A UNIFORMLY CONVEX HEREDITARILY INDECOMPOSABLE BANACH SPACE

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Abstract

We construct a uniformly convex hereditarily indecomposable Banach space, using similar methods as Gowers and Maurey in [GM] and the theory of complex interpolation for a family of Banach spaces of Coifman, Cwikel, Rochberg, Sagher and Weiss ([5a]).

Introduction A *hereditarily indecomposable (or H.I.)* space is an infinite dimensional Banach space such that no subspace can be written as the topological sum of two infinite dimensional subspaces. As an easy consequence, no such space can contain an unconditional basic sequence. This notion also appears as the 'worst' type of subspace of a Banach space in [G]. In [GM], Gowers and Maurey constructed the first known example of a hereditarily indecomposable space. Gowers-Maurey space is reflexive, however it is not uniformly convex. In this article, we provide an example of a uniformly convex hereditarily indecomposable space.

1 A class of uniformly convex Banach spaces

1.1 Definitions

Let c_{00} be the space of sequences of scalars all but finitely many of which are zero. Let $e_1, e_2, \dots$ be its unit vector basis. If $E \subset \mathbb{N}$, then we shall also use the letter E for the projection from c_{00} to c_{00} defined by $E(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i \in E} a_i e_i$. If $E, F \subset \mathbb{N}$, then we write $E < F$ to mean that $\sup E < \inf F$. An *interval* of integers is a subset of $\mathbb{N}$ of the form $\{a, a+1, \dots, b\}$ for some $a, b \in \mathbb{N}$. For N in $\mathbb{N}$, E_N denotes the interval $\{1, \dots, N\}$. The *range* of a vector x in c_{00}, written $ran(x)$, is the smallest interval E such that $Ex = x$. We shall write $x < y$ to mean $ran(x) < ran(y)$. If $x_1 < \dots < x_n$ we shall say that $x_1, \dots, x_n$ are *successive*.

The corresponding notation about range and successive functions will be used for analytic functions with values in c_{00} (the range of such functions is always finite). Let $\mathcal{X}$ be the class of normed spaces of the form $(c_{00}, \|.\|)$, such that $(e_i)_{i=1}^{\infty}$ is a normalized bimonotone basis. By a *block basis* in a space $X \in \mathcal{X}$ we mean a sequence $x_1, x_2, \dots$ of successive non-zero vectors in X (note that such a sequence must be a basic sequence) and by a *block subspace* of a space $X \in \mathcal{X}$ we mean a subspace generated by a block basis.

Let f be the function $\log_2(x+1)$. If $X \in \mathcal{X}$, and all successive vectors $x_1, \dots, x_n$ in X satisfy the inequality $f(n)^{-1} \sum_{i=1}^n \|x_i\| \leq \| \sum_{i=1}^n x_i\|$, then we say that X satisfies a *lower f-estimate*.

Let $q > 1$ in $\mathbb{R}$, q' such that $1/q + 1/q' = 1$. Let $\theta \in \,]0,1[$, and p be the number defined by $1/p = 1 - \theta + \theta/q$.

Let S be the strip $\{z \in \mathbb{C}/Re(z) \in [0,1]\}$, δS its boundary, S_0 the line $\{z/Re(z) = 0\}$, S_1 the line $\{z/Re(z) = 1\}$. Let μ be the Poisson probability measure associated to the point θ for the strip S. We have $\mu(S_0) = 1 - \theta$. Let μ_0 be the probability measure on $\mathbb{R}$ defined by $\mu_0(A) = \mu(iA)/(1-\theta)$, μ_1 be the probability measure on $\mathbb{R}$ defined by $\mu_1(A) = \mu(1+iA)/\theta$. Let $\mathcal{A}_S$ be the set of analytic functions F on S, with values in c_{00}, which are L_1 on δS for $d\mu$ and which satisfy the Poisson integral representation $F(z_0) = \int_{\delta S} F(z) dP_{z_0}(z)$ on S (this is well defined since such functions have finite ranges). If F is analytic and bounded on S, then $F \in \mathcal{A}_S$.

We recall the definition of the interpolation space of a family of N-dimensional spaces from [5a]. Let $\|.\|_z$ for z in S be a family of norms on $\mathbb{C}^N$, equivalent with log-integrable constants, and such that $z \mapsto \|x\|_z$ is measurable for all x in $\mathbb{C}^N$. The interpolation space in θ is defined by the norm $\|x\| = \inf_{F \in \mathcal{A}_S^N,\ F(\theta)=x} (\int_{z \in \delta S} \|F(z)\|_z d\mu(z))$, where $\mathcal{A}_S^N$ denotes the image of the canonical projection from $\mathcal{A}_S$ into the space of functions from S to $\mathbb{C}^N$.

We generalize to the infinite-dimensional case as follows. Let $\{X_z, z \in \delta S\}$ be a family of Banach spaces in $\mathcal{X}$, equipped with norms $\|.\|_z$, such that for all x in c_{00}, the function $z \mapsto \|x\|_z$ is measurable, and such that over vectors of finite range N, the norms $\|.\|_z$ are equivalent with log-integrable constants. Let X_z^N be $E_N X_z$, X^N be the θ-interpolation space of the family X_z^N; the interpolation space of the family in θ is $completion(\cup_{N \in \mathbb{N}} X^N)$.

Now let $\{X_t, t \in \mathbb{R}\}$ be a family of spaces in $\mathcal{X}$, equipped with norms $\|.\|_t$, such that for all t in $\mathbb{R}$, X_t satisfies a f-lower estimate and for all x in c_{00}, the function $t \mapsto \|x\|_t$ is measurable. For vectors of range at most E_N, we have $f(N)^{-1}\|x\|_1 \leq \|x\|_t \leq \|x\|_1$, so that the norms $\|.\|_t$ are equivalent to $\|.\|_1$ with log-integrable constants. We are then allowed to define the θ-interpolation space of the family defined on δS as X_t if $z = it$, l_q if $z = 1 + it$. Let $\mathcal{X}_\theta$ be the class of spaces X obtained in that way.

We shall sometimes use for $z \in \delta S$ the notation $\|.\|_z$, to mean $\|.\|_t$ if $z = it$, and $\|.\|_q$ if $z = 1 + it$. There will be no ambiguity from the context. We shall similarly use the notation $\|.\|_z^*$. The notation X_t^N stands for $E_N X_t$, and X_t^{N*} for $E_N X_t^*$. Also, if not specified, the measure of a subset of $\mathbb{R}$ will be its measure for μ_0.

1.2 Properties of $\mathcal{X}_\theta$

Let X be in $\mathcal{X}_\theta$ and x be in X. Let $\mathcal{A}_\theta(x)$ be the set of functions in $\mathcal{A}_S$ that take the value x at the point θ. Given θ, it is the set of *interpolation functions for* x. By definition, for all x in X, $\|x\| = \inf_{F\in\mathcal{A}_\theta(x)}(\int_{z\in\delta S}\|F(z)\|_z d\mu(z))$. The following theorem is a useful result of [5a].

Theorem 1 *If x is of finite range, there is an interpolation function F for x, that we shall call* minimal for x, *with $ran(F) = ran(x)$ and such that*

$$\|F(it)\|_t = \|x\| \text{ a.e. and } \|F(1+it)\|_q = \|x\| \text{ a.e. .}$$

Lemma 1 *The following formula is also true:*

$$\|x\| = \inf_{F\in\mathcal{A}_\theta(x)} \left(\int_{\mathbb{R}} \|F(it)\|_t d\mu_0(t)\right)^{1-\theta} \left(\int_{\mathbb{R}} \|F(1+it)\|_q d\mu_1(t)\right)^{\theta}.$$

Proof First notice that for any F in $\mathcal{A}_\theta(x)$, by a convexity inequality, the argument in the second infimum is smaller than

$$(1-\theta)\left(\int_{\mathbb{R}} \|F(it)\|_t d\mu_0(t)\right) + \theta\left(\int_{\mathbb{R}} \|F(1+it)\|_q d\mu_1(t)\right)$$

equal to $\int_{z\in\delta S}\|F(z)\|_z d\mu(z)$, so that the second infimum is smaller than the first one.

Now, given $u\in\mathbb{R}$, the map G_u defined on $\mathcal{A}_\theta(x)$ by $G_u(F)(z) = F(z)e^{u(z-\theta)}$ is a bijection on $\mathcal{A}_\theta(x)$. Furthermore, for any u, the expressions

$$\left(\int_{\mathbb{R}} \|(G_u(F)(it)\|_t d\mu_0(t)\right)^{1-\theta} \left(\int_{\mathbb{R}} \|G_u(F)(1+it)\|_q d\mu_1(t)\right)^{\theta}$$

and

$$\left(\int_{\mathbb{R}} \|F(it)\|_t d\mu_0(t)\right)^{1-\theta} \left(\int_{\mathbb{R}} \|F(1+it)\|_q d\mu_1(t)\right)^{\theta}$$

are equal. If we choose a proper u (namely such that $\int_{\mathbb{R}}\|(G_u(F)(it)\|_t d\mu_0(t) = \int_{\mathbb{R}}\|G_u(F)(1+it)\|_q d\mu_1(t)$), this is also equal to $\int_{z\in\delta S}\|G_u(F)(z)\|_z d\mu(z)$. Consequently, the two infima are actually equal.

