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Uniqueness of complex structure and real hereditarily indecomposable Banach spaces

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Abstract

There exists a real hereditarily indecomposable Banach space $X = X(\mathbb{C})$ (respectively $X = X(\mathbb{H})$) such that the algebra $\mathcal{L}(X)/\mathcal{S}(X)$ is isomorphic to $\mathbb{C}$ (respectively to the quaternionic division algebra $\mathbb{H}$).

Up to isomorphism, $X(\mathbb{C})$ has exactly two complex structures, which are conjugate, totally incomparable, and both hereditarily indecomposable. So there exist two Banach spaces which are isometric as real spaces but totally incomparable as complex spaces. This extends results of J. Bourgain and S. Szarek [J. Bourgain, Real isomorphic complex Banach spaces need not be complex isomorphic, Proc. Amer. Math. Soc. 96 (2) (1986) 221–226; S. Szarek, On the existence and uniqueness of complex structure and spaces with "few" operators, Trans. Amer. Math. Soc. 293 (1) (1986) 339–353; S. Szarek, A superreflexive Banach space which does not admit complex structure, Proc. Amer. Math. Soc. 97 (3) (1986) 437–444], and proves that a theorem of G. Godefroy and N.J. Kalton [G. Godefroy, N.J. Kalton, Lipschitz-free Banach spaces, Studia Math. 159 (1) (2003) 121–141] about isometric embeddings of separable real Banach spaces does not extend to the complex case.

The quaternionic example $X(\mathbb{H})$, on the other hand, has unique complex structure up to isomorphism; other examples with a unique complex structure are produced, including a space with an unconditional basis and non-isomorphic to l_2. This answers a question of S. Szarek in [S. Szarek, A superreflexive Banach space which does not admit complex structure, Proc. Amer. Math. Soc. 97 (3) (1986) 437–444].
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1. Introduction

1.1. Isometries between Banach spaces

It is well known that any two real Banach spaces which are isometric must be linearly isometric: this was proved in 1932 by S. Mazur and S. Ulam [18]. In 2003, G. Godefroy and N.J. Kalton also proved that if a separable real Banach space embeds isometrically into a Banach space, then it embeds linearly isometrically into it [10].

It follows from results of J. Bourgain and S. Szarek from 1986 that Mazur–Ulam's result is completely false in the complex case: there exist two Banach spaces which are linearly isometric as real spaces but non-isomorphic as complex spaces [6,19,20]. One of the main results of this paper is the following extension of their result; recall that two spaces are said to be totally incomparable if no infinite dimensional subspace of the one is isomorphic to a subspace of the other:

Theorem 1. *There exist two Banach spaces which are isometric as real spaces, but totally incomparable as complex spaces.*

These spaces are separable, and therefore Theorem 1 provides the first known counterexample to a complex version of the theorem of Godefroy and Kalton.

We shall also show that Theorem 1 is optimal, in the sense that there does not exist a family of more than two Banach spaces which are mutually isomorphic as real spaces but totally incomparable as complex spaces.

1.2. Real hereditarily indecomposable spaces

Our examples are natural modifications of the hereditarily indecomposable Banach space of W.T. Gowers and B. Maurey [13]. Hereditarily indecomposable (or H.I.) Banach spaces were discovered in 1991 by these two authors: a space X is H.I. if no (closed) subspace of X is decomposable (i.e. can be written as a direct sum of infinite-dimensional subspaces). Equivalently, for any two subspaces Y, Z of X and $\epsilon > 0$, there exist $y \in Y$, $z \in Z$ such that $\|y\| = \|z\| = 1$ and $\|y - z\| < \epsilon$. Gowers and Maurey gave the first known example X_{GM} of a H.I. space, both in the real and the complex case. After this result, many other examples of H.I. spaces with various additional properties were given. They are too numerous to be all cited here. We refer to [3] for a list of these examples. Let us mention however the remarkable result of S. Argyros and A. Tolias [5]: for any separable Banach space X not containing ℓ_1, there is a separable H.I. space with a quotient isomorphic to X.

A Banach space X is said to have a Schauder basis $(e_i)_{i\in\mathbb{N}}$ if any vector of X may be written uniquely as an infinite sum $\sum_{i\in\mathbb{N}} \lambda_i e_i$. The basis (e_i) is unconditional if any permutation of (e_i) is again a basis. This is equivalent to saying that there exists $C < +\infty$ such that for any vector written $\sum_{i\in\mathbb{N}} \lambda_i e_i$ in X, and any scalar sequence $(\mu_i)_{i\in\mathbb{N}}$ such that $\forall i \in \mathbb{N}$, $|\mu_i| \leqslant |\lambda_i|$, the inequality $\|\sum_{i\in\mathbb{N}} \mu_i e_i\| \leqslant C\|\sum_{i\in\mathbb{N}} \lambda_i e_i\|$ holds.

