $\simeq_2$ are meager. Then their union is meager as well. We then apply Theorem 2.2. to get a perfect set $P$ avoiding this union, i.e. with the property stated in the definition of perfect Schauder decompositions.

In the second case, assume for example that $\simeq_2$ is non-meager. Being analytic, $\simeq_2$ has the Baire property, so by Theorem 2.3., we may find an element $\alpha \in 2^\omega$ such that, for $\beta$ in a non-meager subset of $2^\omega$, $\mathcal{C}\alpha \simeq \mathcal{C}\beta$. As clearly, the map sending $\beta$ to $\mathcal{C}\beta$ is an homeomorphism on $2^\omega$, we deduce that the $\simeq$-class of $\mathcal{C}\alpha$ is non-meager. A similar proof holds if $\simeq$ or $\simeq_1$ is non-meager. ∎

**Remark 2.5.** In [6], it was shown that in the case of Banach space with a 1-dimensional Schauder decomposition (i.e. with a Schauder basis), a non-meager $\simeq$ class in $2^\omega$ must be comeager. This used the fact that modifying a finite number of vectors of the basis of a Banach space preserves the isomorphism class. We cannot use this fact in the general case of a Schauder decomposition, and in fact the result is false in that case: consider $X = l_1 \oplus (\sum l_2)_{l_2}$. We see that the $\simeq$-class corresponding to $l_2$ is non-meager but not comeager.

We now deduce from Proposition 2.4 the following extension of Theorem 3.4 from [15]. For $X$ and $Y$ Banach spaces, and $K \geq 1$, $X \sim^K Y$ means that $X$ is $K$-isomorphic to $Y$.

**Theorem 2.6.** *Let $X$ be a separable Banach space with an unconditional Schauder decomposition $X = \sum_{n=1}^{+\infty} E_n$ which is not perfect. Then, there exists an integer $N$ and a constant $K$ such that, for every subset $I$ of $[N,\infty)$, $X \sim^K X \oplus (\sum_{n\in I} E_n.)$*

**Proof.** By Proposition 2.4, we assume there is a non-meager $\simeq$-class and we intend to find $K$ and $N$ such that for every subset $I$ of $[N,\infty)$, $X \sim^K X \oplus \sum_{n\in I} E_n$. Let $X(\alpha_0)$ for some $\alpha_0$ be a Banach space in the isomorphism class associated to the non-meager $\simeq$-class. For $n \in \mathbb{N}$, let $\mathcal{A}_n$ be the set of $\alpha$'s such that $X(\alpha) \sim^n X(\alpha_0)$. Then for some $K \in \mathbb{N}$, $\mathcal{A} = \mathcal{A}_K$ is non-meager.

The set $\mathcal{A}$ is analytic, thus has the Baire property, and we deduce that $\mathcal{A}$ is comeager in some basic open set $N(u)$. Here $u = (u(1), \ldots, u(k))$ denotes an element of $2^{<\omega}$. In other words the set $\mathcal{C}$ of $\alpha \in 2^\omega$ such that the concatenation $u^\frown\alpha \in \mathcal{A}$ is comeager in $2^\omega$. We now apply the following classical characterization of comeager subsets of $2^\omega$ which was already used (and proved) in [6]. As $\mathcal{C}$ is comeager in $2^\omega$, then there exists a partition of $\mathbb{N}$ in infinite subsets $M_1$ and $M_2$, subsets $N_1 \subset M_1$ and $N_2 \subset M_2$, such that: for $i = 1, 2$, an element $\alpha$ of $2^\omega$ is in $\mathcal{C}$ whenever for every $n \in M_i$, $\alpha(n) = 1$ if $n \in N_i$ and $\alpha(n) = 0$ if $n \notin N_i$.

Going back to $\mathcal{A}$, we get a partition of $[k+1, +\infty)$ in infinite subsets $A_1$ and $A_2$, subsets $B_1 \subset A_1$ and $B_2 \subset A_2$, such that: for $i = 1, 2$, an element $\alpha$ of $2^\omega$ is in $\mathcal{A}$ whenever

(a) for every $n = 1, \ldots, k$, $\alpha(i) = u(i)$, and

(b) for every $n \in A_i$, $\alpha(n) = 1$ if $n \in B_i$ and $\alpha(n) = 0$ if $n \notin B_i$.

Now let $N = k + 1$, let $I$ be any subset of $[N, \infty)$, and let $Y = \sum_{i \in I} \oplus E_i$.

We let $C_1 = A_1 \cap I$, $C_2 = A_2 \cap I$. The element $\alpha_1$ defined by $\alpha_1(i) = u(i)$, for $i < N$, and $\alpha_1(i) = 1$ if and only if $i \in C_1 \cup B_2$, for $i \geq N$, satisfies (a) and (b), so belongs to $\mathcal{A}$. Denoting by $X(u)$ the finite sum $\sum_{u(i)=1} E_i$, we deduce that

$$X(\alpha_0) \sim X(u) \oplus (\sum_{i \in C_1 \cup B_2} E_i).$$

By unconditionality of the basis,

$$X(\alpha_0) \sim X(u) \oplus (\sum_{i \in B_2} E_i) \oplus (\sum_{i \in C_1} E_i).$$

By the characterization of $\mathcal{A}$ again,

$$X(\alpha_0) \sim X(\alpha_0) \oplus (\sum_{i \in C_1} E_i).$$

Likewise

$$X(\alpha_0) \sim X(\alpha_0) \oplus (\sum_{i \in C_2} E_i).$$

Combining the two, and noting that $C_1$, $C_2$ form a partition of $I$, we get

$$X(\alpha_0) \sim X(\alpha_0) \oplus Y.$$

As $X(\alpha_0)$ is complemented in $X$, we deduce

$$X \sim X \oplus Y.$$

Finally, a look at the proof shows that we can get this isomorphism with an uniform constant depending on $K$ and the constant of unconditionality of the Schauder decomposition. ∎

Concerning Theorem 2.6, note that a stronger conclusion of the form $X \sim X^2$ (as in the case of a space with an unconditional basis) is false: consider a decomposition $X = X_0 \oplus (\sum l_2)_{l_2}$, where $X_0$ is the hereditarily indecomposable Banach space of Gowers and Maurey [12]. There are only four classes of isomorphism of subspaces $X(\alpha)$ (including the classes of $\{0\}$ and $X_0$), but it is not difficult to show that $X$ is not isomorphic to its square.

However, as an immediate consequence of Theorem 2.6, we have:

**Corollary 2.7.** *Suppose that the separable Banach space $X$ has Schauder unconditional decomposition which is not perfect, and assume also that every summand is isomorphic to its square. Then $X$ is isomorphic to its square.*

The next theorem is a variation of Theorem 2.6. We recall that $E_0$ is the equivalence relation defined on $2^\omega$ by $\alpha E_0 \beta$ if and only if $\exists m : \forall n \geq m, \alpha(n) = \beta(n)$. It is the $\leq_B$-lowest Borel equivalence relation above equality on $2^\omega$. For the definition of $\leq_B$: given an equivalence relation $R$ on a Polish space $E$, (resp. $R'$ on $E'$), we say that $(E, R)$ is Borel reducible to $(E', R')$, and write $(E, R) \leq_B (E', R')$, if there is a Borel map $f : E \to E'$, such that for all $x, y$ in $E$, $xRy$ if and only if $f(x)R'f(y)$. We refer to [7] for more about the notion of the relation $\leq_B$ of Borel reducibility between equivalence relations. When a relation reduces $E_0$, in particular there is a perfect set of mutually non-related points.

In [7], a Banach space $X$ was defined to be *ergodic* if the relation $E_0$ is Borel reducible to isomorphism between subspaces of $X$. It was proved by Ferenczi and Rosendal ([6],[18]) that a Banach space $X$ with an unconditional basis which is not ergodic, must be isomorphic to its square, to its hyperplanes and more generally to $X \oplus Y$ for any subspace $Y$ generated by a finite or infinite subsequence of the basis.