Proposition 1 *For all successive vectors $x_1 < \cdots < x_n$ in X,*

$$\frac{1}{f(n)^{1-\theta}}\left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}} \leq \left\|\sum_{i=1}^n x_i\right\| \leq \left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}}.$$

Proof It is enough to prove this in the interpolation space X^N defined above, written in short $(X_t^N, l_q^N)_\theta$, for any $N \geq 1$.

First inequality The unit ball of X_t^N is stable under sums of the form $\sum_{j=1}^n \lambda_j y_j$, where the y_j are successive in the unit ball of X_t^N and $\sum_{j=1}^n |\lambda_j| = 1$.

The unit ball of l_q^N is stable under sums of the form $\sum_{j=1}^n \mu_j z_j$, where the z_j are successive in the unit ball of $l_q^N$and $\sum_{j=1}^n |\mu_j|^q = 1$.

Consequently, the unit ball of X^N is stable under successive sums of the form $\sum_{j=1}^n \lambda_j^{1-\theta} \mu_j^\theta x_j$, where the x_j are in the unit ball of X^N and λ_j and μ_j satisfy the above conditions. Indeed, for every x_j in the unit ball of X^N, let F_j be minimal for x_j; the function F defined by $F(z) = \sum_{j=1}^n \lambda_j^{1-z} \mu_j^z F_j(z)$ is then in $\mathcal{A}_S$ and bounded by 1 *a.e.* on δS, so by definition, $\|F(\theta)\| \leq 1$, that is, $\sum_{j=1}^n \lambda_j^{1-\theta} \mu_j^\theta x_j$ is in the unit ball of X^N.

Now consider any successive vectors x_j in X^N, and apply this stability property to $x_j/\|x_j\|$ and $\lambda_j = \mu_j^q = \|x_j\|^p / \sum_{i=1}^n \|x_i\|^p$. Using the equality $1 - \theta + \theta/q = 1/p$, one finally gets:

$$\Big\|\sum_{j=1}^n x_i\Big\| \leq \left(\sum_{j=1}^n \|x_i\|^p\right)^{\frac{1}{p}}.$$

This inequality will be called the *upper p-estimate for X*.

Second inequality According to [5a], the duality property is true in finite dimension, that is $X^{N*} = ((X_t^N)^*, l_q^{N*})_\theta$. As X_t satisfies a lower f-estimate, so does X_t^N; the dual version of this is that the unit ball of $(X_t^N)^*$ is stable under sums of the form $(1/f(n)) \sum_{j=1}^n y_j^*$, where the y_j^* are successive. As $l_q^{N*} = l_{q'}^N$, we know that its unit ball is stable under successive sums of the form $\sum_{j=1}^n \mu_j z_j^*$, where $\sum_{j=1}^n |\mu_j|^{q'} = 1$. Letting $\lambda_j = 1/f(n)$ for each j, and using the same proof as above, we get that the unit ball of X^{N*} is stable under successive sums of the form $(1/f(n)^{1-\theta}) \sum_{j=1}^n \mu_j^\theta x_j^*$.

Now let x_j be successive vectors in X^N; for $j = 1, \ldots, n$, let x_j^* be successive dual unit vectors such that x_j^* norms x_j (recall that the basis is bimonotone in every X_t, so it is bimonotone in X). We get that $(1/f(n)^{1-\theta}) \sum_{j=1}^n \mu_j^\theta \|x_j\| \leq \| \sum_{j=1}^n x_j\|$. Choosing $\mu_j^{q'} = \|x_j\|^p / \sum_{i=1}^n \|x_i\|^p$ and using the equality $\theta/q' = 1 - 1/p$ gives the desired inequality:

$$\frac{1}{f(n)^{1-\theta}} \left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}} \leq \Big\|\sum_{i=1}^n x_i\Big\|.$$

This inequality will be called the *lower estimate for X*.

Remark Gowers-Maurey's space, and, more generally, spaces satisfying f-lower estimates 'look like' the space l_1 (for successive vectors, the triangular inequality is, up to a logarithmic term, an equality). As the interpolation space of l_1 and l_q is l_p, one expects the space X to 'look like' l_p; the above inequalities show in what sense this is true.

Proposition 2 *The dual space X^* of X is also the interpolation space - as defined at the end of 1.1 - of the family defined on δS as X_t^* if $z = it$ and $l_{q'}$ if $z = 1 + it$.*

Proof Recall that a basis $(x_n)_{n=1}^{\infty}$ of a Banach space is *shrinking* if for every continuous linear functional x^* and every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that the norm of x^* restricted to the span of $x_n, x_{n+1}, \dots$ is at most ϵ. The basis $e_1, e_2, \dots$ is a shrinking basis for X. Indeed, suppose it is not; then we can find $\epsilon > 0$, a norm-1 functional $x^* \in X^*$, and a sequence of successive normalized blocks $x_1, x_2, \dots$ such that $x^*(x_n) \geq \epsilon$ for every n. Then, using the upper p-estimate, we get $n\epsilon \leq x^*(\sum_{i=1}^n x_i) \leq \| \sum_{i=1}^n x_i \| \leq n^{1/p}$, a contradiction if we choose n big enough.

This implies that given x^* in X^*, $\|x^*\|_{X^*} = \lim_{N \to +\infty} \|E_N x^*\|_{X^{N*}}$. But this means that $X^* = completion(\cup_{n \in \mathbb{N}} X^{N*})$; furthermore, according to [5a], X^{N*} is also the interpolation space $((X_t^N)^*, l_{q'}^N)_\theta$; as $(X_t^N)^* = (X_t^*)^N$, we get the desired dual property.

Proposition 3 *The space X is uniformly convex.*

Proof It is enough to prove that any vectors x and y in the unit ball of X^N satisfy the relation $\|\frac{x+y}{2}\| \leq 1 - \delta(\|x - y\|)$ where δ is strictly positive on $]0, +\infty[$ and does not depend on N.

We know by [5a] that for any $r \geq 1$ the norm of a vector x in X_N is given by the formula $\|x\|^r = \inf_{F \in \mathcal{A}_\theta(x)} (\int_{z \in \delta S} \|F(z)\|_z^r d\mu(z))$. As in Lemma 1, we have also:

$$\|x\|^r = \inf_{F \in \mathcal{A}_\theta(x)} \left(\int_{\mathbb{R}} \|F(it)\|_t^r d\mu_0(t) \right)^{1-\theta} \left(\int_{\mathbb{R}} \|F(1+it)\|_q^r d\mu_1(t) \right)^{\theta}.$$

Suppose $q \geq 2$. Then for any vectors a and b in the unit ball of l_q^N, $\|\frac{a+b}{2}\|_q^q \leq 1 - \|\frac{a-b}{2}\|_q^q$ (this Clarkson's inequality can be found in [B]). Now let x and y be in the unit ball of X^N, let F (resp. G) be a minimal interpolation function for x (resp. y) as in Theorem 1. Let us apply the formula with $r = q$:

$$\left\| \frac{x+y}{2} \right\|^q \leq \left(\int_{\mathbb{R}} \left\| \frac{F+G}{2}(it) \right\|_t^q d\mu_0(t) \right)^{1-\theta} \left(\int_{\mathbb{R}} \left\| \frac{F+G}{2}(1+it) \right\|_q^q d\mu_1(t) \right)^{\theta}.$$

The first integral is smaller than 1, so that:

$$\left\|\frac{x+y}{2}\right\|^q \le \left(\int_{\mathbb{R}} \left\|\frac{F+G}{2}(1+it)\right\|_q^q d\mu_1(t)\right)^\theta.$$

Similarly,

$$\left\|\frac{x-y}{2}\right\|^q \le \left(\int_{\mathbb{R}} \left\|\frac{F-G}{2}(1+it)\right\|_q^q d\mu_1(t)\right)^\theta.$$

Adding these two estimates together, and using Clarkson's estimate we get

$$\left\|\frac{x+y}{2}\right\|^{q/\theta} + \left\|\frac{x-y}{2}\right\|^{q/\theta} \le 1.$$

If $q < 2$, there is another estimate in [B]: there is a constant c_q such that for any vectors a and b in the unit ball of l_q^N, $\|\frac{a+b}{2}\|_q \le 1 - c_q\|a-b\|_q^2$. Applying the same method as above, we obtain

$$\left\|\frac{x+y}{2}\right\|^{1/\theta} + c_q\|x-y\|^{2/\theta} \le 1.$$

In both the cases $q \ge 2$ and $q < 2$, the inequalities above are uniform convexity inequalities.

