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MINIMAL SUBSPACES AND ISOMORPHICALLY HOMOGENEOUS SEQUENCES IN A BANACH SPACE

BY

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ABSTRACT

If a Banach space is saturated with subspaces with a Schauder basis, which embed into the linear span of any subsequence of their basis, then it contains a minimal subspace. It follows that any Banach space is either ergodic or contains a minimal subspace.

1. Introduction

The starting point of this article is the solution to the Homogeneous Banach Space Problem given by W. T. Gowers [7] and R. Komorowski — N. Tomczak-Jaegermann [11]. A Banach space is said to be homogeneous if it is isomorphic to its infinite-dimensional closed subspaces; these authors proved that a homogeneous Banach space must be isomorphic to ℓ_2.

Gowers proved that any Banach space must either have a subspace with an unconditional basis or a hereditarily indecomposable subspace. By properties of hereditarily indecomposable Banach spaces, it follows that a homogeneous Banach space must have an unconditional basis (see e.g. [7] for details about this). Komorowski and Tomczak-Jaegermann proved that a Banach space with an unconditional basis must contain a copy of ℓ_2 or a subspace with a successive finite-dimensional decomposition on the basis (2-dimensional if the space has

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finite cotype) which does not have an unconditional basis. It follows that a homogeneous Banach space must be isomorphic to ℓ_2.

While Gowers' dichotomy theorem is based on a general Ramsey-type theorem for block-sequences in a Banach space with a Schauder basis, the subspace with a finite-dimensional decomposition constructed in Komorowski and Tomczak-Jaegermann's theorem can never be isomorphic to a block-subspace. If one restricts one's attention to block-subspaces, the standard homogeneous examples become the sequence spaces c_0 and $\ell_p, 1 \leq p < +\infty$, with their canonical bases; these spaces are well-known to be isomorphic to their block-subspaces. Furthermore, there are classical theorems which characterize c_0 and $\ell_p, 1 \leq p < +\infty$ by means of their block-subspaces. An instance of this is Zippin's theorem ([12] Theorem 2.a.9): a normalized basic sequence is perfectly homogeneous (i.e. equivalent to all its normalized block-sequences) if and only if it is equivalent to the canonical basis of c_0 or some ℓ_p. See also [12] Theorem 2.a.10.

So it is very natural to ask what can be said on the subject of (isomorphic) homogeneity restricted to block-subspaces of a given Banach space with a Schauder basis:

QUESTION 1: *If a Banach space X with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ is isomorphic to its block-subspaces, does it follow that X is isomorphic to c_0 or $\ell_p, 1 \leq p < +\infty$?*

Note that such a basis is not necessarily equivalent to the canonical basis of c_0 or some ℓ_p; take ℓ_2 with a conditional basis.

In the other direction, if a Banach space is not homogeneous, then how many non-isomorphic subspaces must it contain? This question may be asked in the setting of the classification of analytic equivalence relations on Polish spaces by Borel reducibility. This area of research originated from the works of H. Friedman and L. Stanley [6] and independently from the works of L. A. Harrington, A. S. Kechris and A. Louveau [9], and may be thought of as an extension of the notion of cardinality in terms of complexity, when one compares equivalence relations.

If R (resp. S) is an equivalence relation on a Polish space E (resp. F), then it is said that (E, R) is Borel reducible to (F, S) if there exists a Borel map $f\colon E \to F$ such that $\forall x, y \in E, xRy \Leftrightarrow f(x)Sf(y)$. An important equivalence relation is the relation E_0: it is defined on 2^ω by

$$\alpha E_0 \beta \Leftrightarrow \exists m \in \mathbb{N} \forall n \geq m, \alpha(n) = \beta(n).$$

The relation E_0 is a Borel equivalence relation with 2^ω classes and which, furthermore, does not admit a Borel classification by real numbers, that is, there is no Borel map f from 2^ω into $\mathbb{R}$ (equivalently, into a Polish space), such that $\alpha E_0 \beta \Leftrightarrow f(\alpha) = f(\beta)$; such a relation is said to be **non-smooth**. In fact E_0 is the $\leq_B$ minimal non-smooth Borel equivalence relation [9].

There is a natural way to equip the set of subspaces of a Banach space X with a Borel structure, and the relation of isomorphism is analytic in this setting [1]. The relation E_0 appears to be a natural threshold for results about the relation of isomorphism between separable Banach spaces [3], [4], [5], [16]. A Banach space X is said to be **ergodic** if E_0 is Borel reducible to isomorphism between subspaces of X; in particular, an ergodic Banach space has continuum many non-isomorphic subspaces, and isomorphism between its subspaces is non-smooth. The results in [1], [3], [4], [5], [16] suggest that every Banach space non-isomorphic to ℓ_2 should be ergodic, and we also refer to these articles for an introduction to the classification of analytic equivalence relations on Polish spaces by Borel reducibility, and more specifically to complexity of isomorphism between Banach spaces.

Restricting our attention to block-subspaces, the natural question becomes the following:

QUESTION 2: *If X is a Banach space with a Schauder basis, is it true that either X is isomorphic to its block-subspaces or E_0 is Borel reducible to isomorphism between the block-subspaces of X?*

Let us provide some ground for this conjecture by noting that, if we replace isomorphism by equivalence of the corresponding basic sequences, it is completely solved by a result of the author and C. Rosendal using the theorem of Zippin: if X is a Banach space with a normalized basis $(e_n)_{n\in\mathbb{N}}$, then either $(e_n)_{n\in\mathbb{N}}$ is equivalent to the canonical basis of c_0 or $\ell_p, 1 \leq p < +\infty$, or E_0 is Borel reducible to equivalence between normalized block-sequences of X.

A. M. Pelczar has proved that a Banach space which is saturated with subsymmetric sequences contains a minimal subspace [15]. The aim of this article is to prove the isomorphic counterpart of her theorem. The natural generalization is to replace subsymmetric sequences by sequences which are **isomorphically homogeneous**, i.e. such that all subspaces spanned by subsequences are isomorphic. However, it will be enough and more natural with our methods to consider embeddings instead of isomorphisms, which leads us to a stronger result: if a Banach space X is saturated with basic sequences whose linear span embeds in the linear span of any subsequence, then X contains a minimal subspace (Theorem 3).