We can prove a result in the same spirit for Banach spaces with a UFDD, provided we (unessentially) relax the ergodic assumption. We recall that Banach spaces $X$ and $Y$ are *nearly isomorphic*, and write $X \overset{f}{\sim} Y$, if some finite-codimensional subspace of $X$ is isomorphic to a finite-codimensional subspace of $Y$. We shall say that a Banach space is *nearly ergodic* if $E_0$ is Borel reducible to $\overset{f}{\sim}$ between subspaces of $X$. The next proposition shows that when a Banach space is ergodic, then it is nearly ergodic. Note that both imply that there is a perfect set of mutually non (nearly) isomorphic subspaces.

**Proposition 2.8.** *Let $X$ be an ergodic Banach space. Then it is nearly ergodic.*

**Proof.** By definition there exists a Borel map $g$ from $2^\omega$ into the set of subspaces of $X$, such that $\alpha E_0 \beta$ if and only if $g(\alpha) \sim^f g(\beta)$. Denote by $E_f$ the relation defined on $2^\omega$ by $\alpha E_f \beta$ if and only if $g(\alpha) \overset{c}{\sim} g(\beta)$. An $E_f$-class is a countable union of $E_0$-classes and thus is meager. As every $E_f$-class is meager, then by Theorem 2.3, $E_f$ is meager in $(2^\omega)^2$. Every $E_f$-class is also invariant by $E_0$, so by Proposition 14 of [18], the relation $E_0$ is Borel reducible to $E_f$. Combining the map reducing $E_0$ to $E_f$ with $g$, we get a Borel reduction of $E_0$

to $\stackrel{f}{\sim}$ on the set of subspaces of $X$. ■

**Theorem 2.9.** *Let $X$ be a separable Banach space with an unconditional Schauder decomposition $X = \sum_{n=1}^{+\infty} E_n$ such that $E_n$ is of finite dimension for $n > N$, for some $N$. Assume $X$ is not nearly ergodic. Then there exists an integer $k$ and a constant $K$ such that, for every subset $I$ of $[k, \infty)$, $X \sim^K X \oplus (\sum_{n \in I} E_n.)$*

**Proof.** Let $\stackrel{f}{\simeq}$ be the relation induced on $2^\omega$ by near isomorphism between spaces of the form $X(\alpha)$, $\alpha \in 2^\omega$.

First assume $\stackrel{f}{\simeq}$ is meager. Let $1_N$ be the length $N$ sequence $(1, \ldots, 1)$. Then the relation $r$, defined on $2^\omega$ by

$$\alpha r \beta \Leftrightarrow 1_N^\frown \alpha \stackrel{f}{\simeq} 1_N^\frown \beta,$$

is also meager in $(2^\omega)^2$. Furthermore, because $E_n$ is of finite dimension for $n > N$, $r$ is clearly invariant by a finite change of coordinates of the sequences $\alpha$ and $\beta$. According to Proposition 14 of [18], we deduce that $E_0$ is Borel reducible (by some $g$) to $r$. The map $f$ from $2^\omega$ into $2^\omega$, defined by $f(\alpha) = 1_N^\frown g(\alpha)$, is then a Borel reduction of $E_0$ to $\stackrel{f}{\simeq}$, that is, $X$ is nearly ergodic, a contradiction.

So assume $\stackrel{f}{\simeq}$ is non meager. Using Theorem 2.3 as before, we deduce that some $\stackrel{f}{\simeq}$ class is non-meager. This class is a countable union of $\simeq$-classes, so we deduce that some $\simeq$ class is non-meager. We may then proceed as in Theorem 2.6. ■

The result of Kalton about the SBP restricted to spaces with UFDD, for countably primary Banach spaces, mentioned in the introduction, now generalizes to Banach spaces which are either not perfectly decomposable or not nearly ergodic.

**Theorem 2.10.** *Let $X$ be a Banach space with an unconditional basis. Assume $X$ is not perfectly decomposable or not nearly ergodic. Then $X$ has the SBP restricted to spaces with UFDD.*

**Proof.** By Theorem 2.6 or Theorem 2.9, there exists $N \in \mathbb{N}$ such that for every $I \subset [N, +\infty)$, $X \sim X \oplus \sum_{i \in I} \mathbb{R} e_i$. Taking $I = \{N\}$ we see that $X$ is isomorphic to its hyperplanes. Taking $I = [N, +\infty)$ it follows that $X$ is isomorphic to its square.

Now let $Y$ be a Banach space with an UFDD $(E_n)_{n \in \mathbb{N}}$ such that $Y \stackrel{c}{\sim} X$ . Then there exists a space $W$ such that $Y \sim X \oplus W$ and we deduce

$$Y \sim X \oplus W \sim X \oplus X \oplus W \sim X \oplus Y.$$

Also there exists some space $Z$ such that

$$X \sim Z \oplus Y \sim Z \oplus (\sum_{n \in \mathbb{N}} E_n).$$

This UFDD satisfies the hypotheses of Theorem 2.6 or Theorem 2.9. We deduce that for some $K \in \mathbb{N}$, $X \sim X \oplus \sum_{n \geq K} E_n$. As the decomposition is finite-dimensional and $X$ is isomorphic to its hyperplanes, it follows that

$$X \sim X \oplus Y. \quad \blacksquare$$

In [1], Casazza proved that if a Banach space $X$ is an $l_p$-sum of finite-dimensional spaces and is primary, then $X \sim l_p(X)$, and thus $X$ has the SBP. The following Corollary 2.12 extend this result to $l_p$-sums of finite-dimensional spaces which are not perfectly decomposable or not ergodic. We point out here that to prove Corollary 2.12, we need uniformity in the result of Proposition 2.11. This uniformity is one of the results we get which was not proved in the paper of Kalton.

**Proposition 2.11.** *Let $X$ be a Banach space with a UFDD $X = \sum_{n=1}^{+\infty} E_n$. Assume this UFDD is not perfect or the relation $E_0$ is not Borel reducible to near isomorphism between subspaces of $X$ of the form $X(\alpha) = \sum_{\alpha(n)=1} E_n$, $\alpha \in 2^\omega$ (for example $X$ could be non nearly ergodic). Then there exists $N \in \mathbb{N}$ and an infinite sequence of disjoint subsets $(B_k)_{k \in \mathbb{N}}$ of $[N, +\infty)$ such that if $Y = \sum_{n=N}^{+\infty} E_n$, then $Y \sim \sum_{k \in \mathbb{N}}(\sum_{i \in B_k} E_i)$, with $\sum_{i \in B_k} E_i$ uniformly isomorphic to $Y$.*

**Proof.** By Proposition 2.4 or the proof of Theorem 2.9, we know that some $\simeq$-class is non-meager. We start as in the proof of Theorem 2.6, and using the same notation. Let $\alpha_0$ be in some fixed non-meager $\simeq$-class, we may find $K$ such that the set $\mathcal{A}$ of $\alpha$'s such that $X(\alpha) \sim^K X(\alpha_0)$ is non-meager, and thus comeager in some $N(u)$, $u \in 2^\omega$. We note that, by adding a finite sum of spaces $E_i$, and up to modifying the constant $K$ and $\alpha_0$, we may assume that $u(i) = 1$ for all $i \leq |u|$ and also that $\alpha_0 \in N(u)$.

As a preliminary result, let us prove that the non-meager $\simeq$-class we get corresponds to the isomorphism class of $X$. We let $X(u) = \sum_{i \leq |u|} E_i$. Let $N = |u| + 1$. We apply the same proof as in Theorem 2.6 to get, for all $Y = \sum_{i \in I} E_i$, $I \subset [N, +\infty)$:

$$X(\alpha_0) \sim X(\alpha_0) \oplus Y.$$

In particular,

$$X(\alpha_0) \sim X(\alpha_0) \oplus \sum_{i \geq N} E_i,$$

and thus

$$X(u) \oplus X(\alpha_0) \sim X(\alpha_0) \oplus X.$$

On the other hand, we also have for all $Y = \sum_{i \in I} E_i$, $I \subset [N, +\infty)$:

$$X \sim X \oplus Y.$$

Choose $Y$ such that $X(\alpha_0) = X(u) \oplus Y$, then

$$X \oplus X(u) \sim X \oplus X(\alpha_0).$$

Finally, we deduce that $X(u) \oplus X(\alpha_0) \sim X(u) \oplus X$, and as $X(u)$ is finite-dimensional, that $X(\alpha_0) \sim X$.