1.3 l_{p+}^n-averages

Definition 1 Let n be a non-zero integer, C a real number.

Let X be in $\mathcal{X}$. An *l_{1+}^n-average in X with constant C* is a normalized vector $x \in X$ such that $x = \sum_{i=1}^n x_i$ where $x_1 < \cdots < x_n$ are successive vectors and each x_i verifies $\|x_i\| \le Cn^{-1}$.

Let X be in $\mathcal{X}_\theta$. An *l_{p+}^n-average in X with constant C* is a normalized vector $x \in X$ such that $x = \sum_{i=1}^n x_i$ where $x_1 < \cdots < x_n$ are successive vectors and each x_i verifies $\|x_i\| \le Cn^{-1/p}$.

An *l_{1+}^n (resp. l_{p+}^n) -vector* is a non-zero multiple of an l_{1+}^n (resp. l_{p+}^n)-average.

Lemma 2 *Let X be in $\mathcal{X}_\theta$. For every $n \ge 1$, every $C > 1$, every block subspace Y of X contains an l_{p+}^n-average with constant C.*

Proof The proof is the same as in Lemma 3 of [GM]. Suppose the result is false for some Y. Let k be an integer such that $k \log C > (1-\theta)\log f(n^k)$, let $N = n^k$, let $x_1 < \cdots < x_N$ be any sequence of successive norm-1 vectors in Y, and let $x = \sum_{i=1}^N x_i$. For every $0 \le i \le k$ and every $1 \le j \le n^{k-i}$, let $x(i,j) = \sum_{t=(j-1)n^i+1}^{jn^i} x_t$. Thus $x(0,j) = x_j, x(k,1) = x$, and, for $1 \le i \le k$, each $x(i,j)$

is a sum of n successive $x(i-1,j)$'s. By our assumption, no $x(i,j)$ is an l_{p+}^n-vector with constant C. It follows easily by induction that $\|x(i,j)\| \leq C^{-i}n^{i/p}$ and, in particular, that $\|x\| \leq C^{-k}n^{k/p} = C^{-k}N^{1/p}$. However, it follows from the lower estimate in X that $\|x\| \geq N^{1/p}f(N)^{-(1-\theta)}$. This is a contradiction, by choice of k.

Lemma 3 *Let X be in $\mathcal{X}_\theta$. Let $0 < \epsilon < 1/4$. Let $\theta = 1/2$. Let x be an l_{p+}^n-average in X with constant $1+\epsilon$. There exists an interpolation function F for x with $ran(F) = ran(x)$, bounded almost everywhere by $1+\epsilon$, such that except on a set of measure at most $2\sqrt{\epsilon}$, $F(it)$ is an l_{1+}^n-vector in X_t, of norm 1 up to $\sqrt{\epsilon}$, with constant $1+4\sqrt{\epsilon}$.*

Such a function is called *ϵ-representative*, or *representative*, since we shall always consider l_{p+}^n-averages associated to given values of ϵ.

Proof The vector x can be written $\sum_{j=1}^n x_j$ where $x_1 < \cdots < x_n$ are successive vectors and each x_j verifies $\|x_j\| \leq (1+\epsilon)n^{-1/p}$. Let F_j' be a minimal interpolation function for x_j, let F_j be defined by $F_j(z) = n^{-1/p'+z/q'}F_j'(z)$ and let $F = \sum_{j=1}^n F_j$. We show that F is representative for x.

Notice that $F(\theta) = x$, so F is an interpolation function for x, and

$$1 = \|x\| \leq \left(\int_{\mathbb{R}} \|F(it)\|_t d\mu_0(t)\right)^{1-\theta} \left(\int_{\mathbb{R}} \|F(1+it)\|_q d\mu_1(t)\right)^{\theta}.$$

By choice of F, F is bounded by $1+\epsilon$ *a.e.* on δS, so both integrals are smaller than $1+\epsilon$. As a consequence, $\int_{t\in\mathbb{R}} \|F(it)\|_t d\mu_0(t) \geq (1+\epsilon)^{-\theta/(1-\theta)} \geq 1-\epsilon$ (recall that $\theta = 1/2$). As for every t, $\|F(it)\|_t \leq 1+\epsilon$, by a Bienaymé-Tchebitschev estimation, we get that on a set of measure at least $1-2\sqrt{\epsilon}$, $\|F(it)\|_t \geq 1-\sqrt{\epsilon}$.

So on that set, $F(it)$ is of norm 1 up to $\sqrt{\epsilon}$. For each j, $\|F_j(it)\|_t = n^{-1/p'}\|x_j\| \leq (1+\epsilon)/n$; so that $F(it)$ is an l_{1+}^n-vector in X_t with constant $(1+\epsilon)/(1-\sqrt{\epsilon}) \leq 1+4\sqrt{\epsilon}$.

1.4 Rapidly Increasing Sequences

To make a construction similar to the one in [GM], we need definitions of Rapidly Increasing Sequences in X and of special sequences in X^*. We now assume that $\theta = 1/2$.

Definition 2 Let N be a non-zero integer. Let $0 < \epsilon \leq 1$.

Let X be in $\mathcal{X}_\theta$. A sequence $x_1 < \cdots < x_N$ in X is a *Rapidly Increasing Sequence of l_{p+}^n-averages, or R.I.S., of length N with constant* $1+\epsilon$ if x_k is an $l_{p+}^{n_k}$-average with constant $1+\epsilon/n_k$ for each k, $n_1 \geq 4M_f(N/\epsilon)/\epsilon f'(1)$, and $\epsilon/2\ f(n_k)^{1/2} \geq |ran(x_{k-1})|$ for $k = 2, \ldots, N$.

Here $f'(1)$ is the right derivative of f at 1 and M_f is defined on $[1,\infty)$ by $M_f(x) = f^{-1}(36x^2)$.

In spaces X_t, we shall use R.I.S. in Gowers-Maurey sense, that is, sequences of $l_{1+}^{n_k}$-averages with constant $1+\epsilon$ with the same increasing condition as above.

We shall call both kinds "R.I.S." without ambiguity. A R.I.S.-vector is a non-zero multiple of the sum of a R.I.S.. The following proposition links the two kinds of R.I.S..

Lemma 4 *Let X be in $\mathcal{X}_\theta$. Let $0 < \epsilon \leq 1/16$. Let $x_1 < \cdots < x_n$ be a R.I.S. in X with constant $1+\epsilon$, and let $x = \sum_{k=1}^n x_k$. For each k, let F_k be representative for x_k; then $F = F_1 + \cdots + F_n$ is an interpolation function for x, and except on a set of measure at most $4\sqrt{\epsilon}/f(n)$, $F(it)$ is up to $2\sqrt{\epsilon}$ the sum of a R.I.S. in X_t with constant $1 + 4\sqrt{\epsilon}$.*

Proof It is clear that F is an interpolation function for x. According to Lemma 3, for each k, $F_k(it)$ is 'close' to an $l_{1+}^{n_k}$-average, except on a set of measure at most $2\sqrt{\epsilon/n_k}$. The union over k of these sets is of measure at most $\sum_{k=1}^n 2\sqrt{\epsilon/n_k} \leq 4\sqrt{\epsilon/n_1} \leq 4\sqrt{\epsilon}/f(n)$ (this is a consequence of the increasing condition and the lower bound for n_1 in the definition of the R.I.S.).

Now let t be in this union. For every k, let $|F|_k(it)$ denote the normalization of $F_k(it)$; $|F|_k(it)$ is an $l_{1+}^{n_k}$-average with constant $1 + 4\sqrt{\epsilon/n_k}$. The sequence $|F|_1(it) < \cdots < |F|_n(it)$ is a R.I.S. in X_t, with constant $\sup_k(1 + 4\sqrt{\epsilon/n_k}) \leq 1 + 4\sqrt{\epsilon}$ (because $1 + 4\sqrt{\epsilon} > 1 + \epsilon$, the increasing condition is indeed verified).

It remains to show that $F(it)$ and the sum of the $|F|_k(it)$ are equal up to $2\sqrt{\epsilon}$; and indeed $\|F(it) - \sum_{k=1}^n |F|_k(it)\|_t \leq \sum_{k=1}^n |1 - \|F_k(it)\||_t \leq \sum_{k=1}^n \sqrt{\epsilon/n_k} \leq 2\sqrt{\epsilon}$, so that the proof is complete.