In combination with a result of C. Rosendal [16], it follows that if X is a Banach space with a Schauder basis, then either E_0 is Borel reducible to isomorphism between block-subspaces of X, or X contains a block-subspace which is block-minimal (i.e. embeds as a block-subspace of any of its block-subspaces), Corollary 16. This improves a result of [5] which states that a Banach space contains continuum many non-isomorphic subspaces or a minimal subspace.

We think that our methods also have an intrisic interest. They show how to combine non-trivial combinatorial results concerning subsequences (mainly the infinite Ramsey Theorem and its consequences), and combinatorial methods about block-sequences (mainly in the spirit of Gowers' Theorem). Given a basic sequence such that all subsequences satisfy some embedding property, we shall indeed use Silver's infinite Ramsey Theorem to pass to a subsequence for which there is a continuous map producing witnesses for these embeddings (Lemma 7). This will provide the first step of an induction, which, as in [15], or as Maurey's proof of Gowers' dichotomy theorem [14], is based on some stabilization process for block-subspaces. Our theorem will follow.

Combinatorial methods about subsequences or about block-sequences are often used in Banach space theory; but they are less frequently combined. Hopefully, our methods could lead to other applications in that area.

2. Notation

Let X be a Banach space with a Schauder basis $(e_n)_{n\in\mathbb{N}}$. If $(x_n)_{n\in J}$ is a finite or infinite block-sequence of X then $[x_n]_{n\in J}$ will stand for its closed linear span. We shall also use some standard notation about finitely supported vectors on $(e_n)_{n\in\mathbb{N}}$, for example, we shall write $x < y$ and say that x and y are successive when $\max(supp(x)) < \min(supp(y))$. The set of normalized block-sequences in X, i.e. sequences of successive blocks in X, is denoted $bb(X)$.

Let $\mathbb{Q}(X)$ be the set of non-zero blocks of the basis (i.e. finitely supported vectors) which have rational coordinates on $(e_n)_{n\in\mathbb{N}}$ (or coordinates in $\mathbb{Q}+i\mathbb{Q}$ if we deal with a complex Banach space). We denote by $bb_{\mathbb{Q}}(X)$ the set of block-bases of vectors in $\mathbb{Q}(X)$, and by $\mathcal{G}_{\mathbb{Q}}(X)$ the corresponding set of block-subspaces of X.

The notation $bb_{\mathbb{Q}}^{<\omega}(X)$ (resp. $bb_{\mathbb{Q}}^{n}(X)$) will be used for the set of finite (resp. length n) block-sequences with vectors in $\mathbb{Q}(X)$; the set of finite block-subspaces generated by block-sequences in $bb_{\mathbb{Q}}^{<\omega}(X)$ will be denoted by $Fin_{\mathbb{Q}}(X)$.

We shall consider $bb_{\mathbb{Q}}(X)$ as a topological space, when equipped with the product of the discrete topology on $\mathbb{Q}(X)$. As $\mathbb{Q}(X)$ is countable, this turns

$bb_{\mathbb{Q}}(X)$ into a Polish space. Likewise, $\mathbb{Q}(X)^\omega$ is a Polish space.

For a finite block sequence $\tilde{x} = (x_1, \dots, x_n) \in bb_{\mathbb{Q}}^{<\omega}(X)$, we denote by $N_{\mathbb{Q}}(\tilde{x})$ the set of elements of $bb_{\mathbb{Q}}(X)$ whose first n vectors are $(x_1, \dots, x_n)$; this is the basic open set associated to $\tilde{x}$.

The set $[\omega]^\omega$ is the set of increasing sequences of integers, which we sometimes identify with infinite subsets of ω. It is equipped with the product of the discrete topology on ω. The set $[\omega]^{<\omega}$ is the set of finite increasing sequences of integers. If $a = (a_1, \dots, a_k) \in [\omega]^{<\omega}$, then $[a]$ stands for the basic open set associated to a, that is the set of increasing sequences of integers of the form $\{a_1, \dots, a_k, n_{k+1}, n_{k+2}, \dots\}$. If $A \in [\omega]^\omega$, then $[A]^\omega$ is the set of increasing sequences of integers in A (where A is seen as a subset of ω).

We recall that two basic sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are said to be **equivalent** if the map T: $[x_n]_{n\in\mathbb{N}} \to [y_n]_{n\in\mathbb{N}}$ defined by $T(x_n) = y_n$ for all $n \in \mathbb{N}$ is an isomorphism. For $C \geq 1$, they are **C-equivalent** if $\|T\|\|T^{-1}\| \leq C$. A basic sequence is said to be **(C-)subsymmetric** if it is (C-)equivalent to all its subsequences.

We shall sometimes use "standard perturbation arguments" without being explicit. This expression will refer to one of the following well-known facts about block-subspaces of a Banach space X with a Schauder basis. Any basic sequence (resp. block-basic sequence) in X is an arbitrarily small perturbation of a basic sequence in $\mathbb{Q}(X)^\omega$ (resp. block-basic sequence in $bb_{\mathbb{Q}}(X)$), and in particular is $1+\epsilon$-equivalent to it, for arbitrarily small $\epsilon > 0$. Any subspace of X has a subspace which is an arbitrarily small perturbation of a block-subspace of X (and in particular, with $1+\epsilon$-equivalence of the corresponding bases, for arbitrarily small $\epsilon > 0$). If X is reflexive, then any basic sequence in X has a subsequence which is a perturbation of a block-sequence of X (and, in particular, is $1+\epsilon$-equivalent to it, for arbitrarily small $\epsilon > 0$).

We shall also use the fact that any Banach space contains a basic sequence.