So, modifying $K$ if necessary, we may assume that $\alpha_0(i) = 1$, $\forall i$, that is $X(\alpha_0) = X$. Now we prove the result about the decomposition. We note that the characterization of comeager subsets of $2^\omega$ in terms of partitions of $\mathbb{N}$, that we used in Theorem 2.6, can be generalized to an infinite partition (see [6] about this). So from the fact that the set of $\alpha$'s such that $X \sim^K X(\alpha)$ is comeager in $N(u)$, we get a sequence $(B_n)_{n\in\mathbb{N}}$ of disjoint subsets of $[N, +\infty)$ such that:

(a) $X \sim^K X(u) \oplus \sum_{i\in B_k} E_i$, for all $k \in \mathbb{N}$, and

(b) $X \sim X(u) \oplus \sum_{k\in\mathbb{N}}(\sum_{i\in B_k} E_i)$.

Let $Y = \sum_{n=N}^{+\infty} E_n$. From (b), we have that

$$X(u) \oplus Y = X \sim X(u) \oplus \sum_{k\in\mathbb{N}}(\sum_{i\in B_k} E_i),$$

and thus as $X(u)$ is finite-dimensional,

$$Y \sim \sum_{k\in\mathbb{N}}(\sum_{i\in B_k} E_i).$$

From (a), we get

$$X(u) \oplus Y = X \sim^K X(u) \oplus (\sum_{i\in B_k} E_i), \forall k \in \mathbb{N},$$

and so,

$$Y \sim (\sum_{i\in B_k} E_i), \forall k \in \mathbb{N},$$

where the constant of isomorphism depends only on $K$ and on $\dim X(u)$. ■

**Corollary 2.12.** *Let $Z = c_0$ or $l_p$ for $1 \leq p < +\infty$. Let $X = (\sum_{n\in\mathbb{N}} E_n)_Z$, where for each $n$, $E_n$ is finite-dimensional. Assume this UFDD is not perfect or $X$ is not ergodic. Then $X \sim (\sum X)_Z$. So by Pelczynski's Decomposition Method, $X$ has the SBP.*

**Proof.** Let $X$ be as in the hypotheses. First note that any $l_p$-sum, and in particular $X$ or any subspace of $X$ of the form $X(\alpha), \alpha \in 2^\omega$, contains a complemented copy of $l_p$ and so, is isomorphic to its hyperplanes. In particular, two subspaces $X(\alpha)$ and $X(\beta)$, $\alpha, \beta \in 2^\omega$, are nearly isomorphic if and only if they are isomorphic, and so the relation $E_0$ is Borel reducible to near isomorphism between these subspaces if and only if it is Borel reducible to isomorphism between them. This implies that the hypotheses of Proposition 2.11 are satisfied.

So the conclusion of Proposition 2.11 holds, and we note that the copies are uniform and that the infinite direct sum is, in this case, an $l_p$-sum. So for some $N \in \mathbb{N}$, $Y = \sum_{n\geq N} E_n$ satisfies $Y \sim l_p(Y)$. As $X$ is isomorphic to its hyperplanes, it follows that $X \sim Y$, and so $X \sim l_p(X)$. ■

## 3. A continuum of mutually non-isomorphic Banach spaces which are complemented in each other.

We now turn to the main theorem of this paper, which provides a family of cardinality the continuum of mutually non-isomorphic separable Banach spaces which are complemented in each other (Theorem 3.9). In order to present this family of Banach spaces, we need to fix some notation and background from [13].

### 3.1. Preliminaries.

Let $c_{00}$ be the vector space of all complex sequences which are eventually 0. Let $(e_n)_{n\in\mathbb{N}}$ be the standard basis of $c_{00}$. Given a vector $a = \sum a_n e_n$ its *support*, denoted supp(a), is the set of $n$ such that $a_n \neq 0$. Given subsets $E$, $F$ of $\mathbb{N}$, we say that $E < F$ if every element of $E$ is less than every element of $F$. If $x$, $y \in c_{00}$, we say that $x < y$ if $\text{supp}(x) < \text{supp}(y)$. If $x_1 < x_2 < \cdots < x_n$, then we say that the vectors $x_1$, $x_2$, $\cdots$ $x_n$ are *successive*. An infinite sequence of sucessive non-zero vectors is also called a *block basis* and a subspace generated by a block basis is a *block subspace*.

Given a subset $E$ of $\mathbb{N}$ and a vector $a$ as above, we write $Ea$ for the vector $\sum_{n\in E} a_n e_n$. An interval of integers is a set of the form $\{m, m+1, \cdots, n\}$, where $m, n \in \mathbb{N}$. The *range* of a vector $x$, written $ran(x)$, is the smallest interval containing supp(x).

The following set of functions was first defined by Schlumprecht in [19] except for condition (vi) which was added in [13]. $\mathcal{F}$ denotes the set of functions $f : [1, \infty) \to [1, \infty)$ satisfying the following conditions.

(i) $f(1) = 1$ and $f(x) < x$ for every $x > 1$;
(ii) $f$ is strictly increasing and tends to infinity;
(iii) $\lim_{x\to\infty} x^{-q} f(x) = 0$ for every $q > 0$;
(iv) the function $x/f(x)$ is concave and non-decreasing;
(v) $f(xy) \leq f(x)f(y)$ for every $x, y \geq 1$;
(vi) the right derivate of $f$ at 1 is positive.

Let $\mathcal{X}$ stand for the set of normed spaces $(c_{00}, \|.\|)$ such that the sequence $(e_n)_{n\in\mathbb{N}}$ is a normalized bimonotone basis, this means that $\|Ex\| \leq \|x\|$ for every vector $x \in c_{00}$ and every interval $E$. Given $X \in \mathcal{X}$ and $f \in \mathcal{F}$, it is said that $X$ *satisfies a lower f-estimate* if, given any vector $x \in X$ and any sequence of intervals $E_1 < E_2 < \cdots < E_n$, we have $\|x\| \geq f(n)^{-1} \sum_{i=1}^{n} \|E_i x\|$.

Given two infinite subsets $A$ and $B$ of $\mathbb{N}$, Gowers and Maurey define the *spread from $A$ to $B$* to be the map $S_{A,B} : c_{00} \to c_{00}$ defined as follows. Let $\rho$ be the order-preserving bijection from $A$ to $B$, then $S_{A,B}(e_n) = e_{\rho(n)}$ when $n \in \mathbb{N}$, and $S_{A,B}(e_n) = 0$ otherwise.

Given any set $\mathcal{S}$ of spreads, they say that it is a *proper set* it it is closed under composition and taking adjoints and if for every $(i, j) \neq (k, l)$, there are only finitely many spreads $S \in \mathcal{S}$ for which $e_i^*(Se_j) \neq 0$ and $e_k^*(Se_l) \neq 0$.

For every Banach space $X$ satisfying a lower $f$-estimate for some $f \in \mathcal{F}$, and for every subspace $Y$ of $X$ generated by a block basis, they define a seminorm $|||.|||$ on the set $L(Y,X)$ of linear mappings from $Y$ to $X$ as follows. For $X \in \mathcal{X}$, and every integer $N \geq 1$, consider the equivalent norm on $X$ defined by

$$\|x\|_{(N)} = \sup \sum_{i=1}^{N} \|E_i x\|,$$

where the supremum is extended to all sequences $E_1, E_2, \cdots, E_N$ of successive intervals. Let $\mathcal{L}(Y)$ be the set of sequences $(x_n)_{n\in\mathbb{N}}$ of successive vectors in $Y$ such that $\|x_n\|_{(n)} \leq 1$, for every $n \in \mathbb{N}$. Now let

$$|||T||| = \sup_{x\in\mathcal{L}(Y)} \lim\ \sup_n \|T(x_n)\|.$$

Finally, we also recall that two Banach spaces $X$ and $Y$ are said to be totally incomparable if no infinite dimensional subspace of $X$ is isomorphic to a subspace of $Y$.

### 3.2. The main result.

The principal purpose of this section is to prove Theorem 3.9. Before that, we need some auxiliary results which are similar to the ones involved in the construction by Gowers and Maurey of the Banach space $X_1$ isomorphic to its cube but not to its square. We shall improve these results in two directions. First we shall need to have a family of spaces $X_r$ constructed on the model of $X_1$, for $1/2 \leq r \leq 1$, and we shall take care that for $r \neq r'$, $X_r$ and $X_{r'}$ are totally incomparable spaces. For this we shall have to write new versions of some technical lemmas in [13]. Then, we shall need to know that each $X_r$ is not nearly isomorphic to its square (instead of just non-isomorphic). This requires a little bit of extra care in the proofs as well.