Special sequences The trick is to define special sequences of *dual interpolation functions*. Thus, by a Gowers-Maurey construction, we obtain spaces X_t that "look like" Gowers-Maurey's space and such that the special property of the X_t is somehow uniform on t; more precisely, we build spaces X_t - and the related X - and a space Δ of dual interpolation functions such that Δ is countable, stable under 'Schlumprecht's operation' and under taking special functions, and such that any vector in the unit ball of X^* has an almost minimal interpolation function in Δ. This construction, and the proof that X is hereditarily indecomposable, are developped in the next two parts.

2 Construction of a space X in $\mathcal{X}_\theta$

2.1 Construction of spaces X_t

Let $J = \{j_1, j_2, \ldots\}$, where $(j_n)_{n\in\mathbb{N}}$ is a sequence of integers such that $f(j_1) > 256$ and $\log\log\log j_n > 4(j_{n-1})^2$ for $n > 1$. Let $K = \{j_1, j_3, j_5, \ldots\}$ and $L = \{j_2, j_4, j_6, \ldots\}$. Let $\{L_m, m \in \mathbb{N}^*\}$ be a partition of L with every L_m infinite.

For $r \in [1, +\infty]$, let $B(l_r)$ denote the unit ball of $l_r \cap c_{00}$. For $N \geq 1$ and $z \in \mathbb{C}$, let $f(N,z) = f(N)^{1-z} N^{z/q'}$ and $g(N,z) = \sqrt{f(N)}^{1-z} N^{z/q'}$.

Definition 3 Given a subset D of $\mathcal{A}_S$, for every $N > 0$, the set of *N-Schlumprecht sums in D*, written $B_N(D)$, is the set of functions of the form $f(N,z)^{-1}\Sigma_{i=1}^N F_i$, where the F_i are successive in D. A *Schlumprecht sum in D* is a N-Schlumprecht sum in D for some $N > 0$. Let $B(D)$ be the set of Schlumprecht sums. If D is countable, given an injection τ from $\cup_{m\in\mathbb{N}} B(D)^m$ to $\mathbb{N}$, and an integer k, a *special function* in D, for τ, with length k, is a function of the form $g(k,z)^{-1}\Sigma_{j=1}^k G_j$, with $G_j \in B_{n_j}(D), G_1 < \cdots < G_k$, $n_1 = j_{2k}$ and $n_j = \tau(G_1, \ldots, G_{j-1})$ for $j = 2, \ldots, k$; $G_1, \ldots, G_k$ is a *special sequence in D*.

Here, it does not seem possible to define the set of special functions before defining the spaces X_t as in [GM], so we build them at the same time by induction.

Step 1 For every t in $\mathbb{R}$, let $D_1(t) = B(l_1)$. Let $\mathcal{D}_1$ be the set of functions in $\mathcal{A}_S$ with values in $D_1(t)$ for almost every it and in $B(l_{q'})$ almost everywhere on S_1. Let Δ_1 be a countable set of functions in $\mathcal{A}_S$, dense in $\mathcal{D}_1$ for the L_1-norm (namely $\|F\| = \int_{z\in\delta S} \|F(z)\|_1 d\mu(z)$). For this first step, we may assume that all functions in Δ_1 are continuous. Let σ_1 be an injection from $\cup_{m\in\mathbb{N}}(\Delta_1)^m$ to L_1, the first subset of L in the partition $\{L_m, m \in \mathbb{N}^*\}$. Let $S_0^1 = \mathbb{R}$.

Step n We are given a set of sequences $D_{n-1}(t)$ for every t in $\mathbb{R}$, a set $\mathcal{D}_{n-1}$ of functions in $\mathcal{A}_S$, a countable set Δ_{n-1} of functions in $\mathcal{A}_S$ defined everywhere on S_0, a subset S_0^{n-1} of $\mathbb{R}$ of measure 1 (that stands for the set of 'significative' values of the functions in Δ_{n-1}), and an injection σ_{n-1} from $\cup_{m\in\mathbb{N}}(\Delta_{n-1})^m$ to $L_1 \cup \ldots \cup L_{2n-3}$.

Then let $\Delta'_n = B(\Delta_{n-1}) \cup \{EF, E \textit{ interval}, F \in \Delta_{n-1}\}$. Let τ_{n-1} be an injection from $\cup_{m\in\mathbb{N}}({\Delta'}_n^m \setminus \Delta_{n-1}^m)$ to L_{2n-2}.

Let $\mathcal{S}_{n-1}$ be the set of special functions in Δ_{n-1}, for τ_{n-1}, with length in K. For every t in $\mathbb{R}$, let $D_n^S(t)$ be the sets of sequences of the form $f(N)^{-1}\Sigma_{i=1}^N x_i$ where the x_i are successive in $D_{n-1}(t)$, $D_n^I(t)$ be the set of sequences Ex where E is an interval and x is in $D_{n-1}(t)$; if $t \in S_0^{n-1}$, let $D_n^s(t)$ be the set of sequences of the form $G(it)$ where $G \in \mathcal{S}_{n-1}$, otherwise, let $D_n^s(t) = \emptyset$. Let $D'_n(t) = D_n^S(t) \cup D_n^I(t) \cup D_n^s(t)$ and let $D_n(t) = \textit{conv}(\cup_{|\lambda|=1}\lambda {D'}_n(t))$. Let $\mathcal{D}_n$ be the set of functions in $\mathcal{A}_S$ with values in $D_n(t)$ for almost every it and in $B(l_{q'})$ almost everywhere on S_1.

We complete Δ'_n in Δ_n countable set of functions in $\mathcal{A}_S$, dense in $\mathcal{D}_n$ for the L_1-norm. There is a subset $S_0^n \subset S_0^{n-1}$ of $\mathbb{R}$ of measure 1 such that $F(it)$ is indeed in $D_n(t)$ for all F in Δ_n and for all t in S_0^n. With an injection τ'_{n-1}, from $\cup_{m\in\mathbb{N}}(\Delta_n^m \setminus {\Delta'}_n^m)$ to L_{2n-1}, we obtain an injection σ_n, from $\cup_{m\in\mathbb{N}}(\Delta_n)^m$ to $L_1 \cup \ldots \cup L_{2n-1}$.

Definition of X_t It is easy to verify that the sequences $D_n(t)$ for every t in $\mathbb{R}$, $\mathcal{D}_n$ and Δ_n are increasing, that the sequence S_0^n is decreasing and that for every n, σ_n coincides with σ_{n-1} on its set of definition.

We then define $D_t = \cup_{n\in\mathbb{N}} D_n(t)$ for every t in $\mathbb{R}$, $\mathcal{D} = \cup_{n\in\mathbb{N}}\mathcal{D}_n$, $\Delta = \cup_{n\in\mathbb{N}}\Delta_n$, $S_0^\infty = \cap_{n\in\mathbb{N}} S_0^n$ and σ the injection from $\cup_{m\in\mathbb{N}}\Delta^m$ to L whose restrictions are the σ_n.

Finally for every t in $\mathbb{R}$, we define the space X_t by its norm on c_{00}:

$$\forall x \in c_{00}, \|x\|_t = \sup_{y\in D(t)} |< x, y >|.$$

2.2 Properties of $\mathcal{D}$ and Δ

Proposition 4

(a) For every t in $\mathbb{R}$, $B(l_1) \subset D(t) \subset B(l_\infty)$.

(b) The set Δ is countable, dense in $\mathcal{D}$, stable under interval projections and Schlumprecht sums in Δ.

(c) For every t in $\mathbb{R}$, the set $D(t)$ is convex, stable under interval projections, multiplication by a scalar of modulus 1 (or balanced*), and sums of the form $f(N)^{-1}\Sigma_{i=1}^N x_i$, with $x_i \in D(t)$ and $x_1 < \cdots < x_N$.*

(d) The set $\mathcal{D}$ is convex, balanced, stable under interval projections, Schlumprecht sums in $\mathcal{D}$, and under taking special functions in Δ for σ with length in K.

Proof

(a) The left inclusion is a consequence of the facts that $D_1(t) = B(l_1)$ and that $D_n(t)$ is increasing; for the right inclusion, notice that by induction, $D_n(t) \subset B(l_\infty)$ for every n.

(b) The set Δ is countable as a countable union of countable sets; it is dense in $\mathcal{D}$ because for every n, Δ_n is dense in $\mathcal{D}_n$; the stability property under interval projections and Schlumprecht sums is ensured because for every n, Δ_n contains Δ'_n, the set of projections and sums from Δ_{n-1}.

(c) The set $D(t)$ is convex as an increasing union of convex sets; the stability properties are ensured by the definition of $D'_n(t)$ and $D_n(t)$ from $D_{n-1}(t)$.

(d) The set $\mathcal{D}_n$ is the set of functions with values in the convex, balanced, and interval projection stable sets $D_n(t)$ and $B(l_{q'})$ on δS, so that it is convex, balanced and stable under interval projections; and so is $\mathcal{D}$.