Classical spaces, such as c_0, l_p for $1 \leqslant p < +\infty$, L_p for $1 < p < +\infty$, and Tsirelson's space T have unconditional bases; or contain subspaces with an unconditional basis in the case of $C([0,1])$ or L_1. The H.I. spaces of Gowers and Maurey were the first known examples of spaces not containing any unconditional basic sequence, thus answering an old open question by the negative. The importance of H.I. spaces also stems from the famous Gowers' dichotomy theorem [12]: any Banach space contains either a subspace with an unconditional basis or a H.I.

subspace. In some sense, H.I. spaces capture even more of the general structure of all separable Banach spaces than classical spaces. By [2], any Banach space containing copies of all separable (reflexive) H.I. spaces must be universal for the class of separable Banach spaces (i.e. must contain an isomorphic copy of any separable Banach space). On the other hand, all spaces with an unconditional basis may be embedded into the unconditional universal space U of Pełczyński (see e.g. [16] about this). The space U has an unconditional basis and thus is certainly not universal (for example, it does not contain L_1 nor a H.I. subspace).

H.I. spaces have interesting properties linked to the space of operators defined on them. An operator $s \in \mathcal{L}(Y, Z)$ is strictly singular if no restriction of s to an infinite-dimensional subspace of Y is an isomorphism into Z. Equivalently, for any $\epsilon > 0$, any subspace Y' of Y, there exists $y \in Y'$ such that $\|s(y)\| \leqslant \epsilon\|y\|$. The ideal of strictly singular operators is denoted $\mathcal{S}(Y, Z)$. It is a perturbation ideal for Fredholm operators, we refer to [16] about this. Gowers and Maurey proved that any complex H.I. space X has what we shall call the *$\lambda Id + S$-property*, i.e. any operator on X is of the form $\lambda Id + S$, where λ is scalar and S strictly singular. Note however that spaces with the $\lambda Id + S$-property which are far from being H.I. were also constructed [4]. It follows from this property that any operator on X is either strictly singular or Fredholm with index 0, and so X cannot be isomorphic to a proper subspace (thus the existence of X_{GM} answers the old hyperplane's problem, which had been solved previously by Gowers [11]). Then in [7] the author extended the result: if X is complex H.I. and Y is a subspace of X, then every operator from Y into X is of the form $\lambda i_{Y,X} + s$, where λ is scalar, $i_{Y,X}$ the canonical injection of Y into X, and $s \in \mathcal{S}(Y, X)$. This property of operators characterizes complex H.I. spaces.

When X is real the situation is more involved. From now on, X_{GM} denotes the real version of the H.I. space of Gowers and Maurey, as opposed to the complex version $X_{\mathrm{GM}}^{\mathbb{C}}$. The real space X_{GM} has the property that for any $Y \subset X_{\mathrm{GM}}$, any operator from Y to X_{GM} is of the form $\lambda i_{Y,X_{\mathrm{GM}}} + s$, where s is strictly singular [13] (note that this property of operators implies the H.I. property). In general, a real H.I. space X must satisfy that for all $Y \subset X$, $\dim \mathcal{L}(Y, X)/\mathcal{S}(Y, X) \leqslant 4$ [8]. The converse is false: the space $X = X_{\mathrm{GM}} \oplus X_{\mathrm{GM}}$ is not H.I. but for any $Y \subset X$, $\dim \mathcal{L}(Y, X)/\mathcal{S}(Y, X) \leqslant 4$ (this is proved in Remark 7). Also when X is real H.I., the algebra $\mathcal{L}(X)/\mathcal{S}(X)$ is a division algebra isomorphic to $\mathbb{R}$, $\mathbb{C}$ or the quaternionic division algebra $\mathbb{H}$ [8]. This implies easily, by continuity of the Fredholm index, that any operator on X is either strictly singular or Fredholm with index 0 (this was already proved in [13]), and so X is not isomorphic to a proper subspace.