The proof of Lemma 3.1 is implicit in [13], page 864, and in the proof of [13], Lemma 9, in the case $f(x) = \log_2(x+1)$.

**Lemma 3.1.** *Let $f \in \mathcal{F}$ and $J \subset \mathbb{N}$ be such that, if $m, n \in \mathbb{N}$, $m < n$, then log log log $n \geq 4m^2$. Write $J$ in increasing order as $\{j_1, j_2, \cdots\}$ and let $K = \{j_1, j_3, j_5, \cdots\}$. Suppose that*

(a) $f^{1/2} \in \mathcal{F}$;
(b) $f(j_1) > 256$;
(c) *exp exp* $j_n < f^{-1}(f(j_m))^{1/2},\ \forall m, n \in \mathbb{N}, m < n$;
(d) $16 f(x^{1/2}) \geq f(x),\ \forall x \geq 0$;
(e) $f(j_n)^{3/2} > f(j_n^3),\ \forall n \in \mathbb{N}$;
(f) *For every $N \in J \setminus K$ and $x_0$ in the interval* [log N, exp N], *the function given by the tangent to $(x \to x/f(x))$ at $x_0$ is at least $x/f^{1/2}(x)$ for all positive $x$ outside the interval* [log log $N$, exp exp $N$].

*Then for every $K_0 \subset K$, there is a function $g \in \mathcal{F}$ such that $f \geq g \geq f^{1/2}$, $g(k) = f(k)^{1/2}$ whenever $k \in K_0$ and $g(x) = f(x)$ whenever $N \in J \setminus K_0$ and $x$ in the interval* [log N, exp N].

The next lemmas extend some technical results of [13] concerning $f(x) = \log_2(x+1)$ to the case of $f(x) = (log_2(x+1))^r$, $r \in [1/2, 1]$. Their calculus proofs are postponed until Section 5.

**Lemma 3.2.** *Let $f_r$ be defined on $[1, +\infty)$ by $f_r(x) = (log_2(x+1))^r$ for every $r \in (0, 1]$. Then $f_r$ belongs to the class $\mathcal{F}$.*

**Lemma 3.3.** *There exists $J \subset \mathbb{N}$ such that writing it in increasing order as $\{j_1, j_2, \cdots\}$ and letting $K = \{j_1, j_3, j_5, \cdots\}$, we have that for every $K_0 \subset K = (j_{2i-1})_{i \in \mathbb{N}}$ and every $r \in [1/2, 1]$, there is a function $g_r \in \mathcal{F}$ such that $f_r \geq g_r \geq {f_r}^{1/2}$, $g_r(k) = f_r(k)^{1/2}$ whenever $k \in K_0$ and $g_r(x) = f_r(x)$ whenever $N \in J \setminus K_0$ and $x$ in the interval* [log N, exp N].

The case $r = 1$ in the following theorem is the main result of [8], see [8, Theorem 5]. By using Lemma 3.3 instead of [6, Lemma 6] in the argumentation of [8] we obtain:

**Theorem 3.4.** *Let $\mathcal{S}$ be a proper set of spreads and $r \in [1/2, 1]$. There exists a Banach space $X_r(\mathcal{S})$ satisfying a lower $f_r$-estimate with the following three properties.*

(i) *For every $x \in X_r(\mathcal{S})$ and every $S_{A,B} \in \mathcal{S}$, $\|S_{A,B}x\| \leq \|x\|$ and therefore $\|S_{A,B}x\| = \|x\|$ if supp(x) $\subset A$.*

(ii) *If $Y$ is a subspace of $X_r(\mathcal{S})$ generated by a block basis, then every operator from $Y$ to $X_r(\mathcal{S})$ is in the $|||.|||$-closure of the set of restrictions to $Y$ of the operators in the algebra $\mathcal{A}$ generated by $\mathcal{S}$. In particular, all operators on $X_r(\mathcal{S})$ are $|||.|||$-perturbations of operators in $\mathcal{A}$.*

(iii) *The seminorm $|||.|||$ satisfies the algebra inequality $|||UV||| \leq |||U|||\,|||V|||$.*

We are now in position to define our family of totally incomparable versions of the 'cube but not square' space $X_1$ of Gowers and Maurey. The definitions follow the ones in [13]. For $i = 0, 1, 2$ let $A_i$ be the set of positive integers equal $i+1$ (mod 3), let $S'_i$ be the spread from $\mathbb{N}$ to $A_i$ and $\mathcal{S}'$ be the semigroup generated by $S'_0$, $S'_1$ and $S'_2$ and their adjoints. In [13], page 560, it was shown that $\mathcal{S}'$ is a proper set. Given $r \in [1/2, 1]$, the Banach space we are interested in is the space $X_r(\mathcal{S}')$ obtained by Theorem 3.4. As in [13] it will be useful to define it slightly less directly as follows.

Let $\mathcal{T}$ be the ternary tree $\cup_{n=0}^{\infty}\{0, 1, 2\}^n$. Let $Y_{00}$ be the vector space of finitely supported scalar sequences indexed by $\mathcal{T}$ (including the empty sequence). Denote the canonical basis of $Y_{00}$ by $(e_t)_{t \in \mathcal{T}}$, write $e$ for $e_\emptyset$. If $s, t \in \mathcal{T}$, let $s^\frown t$ stand for the concatenation of s and t. We shall now describe some operators on $Y_{00}$.

Let $S_i$ for $i = 0, 1, 2$ be defined by their action on the basis as follows: $S_i e_t = e_{t^\frown i}$. The adjoint $S^*$ acts in the following way: $S_i^* e_t = e_s$ if t is of the form $t = s^\frown i$ and $S_i^* e_t = 0$ otherwise.

If $P$ denotes the natural rank one projection on the line $\mathbb{R}e$, then we denote by $\mathcal{I}$ and $\mathcal{A}$ respectively the proper set generated by $S_0$, $S_1$ and $S_1$, and the algebra generated by this proper set. Strictly speaking, $\mathcal{I}$ is not a proper set, but it is easy to embed $\mathcal{I}$ in $\mathbb{N}$ so that the maps $S_0$, $S_1$ and $S_1$ become spreads as defined earlier.

In order to obtain the space $X_r(\mathcal{S}')$, consider the subset $\mathcal{T}'$ of $\mathcal{T}$ consisting of all words $t \in \mathcal{T}$ that do not start with 0 (including the empty sequence). We modify the definition of $S_0$ slightly, by letting $S_0' e$ equal $e$ instead of $e_0$. The operators $S_1'$ and $S_2'$ are defined exactly as $S_1$ and $S_2$ were.

To each $s = (i_1, \cdots i_n) \in \mathcal{T}'$ we can associate the integer $n_s = 3^{n-1} i_1 + \cdots + 3i_{n-1} + i_n + 1$, with $n_\emptyset = 1$, and this defines a bijection between $\mathcal{T}'$ and $\mathbb{N}$. The operators $S_0'$, $S_1'$ and $S_2'$ coincide with the spreads on $c_{00}$ defined earlier, so we can define $\mathcal{S}'$ to be the proper set they generate and obtain the space $X_r(\mathcal{S}')$.

Let $Y$ be the completion of $Y_{00}$ equipped with the $l_1$ norm, in other words let $Y = l_1(\mathcal{T})$ and let $\mathcal{E}$ denote the norm closure of $\mathcal{A}$ in $L(Y)$. Let also $\mathcal{J}$ the closed two-sided ideal in $\mathcal{E}$ generated by $P$. Let $\mathcal{O}$ denote the quotient algebra $\mathcal{E}/\mathcal{J}$.

Now we consider the algebra $\mathcal{A}'$ generated by $\mathcal{S}'$. In [13], Lemma 20, it was proved that $|||.|||$ is a norm on $\mathcal{A}'$. If we write $\mathcal{G}$ for the $|||.|||$-completion of $\mathcal{A}'$ then Theorem 3.4 implies that $\mathcal{G}$ is a Banach algebra and there exists a unital algebra homomorphism $\phi$ from $L(X_r(\mathcal{S}'))$ to $\mathcal{G}$ ([13], page 550).