To show the Schlumprecht stability property, it is enough, given successive functions $F_1 < \cdots < F_N$ in $\mathcal{D}_{n-1}$, to show that $F = f(N,z)^{-1}\Sigma_{j=1}^N F_j$ is in $\mathcal{D}_n$. For each j, $F_j(it)$ is in $D_{n-1}(t)$ almost everywhere. The set of $t \in \mathbb{R}$ such that this happens for every j is still of measure 1. On this set, $F(it) = (f(N)N^{-1/q'})^{it}(1/f(N))\sum_{j=1}^N F_j(it)$ is in $D_n(t)$, by the definition of $D_n(t)$. In the same way, almost everywhere on S_1, $F_j(1+it)$ is in $B(l_{q'})$ for every j, so that $F(1+it)$ is in $B(l_{q'})$ too. By definition, this means that F is in $\mathcal{D}_n$.

To show the special property, first notice that a special function G in Δ is a special function in Δ_n for some n in $\mathbb{N}$. It follows that $G(it) \in D_{n+1}(t)$ for every t in S_0^n, that is, almost everywhere; furthermore, $G(1+it)$ is in $B(l_{q'})$ almost everywhere; so G is in $\mathcal{D}_{n+1}$.

Lemma 5 *Let $\mathcal{S}$ be the set of functions in $\mathcal{A}_S$ with values in $D(t)$ for almost every it and in $B(l_{q'})$ almost everywhere on S_1. Then $\mathcal{D}$ is dense in $\mathcal{S}$ for the L_1-norm.*

Proof Let F be in $\mathcal{S}$, $0 < \epsilon < 1$. Let N be such that $ran(F) \subset E_N$.

We recall Havin lemma from [P] in a rougher version. Furthermore, we state it on S instead of on the unit disk of $\mathbb{C}$ (the two versions are equivalent using a conformal mapping).

Lemma For every $\epsilon' > 0$, there exists $\delta > 0$ such that for every subset e of δS with $\mu(e) \leq \delta$, there exists g_e in $H^\infty(S)$ with $|g_e| \leq 1$ *a.e.* on δS, $\sup_{z \in e} |g_e(z)| \leq \epsilon'$, and $\int_{\delta S} |g_e(z) - 1| d\mu(z) \leq \epsilon'$.

Now let δ be associated to $\epsilon' = \epsilon/N$. The sequence $(\{t : F(it) \in D_n(t)\})_{n \in \mathbb{N}}$ is increasing and its union is of measure 1 for μ_0, so there exists n such that $T = \{t / F(it) \in D_n(t)\}$ is of measure at least $1 - \delta$. For μ, $\delta S \setminus iT$ is of measure at most $\delta(1-\theta) \leq \delta$. Let H be the function $g_{\delta S \setminus iT}$. Let $\tilde{F} = H.F$. The function $\tilde{F}$ is in $\mathcal{A}_S$. Furthermore, $\tilde{F}(1+it)$ is in $B(l_{q'})$ *a.e.* on S_1, $\tilde{F}(it)$ is in $D_n(t)$ *a.e.* on T; this last assertion is also true on $S_0 \setminus T$, because almost everywhere on this set, $\tilde{F}(it)$ is in $1/N \ D(t)$ and because, for functions of range at most E_N, we have the following inclusions: $1/N \ D(t) \subset 1/N \ B(l_\infty) \subset B(l_1) \subset D_n(t)$. This proves that $\tilde{F}$ is in $\mathcal{D}_n$.

It remains to show that F and $\tilde{F}$ are close, and indeed:

$$\int_{\delta S} \|(F - \tilde{F})(z)\|_1 d\mu(z) \leq N \int_{\delta S} |H(z) - 1| d\mu(z) \leq \epsilon.$$

2.3 Definition of X

For every x in c_{00}, the function $t \mapsto \|x\|_t$ is measurable. To see it, it is enough to prove that the restriction of the function to S_0^∞ is measurable. We prove this by induction on $|ran(x)|$. Remember that $\|x\|_t = \sup_{y \in D(t)} | < x, y > |$. Now let y be in $D(t)$; either y is, up to multiplication by a scalar of modulus 1, the value in it of the projection of a special function, and there are countably many of them; or y is a n-Schlumprecht sum with $n > 1$ so that $| < x, y > | \leq (1/f(n)) \sum_{j=1}^n \|\mathcal{E}_j x\|_t$, where $\mathcal{E}_1 < \cdots < \mathcal{E}_n$ are successive intervals; or y is in $B(l_1)$. Finally,

$$\|x\|_t = \|x\|_\infty \bigvee \sup_{G \ special, E} | < x, EG(it) > | \bigvee \sup_{n \geq 2, \mathcal{E}_1 < \cdots < \mathcal{E}_n} \frac{1}{f(n)} \sum_{j=1}^n \|\mathcal{E}_j x\|_t.$$

We may restrict the last sup to intervals $\mathcal{E}_j$ that do not contain $ran(x)$; $t \mapsto \|x\|_t$ is then the supremum of a countable family of measurable functions by the induction hypothesis, so it is a measurable function.

Furthermore, it follows from the stability property of $D(t)$ that for every t in $\mathbb{R}$, X_t satisfies a lower-f estimate. We can then define a Banach space X in $\mathcal{X}_\theta$ as in the first part of this article.

Lemma 6 *Let $F^* \in \mathcal{D}$. Then $F^*(\theta)$ is in the unit ball of X^*.*

Proof First notice that if we restrict them to finite range vectors, it is a consequence of their convexity and of the definition of $\|.\|_t$ that the unit ball of X_t^* and $\overline{D(t)}$ coincide. Now, given F^* in $\mathcal{D}$, it is of finite range. For almost every t, $F^*(it) \in D(t)$, so that by the previous remark, $\|F^*(it)\|_t^* \leq 1$. Furthermore, $\|F^*(1+it)\|_{q'} \leq 1$, so by Proposition 2, $\|F^*(\theta)\|^* \leq 1$.

We need to recall some definitions and properties of [GM]. Let $\mathcal{F}$ be Schlumprecht's space of functions (the explicit definition is in [GM]; just think of these functions as log-like). We notice that f and $\sqrt{f} \in \mathcal{F}$. Given X in $\mathcal{X}$, given g in $\mathcal{F}$, a functional x^* in X^* is an $(M,g)-form$ if $\|x^*\|^* \leq 1$ and $x^* = \sum_{j=1}^{M} x_j^*$ for some sequence $x_1^* < \cdots < x_M^*$ of successive functionals such that $\|x_j^*\|^* \leq g(M)^{-1}$ for each j.

Let $K_0 \subset K$, and let us define a function $\phi : [1,\infty) \mapsto [1,\infty)$ as

$$\phi(x) = \sqrt{f(x)} \ \text{ if } x \in K_0, \ \phi(x) = f(x) \ \text{ otherwise.}$$

We now state two lemmas that are a slight modification of Lemma 7 of [GM] for the first one and a mixture of Lemmas 8 and 9 of [GM] for the second one.

Then we prove that the property of minimality of the R.I.S. (Lemma 10 of [GM]) is true in every X_t, and then that it can be extended to X.

Lemma 7 *Let $f, g \in \mathcal{F}$ with $g \geq \sqrt{f}$, let $X \in \mathcal{X}$ satisfy a lower f-estimate, let $0 < \epsilon \leq 1$, let $x_1 < \cdots < x_N$ be a R.I.S. in X for f with constant $1+\epsilon$, and let $x = \sum_{i=1}^{N} x_i$. Suppose that*

$$\|Ex\| \leq 1 \vee \sup\{|x^*(Ex)| : M \geq 2, x^* \text{ is an } (M,g)-form\}$$

for every interval E. Then $\|x\| \leq (1+2\epsilon)Ng(N)^{-1}$.

Lemma 8 *Given $K_0 \subset K$, there is a function $g : [1,\infty) \mapsto [1,\infty)$ such that: $g \in \mathcal{F}$, $\sqrt{f} \leq g \leq \phi \leq f$, and if $N \in J \setminus K_0$, then $g = f$ on the interval $[\log N, \exp N]$.*

Lemma 9 *Let $t \in \mathbb{R}$. Let $N \in L$, let $n \in [\log N, \exp N]$, let $0 < \epsilon \leq 1$, and let $x_1 < \cdots < x_n$ be a R.I.S. in X_t with constant $1+\epsilon$. Then*

$$\Big\| \sum_{i=1}^{n} x_i \Big\|_t \leq (1+2\epsilon)n/f(n).$$

Proof The space $X_t \in \mathcal{X}$ satisfies a lower f-estimate.