We will show that each of the values 2 and 4 for $\dim \mathcal{L}(X)/\mathcal{S}(X)$ is indeed possible. We shall build versions of X_{GM} for which the algebra $\mathcal{L}(X)/\mathcal{S}(X)$ is isomorphic to $\mathbb{C}$ or $\mathbb{H}$. Furthermore, the complex and the quaternionic examples satisfy $\dim \mathcal{L}(Y, X)/\mathcal{S}(Y, X) = 2$ and 4 respectively, for any subspace Y of X:

Theorem 2. *There exists a real H.I. Banach space $X(\mathbb{C})$ such that for any subspace Y of $X(\mathbb{C})$, $\dim \mathcal{L}(Y, X(\mathbb{C}))/\mathcal{S}(Y, X(\mathbb{C})) = 2$, and such that the algebra $\mathcal{L}(X(\mathbb{C}))/\mathcal{S}(X(\mathbb{C}))$ is isomorphic to the complex field $\mathbb{C}$.*

There exists a real H.I. Banach space $X(\mathbb{H})$ such that for any subspace Y of $X(\mathbb{H})$, $\dim \mathcal{L}(Y, X(\mathbb{H}))/\mathcal{S}(Y, X(\mathbb{H})) = 4$, and the algebra $\mathcal{L}(X(\mathbb{H}))/\mathcal{S}(X(\mathbb{H}))$ is isomorphic to $\mathbb{H}$, the algebra of quaternions.

The initial idea of the construction of H.I. spaces X such that $\mathcal{L}(X)/\mathcal{S}(X)$ is complex or quaternionic was given to the author by his Ph.D. advisor B. Maurey in 1996, but was not checked at that time. Recently the question was asked the author by D. Kutzarova and S. Argyros and the interest in this subject was revived by the survey paper of Argyros [3], see also [5], and this motivated the author into constructing explicitly the examples of Theorem 2. Since their algebras of operators are well described, and since they are naturally equipped with a $\mathbb{C}$-linear structure, these spaces are natural examples to study when looking for complex structure properties of Banach spaces. This led to new results concerning the relation between different complex structures on a general Banach space.

1.3. *Complex structures on a real Banach space*

In the following, spaces and subspaces are supposed infinite-dimensional and closed, unless specified otherwise. If X is a complex Banach space, with scalar multiplication denoted $(\lambda + i\mu)x$ for $\lambda, \mu \in \mathbb{R}$ and $x \in X$, its conjugate $\overline{X}$ is defined as X equipped with the scalar multiplication $(\lambda + i\mu).x := (\lambda - i\mu)x$. Note that X and $\overline{X}$ are isometric as real spaces.

When X is a real Banach space, a *complex structure* X^I on X is X seen as a complex space with the law

$$\forall \lambda, \mu \in \mathbb{R}, \quad (\lambda + i\mu).x = (\lambda Id + \mu I)(x),$$

where I is some operator on X such that $I^2 = -Id$, and renormed with the equivalent norm

$$|||x||| = \sup_{\theta \in \mathbb{R}} \|\cos\theta x + \sin\theta I x\|.$$

Note that the conjugate of X^I is the complex structure X^{-I}.

We shall sometimes refer to the complex structures on a complex Banach X, meaning by that the complex structures on X seen with its $\mathbb{R}$-linear structure.

It is known that complex structures do not always exist on a Banach space, even on a uniformly convex one [20], and the H.I. space of Gowers and Maurey [13] and related constructions (e.g. those of [14]) provide various examples of this situation. Gowers also constructed a space with an unconditional basis on which every operator is a strictly singular perturbation of a diagonal operator [11,14], and which therefore does not admit complex structure (this answers Pb 7.1 in [20]).

Concerning uniqueness, there are few results in the literature. The space ℓ_2 was the only space which was known to have a unique complex structure. In the other direction, it follows from local random techniques of J. Bourgain [6], S. Szarek [19,20], that there exists a complex Banach space not isomorphic to its conjugate; the space is an l_2-sum of finite-dimensional spaces which are far in the Banach–Mazur distance from their conjugates. Therefore there exist spaces which are isometric as real spaces but not isomorphic as complex spaces. Later on, N.J. Kalton [15] constructed a simple example defined as a twisted Hilbert space. Recently, R. Anisca [1] constructed a subspace of L_p, $1 \leqslant p < 2$, which has the same property, and moreover admits continuum many non-isomorphic complex structures. Note that these examples fail to have an unconditional basis, since when a complex Banach space X has an unconditional basis (e_n), the map c defined by $c(\sum_{n\in\mathbb{N}} \lambda_n e_n) = \sum_{n\in\mathbb{N}} \overline{\lambda_n} e_n$ is a $\mathbb{C}$-linear isomorphism from X onto $\overline{X}$.