In [13], Lemma 25, it was shown that there is a norm-one algebra homomorphism $\theta$ from $\mathcal{G}$ to $\mathcal{O}$.

Finally, let $V$ be an arbitrary set and $\psi : V \to V$ a function, we denote by $\psi_3$ the function from the set of matrices $M_3(V)$ to $M_3(V)$ given by $\psi_3((v_{i,j})_{1\le i,j\le 3}) = (\psi(v_{i,j}))_{1\le i,j\le 3}$, for every $(v_{i,j})_{1\le i,j\le 3} \in M_3(V)$.

**Remark 3.5.** Gowers and Maurey proved that $X_1(\mathcal{S}')$ is isomorphic to its cube $X_1(\mathcal{S}')^3$ ([13], page 563). Likewise $X_r(\mathcal{S}')$ is isomorphic to its cube $X_r(\mathcal{S}')^3$. Furthermore, it is important to note that the norms of the projections involved in $X_r(\mathcal{S}') \overset{c}{\hookrightarrow} X_r(\mathcal{S}')^2$ and $X_r(\mathcal{S}')^2 \overset{c}{\hookrightarrow} X_r(\mathcal{S}')$ do not depend on the number $r \in [1/2, 1]$. Indeed, letting

$$X_j = \{\sum_{i=1}^{\infty} x_{3i+j}\ e_{3i+j} \in X_r(\mathcal{S}')\}$$

for every $j = 0, 1, 2$, we have by Theorem 3.4 (i)

(a) $X_j$ is isometric to $X_r(\mathcal{S}')$ for every $j = 0, 1, 2$.

(b)$X_r(\mathcal{S}') = X_0 \oplus X_1 \oplus X_2$.

(c) The operator $P_r$ from $X_r(\mathcal{S}')$ onto $X_0 \oplus X_1$ defined by $P_r(x) = S_{\mathbb{N},A_0}(x) + S_{\mathbb{N},A_1}(x)$ is a projection with $\|P_r\| \le 2$.

(d) The operator $Q_r$ from $X_0 \oplus X_1$ onto $X_0$ defined by $Q_r = S_{\mathbb{N},A_0}(x)$ is a projection with $\|Q_r\| = 1$.

**Lemma 3.6.** *$X_r(\mathcal{S}')$ is nearly isomorphic to $X_r(\mathcal{S}')^2$ for no $r \in [1/2, 1]$.*

**Proof.** It is inspired by [13], Theorem 26, where it was proved that $X_1(\mathcal{S}')$ is not isomorphic to its square $X_1(\mathcal{S}')^2$. Given $r \in [1/2, 1]$, denote $X_r(\mathcal{S}')$ by $X$ and assume that some finite codimensional subspace of $X$ is isomorphic to a finite codimensional subspace of $X^2$. We consider the two possible cases.

*First case:* $X \sim X^2 \oplus F$, for some finite-dimensional space $F$. Let $U$ be an isomorphism from $X$ onto $X^2 \oplus F$, and assume without loss of generality that $F \subset X$. Write $U = (U_1, U_2, U_3)$, where $U_1 \in L(X)$, $U_2 \in L(X)$ and $U_3 \in L(X, F)$.

Let $H$ be such that $X = F \oplus H$. Thus there exists an isomorphism $V$ from $X^2$ onto $H$. Defining $V_1(z) = V(x, 0)$ and $V_2(x) = V(0, x)$, for every $x \in X$, we have that $V_1 \in L(X)$, $V_2 \in L(X)$ and $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$, for every $(x_1, x_2) \in X^2$.

Next consider the isomorphism $\Psi : X \oplus X^2 \to X^2 \oplus F \oplus H$ defined by

$$\Psi(x_1, (x_2, x_3)) = U(x_1) + V(x_2, x_3)$$

Its matrix as linear map from $X^3$ to $X^3$ is given by

$$A = \begin{pmatrix} U_1 & 0 & 0 \\ U_2 & 0 & 0 \\ U_3 & V_1 & V_2 \end{pmatrix}$$

where $U_3$ is seen as an operator from $X$ into $X$. Since $A$ is an invertible element of $M_3(L(X))$, it follows that $\theta_3\Phi_3(A)$ is invertible in $M_3(\mathcal{O})$. Therefore, by [13], Corollary 24, there exists an invertible element $B$ in $M_3(\mathcal{E})$ such that $\Pi_3(B) = \theta_3\Phi_3(A)$, where $\Pi$ is the canonical application from $\mathcal{O}$ onto $\mathcal{E}/\mathcal{J}$.

As was noted in [13], page 550, the Kernel of $\phi$ is the set of $T \in L(X)$ satisfying $|||T||| = 0$. The basis $(e_n)_{n\in\mathbb{N}}$ being shrinking ([13], page 551), the Kernel of $\phi$ contains the compact operators. Indeed for any $x = (x_n)_{n\in\mathbb{N}} \in \mathcal{L}(Y)$, the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded and converges weakly to 0. So if $T$ is compact, $(T(x_n))_{n\in\mathbb{N}}$ converges in norm to 0. In particular, $\phi(U_3) = 0$, because $U_3$ is of finite rank.

On the other hand, according to [13], page 564, $\mathcal{J}$ consists exactly of the compact $w^*$-continuous operators on $l_1$. Hence

$$B = \begin{pmatrix} u & c_1 \\ c_2 & v \end{pmatrix}$$

where $u \in M_{2,1}(\mathcal{E})$, $v \in M_{1,2}(\mathcal{E})$, $c_1 \in M_{1,1}(\mathcal{E})$ and $c_2 \in M_{2,2}(\mathcal{E})$, with $c_1$ and $c_2$ compacts. It follows from [17], page 80, that the operator $D$ below defined is Fredholm

$$D = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

Consequently $u$ and $v$ are also Fredholm operators; which is absurd because there exists no Fredholm element in $M_{2,1}(\mathcal{E})$ ([13], Lemma 21).

*Second case:* $X^2 \sim X \oplus F$ for some finite-dimensional space $F$.

In this case, we would have $X^3 \sim X^2 \oplus F$. According to Remark 3.5, $X \sim X^3$. Thus by the first case we also would obtain a contradiction. ■

We recall the definition of *Rapidly Increasing Sequences* given in [13]. For $X \in \mathcal{X}$, $x \in X$ and every integer $N \geq 1$, recall that

$$\|x\|_{(N)} = \sup \sum_{i=1}^{N} \|E_i x\|,$$

where the supremum is extended to all sequences $E_1, E_2, \cdots, E_N$ of successive intervals.

For $0 < \epsilon \leq 1$ and $f \in \mathcal{F}$, we say that a sequence $x_1, x_2, \cdots, x_N$ of successive vectors *satisfies the $RIS(\epsilon)$ condition for the function $f$* if there is a sequence $(2N/f'(1))f^{-1}(N^2/\epsilon^2) < n_1 < \cdots < n_N$ of integers (where f'(1) is the right derivate in 1) such that $\|x_i\|_{(n_i)} \leq 1$ for each $i = 1, \cdots, N$ and

$$\epsilon f(n_i)^{1/2} > |ran(\sum_{j=1}^{i-1} x_j)|$$

for every $i = 2, \cdots, N$ ([13], page 546).

**Lemma 3.7.** *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be proper sets of spreads and $r, s \in [1/2, 1]$, with $r \neq s$. Then the Banach spaces $X_r(\mathcal{S}_1)$ and $X_s(\mathcal{S}_2)$ are totally incomparable.*

**Proof.** Fix $r > s$ in $[1/2, 1]$ and suppose that $X_r(\mathcal{S}_1)$ and $X_s(\mathcal{S}_2)$ are not totally incomparable. Thus, by a standard perturbation argument, we may find an infinite sequence of successive non-zero vectors $(z_n)_{n\in\mathbb{N}}$ in $X_r(\mathcal{S}_1)$, and an isomorphism $T$ from $\overline{\text{span}}\{z_n : n \in \mathbb{N}\}$ into $X_s(\mathcal{S}_2)$, such that $(T(z_n))_{n\in\mathbb{N}}$ is successive in $X_s(\mathcal{S}_2)$.