Let x be the sum of the R.I.S. $x_1 < \cdots < x_n$. Let E be any interval. Let ϕ be the function defined above in the case $K_0 = K$ and g associated to ϕ by Lemma 8. Let x^* be a functional in $D(t)$. If x^* is in $D_1(t)$, then $|x^*(Ex)| \leq 1$. Else there exists $m \geq 2$ such that x^* is in $D_m(t) \setminus D_{m-1}(t)$; then, by definition of $D_m(t)$, either x^* is an $(M, f) - form$ with $M \geq 2$ or x^* is an $(M, \sqrt{f}) - form$ with $M \in K$; since $g \leq \phi$, it follows that x^* is an $(M, g) - form$ with $M \geq 2$. Consequently,

$$\|Ex\|_t \leq 1 \vee \sup\{|x^*(Ex)| : M \geq 2,\ x^* \ is\ an\ (M,g) - form\}$$

Since $g \in \mathcal{F}$ and $g \geq \sqrt{f}$, all the hypotheses of Lemma 7 are satisfied. It follows that $\| \sum_{i=1}^{n} x_i \|_t \leq (1+2\epsilon)n/g(n)$. By Lemma 8, $g(n) = f(n)$, which proves our statement.

Lemma 10 *Let X be the space defined at the beginning of 2.3. Let $N \in L$, let $n \in [\log N, \exp N]$, let $0 < \epsilon \leq 1/16$, and let $X_1 < \cdots < X_n$ be a R.I.S. in X with constant $1+\epsilon$. Then*

$$\Big\| \sum_{i=1}^{n} X_i \Big\| \leq (1+10\sqrt{\epsilon})n^{1/p}/f(n)^{1-\theta}.$$

Proof Let F_k be representative for X_k, and $F = F_1 + \ldots + F_n$. We know that F is an interpolation function for $X_1 + \cdots + X_k$ so

$$\Big\| \sum_{i=1}^{n} X_i \Big\| \leq \left(\int_{\mathbb{R}} \|F(it)\|_t d\mu_0(t) \right)^{1-\theta} \left(\int_{\mathbb{R}} \|F(1+it)\|_q d\mu_1(t) \right)^{\theta}.$$

For the second integral, the following estimate holds:

$$\int_{\mathbb{R}} \|F(1+it)\|_q d\mu_1(t) \leq (1+\epsilon)n^{1/q}.$$

According to Lemma 4, there is a set A of measure at most $4\sqrt{\epsilon}/f(n)$ such that on $\mathbb{R} \setminus A$, $F(it)$ is up to $2\sqrt{\epsilon}$ the sum x_t of a R.I.S. in X_t. On $\mathbb{R} \setminus A$, $\|F(it)\|_t \leq \|x_t\|_t + 2\sqrt{\epsilon}$; furthermore, x_t is a R.I.S. in X_t with constant $1+4\sqrt{\epsilon}$, so that by Lemma 9, $\|x_t\|_t \leq (1+8\sqrt{\epsilon})n/f(n)$. On A, we have only $\|F(it)\|_t \leq (1+\epsilon)n$.

Gathering these estimates, we get:

$$\int_{\mathbb{R}} \|F(it)\|_t d\mu_0(t) \leq [(1+8\sqrt{\epsilon})\frac{n}{f(n)} + 2\sqrt{\epsilon}] + \frac{4\sqrt{\epsilon}}{f(n)}\ (1+\epsilon)n \leq (1+15\sqrt{\epsilon})\frac{n}{f(n)}.$$

Going back to the R.I.S. $X_1 < \cdots < X_n$, we have

$$\Big\|\sum_{i=1}^{n} X_i\Big\| \leq (1+15\sqrt{\epsilon})^{1-\theta}(1+\epsilon)^{\theta}\frac{n^{1-\theta+\theta/q}}{f(n)^{1-\theta}} \leq (1+10\sqrt{\epsilon})\frac{n^{1/p}}{f(n)^{1-\theta}}.$$

Lemma 11 *Let $t \in \mathbb{R}$. Let $N \in L$, let $0 < \epsilon < 1/4$, let $M = N^{\epsilon}$ and let $x_1 < \cdots < x_N$ be a R.I.S. in X_t with constant $1+\epsilon$. Then $\sum_{i=1}^{N} x_i$ is an l_{1+}^{M}-vector in X_t with constant $1+4\epsilon$.*

Proof It is the same as the one of Lemma 11 in [GM]. Let $m = N/M$, let $x = \sum_{i=1}^{N} x_i$ and for $1 \leq j \leq M$ let $y_j = \sum_{i=(j-1)m+1}^{jm} x_i$. Then each y_j is the sum of a R.I.S. of length m with constant $(1+\epsilon)$. By Lemma 9 we have $\|y_j\|_t \leq (1+2\epsilon)mf(m)^{-1}$ for every j while $\|\sum_{j=1}^{m} y_j\|_t = \|x\| \geq Nf(N)^{-1}$. It follows that x is an l_{1+}^{M}-vector in X_t with constant at most $(1+2\epsilon)f(N)/f(m)$. But $m = N^{1-\epsilon}$ so $f(N)/f(m) \leq (1-\epsilon)^{-1}$. The result follows.

Lemma 12 *Let $\epsilon_0 = 1/10$. Let $k \in K$ and $F_1^*, \ldots, F_k^*$ be a special sequence of length k, with $F_i^* \in B_{M_i}(\Delta)$. Let $t \in S_0^{\infty}$. Let $x_1 < \cdots < x_k$ a sequence of successive vectors in X_t, where every x_i is a normalized R.I.S.-vector of length M_i and constant $1+\epsilon_0/4$. Suppose $ran(F_i^*) \subset ran(x_i)$ for $i = 1, \ldots, k$, and $1/2\ \epsilon_0 f(M_i^{\epsilon_0/4})^{1/2} \geq |ran(x_{i-1})|$ for $i = 2, \ldots, k$.*

If for every interval E, $|(\sum_{i=1}^{k} F_i^*(t))(\sum_{i=1}^{k} Ex_i)| \leq 4$, then

$$\Big\|\sum_{i=1}^{k} x_i\Big\|_t \leq (1+2\epsilon_0)k/f(k).$$

Proof First we recall two lemmas of [GM].

Lemma GM4 Let $M, N \in \mathbb{N}$ and $C \geq 1$, let $X \in \mathcal{X}$, let $x \in X$ be an l_{1+}^{N}-vector with constant C and let $\mathcal{E}_1 < \cdots < \mathcal{E}_M$ be a sequence of intervals. Then $\sum_{j=1}^{M} \|\mathcal{E}_j x\| \leq C(1+2M/N)\|x\|$.

Lemma GM5 Let $f, g \in \mathcal{F}$ with $g \geq f^{1/2}$ and let $X \in \mathcal{X}$ satisfy a lower f-estimate. Let $0 < \epsilon \leq 1$, let $x_1 < \cdots < x_N$ be a R.I.S. in X with constant $1+\epsilon$ and let $x = \sum_{i=1}^{N} x_i$. Let $M \geq M_f(N/\epsilon)$, let x^* be an (M, g)-form and let E be any interval. Then $|x^*(Ex)| \leq 1+2\epsilon$.

According to Lemma 11, each x_i is an $l_{1+}^{N_i}$-average with constant $1+\epsilon_0$, where $N_i = M_i^{\epsilon_0/4}$. The increasing condition and the lower bound for M_1 ensure that $x_1 < \cdots < x_k$ is a R.I.S. in X_t of length k with constant $1+\epsilon_0$.

To prove this Lemma we shall apply Lemma 7. First, we show that if $G_1^*, \ldots, G_k^*$ is any special sequence in Δ of length k and E is any interval, then $|z^*(Ex)| < 1$, where z^* is the $(k, \sqrt{f})$-form $f(k)^{-1/2} \sum_{i=1}^k z_i^*$ with $z_j^* = G_j^*(it)$, and $x = \sum_{i=1}^k x_i$.

Indeed, let s be maximal such that $G_s^* = F_s^*$ or zero if no such s exists. Suppose now $i \neq j$ or one of i, j is greater than $s+1$. Then since σ is an injection, we can find $L_1 \neq L_2 \in L$ such that z_i^* is an (L_1, f)-form and x_j is the normalized sum of a R.I.S. of length L_2 and also an $l_{1+}^{L_2'}$-average with constant $1+\epsilon_0$, where $L_2' = L_2^{\epsilon_0/4}$. We can now use Lemmas GM4 and GM5 to show that $|z_i^*(Ex_j)| < k^{-2}$.

If $L_1 < L_2$, it follows from the lacunarity of L that $L_1 < L_2'$. We know that $L_1 \geq j_{2k}$ since L_1 appears in a special sequence of length k. Lemma GM4 thus gives $|z_i^*(Ex_j)| = |(Ez_i^*)(x_j)| \leq 3(1+\epsilon_0)/f(L_1)$. The conclusion in this case now follows from the fact that $f(l) \geq 4k^2$ when $l \geq j_{2k}$.