Finally, we recall the definition of unconditionality for basic sequences: a Schauder basis $(e_n)_{n\in\mathbb{N}}$ of a Banach space X is said to be unconditional if there is some $C \geq 1$ such that for any $I \subset \mathbb{N}$, any norm 1 vector $x = \sum_{n\in\mathbb{N}} a_n e_n \in X$, $\|\sum_{n\in I} a_n e_n\| \leq C$.

3. The main result

We recall different notions of minimality for Banach spaces. A Banach space X is said to be **(C-)minimal** if it (C-)embeds into any of its subspaces. If X has a Schauder basis $(e_n)_{n\in\mathbb{N}}$, then it is said to be **block-minimal** if every block-subspace of X has a further block-subspace which is isomorphic to X, and is said to be **equivalence block-minimal** if every block-sequence of $(x_n)_{n\in\mathbb{N}}$ has a further block-sequence which is equivalent to $(x_n)_{n\in\mathbb{N}}$.

The theorem of Pelczar [15] states that a Banach space which is saturated with subsymmetric sequences must contain an equivalence block-minimal subspace with a Schauder basis.

A basic sequence **embeds (resp. C-embeds) into its subsequences** if its linear span embeds (resp. C-embeds) into the linear span of any of its subsequences. We may now state our isomorphic version of Pelczar's theorem:

THEOREM 3: *A Banach space which is saturated with basic sequences which embed into their subsequences contains a minimal subspace.*

We first prove two uniformity lemmas. For $N \in \mathbb{N}$ let $d_c(N)$ denote an integer such that if X is a Banach space with a basis $(e_n)_{n\in\mathbb{N}}$ with basis constant c, and $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are normalized block-basic sequences of X such that $x_n = y_n$ for all $n > N$, then $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are $d_c(N)$-equivalent. We leave as an exercise to the reader to check that such an integer exists.

LEMMA 4: *Let $(x_n)_{n\in\mathbb{N}}$ be a basic sequence in a Banach space which embeds into its subsequences. Then there exists $C \geq 1$ and a subsequence of $(x_n)_{n\in\mathbb{N}}$ which C-embeds into its subsequences.*

Proof: Let $(x_n)_{n\in\mathbb{N}}$ be a basic sequence which embeds into its subsequences, and let c be its basis constant. It is clearly enough to find a subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $C \geq 1$ such that $(x_n)_{n\in\mathbb{N}}$ C-embeds into any subsequence of $(y_n)_{n\in\mathbb{N}}$ (with the obvious definition).

Assuming the conclusion is false, we construct by induction a sequence of subsequences $(x_n^k)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, such that for all $k \in \mathbb{N}$, $(x_n^k)_{n\in\mathbb{N}}$ is a subsequence of $(x_n^{k-1})_{n\in\mathbb{N}}$ such that $(x_n)_{n\in\mathbb{N}}$ does not $kd_c(k)$-embed into $(x_n^k)_{n\in\mathbb{N}}$.

Let $(y_n)_{n\in\mathbb{N}}$ be the diagonal subsequence of $(x_n)_{n\in\mathbb{N}}$ defined by $y_n = x_n^n$. Then $(x_n)_{n\in\mathbb{N}}$ does not $kd_c(k)$-embed into $(x_1^k, \ldots, x_{k-1}^k, y_k, y_{k+1}, \ldots)$. So $(x_n)_{n\in\mathbb{N}}$ does not k-embed in $(y_n)_{n\in\mathbb{N}}$. Now k was arbitrary, so this contradicts our hypothesis. ∎

LEMMA 5: *Let X be a Banach space which is saturated with basic sequences which embed into their subsequences. Then there exists a subspace Y of X with a Schauder basis, and a constant $C \geq 1$ such that every block-sequence of Y (resp. in $bb_{\mathbb{Q}}(Y)$) has a further block-sequence (resp. in $bb_{\mathbb{Q}}(Y)$) which C-embeds into its subsequences.*

Proof: A space which is spanned by a basic sequence which embeds into its subsequences must in particular embed into its hyperplanes, so is isomorphic to a proper subspace; by [8] Corollary 19 and Theorem 21, such a space cannot be hereditarily indecomposable. Thus X does not contain a hereditarily indecomposable subspace; otherwise, some further subspace would be hereditarily indecomposable (since the H.I. property is hereditary) and spanned by a basic sequence which embeds in its subsequences (by the saturation property).

So by Gowers' dichotomy theorem, we may assume X has an unconditional basis (let c be its basis constant). If c_0 or ℓ_1 embeds into X then we are done, so by the classical theorem of James, we may assume X is reflexive. Thus by standard perturbation arguments, every normalized block-sequence in X has a further normalized block-sequence in X which embeds into its subsequences (here we also used the obvious fact that if a basic sequence $(x_n)_{n\in\mathbb{N}}$ embeds into its subsequences, then so does any subsequence of $(x_n)_{n\in\mathbb{N}}$).

Assuming the conclusion is false, we construct by induction a sequence of block-sequences $(x_n^k)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, for $k \in \mathbb{N}$, such that for all $k \in \mathbb{N}$, $(x_n^k)_{n\in\mathbb{N}}$ is a block-sequence of $(x_n^{k-1})_{n\in\mathbb{N}}$ such that no block-sequence of $(x_n^k)_{n\in\mathbb{N}}$ $kd_c(k)^2$-embeds into its subsequences.

Let $(y_n)_{n\in\mathbb{N}}$ be the diagonal block-sequence of $(x_n)_{n\in\mathbb{N}}$ defined by $y_n = x_n^n$, and let $(z_n)_{n\in\mathbb{N}}$ be an arbitrary block-sequence of $(y_n)_{n\in\mathbb{N}}$.

Then $(x_1^k, \ldots, x_{k-1}^k, z_k, z_{k+1}, \ldots)$ is a block-sequence of $(x_n^k)_{n\in\mathbb{N}}$ and so does not $kd_c(k)^2$-embed into its subsequences. So $(z_n)_{n\in\mathbb{N}}$ does not k-embed into its subsequences — this is true as well of its subsequences. As k was arbitrary, we deduce from Lemma 4 that $(z_n)_{n\in\mathbb{N}}$ does not embed into its subsequences. As $(z_n)_{n\in\mathbb{N}}$ was an arbitrary block-sequence of $(y_n)_{n\in\mathbb{N}}$, this contradicts our hypothesis.