The real spaces $X(\mathbb{C})$ and $X(\mathbb{H})$ possess an operator J such that $J^2 = -Id$; the associated complex structures are H.I. In fact the space $X(\mathbb{C})$ is "morally" the same as the complex version of Gowers–Maurey's space $X_{\mathrm{GM}}^{\mathbb{C}}$ seen with its $\mathbb{R}$-linear structure, and the results stated in the first part of Theorem 2 and their consequences about complex structures for $X(\mathbb{C})$ are valid for $X_{\mathrm{GM}}^{\mathbb{C}}$. There is however a technical difference in the fact that the special functionals used in the definition of the norm in our construction of $X(\mathbb{C})$ are $\mathbb{R}$-linear but not $\mathbb{C}$-linear when viewed in the complex setting (see Lemma 13 which prevents this).

We show that $X(\mathbb{C})$, with the complex structure $X^J(\mathbb{C})$ associated to some canonical operator J, is totally incomparable with its conjugate. Therefore we have examples of Banach spaces which are isometric as real spaces but totally incomparable as complex spaces. Furthermore, it turns out that $X^J(\mathbb{C})$ and its conjugate are the only complex structures on $X(\mathbb{C})$ up to isomorphism. Note that in [17], B. Maurey remarked, without proof, that the space $X_{\mathrm{GM}}^{\mathbb{C}}$ is not isomorphic to its conjugate.

The space $X(\mathbb{H})$, on the other hand, admits a unique complex structure up to isomorphism. This answers a natural question of Szarek from [20]: he asked whether the Hilbert space was the only space with unique complex structure.

Theorem 3. *There exists a real H.I. Banach space which admits exactly two complex structures up to isomorphism. Moreover, these two complex structures are conjugate and totally incomparable.*

There also exists a real H.I. Banach space with unique complex structure up to isomorphism.

We shall see that the space $X(\mathbb{C})$ is in some sense the only possible example of space with totally incomparable complex structures: if a space X admits two totally incomparable complex structures, then these structures must be conjugate up to isomorphism and both saturated with H.I. subspaces. It follows that there cannot be more than two mutually totally incomparable structures on a Banach space. We shall also note that for any $n \in \mathbb{N}$, the space $X(\mathbb{C})^n$ is an example of a space with exactly $n+1$ complex structures up to isomorphism.

Our constructions are different from those from [1,6,15,19], although as noted by Maurey in [17], there are some similarities between the "few operators" properties of Gowers–Maurey's spaces and the "few operators" properties of the finite-dimensional spaces glued together in Bourgain–Szarek's example. In fact, our spaces are quite close to the examples of [14], where spaces X are constructed such that the quotient algebra $\mathcal{L}(X)/\mathcal{S}(X)$ is isomorphic to a given algebra $\mathcal{A}$, under certain conditions on $\mathcal{A}$.

The complex structure properties of the spaces $X(\mathbb{C})$ and $X(\mathbb{H})$ follow directly from their few operators properties. For example, J and $-J$ are, up to strictly singular operators, the only operators on $X(\mathbb{C})$ whose square is $-Id$.

Some general results concerning the relation between complex structures on any given Banach space are obtained, with some applications to classical spaces such as the ℓ_p-spaces. We deduce that if X is a Banach space, then either all complex structures on X satisfy P, or all complex structures on X fail P, when P is any of the following properties: containing an unconditional basic sequence, being unconditionally saturated, containing a H.I. subspace, being saturated with H.I. subspaces, being H.I. A general method is also provided to study the isomorphism classes of complex structures on a given Banach space X by studying some group of invertible elements of $\mathcal{L}(X)/\mathcal{S}(X)$. The main tool for these results is the theory of Fredholm operators; we refer to [16] for the few Fredholm theory results we shall need.

As an application, we prove that whenever $\{X_i,\ 1 \leqslant i \leqslant N\}$ is a family of pairwise totally incomparable real Banach spaces with the $\lambda Id + S$-property, and n_i, $1 \leqslant i \leqslant N$, are integers, the direct sum $\sum_{1 \leqslant i \leqslant N} \oplus X_i^{2n_i}$ has a unique complex structure up to isomorphism. This provides additional examples of spaces not isomorphic to a Hilbert space, which have a unique complex structure. We also provide an unconditional version of the space $X(\mathbb{C})$:

Theorem 4. *There exists a real Banach space with an unconditional basis, non-isomorphic to l_2, with unique complex structure up to isomorphism.*

Finally, note that a complex Banach space, which is H.I. as a real space, is always complex H.I. Indeed if it contained two $\mathbb{C}$-linear subspaces in a direct sum, these would in particular form a direct sum of $\mathbb{R}$-linear subspaces. We shall show that the converse is false (the complexification of the real X_{GM} will do).