If $L_2 < L_1$, we apply Lemma GM5 in X_t with $\epsilon = 1$ to the non-normalized sum x_j' of the R.I.S. the normalized sum of which is x_j. The definition of L gives us that $M_f(L_2) < L_1$, so Lemma GM5 gives $|z_i^*(Ex_j')| \leq 3$. It follows from the lower f-estimate in X_t that $\|x_j'\| \geq L_2/f(L_2)$. The conclusion now follows because $l \geq j_{2k}$ implies that $f(l)/l \leq 1/4k^2$.

Now choose an interval E' such that

$$\Big|(\sum_{i=1}^{s} z_i^*)(Ex)\Big| = \Big|(\sum_{i=1}^{k} F_i^*(it))(E'x)\Big| \leq 4.$$

It follows that

$$\Big|(\sum_{i=1}^{k} z_i^*)(Ex)\Big| \leq 4 + |z_{s+1}^*(x_{s+1})| + k^2.k^{-2} \leq 6.$$

We finally obtain that $|z^*(Ex)| \leq 6f(k)^{-1/2} < 1$ as claimed.

Now let ϕ' be the function

$$\phi'(x) = \sqrt{f(x)} \ \textit{if} \ x \in K, x \neq k, \phi'(x) = f(x) \ \textit{otherwise}.$$

Let g' be the function obtained from ϕ' by Lemma 8 in the case $K_0 = K \setminus \{k\}$; we know that $g'(l) = f(l)$ for every $l \in L \cup \{k\}$.

It follows from what we have just shown about special sequences of length k that for every interval E,

$$\|Ex\|_t \leq 1 \vee \sup\{|x^*(Ex)| : M \geq 2, x^* \ \textit{is an} \ (M, g') - \textit{form}\}.$$

Since x is the sum of a R.I.S., Lemma 7 implies that $\|x\|_t \leq (1+2\epsilon_0)kg'(k)^{-1} = (1+2\epsilon_0)k/f(k)$.

3 X is hereditarily indecomposable

Let Y and Z be two infinite-dimensional subspaces of X. We want to show that their sum is not a topological sum. Let $\delta > 0$. We shall build two vectors $y \in Y$ and $z \in Z$ such that $\delta\|y+z\| > \|y-z\|$.

Let $\epsilon_0 = 1/10$. Let $k \in K$ be an integer such that $1/4 < \epsilon_0\ k^{1/p}/f(k)^{1-\theta}$ and $2/\sqrt{f(k)}^{1-\theta} < \delta$, and let $\epsilon > 0$ be such that $\sqrt{\epsilon} \leq \epsilon_0/4kf(k)$. We may assume that both Y and Z are spanned by block bases. By Lemma 2, Y and Z contain, for every $N \in \mathbb{N}$, an l_{p+}^N-average with constant $1+\epsilon$. We now build a sequence $(x_j)_{j=1}^k$ in X by iteration.

First step Let $x_1 \in Y$ be a R.I.S.-vector of norm 1, constant $1+\epsilon$ and length $M_1 = j_{2k}$; we have $M_1^{\epsilon_0/4} = N_1 \geq 4M_f(k/\epsilon_0)/\epsilon_0 f'(1)$. Let $x_{11} < \cdots < x_{1M_1}$ be the R.I.S. whose normalized sum is x_1: there exists λ_1 such that $\lambda_1 x_1 = x_{11} + \cdots + x_{1M_1}$. Applying the lower estimate in X and Lemma 10, we get

$$M_1^{1/p}/f(M_1)^{1-\theta} \leq \|\lambda_1 x_1\| \leq (1+10\sqrt{\epsilon})M_1^{1/p}/f(M_1)^{1-\theta}.$$

so that $\lambda_1 = M_1^{1/p}/f(M_1)^{1-\theta}$ up to the multiplicative factor $1+10\sqrt{\epsilon}$.

Now we associate to x_{1m}:

- a representative function F_{1m} for x_{1m};
- a vector x_{1m}^* in X^* that norms x_{1m} and with $ran(x_{1m}^*) \subset ran(x_{1m})$;
- a minimal interpolation function F_{1m}^* for x_{1m}^*; it exists because of Proposition 2 and because, as x_{1m}^* is of finite range, Theorem 1 applies.

The function F_{1m}^* is in $\overline{\mathcal{S}}$. Indeed, remember that if we restrict them to finite range vectors, the unit ball of X_t^* and $\overline{D(t)}$ coincide; so by the convexity of $D(t)$, for every $\nu > 0$, the function $F_{1m}^*/(1+\nu)$ takes its values in $D(t)$ for almost every it; as it takes its values in $B(l_{q'})$ *a.e.* on S_1, it is in $\mathcal{S}$, which ends the proof.

By Lemma 5, F_{1m}^* can be approached by a function $\mathcal{F}_{1m}^*$ in Δ (and because of the interval projection stability of Δ, we may assume that $ran(\mathcal{F}_{1m}^*) \subset ran(F_{1m}^*)$). More precisely, we suppose that $\mathcal{F}_{1m}^*$ is close to F_{1m}^* up to $\epsilon/(1+\epsilon)$ for the norm $\int_{z\in\delta S}\|.\|_z^* d\mu(z)$ (over functions of finite range, this norm is equivalent to the norm $\int_{z\in\delta S}\|.\|_1 d\mu(z)$ first introduced).

Lastly, we define two functions:

Let $\mathcal{F}_1^* = f(M_1,z)^{-1}\sum_{m=1}^{M_1}\mathcal{F}_{1m}^*$. It belongs to Δ. Let $x_1^* = \mathcal{F}_1^*(\theta)$.

Let $F_1 = f(M_1,z)M_1^{-1}\sum_{m=1}^{M_1}F_{1m}$.

Iteration Let $M_2 = \sigma(\mathcal{F}_1^*) \in L$. We may assume we chose $\mathcal{F}_1^*$ such that $1/2\ \epsilon_0 f(M_2^{\epsilon_0/4})^{1/2} \geq |ran(F_1)|$ (by choosing a function $\mathcal{F}_{11}^*$ such that M_2 is big enough). Let $x_2 \in Z$ be a R.I.S.-vector of norm 1, constant $1+\epsilon$ and length M_2, and $x_2 > x_1$; and repeat the above construction. By iterating it, we obtain for $j = 1, \ldots, k$ sequences $F_j, x_j, \mathcal{F}_j^*, x_j^*$ such that:

- $x_j \in Y$ when j is odd, $x_j \in Z$ otherwise.
- $\|x_j\| = 1$ for every j and $\|x_j^*\|^* \leq 1$.
- $x_j = F_j(\theta)$ up to $10\sqrt{\epsilon}$ and $x_j^* = \mathcal{F}_j^*(\theta)$.
- $\mathcal{F}_1^*, \ldots, \mathcal{F}_k^*$ is a special sequence of length k.
- For $j = 2, \ldots, k$, $1/2\ \epsilon_0 f(M_j^{\epsilon_0/4})^{1/2} \geq |ran(F_{j-1})|$.
- For every j, $< \mathcal{F}_j^*(\theta), F_j(\theta) >= 1$ up to ϵ.
- For every j, except on $\mathcal{J}_j$ of measure at most $2\sqrt{\epsilon}$, $< \mathcal{F}_j^*(it), F_j(it) >= 1$ up to $2\sqrt{\epsilon}$.
- For every j, except on $\mathcal{J}_j'$ of measure at most $4\sqrt{\epsilon}/f(M_j)$, $F_j(it)$ is up to $10\sqrt{\epsilon}$ the normalized sum of a R.I.S. with constant $1 + 4\sqrt{\epsilon} \leq 1 + \epsilon_0/4$.

Proof Only the last three points are not obvious.

First point For F and F^* in $\mathcal{A}_S$, define $< F, F^* >$ to be $\int_{z \in \delta S} < F(z), F^*(z) > d\mu(z)$, and notice that this is equal to $< F(\theta), F^*(\theta) >$ by analyticity. Now for every j,

$$< \mathcal{F}_j^*, F_j >= \frac{1}{M_j} \sum_{m=1}^{M_j} < \mathcal{F}_{jm}^*, F_{jm} > .$$

If we replace each $\mathcal{F}_{jm}^*$ by F_{jm}^*, the sum is equal to

$$\frac{1}{M_j} \sum_{m=1}^{M_j} < x_{jm}^*, x_{jm} >= 1.$$

The error we make by doing this is $|1/M_j \sum_{m=1}^{M_j} < \mathcal{F}_{jm}^* - F_{jm}^*, F_{jm} > |$, smaller than

$$\frac{1}{M_j} \sum_{m=1}^{M_j} \int_{z \in \delta S} \|(\mathcal{F}_{jm}^* - F_{jm}^*)(z)\|_z^* \|F_{jm}(z)\|_z d\mu(z) \leq \frac{1}{M_j} \sum_{m=1}^{M_j} \frac{\epsilon}{1+\epsilon}(1+\epsilon) \leq \epsilon$$

(we recall that as F_{jm} is representative for x_{jm}, $\|F_{jm}(z)\|_z \leq 1 + \epsilon$ *a.e.*).