By standard perturbation arguments, we deduce from this the stated result with block-sequences in $bb_{\mathbb{Q}}(Y)$. ∎

Recall that $\mathbb{Q}(X)^\omega$ is equipped with the product of the discrete topology on $\mathbb{Q}(X)$, which turns it into a Polish space.

Definition 6: Let X be a Banach space with a Schauder basis, and let $(x_n)_{n\in\mathbb{N}} \in \mathbb{Q}(X)^\omega$. We shall say that $(x_n)_{n\in\mathbb{N}}$ **continuously embeds (resp. C-continuously embeds) into its subsequences** if there exists a continuous map ϕ: $[\omega]^\omega \to \mathbb{Q}(X)^\omega$ such for all $A \in [\omega]^\omega$, $\phi(A)$ is a sequence of vectors in $[x_n]_{n\in A} \cap \mathbb{Q}(X)$ which is equivalent (resp. C-equivalent) to $(x_n)_{n\in\mathbb{N}}$.

This definition depends on the Banach space X in which we pick the basic sequence $(x_n)_{n\in\mathbb{N}}$; this will not cause us any problem, as it will always be clear which is the underlying space X.

The interest of this notion stems from the following lemma, which was essentially obtained by Rosendal as part of the proof of [16], Theorem 11. To prove it, we shall need the following fact, which is well-known to descriptive set theoricians. The algebra $\sigma(\Sigma_1^1)$ is the σ-algebra generated by analytic sets. For any $\sigma(\Sigma_1^1)$-measurable function from $[\omega]^\omega$ into a metric space, there exists $B \in [\omega]^\omega$ such that the restriction of f to $[B]^\omega$ is continuous.

Indeed, by Silver's Theorem 21.9 in [10], any analytic set in $[\omega]^\omega$ is completely Ramsey, and so any $\sigma(\Sigma_1^1)$ set in $[\omega]^\omega$ is (completely) Ramsey as well (use, for example, [10] Theorem 19.14). One concludes using the proof of [13] Theorem 9.10 which only uses the Ramsey-measurability of the function.

LEMMA 7: *Let X be a Banach space with a Schauder basis, let $(x_n)_{n\in\mathbb{N}} \in bb_{\mathbb{Q}}(X)$ be a block-sequence which C-embeds into its subsequences, and let ϵ be positive. Then some subsequence of $(x_n)_{n\in\mathbb{N}}$ $C+\epsilon$-continuously embeds into its subsequences.*

Proof: By standard perturbation arguments, we may find for each $A \in [\omega]^\omega$ a sequence $(y_n)_{n\in\mathbb{N}} \in \mathbb{Q}(X)^\omega$ such that $y_n \in [x_k]_{k\in A}$ for all $n \in \mathbb{N}$, and such that the basic sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are $C+\epsilon$-equivalent. The set $P \subset [\omega]^\omega \times \mathbb{Q}(X)^\omega$ of couples $(A,(y_n))$ with this property is Borel (even closed), so by the Jankov–von Neumann Uniformization Theorem (Theorem 18.1 in [10]), there exists a C-measurable selector f: $[\omega]^\omega \to \mathbb{Q}(X)^\omega$ for P. By the fact before this lemma, there exists $B \in [\omega]^\omega$ such that the restriction of f to $[B]^\omega$ is continuous. Write $B = (b_k)_{k\in\mathbb{N}}$ where $(b_k)_k$ is increasing. By composing f with the obviously continuous maps ψ_B: $[\omega]^\omega \to [B]^\omega$, defined by $\psi_B((n_k)_{k\in\mathbb{N}}) = (b_{n_k})_{k\in\mathbb{N}}$, and μ_B: $\mathbb{Q}(X)^\omega \to \mathbb{Q}(X)^\omega$, defined by $\mu_B((y_n)_{n\in\mathbb{N}}) = (y_{b_n})_{n\in\mathbb{N}}$, we obtain a continuous map ϕ: $[\omega]^\omega \to \mathbb{Q}(X)^\omega$ which indicates that $(x_n)_{n\in B} C+\epsilon$-continuously embeds into its subsequences. ∎

We now start the proof of Theorem 3. So we consider a Banach space X which is saturated with basic sequences which embed into their subsequences and wish to find a minimal subspace in X.

By Lemma 5 and Lemma 7, we may assume that X is a Banach space with a Schauder basis and that there exists $C \geq 1$ such that every block-sequence in $bb_{\mathbb{Q}}(X)$ has a further block-sequence in $bb_{\mathbb{Q}}(X)$ which C-continuously embeds into its subsequences.

For the rest of the proof X and $C \geq 1$ are fixed with this property. Recall that the set of block-subspaces of X which are generated by block-sequences in $bb_{\mathbb{Q}}(X)$ is denoted by $\mathcal{G}_{\mathbb{Q}}(X)$; the set of finite block-subspaces which are generated by block-sequences in $bb_{\mathbb{Q}}^{<\omega}(X)$ is denoted by $Fin_{\mathbb{Q}}(X)$. If $n \in \mathbb{N}$ and $F \in Fin_{\mathbb{Q}}(X)$, we write $n \leq F$ to mean that $n \leq \min(supp(x))$ for all $x \in F$.

We first express the notion of continuous embedding in terms of a game. For $L = [l_n]_{n\in\mathbb{N}}$ with $(l_n)_{n\in\mathbb{N}} \in bb_{\mathbb{Q}}(X)$, we define an "asymptotic" game H_L as follows. A k-th move for Player 1 is some $n_k \in \mathbb{N}$. A k-th move for Player 2 is some $(F_k, y_k) \in Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X)$, with $n_k \leq F_k \subset L$ and $y_k \in \Sigma_{j=1}^{k} F_j$.

Player 2 wins the game H_L if $(y_n)_{n\in\mathbb{N}}$ is C-equivalent to $(l_n)_{n\in\mathbb{N}}$.