For $N$ in $K \subset J$, we may then block $(z_n)_{n\in\mathbb{N}}$ to construct a sequence $x_1, x_2, \cdots, x_N$ satisfying the R.I.S(1) condition for the function $f_r$ with $\|x_i\| \geq 1/2$ ([13], Lemma 4). Now putting $x = \sum_{n=1}^{N} x_n$ we obtain by the lemma analogous to Lemma 7 in [13] that

$$\|x\| \leq \frac{4N}{f_r(N)}.$$

Consequently

$$\|T(x)\| \leq \|T\| \frac{4N}{f_r(N)}.$$

On the other hand, since $T(x_n)$ is successive in $X_s(\mathcal{S}_2)$ and $X_s(\mathcal{S}_2)$ satisfies a lower $f_s$-estimate we deduce that

$$\|T(x)\| \geq \frac{1}{f_s(N)} \sum_{n=1}^{N} \|T(x_n)\| \geq \frac{1}{f_s(N)} \frac{1}{\|T^{-1}\|} \sum_{n=1}^{N} \|x_n\| \geq \frac{N}{2} \frac{1}{f_s(N)} \frac{1}{\|T^{-1}\|}.$$

It follows that

$$(\log_2(N+1))^{r-s} \leq 8 \, \|T\| \, \|T^{-1}\|,$$

which is a contradiction for $N$ large enough. ∎

The last ingredient for our proof is the following standard lemma from [14].

**Lemma 3.8.** *Let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of Banach spaces. Suppose that the Banach space $X$ is isomorphic to a subspace of $l_2(Y_i)_{i\in\mathbb{N}}$. Then some subspace of $X$ is isomorphic to a subspace of $Y_n$ for some $n \in \mathbb{N}$ or $l_2$ is isomorphic to a subspace of $X$.*

We are now ready to prove our main result.

**Theorem 3.9.** *There exists a family of separable Banach spaces $(X_\alpha)_{\alpha\in\{1,2\}^\omega}$ such that*
*(a) $X_\alpha \overset{c}{\hookrightarrow} X_\beta$ for every $\alpha$ and $\beta$ in $\{1,2\}^\omega$.*
*(b) $X_\alpha \not\simeq X_\beta$ for every $\alpha \neq \beta$.*

**Proof.** Let $\mathcal{S}'$ be the spread considered after Theorem 3.4. We pick a sequence $(r_n)_{n\in\mathbb{N}}$ of numbers in $[1/2, 1]$, with $r_m \neq r_n$ whenever $m \neq n$ and define $Z_n = X_{r_n}(\mathcal{S}')$, for every $n \in \mathbb{N}$.

We then define for $\alpha \in \{1,2\}^\omega$, the following $l_2$-sum of Banach spaces:

$$X_\alpha = (\sum_{n=1}^{\infty} Z_n^{\alpha(n)})_2.$$

It follows from Remark 3.5 that any two such spaces are complemented in each other. We next assume that there exists an isomorphism $T$ from $X_\beta$ onto $X_\alpha$ and intend to prove that $\alpha = \beta$. Let $P_1$ be the canonical projection from $X_\alpha$ onto $Z_1^{\alpha(1)}$, and $Q_1$ be the canonical projection from $X_\alpha$ onto $(\sum_{n=2}^{\infty} Z_n^{\alpha(n)})_2$. We define analogously $P_1'$ and $Q_1'$ to be the projections corresponding to the space $X_\beta$.

We now claim that $S = Q_1 T_{|Z_1^{\beta(1)}}$ is strictly singular, that is, $S$ cannot be an isomorphism on any infinite dimensional subspace of $Z_1^{\beta(1)}$.

Indeed, otherwise, there exists an infinite dimensional subspace $Z$ of $Z_1^{\beta(1)}$, such that $Q_1 T_{|Z}$ is an isomorphism into, so $Z$ is isomorphic to some subspace of $(\sum_{n=2}^{\infty} Z_n^{\alpha(n)})_2$. Since $Z_1^{\beta(1)}$ has no unconditional basic sequence ([13], page 567), it contains no subspace isomorphic to $l_2$ and therefore by Lemma 3.8, we deduce that $Z$ contains an infinite dimensional subspace which is isomorphic to a subspace of $Z_n^{\alpha(n)}$ for some $n \geq 2$, contradicting Lemma 3.7.

We then define the operators $U : Z_1^{\beta(1)} \to Z_1^{\alpha(1)}$ and $V : Z_1^{\alpha(1)} \to Z_1^{\beta(1)}$ by

$$U(x) = P_1 T(x) \text{ and } V(x) = P_1' T^{-1}(x).$$

We consider $VU \in L(Z_1^{\beta(1)})$. For any $x \in Z_1^{\beta(1)}$, we deduce that

$$VU(x) = VP_1T(x) = V(Id_{X_\alpha} - Q_1)T(x) = V(T(x) - Q_1T(x)) = VT(x) - VS(x),$$

therefore

$$VU(x) = P_1'(x) - VS(x) = x - VS(x),$$

that is, $VU = Id_{Z_1^{\beta(1)}} + s$, where $s$ is strictly singular.

Now by symmetry, we also have that $UV = Id_{Z_1^{\alpha(1)}} + s'$, where $s'$ is strictly singular. Then $UV$ and $VU$ are Fredholm operators ([17], page 80). It follows that $U$ and $V$ are isomorphisms on finite codimensional subspaces and have finite dimensional cokernels, hence are Fredholm. Consequently $Z_1^{\alpha(1)}$ and $Z_1^{\beta(1)}$ have isomorphic finite codimensional subspaces, and by Lemma 3.6, this means that $\alpha(1) = \beta(1)$. This proof can be repeated for an arbitrary $n \in \mathbb{N}$, so we conclude that $\alpha = \beta$. ■

## 4. Some remarks and problems.

**Problem 4.1.** As already said, it comes easily from the properties of the space $X_1(\mathcal{S}_p)$ of Gowers and Maurey mentioned in the introduction that it satisfies $SBI(X_1(\mathcal{S}_p)) = p$. Does there exist a Banach space $X$ with $SBI(X) = \aleph_0$? In particular, is the space $X_E$ defined in [9] such a space?

**Problem 4.2.** $SBI(X) = 2^{\aleph_0}$ is the highest possible value for a separable Banach space $X$. The next step concerning separable Banach spaces should rather be expressed in terms of complexity of the relation of isomorphism. We refer to [5] or [18] for a survey about the notion of relative complexity of analytic equivalence relations on Polish spaces, applied to isomorphism between separable Banach spaces. How complex can be an equivalence relation $R$ on $2^\omega$, which is Borel reducible to isomorphism between separable Banach spaces, with the condition that the image of the reducing map is formed by Banach spaces which are all complemented in each other?

As a consequence of Lemma 3.7, we also derive the following proposition. We recall that a property $P$ of a Banach space is said to be a *three-space property* if whenever a Banach space $X$ has a subspace $Y$ which satisfies $P$ and such that $X/Y$ satisfies $P$, it follows that $X$ satisfies $P$. See [3] for a survey on three-space problems.

**Proposition 4.3.** *Failing the Schroeder-Bernstein Property is not a three-space property.*

**Proof.** Let $S$ be the right shift on $c_{00}$, that is, $S : c_{00} \to c_{00}$ is given by $S(a_1, a_2, \cdots) = (0, a_1, a_2, \cdots)$. Denote by $\mathcal{S}_2$ and $\mathcal{S}_3$ the proper set generated by $S^2$ and $S^3$ respectively. Consider the Banach spaces $X = X_1(\mathcal{S}_2)$ and $Y = X_{1/2}(\mathcal{S}_3)$ given by Theorem 3.4. By Lemma 3.7, $X$ and $Y$ are totally incomparable spaces.

We know by Theorem 19 in [12] and the remarks after this theorem that:

(a) $X$ is isomorphic to its subspaces of codimensions two but not to its hyperplanes.

(b) Two finite-codimensional subspaces of $Y$ are isomorphic if and only if their codimensions are equal mod 3.

(c) Every complemented subspace of $X$ (resp. $Y$) has finite dimension or codimension in $X$ (resp. $Y$).