Second point Let F_j^* be the function $f(M_1,z)^{-1}\sum_{m=1}^{M_1} F_{jm}^*$. It is easy to see that

$$1 = \int_{z\in\delta S} < F_j^*(z), F_j(z) > d\mu(z),$$

while

$$< F_j^*(z), F_j(z) >\leq 1+\epsilon \ a.e. \ .$$

By a Bienaymé-Tchebitschev estimation, except on a set of measure at most $\sqrt{\epsilon}$, $< F_j^*(z), F_j(z) >= 1$ up to $\sqrt{\epsilon}$. Furthermore, we know that

$$\int_{z\in\delta S} | < (\mathcal{F}_j^* - F_j^*)(z), F_j(z) > | d\mu(z) \leq \epsilon$$

so that except on a set of measure at most $\sqrt{\epsilon}$, $< (\mathcal{F}_j^* - F_j^*)(z), F_j(z) >= 0$ up to $\sqrt{\epsilon}$.

Adding these two estimates completes the proof.

Third point For each m, F_{jm} is representative for x_{jm}, so by Lemma 4, except on a set $\mathcal{J}_j'$ of measure $4\sqrt{\epsilon}/f(M_j)$, we have

$$\Big\| \sum_{m=1}^{M_j} F_{jm}(it) - x_t \Big\|_t \leq 2\sqrt{\epsilon},$$

where x_t is the sum of a R.I.S. in X_t with constant $1+4\sqrt{\epsilon}$. So

$$\left\| F_j(it) - \left(\frac{M_j^{1/q'}}{f(M_j)}\right)^{it} \frac{f(M_j)}{M_j} x_t \right\|_t \leq 2\sqrt{\epsilon}\frac{f(M_j)}{M_j} \leq 2\sqrt{\epsilon}.$$

The proof follows, because by Lemma 9,

$$M_j/f(M_j) \leq \|x_t\|_t \leq (1+8\sqrt{\epsilon})M_j/f(M_j),$$

so $f(M_j)/M_j \ x_t$ is up to $8\sqrt{\epsilon}$ a *normalized* R.I.S.-vector.

Estimation of $\|\sum_{j=1}^k x_j\|$ Let $\mathcal{G}^* = g(k,z)^{-1}\Sigma_{j=1}^k \mathcal{F}_j^*$. Since for every j, $\mathcal{F}_j^* \in \Delta$, and k is in K, $\mathcal{G}^*$ is in $\mathcal{D}$ and by Lemma 6, $x^* = \mathcal{G}^*(\theta)$ is in the unit ball of X^*.

So $\|\sum_{j=1}^k F_j(\theta)\| \geq x^*(\sum_{j=1}^k F_j(\theta)) \geq (1-\epsilon)k^{1/p}/\sqrt{f(k)}^{1-\theta}$, and

$$\Big\|\sum_{j=1}^k x_j\Big\| \geq (1-\epsilon_0)k^{1/p}/\sqrt{f(k)}^{1-\theta} - 1/4 \geq (1-2\epsilon_0)k^{1/p}/\sqrt{f(k)}^{1-\theta}.$$

(the 1/4 is the error we made by replacing the x_j's by the $F_j(\theta)$'s).

Estimation of $\|\sum_{j=1}^{k}(-1)^{j-1}x_j\|$ Let $\mathcal{J}$ be the union of the $\mathcal{J}_j$'s and the $\mathcal{J}_j'$'s. The set $\mathcal{J}$ is of measure at most $6k\sqrt{\epsilon}$.

For every t in $\mathbb{R}\setminus\mathcal{J}$, for every interval E, let us evaluate

$$\left|\left(\sum_{j=1}^{k}\mathcal{F}_j^*(it)\right)\left(\sum_{j=1}^{k}(-1)^{j-1}EF_j(it)\right)\right|.$$

This is a sum of at most k scalars. Those who come from terms of range included in E are equal to $(-1)^{j-1}$ up to $2\sqrt{\epsilon}$, so that their sum is $-1, 0$ or 1 up to $2k\sqrt{\epsilon}$; two others can come from terms whose range intersects E, they are bounded in modulus by $1+10\sqrt{\epsilon}$; the others are equal to 0. So the sum is smaller than $1+2k\sqrt{\epsilon}+2(1+10\sqrt{\epsilon})\leq 3+3k\sqrt{\epsilon}$.

For every j, $F_j(it)$ is up to $10\sqrt{\epsilon}$ a R.I.S. vector $x_j(t)$. The $(-1)^{j-1}x_j(t)$'s satisify the hypotheses of Lemma 12: the increasing condition is satisfied, and for every interval E,

$$\left|\left(\sum_{j=1}^{k}\mathcal{F}_j^*(it)\right)\left(\sum_{j=1}^{k}(-1)^{j-1}Ex_j(t)\right)\right|\leq 3+3k\sqrt{\epsilon}+10k\sqrt{\epsilon}\leq 4.$$

It then follows from the conclusion of Lemma 12 and the relation between $F_j(t)$ and $x_j(t)$ that

$$\left\|\sum_{j=1}^{k}(-1)^{j-1}F_j(it)\right\|_t\leq(1+2\epsilon_0)k/f(k)+10k\sqrt{\epsilon}.$$

It follows that

$$\int_{\mathbb{R}\setminus\mathcal{J}}\left\|\sum_{j=1}^{k}(-1)^{j-1}F_j(it)\right\|_t d\mu_0(t)\leq(1+2\epsilon_0)k/f(k)+10k\sqrt{\epsilon}.$$

We now want to estimate the integral of this same norm on $\mathcal{J}$. It is enough, by a triangular inequality, to evaluate $\int_{t\in\mathcal{J}}\|F_j(it)\|_t d\mu_0(t)$. If t belongs to $\mathcal{J}_j'$, by a triangular inequality, $\|F_j(it)\|_t\leq(1+\epsilon)f(M_j)$, but recall that $\mathcal{J}_j'$ is of measure at most $4\sqrt{\epsilon}/f(M_j)$; else, $F_j(it)$ is up to $10\sqrt{\epsilon}$ a normalized R.I.S. vector, so that $\|F_j(it)\|_t\leq 1+10\sqrt{\epsilon}$, and this on a set of measure less than $6k\sqrt{\epsilon}$. Finally,

$$\int_{\mathcal{J}}\|F_j(it)\|_t d\mu_0(t)\leq 6k\sqrt{\epsilon}(1+10\sqrt{\epsilon})+\frac{4\sqrt{\epsilon}}{f(M_j)}(1+\epsilon)f(M_j)\leq 7k\sqrt{\epsilon}.$$

and

$$\int_{\mathcal{J}}\left\|\sum_{j=1}^{k}(-1)^{j-1}F_j(it)\right\|_t d\mu_0(t)\leq 7k^2\sqrt{\epsilon}.$$

It follows from these two estimates that

$$\int_{\mathbb{R}} \Big\| \sum_{j=1}^{k} (-1)^{j-1} F_j(it) \Big\|_t d\mu_0(t) \leq (7k^2+10k)\sqrt{\epsilon}+(1+2\epsilon_0)\frac{k}{f(k)} \leq (1+4\epsilon_0)\frac{k}{f(k)}.$$

Furthermore, almost everywhere on S_1,

$$\Big\| \sum_{j=1}^{k} (-1)^{j-1} F_j(1+it) \Big\|_q \leq (1+\epsilon)k^{1/q},$$

so that, by Lemma 1,

$$\Big\| \sum_{j=1}^{k} (-1)^{j-1} F_j(\theta) \Big\| \leq (1+3\epsilon_0)k^{1/p}/f(k)^{1-\theta},$$

and

$$\Big\| \sum_{j=1}^{k} (-1)^{j-1} x_j \Big\| \leq (1+3\epsilon_0)k^{1/p}/f(k)^{1-\theta} + 1/4 \leq (1+4\epsilon_0)k^{1/p}/f(k)^{1-\theta}.$$

Conclusion Let $y \in Y$ be the sum of the x_j with odd indices, $z \in Z$ be the sum of the x_j with even indices. By the above estimates and by choice of k, they satisfy $\delta\|y+z\| > \|y-z\|$. As δ is arbitrary and so are Y and Z, X is hereditarily indecomposable.

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