We claim the following:

Lemma 8: *Let X be a Banach space with a Schauder basis, and let $(l_n)_{n\in\mathbb{N}} \in bb_{\mathbb{Q}}(X)$ be a block-sequence which C-continuously embeds into its subsequences. Let $L = [l_n]_{n\in\mathbb{N}}$. Then Player 2 has a winning strategy in the game H_L.*

Proof: Let ϕ be the continuous map in Definition 6. We describe a winning strategy for Player 2 by induction.

We assume that Player 1's moves were $(n_i)_{i\leq k-1}$ and that the $k-1$ first moves prescribed by the winning strategy for Player 2 were $(F_i, y_i)_{i\leq k-1}$, with F_i of the form $[l_{n_i}, \ldots, l_{m_i}]$, $n_i \leq m_i$, for all $i \leq k-1$; letting

$$a_{k-1} = [n_1, m_1] \cup \cdots \cup [n_{k-1}, m_{k-1}] \in [\omega]^{<\omega},$$

we also assume that $\phi([a_{k-1}]) \subset N_{\mathbb{Q}}(y_1, \ldots, y_{k-1})$. We now describe the k-th move of the winning strategy for Player 2.

Let n_k be a k-th move for Player 1. We may clearly assume that $n_k > m_{k-1}$. Let $A_k = \bigcup_{i\leq k-1}[n_i, m_i] \cup [n_k, +\infty) \in [\omega]^{\omega}$. The sequence $\phi(A_k)$ is of the form $(y_1, \ldots, y_{k-1}, y_k, z_{k+1}, \ldots)$ for some $y_k, z_{k+1}, \ldots$ in $\mathbb{Q}(X)$. By continuity of ϕ in A_k there exists $m_k > n_k$ such that, if $a_k = [n_1, m_1] \cup \cdots \cup [n_k, m_k] \in [\omega]^{<\omega}$, then $\phi([a_k]) \subset N_{\mathbb{Q}}(y_1, \ldots, y_k)$. We may assume that $\max(supp(l_{m_k})) \geq \max(supp(y_k))$; so as $y_k \in [l_i]_{i\in A_k}$, we have that $y_k \in \bigoplus_{j=1}^{k}[l_i]_{i\in[n_j,m_j]}$. So $(F_k, y_k) = ([l_{n_k}, \ldots, l_{m_k}], y_k)$ is an admissible k-th move for Player 2 for which the induction hypotheses are satisfied.

Repeating this by induction we obtain a sequence $(y_n)_{n\in\mathbb{N}}$ which is equal to $\phi(A)$, where $A = \bigcup_{k\in\mathbb{N}}[n_k, m_k]$, and so which is, in particular, C-equivalent to $(l_n)_{n\in\mathbb{N}}$. ∎

Definition 9: Given L, M two block-subspaces in $\mathcal{G}_{\mathbb{Q}}(X)$, define the game $G_{L,M}$ as follows. A k-th move for Player 1 is some $(x_k, n_k) \in \mathbb{Q}(X) \times \mathbb{N}$, with $x_k \in L$, and $x_k > x_{k-1}$ if $k \geq 2$. A k-th move for Player 2 is some $(F_k, y_k) \in Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X)$ with $n_k \leq F_k \subset M$ and $y_k \in F_1 \oplus \cdots \oplus F_k$ for all $k \in \mathbb{N}$.

$$G_{L,M} \quad \begin{array}{lcccccc} 1: & x_1, n_1 & & x_2, n_2 & & \cdots & \\ 2: & & F_1, y_1 & & F_2, y_2 & & \cdots \end{array}$$

Player 2 wins $G_{L,M}$ if $(y_n)_{n\in\mathbb{N}}$ is C-equivalent to $(x_n)_{n\in\mathbb{N}}$.

The following easy fact will be needed in the next lemma: if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are C-equivalent basic sequences, then any block-sequence of $(x_n)_{n\in\mathbb{N}}$ of the form $(\sum_{i\in I_n} \lambda_i x_i)_{n\in\mathbb{N}}$ is C-equivalent to $(\sum_{i\in I_n} \lambda_i y_i)_{n\in\mathbb{N}}$.

LEMMA 10: *Assume $(l_n)_{n\in\mathbb{N}}$ is a block-sequence in $bb_{\mathbb{Q}}(X)$ which C-continously embeds into its subsequences, and let $L = [l_n, n \in \mathbb{N}]$. Then Player 2 has a winning strategy in the game $G_{L,L}$.*

Proof: Assume the first move of Player 1 was (x_1, n_1); write $x_1 = \sum_{j\leq k_1} \lambda_j l_j$. Letting in the game H_L Player 1 play the integer n_1, k_1 times, the winning strategy of Lemma 8 provides moves $(F_1^1, z_1), \ldots, (F_1^{k_1}, z_{k_1})$ for Player 2 in that game. We let $y_1 = \sum_{j\leq k_1} \lambda_j z_j$, and $F_1 = \sum_{j=1}^{k_1} F_1^j$. In particular, $n_1 \leq F_1 \subset L$ and $y_1 \in F_1$.

We describe the choice of F_p and y_p at step p. Assuming the p-th move of Player 1 was (x_p, n_p), we write $x_p = \sum_{k_{p-1}<j\leq k_p} \lambda_j l_j$. Letting in the game H_L Player 1 play $k_p - k_{p-1}$ times the integer n_p, the winning strategy of Lemma 8 provides moves $(F_p^{k_{p-1}+1}, z_{k_{p-1}+1}), \ldots, (F_p^{k_p}, z_{k_p})$ for Player 2 in that game. We let $y_p = \sum_{k_{p-1}<j\leq k_p} \lambda_j z_j$, and $F_p = \sum_{k_{p-1}<j\leq k_p} F_p^j$. In particular, $n_p \leq F_p \subset L$ and $y_p \in \sum_{j=1}^{p} F_j$.