Clearly from (a) and (b) $X$ and $Y$ do not have the SBP. In fact $SBI(X) = 2$ and $SBI(Y) = 3$. So, to prove the Proposition it suffices to show that $X \oplus Y$ has the SBP.

Suppose then that $Z \overset{c}{\hookrightarrow} X \oplus Y$ and $X \oplus Y \overset{c}{\hookrightarrow} Z$ for some Banach space $Z$. Since $X$ and $Y$ are totally incomparable spaces, by Theorem 23 in [20], we have that $Z = Z_1 \oplus Z_2$, where $Z_1 \overset{c}{\hookrightarrow} X$ and $Z_2 \overset{c}{\hookrightarrow} Y$. Moreover, $X \overset{c}{\hookrightarrow} Z_1 \oplus Z_2$ and $X$ and $Z_2$ are totally incomparable spaces. Consequently, again by Theorem 23 in [20], we conclude that $Z_1$ is an infinite dimensional space. According to (c) we deduce that $Z_1 \sim X$ or $Z_1 \sim X \oplus \mathbb{R}$. In the same way, we obtain that $Z_2 \sim Y$, $Z_2 \sim Y \oplus \mathbb{R}$ or $Z_2 \sim Y \oplus \mathbb{R}^2$.

Therefore $Z$ is isomorphic to one of the following spaces: $X \oplus Y$, $X \oplus Y \oplus \mathbb{R}$, $X \oplus Y \oplus \mathbb{R}^2$ or $X \oplus Y \oplus \mathbb{R}^3$. Hence, to see that $Z$ is isomorphic to $X \oplus Y$, it is enough to show that $X \oplus Y$ is isomorphic to its hyperplanes. But this is true because of (a) and (b). Indeed,

$$X \oplus Y \oplus \mathbb{R} \sim (X \oplus \mathbb{R}^2) \oplus Y \oplus \mathbb{R} \sim X \oplus (Y \oplus \mathbb{R}^3) \sim X \oplus Y. \quad \blacksquare$$

Proposition 4.3 leads naturally to the following problems:

**Problem 4.4.** Assume $X$ is a Banach space such that $X^2$ has the SBP. Does it follow that $X$ has the SBP?

**Problem 4.5.** Is the SBP a three-space property?

A partial answer to Problem 4.5 was given by Casazza in [2]. He noticed that if $X$ and $Y$ have the SBP, and are totally incomparable spaces, then $X \oplus Y$ has the SBP.

As another immediate consequence of Theorem 3.4 and Lemma 3.7, we obtain the following result of I. Gasparis [10]. We recall that a Banach space $X$ is hereditarily indecomposable if no closed subspace $Y$ of $X$ contains a pair of infinite dimensional closed subspaces $M$ and $N$ such that $Y = M \oplus N$.

**Corollary 4.6. (I. Gasparis)** *There exists a family of cardinality the continuum of separable totally incomparable hereditarily indecomposable Banach spaces.*

**Proof.** Let $\mathcal{S} = Id$ be the identity of $c_{00}$. Then, by [13], section 4.1, and using Lemma 3.7 we see that the spaces $(X_r(Id))_{r\in[1/2,1]}$ given by Theorem 3.4 are a continuum of totally incomparable hereditarily indecomposable Banach spaces. $\blacksquare$

**Remark 4.7.** Gasparis gets in fact a stronger result, that is, a continuum of asymptotically $l_1$ hereditarily indecomposable Banach spaces. There is an even simpler way to obtain the result of Gasparis, if one doesn't care for the asymptotically $l_1$ part. In [4] it was constructed a family $X_p, 1 < p < +\infty$ of (uniformly convex) hereditarily indecomposable Banach spaces with a Schauder basis. Each $X_p$ satisfies the following norm inequality for successive vectors $x_1, \dots, x_n$ on the basis:

$$\frac{1}{f(n)^{1/2}}(\sum_{k=1}^{n} \|x_k\|^p)^{\frac{1}{p}} \leq \| \sum_{k=1}^{n} x_k\| \leq (\sum_{k=1}^{n} \|x_k\|^p)^{\frac{1}{p}},$$

where as before, $f$ is defined by $f(x) = \log_2(x+1)$. By the same argument as in Lemma 3.7, it follows from this inequality that for $p \neq p'$, $X_p$ and $X_{p'}$ are totally incomparable spaces.

## 5. Appendix.

In this appendix, we give the proofs of Lemmas 3.2 and 3.3 which were postponed in Section 3.

**Proof of Lemma 3.2.** Since $f(x) = \log_2(x+1)$ is in $\mathcal{F}$, it follows that (i), (ii), (iii), (v) and (vi) of the definition of $\mathcal{F}$ hold also for $f_r(x) = f^r(x)$. To show that $f_r$ belongs to $\mathcal{F}$, it only remains to show that the second derivate of the function $F(x) = x/f_r(x)$ is negative on $[1, \infty)$. We have,

$F"(x) = \frac{r(\log_2(x+1))^r \log_2 e}{(x+1)^2(\log_2(x+1))^{2(r+1)}} \left(-x\log_2(x+1) - 2\log_2(x+1) + (r+1)x\log_2 e\right).$

Consider $D(x) = -x\log_2(x+1) - 2\log_2(x+1) + (r+1)x\log_2 e$. Then $D(1) < 0$, because $r \leq 1$ and $e^2 < 8$. Thus it is enough to prove that $D'(x) < 0$ for all $x \geq 1$. Compute

$D'(x) = \frac{1}{x+1}\left(-(x+1)\log_2(x+1) - x\log_2 e - 2\log_2 e + (r+1)(x+1)\log_2 e\right).$

Let $H(x) = -(x+1)\log_2(x+1) - x\log_2 e - 2\log_2 e + (r+1)(x+1)\log_2 e$.

In particular, $H(1) = -2 + (2r-1)\log_2 e < 0$ since $r \leq 1$. Also, $H'(x) = -\log_2(x+1) + (r-1)\log_2 e < 0$, $\forall x \geq 1$, because $r \leq 1$. Therefore $H(x) < 0$ for all $x \geq 1$. ■

To make clear the proof of Lemma 3.3, we stand out some simple inequalities.
(1) $16\log_2(x^{1/2}+1) > \log_2(x+1), \ \forall x \geq 1$.
(2) $(\log_2(x+1))^{3/2} > \log_2(x^3+1), \ \forall x \geq 1048576$.
(3) $\log_2(x+1) \leq x^{1/4}, \ \forall x \geq 32^4$.
(4) $x^{1/4} \leq (x/2)^{1/2}, \ \forall x \geq 32^4$.
(5) $64x^3 < e^{x/2}, \ \forall x \geq 18$.
(6) $x+1 < e^{x/2}, \ \forall x \geq 6$.

**Proof of Lemma 3.3.** Let $J = (j_n)_{n\in\mathbb{N}}$ be a subset of $\mathbb{N}$ such that

$$j_1 > 10^{10^{10^{400}}} \text{ and } j_{n+1} > 10^{10^{10^{4(j_n)^2}}}, \forall n \in \mathbb{N}.$$

It suffices to verify that the hypothesis of Lemma 3.1 are satisfied for this $J$ and for every $f_r$, with $r \in [1/2, 1]$.

Fix $r \in [1/2, 1]$. By Lemma 3.2 $f_r \in \mathcal{F}$ and clearly log log log $n \geq 4m^2$, $\forall m, n \in J$, $m < n$.

(a) Since $f_r^{1/2} = f_{r/2}$, Lemma 3.2 implies that $f_r^{1/2} \in \mathcal{F}$.

(b) $f_r(j_1) > 256$ if and only if $j_1 > 2^{256^{1/r}} - 1$. Since $r \geq 1/2$ this is clearly true.

(c) $exp\ exp\ j_n < f_r^{-1}(f_r(j_m))^{1/2}$ if and only if $\log_2(1+e^{e^{j_n}}) < (\log_2(1+j_m))^{1/2}$. But this last inequality follows from the definition of $J$.

(d) $16 f_r(x^{1/2}) > f_r(x)$ if and only if $16^{1/r}\log_2(x^{1/2}+1) > \log_2(x+1)$. Since $r \leq 1$, this follows from (1).

(e) $f_r(p)^{3/2} > f_r(p^3)$ if and only if $(\log_2(p+1))^{3/2} > \log_2(p^3+1)$. This is a consequence of (2).

(f) Fix $N \in J \setminus K$ and $x_0$ in the interval [log N, exp N]. The equation of the tangent t(x) to $x/f_r(x)$ at $x_0$ is

$$t(x) = \frac{x_0}{f_r(x_0)} + \frac{1}{f_r(x_0)}(1 - \frac{r x_0}{(x_0+1)\log(x_0+1)})(x - x_0).$$

*Claim 1.*

$$\frac{x_0}{2(\log_2(x_0+1))^{r+1}} \leq t(x),\ \forall x \geq 0. \tag{7}$$

Indeed, since $t(0) = (r x_0^2 \log_2 e)/(x_0+1)(\ln(x_0+1))^{r+1}$ and $2 \leq e^{\frac{2rx_0}{x_0+1}}$, because $j_n \leq x_0$, $\log 2 \leq j_n/(j_n+1)$, for every $n \in \mathbb{N}$ and $r \geq 1/2$, we deduce that

$$\frac{x_0}{2(\log_2(x_0+1))^{r+1}} \leq t(0). \tag{8}$$

Moreover, $r \leq 1$ implies that the angular coefficient of t(x) is positive, hence $t(0) \leq t(x)$, $\forall x \geq 0$. So by (7) we conclude (8).

*Claim 2:* If $x < \log \log N$, then

$$\frac{x}{f_r(x)^{1/2}} \leq \frac{x_0}{2(\log_2(x_0+1))^{r+1}}. \tag{9}$$

Indeed, let $c = x_0/2(\log_2(x_0+1))^{r+1}$. Consequently (9) holds if and only if $x^{2/r} \leq c^{2/r} \log_2(x+1)$.

Let $d(x) = c^{2/r} \log_2(x+1) - x^{2/r}$.

$d(1) \geq 0$ if and only if $\log_2(x_0+1) \leq (x_0/2)^{1/(r+1)}$. Since $r \leq 1$, the last inequality is true because of (3) and (4).

Furthermore $d'(x) > 0$ if and only if $x^{(2-r)/r}(x+1)\ 2/r < c^{2/r} \log_2 e$, that is

$$x^{(2-r)/r}(x+1)\,\frac{2}{r} < \left(\frac{{x_0}^{1/2(r+1)}}{\log_2(x_0+1)}\right)^{4/r} {x_0}^{2/(r+1)}\,(\log_2(x_0+1))^{2(1-r)/r}\,\frac{\log_2 e}{2^{2/r}}. \tag{10}$$

Since $1/2 \leq r \leq 1$, it follows that $(2-r)/r \leq 3$ and $2/r \leq 4$. So, the first side of (10) is less than or equal to $4x^3(x+1)$. On the other hand, again since $1/2 \leq r \leq 1$, we know that $1 \leq 4/r$, $1 \leq 2/(r+1)$, $0 \leq 2(1-r)/r$ and $16 \leq 1/2^{2/r}$. Moreover, by (7), $1 \leq {x_0}^{1/2(r+1)}/\log_2(x_0+1)$. Hence the second side of (10) is greater than or equal to $x_0/16$.

Therefore to prove (10) it suffices to show that $64(x+1)x^3 < x_0$.

To see this, suppose that $x < 18$, thus

$$64(x+1)x^3 \leq (\ln N)^{1/2}\ (\ln N)^{1/2} < x_0.$$

Now assume that $x \geq 18$. Hence by (5) and (6) we have

$$64(x+1)x^3 \leq e^{x/2}e^{x/2} < x_0.$$

*Claim 3:* If $\exp\exp N \leq x$, then $x \geq 2x_0$.

Indeed $2x_0 \leq 2\exp N \leq \exp\exp N \leq x$.

*Claim 4:* For every $x \geq 2x_0$, we have

$$\frac{x}{4f_r(x_0)} \leq t(x).$$

Indeed, consider $d(x) = t(x) - x/4f_r(x_0)$. Thus $d(2x_0) > 0$ because $rx_0/(x_0+1) < 3/2\log(x_0+1)$ and $d'(x) > 0$ because $rx_0/(x_0+1) < 3/4\log(x_0+1)$.

*Claim 5:* If $\log\log N \leq x$, then

$$\frac{x}{f_r(x)^{1/2}} \leq \frac{x}{4f_r(x_0)}. \tag{11}$$

Indeed, (11) holds if and only if

$$16(\log_2(x_0+1))^2 \leq \log_2(x+1).$$

But this is true by the definition of $J$, because

$$16(\log_2(x_0+1))^2 \leq 16(\log_2(e^N+1))^2 \leq \log_2(e^{e^N}+1) \leq \log_2(x+1).$$

Finally, by Claims (1), (2), (3), (4) and (5) we deduce that $x/f_r(x)^{1/2} \leq t(x)$, for every $x$ outside the interval $[\log\log N,\ \exp\exp N]$. ∎

**Acknowledgements:** We wish to thank C. Rosendal for answering many of our doubts about descriptive set theory results.

# References


[1] P. G. Casazza, *Finite dimensional decompositions in Banach spaces*, Contemp. Math, **52** (1986), 1-29.

[2] P. G. Casazza, *The Schroeder-Bernstein property for Banach space*, Contemp. Math. **85** (1989), 61-78.

[3] J. M. F. Castillo and M. González. *Three-space problems in Banach spaces theory.* Lecture Notes in Mathematics. **1667** (1997).

[4] V. Ferenczi, *A uniformly convex and hereditarily indecomposable Banach space.* Israel J. Math. **102** (1997), 199–225.

[5] V. Ferenczi and E. M. Galego, *Some equivalence relations which are Borel reducible to isomorphism between Banach spaces*, preprint.

[6] V. Ferenczi and C. Rosendal, *On the number of non-isomorphic subspaces of a Banach space*, preprint.

[7] V. Ferenczi and C. Rosendal, *Ergodic Banach spaces*, preprint.

[8] E. M. Galego, *On solutions to the Schroeder-Bernstein problem for Banach spaces.* Arch. Math. (Basel) **79** (2002), no. 4, 299–307.

[9] E. M. Galego, *The Schroeder-Bernstein index for Banach spaces.* Studia Math. To appear.

[10] I. Gasparis, *A continuum of totally incomparable hereditarily indecomposable Banach spaces.* Studia Math. **151** (2002), no. 3, 277–298.

[11] W.T. Gowers, *A solution to the Schroeder-Bernstein problem for Banach spaces*, Bull. London Math. Soc. **28** (1996), 297-304.

[12] W.T. Gowers and B. Maurey, *The unconditional basic sequence problem.* J. Amer. Math. Soc. **6** (1993), no. 4, 851–874.

[13] W.T. Gowers and B. Maurey, *Banach spaces with small spaces of operators.* Math. Ann. **307** (1997), 543-568.

[14] R. Herman and R. Whitley, *An example concerning reflexivity.* Studia Math. **28** (1966/67) 289–294.

[15] N.J. Kalton, *A remark on Banach spaces isomorphic to their squares.* Function spaces (Edwardsville, IL, 1998), 211–217, Contemp. Math., **232**, Amer. Math. Soc., Providence, RI, 1999.

[16] A. S. Kechris. *Classical descriptive set theory.* Graduate Texts in Mathematics, 156. Springer-Verlag, New York, 1995.

[17] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I, Sequence spaces*, Springer-Verlag New York Heidelberg Berlin. 1979.

[18] C. Rosendal. *Incomparable, non isomorphic and minimal Banach spaces*, preprint.

[19] T. Schlumprecht, *An arbitrarily distortable Banach space.* Israel J. Math. **76** (1991), no. 1-2, 81–95.

[20] P. Wojtaszczyk. *On projections and unconditional bases in direct sums of Banach spaces.* II. Studia Math. **62** (1978), 193-201.

Equipe d'Analyse Fonctionnelle, Université Paris 6
Boite 186, 4, Place Jussieu, 75252, Paris Cedex 05, France
e-mail: ferenczi@ccr.jussieu.fr
And
Department of Mathematics, IME, University of São Paulo
São Paulo 05315-970 Brazil
e-mail: eloi@ime.usp.br, ferenczi@ime.usp.br