Finally, by construction, $(z_n)_{n\in\mathbb{N}}$ is C-equivalent to $(l_n)_{n\in\mathbb{N}}$. It follows that $(y_p)_{p\in\mathbb{N}}$ is C-equivalent to $(x_p)_{p\in\mathbb{N}}$. ∎

The non-trivial Lemma 10 will serve as the first step of a final induction which is on the model of the demonstration of Pelczar in [15] (note that there, the first step of the induction was straightforward). The rest of our reasoning will now be along the lines of her work, with the difference that we chose to express the proof in terms of games instead of using trees, and that we needed the moves of Player 2 to include the choice of finite-dimensional subspaces F_n's in which to pick the vectors y_n's. This is due to the fact that the basic sequence which witnesses the embedding of X into a given subspace generated by a subsequence is not necessarily successive on the basis of X.

Let L, M be block-subspaces in $\mathcal{G}_{\mathbb{Q}}(X)$. Let

$$a \in bb_{\mathbb{Q}}^{<\omega}(X) \quad \text{and} \quad b \in (Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X))^{<\omega}$$

be such that $|a| = |b|$ or $|a| = |b| + 1$ (here $|x|$ denotes as usual the length of the finite sequence x). Such a couple (a, b) will be called a **state** of the game $G_{L,M}$ and the set of states will be written $St(X)$. It is important to note that $St(X)$ is countable. The empty sequence in $bb_{\mathbb{Q}}^{<\omega}(X)$ (resp. $(Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X))^{<\omega}$) will be denoted by $\emptyset$.

We define $G_{L,M}(a, b)$ intuitively as "the game $G_{L,M}$ starting from the state (a, b)". Precisely, if $|a| = |b|$, then write $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_p)$, with $b_i = (B_i, \beta_i)$ for $i \leq p$.

A k-th move for Player 1 is $(x_k, n_k) \in \mathbb{Q}(X) \times \mathbb{N}$, with $x_k \in L$, $x_1 > a_p$ if $k = 1$ and $a \neq \emptyset$, and $x_k > x_{k-1}$ if $k \geq 2$. A k-th move for Player 2 is $(F_k, y_k) \in Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X)$ with $n_k \leq F_k \subset M$ and $y_k \in B_1 \oplus \cdots \oplus B_p \oplus F_1 \oplus \cdots \oplus F_k$ for all k.

$$G_{L,M}(a,b),\ |a| = |b| \qquad \begin{array}{lllll} 1: & x_1, n_1 & & x_2, n_2 & & \cdots \\ 2: & & F_1, y_1 & & F_2, y_2 & & \cdots \end{array}$$

Player 2 wins $G_{L,M}(a, b)$ if the sequence $(\beta_1, \ldots, \beta_p, y_1, y_2, \ldots)$ is C-equivalent to the sequence $(a_1, \ldots, a_p, x_1, x_2, \ldots)$.

Now if $|a| = |b| + 1$, then write $a = (a_1, \ldots, a_{p+1})$ and $b = (b_1, \ldots, b_p)$, with $b_i = (B_i, \beta_i)$ for $i \leq p$.

A first move for Player 1 is $n_1 \in \mathbb{N}$. A first move for Player 2 is $(F_1, y_1) \in Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X)$ with $n_1 \leq F_1 \subset M$ and $y_1 \in B_1 \oplus \cdots \oplus B_p \oplus F_1$.

For $k \geq 2$, a k-th move for Player 1 is $(x_k, n_k) \in \mathbb{Q}(X) \times \mathbb{N}$, with $x_k \in L$, $x_2 > a_{p+1}$ if $k = 2$, and $x_k > x_{k-1}$ if $k > 2$; a k-th move for Player 2 is $(F_k, y_k) \in Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X)$ with $n_k \leq F_k \subset M$ and $y_k \in B_1 \oplus \cdots \oplus B_p \oplus F_1 \oplus \cdots \oplus F_k$.

$$G_{L,M}(a,b),\ |a| = |b|+1 \qquad \begin{array}{lllll} 1: & n_1 & & x_2, n_2 & & \cdots \\ 2: & & F_1, y_1 & & F_2, y_2 & & \cdots \end{array}$$

Player 2 wins $G_{L,M}(a, b)$ if the sequence $(\beta_1, \ldots, \beta_p, y_1, y_2, \ldots)$ is C-equivalent to the sequence $(a_1, \ldots, a_p, a_{p+1}, x_2, \ldots)$.

We shall use the following classical stabilization process, called "zawada" in [15]; see also the proof by B. Maurey of Gowers' dichotomy theorem [14]. We define the following order relation on $\mathcal{G}_{\mathbb{Q}}(X)$: for $M, N \in \mathcal{G}_{\mathbb{Q}}(X)$, with

$M = [m_i]_{i\in\mathbb{N}}, (m_i)_{i\in\mathbb{N}} \in bb_{\mathbb{Q}}(X)$, write $M \subset^* N$ if there exists $p \in \mathbb{N}$ such that $m_i \in N$ for all $i \geq p$.

Let τ be a map defined on $\mathcal{G}_{\mathbb{Q}}(X)$ with values in the set 2^{Σ} of subsets of some countable set Σ. Assume the map τ is monotonous with respect to $\subset^*$ on $\mathcal{G}_{\mathbb{Q}}(X)$ and to inclusion on 2^{Σ}. Then by [15] Lemma 2.1, there exists a block-subspace $M \in \mathcal{G}_{\mathbb{Q}}(X)$ which is stabilizing for τ, i.e. $\tau(N) = \tau(M)$ for every $N \subset^* M$.

We now define a map τ: $\mathcal{G}_{\mathbb{Q}}(X) \to 2^{St(X)}$ by $(a,b) \in \tau(M)$ iff there exists $L \subset^* M$ such that Player 2 has a winning strategy for the game $G_{L,M}(a,b)$.

LEMMA 11: *Let M' and M be in $\mathcal{G}_{\mathbb{Q}}(X)$. If $M' \subset^* M$ then $\tau(M') \subset \tau(M)$.*

Proof: Let $M' \subset^* M$, let $(a,b) \in \tau(M')$, and let $L \subset^* M'$ be such that Player 2 has a winning strategy in $G_{L,M'}(a,b)$. Let m be an integer such that for any $x \in \mathbb{Q}(X)$, $x \in M'$ and $\min(supp(x)) \geq m$ implies $x \in M$. We describe a winning strategy for Player 2 in the game $G_{L,M}(a,b)$: assume Player 1's p-th move was (n_p, x_p) (or just n_1 if it was the first move and $|a| = |b| + 1$); without loss of generality $n_p \geq m$. Let (F_p, y_p) be the move prescribed by the winning strategy for Player 2 in $G_{L,M'}(a,b)$. Then $F_p \geq n_p \geq m$ and $F_p \subset M'$, so $F_p \subset M$. The other conditions are satisfied to ensure that we have described the p-th move of a winning strategy for Player 2 in the game $G_{L,M}(a,b)$. It remains to note that $L \subset^* M$ as well as to conclude that $(a,b) \in \tau(M)$. ∎

By the stabilization lemma, there exists a block-subspace $M_0 \in \mathcal{G}_{\mathbb{Q}}(X)$ such that for any $M \subset^* M_0$, $\tau(M) = \tau(M_0)$.

For $L, M \in \mathcal{G}_{\mathbb{Q}}(X)$ we write $L =^* M$ if $L \subset^* M$ and $M \subset^* L$.

We now define a map ρ: $\mathcal{G}_{\mathbb{Q}}(X) \to 2^{St(X)}$ by $(a,b) \in \rho(M)$ iff there exists $L =^* M$ such that Player 2 has a winning strategy for the game $G_{L,M_0}(a,b)$.

LEMMA 12: *Let M' and M be in $\mathcal{G}_{\mathbb{Q}}(X)$. If $M' \subset^* M$ then $\rho(M') \supset \rho(M)$.*

Proof: Let $M' \subset^* M$, let $(a,b) \in \rho(M)$, and let $L =^* M$ be such that Player 2 has a winning strategy in $G_{L,M_0}(a,b)$. Define $L' = M' \cap L$. As $L' \subset L$, it follows immediately that Player 2 has a winning strategy in the game $G_{L',M_0}(a,b)$. It is also clear that $L' =^* M'$, so $(a,b) \in \rho(M')$. ∎

So there exists a block-subspace $M_{00} \in \mathcal{G}_{\mathbb{Q}}(X)$ of M_0 which is stabilizing for ρ, i.e. for any $M \subset^* M_{00}$, $\rho(M) = \rho(M_{00})$.

LEMMA 13: $\rho(M_{00}) = \tau(M_{00}) = \tau(M_0)$.

Proof: First it is obvious by definition of M_0 that $\tau(M_{00}) = \tau(M_0)$.

Let $(a,b) \in \rho(M_{00})$. There exists $L =^* M_{00}$ such that Player 2 has a winning strategy in $G_{L,M_0}(a,b)$; as $L \subset^* M_0$, this implies that $(a,b) \in \tau(M_0)$.

Let $(a,b) \in \tau(M_{00})$. There exists $L \subset^* M_{00}$ such that Player 2 has a winning strategy in $G_{L,M_{00}}(a,b)$. As $M_{00} \subset M_0$, this is a winning strategy for $G_{L,M_0}(a,b)$ as well. This implies that $(a,b) \in \rho(L)$ and, by the stablization property for ρ, $(a,b) \in \rho(M_{00})$. ∎

We now turn to the concluding part of the proof of Theorem 3. By our assumption about X just before Definition 9, there exists a block-sequence $(l_n)_{n\in\mathbb{N}}$ of $bb_{\mathbb{Q}}(X)$ which is contained in M_{00}, and C-continuously embeds into its subsequences, and without loss of generality assume that $L_0 := [l_n, n \in \mathbb{N}] = M_{00}$. We fix an arbitrary block-subspace M of L_0 generated by a block-sequence in $bb_{\mathbb{Q}}(X)$ and we shall prove that L_0 embeds into M. By standard perturbation arguments this implies that L_0 is minimal.

We construct by induction a subsequence $(a_n)_{n\in\mathbb{N}}$ of $(l_n)_{n\in\mathbb{N}}$, a sequence $b_n = (F_n, y_n) \in (Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X))^\omega$ such that $F_n \subset M$ and $y_n \in F_1 \oplus \cdots \oplus F_n$ for all $n \in \mathbb{N}$, and such that $((a_n)_{n\le p}, (F_n, y_n)_{n\le p}) \in \rho(L_0)$ for all $p \in \mathbb{N}$.

By Lemma 10, Player 2 has a winning strategy in G_{L_0,L_0}, and so in particular $(\emptyset, \emptyset) \in \rho(L_0)$ (recall that $\emptyset$ denotes the empty sequence in the sets corresponding to the first and second coordinates). This takes care of the first step of the induction.

Assume $(a,b) = ((a_n)_{n\le p-1}, (F_n, y_n)_{n\le p-1})$ is a state such that $(a_n)_{n\le p-1}$ is a finite subsequence of $(l_n)_{n\in\mathbb{N}}$, such that $F_n \subset M$ and $y_n \in F_1 \oplus \cdots \oplus F_n$ for all $n \le p-1$, and such that $(a,b) \in \rho(L_0)$.

As (a,b) belongs to $\rho(L_0)$, there exists $L =^* L_0$ such that Player 2 has a winning strategy in the game $G_{L,M_0}(a,b)$. In particular, $L_0 \subset^* L$ so we may choose m_p large enough such that $l_{m_p} > a_{p-1}$ and $l_{m_p} \in L$; we let Player 1 play $a_p = l_{m_p}$. Player 2 has a winning strategy in the game $G_{L,M_0}(a',b)$, where $a' = (a_n)_{n\le p}$. In other words, (a',b) belongs to $\rho(L_0)$. Now $\rho(L_0) = \tau(M)$, so there exists $L \subset^* M$ such that Player 2 has a winning strategy in the game $G_{L,M}(a',b)$. Let Player 1 play any integer n_p, and (F_p, y_p) with $F_p \subset M$ and $y_p \in F_1 \oplus \cdots \oplus F_p$ be a move for Player 2 prescribed by that winning strategy in response to n_p. Once again, Player 2 has a winning strategy in $G_{L,M}(a',b')$, with $b' = (F_n, y_n)_{n\le p}$; i.e. $(a',b') \in \tau(M) = \rho(L_0)$.

To conclude, note that $(a_n, b_n)_{n\le p} \in \rho(L_0)$ implies in particular that $(a_n)_{n\le p}$ and $(y_n)_{n\le p}$ are C-equivalent, and this is true for any $p \in \mathbb{N}$, so $(a_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are C-equivalent. Hence $[a_n]_{n\in\mathbb{N}}$ C-embeds into M. Now $(a_n)_{n\in\mathbb{N}}$ is a subsequence of $(l_n)_{n\in\mathbb{N}}$, so by our hypothesis, L_0 C-embeds into $[a_n]_{n\in\mathbb{N}}$ and thus C^2-embeds in M, and this concludes the proof of Theorem 3.

As a consequence of our proof we obtain a uniform version of Theorem 3:

THEOREM 14: *Let $C \geq 1$ and let $\epsilon > 0$. If a Banach space is saturated with basic sequences which C-embed into their subsequences, then it contains a $C^2+\epsilon$-minimal subspace.*

4. Corollaries and remarks

D. Kutzarova drew our attention to the dual T^* of Tsirelson's space; it is minimal [2], but contains no block-minimal block-subspace (use, e.g., [2] Proposition 2.4 and Corollary 7.b.3 in their T^* versions, with Remark 1 after [2] Proposition 1.16). So Theorem 3 applies to situations where Pelczar's theorem does not. On the other hand, we do have (recall that a basic sequence $(x_n)_{n\in\mathbb{N}}$ is isomorphically homogeneous if all subspaces spanned by subsequences of $(x_n)_{n\in\mathbb{N}}$ are isomorphic):

COROLLARY 15: *A Banach space with a Schauder basis which is saturated with isomorphically homogeneous basic sequences contains a block-minimal block-subspace.*

Proof: Let X have a Schauder basis and be saturated with isomorphically homogeneous basic sequences. By the beginning of the proof of Lemma 5, we may assume X is reflexive. By Theorem 3, there exists a minimal subspace Y in X, which is a block-subspace if you wish; passing to a further block-subspace assume furthermore that Y has an isomorphically homogeneous basis. Take any block-subspace Z of $Y = [y_n]_{n\in\mathbb{N}}$; then Y embeds into Z. By reflexivity and standard perturbation results, some subsequence of $(y_n)_{n\in\mathbb{N}}$ spans a subspace which embeds as a block-subspace of Z. As $(y_n)_{n\in\mathbb{N}}$ is isomorphically homogeneous, this means that Y embeds as a block-subspace of Z. ∎

We recall that a Banach space is said to be ergodic if the relation E_0 is Borel reducible to the relation of isomorphism between its subspaces.

COROLLARY 16: *A Banach space is ergodic or contains a minimal subspace.*

Proof: We prove the stronger result that if X is a Banach space with a Schauder basis, then either E_0 is Borel reducible to isomorphism between block-subspaces of X or X contains a block-minimal block-subspace.

Assume E_0 is not Borel reducible to isomorphism between block-subspaces of X. By [16] Theorem 19, any block-sequence in X has an isomorphically homogeneous subsequence. In particular, X is saturated with isomorphically homogeneous sequences, so apply Corollary 15. ∎

COROLLARY 17: *A Banach space X contains a minimal subspace or the relation E_0 is Borel reducible to the relation of biembeddability between subspaces of X.*

Proof: Note that the relation $\sim^{emb}$ of biembeddability between subspaces of X is analytic. By [16] Theorem 15, if E_0 is not Borel reducible to biembeddability between subspaces of X, then every basic sequence in X has a subsequence $(x_n)_{n\in\mathbb{N}}$ which is homogeneous for the relation between subsequences corresponding to $\sim^{emb}$, that is, for any subsequence $(x_n)_{n\in I}$ of $(x_n)_{n\in\mathbb{N}}$, $[x_n]_{n\in I} \sim^{emb} [x_n]_{n\in\mathbb{N}}$. This means that $(x_n)_{n\in\mathbb{N}}$ embeds into its subsequences. So X is saturated with basic sequences which embed into their subsequences. ∎

We conclude with a remark about the proof of Theorem 3. The sequences $(m_p)_{p\in\mathbb{N}} \in [\omega]^\omega$ (associated to a subsequence of $(l_n)_{n\in\mathbb{N}}$) and $b_p = (F_p, y_p) \in (Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X))^\omega$ (with $(y_p)_{p\in\mathbb{N}}$ C-equivalent to $(l_{m_p})_{p\in\mathbb{N}}$) in our final induction may clearly be chosen with $F_p \subset M_p$ for all p, for an arbitrary sequence $(M_p)_{p\in\mathbb{N}}$ of block-subspaces of L_0. Also, $(l_n)_{n\in\mathbb{N}}$ C-continuously embeds into its subsequences, i.e. there is a continuous map f: $[\omega]^\omega \to bb_{\mathbb{Q}}(X)$ such that $f(A)$ is C-equivalent to $(l_n)_{n\in\mathbb{N}}$ for all $A \in bb_{\mathbb{Q}}(X)$.

By combining these two facts, it is easy to see that Player 2 has a winning strategy to produce a sequence $(y_n)_{n\in\mathbb{N}}$ which is C^2-equivalent to $(l_n)_{n\in\mathbb{N}}$, in a "modified" Gowers' game, where a p-th move for Player 1 is a block-subspace $Y_p \in \mathcal{G}_{\mathbb{Q}}(X)$ with $Y_p \subset L_0$, and a p-th move for Player 2 is a couple $(F_p, y_p) \in (Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X))^\omega$ with $F_p \subset Y_p$ and $y_p \in F_1 \oplus \cdots \oplus F_p$.

This is an instance of a result with a Gowers-type game where Player 2 is allowed to play sequences of vectors which are not necessarily block-basic sequences.

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