2. Preliminaries on H.I. spaces

The following was proved in [7,8].

Theorem 5. *(See [7,8].) Let X be a real H.I. space. Then there exists a division algebra E which is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and, for $Y \subset X$, linear embeddings i_Y of $E_Y = \mathcal{L}(Y,X)/\mathcal{S}(Y,X)$ into E. Let $\leqslant$ be the relation between subspaces of X defined by $Z \leqslant Y$ iff Z embeds into Y by an isomorphism of the form $i_{ZX} + s$, where $s \in \mathcal{S}(Z,X)$. For any $Z \leqslant Y$, the canonical restriction map modulo strictly singular operators $p_{YZ} : E_Y \to E_Z$ embeds E_Y into E_Z and satisfies $i_Y = i_Z p_{YZ}$. The algebra E is actually the inductive limit of the system (E_Y, p_{YZ}) under $\leqslant$, which is a filter relation. Furthermore the map i_X embeds $\mathcal{L}(X)/\mathcal{S}(X)$ as a subalgebra of E.*

A technical lemma (Lemma 2 in [8]) will be very useful. Given a Banach space X, a subspace Y of X is defined in [8] to be *quasi-maximal* in X if $Y + Z$ for Z infinite-dimensional in X is never a direct sum. Equivalently the quotient map from X onto X/Y is strictly singular. An obvious remark is that a space X is H.I. if and only if any subspace of X is quasi-maximal in X.

Lemma 6. *(See [8].) Let X be a Banach space, let T be an operator from X into some Banach space and let Y be quasi-maximal in X. Then T is strictly singular if and only if $T|_Y$ is strictly singular.*

In particular, if X is H.I. and Y a subspace of X, then T is strictly singular if and only if $T|_Y$ is strictly singular.

The "filter property" will denote the fact that if X is H.I. and Y, Z are subspaces of X, then there exists a subspace W such that $W \leqslant Y$ and $W \leqslant Z$—in particular W embeds into Y and Z (Lemma 1 in [7]).

We recall that a space X is said to be HD_n if X contains at most and exactly n infinite-dimensional subspaces in a direct sum [8]. For example, a space is HD_1 if and only if it is H.I.

We finally recall that X_{GM} denotes the real H.I. space of Gowers and Maurey. The following remark shows that real H.I. spaces are not characterized by the property that $\dim \mathcal{L}(Y,X)/\mathcal{S}(Y,X) \leqslant 4$ for all subspaces Y of X.

Remark 7. Let $X = X_{\mathrm{GM}} \oplus X_{\mathrm{GM}}$. For any $Y \subset X$, $\dim \mathcal{L}(Y,X)/\mathcal{S}(Y,X) \leqslant 4$.

Proof. By [8, Corollary 2], X is HD_2 as a direct sum of two H.I. spaces. Let Y be an arbitrary subspace of X, then Y is either H.I. or HD_2. Let d be the dimension of $\mathcal{L}(Y,X)/\mathcal{S}(Y,X)$. Assume Y is HD_2. Then Y contains a direct sum of H.I. spaces $Z_1 \oplus Z_2$. By [8, Corollary 3], passing to subspaces and by the filter property, we may assume that Z_1 and Z_2 are isomorphic and embed into X_{GM}. Since $\dim \mathcal{L}(Z_1, X_{\mathrm{GM}})/\mathcal{S}(Z_1, X_{\mathrm{GM}}) = 1$, we deduce that $\dim \mathcal{L}(Z_1 \oplus Z_2, X)/\mathcal{S}(Z_1 \oplus Z_2, X) \leqslant 4$.

Since $Z_1 \oplus Z_2$ is HD_2 [8, Corollary 2], it is quasi-maximal in Y, so the restriction map r defined from $\mathcal{L}(Y, X)/\mathcal{S}(Y, X)$ into $\mathcal{L}(Z_1 \oplus Z_2, X)/\mathcal{S}(Z_1 \oplus Z_2, X)$ by $r(\tilde{T}) = \widetilde{T|_{Z_1\oplus Z_2}}$ is well defined and injective (Lemma 6)—here $\tilde{T}$ denotes the class of T modulo strictly singular operators. It follows that

$$d \leqslant \dim \mathcal{L}(Z_1 \oplus Z_2, X)/\mathcal{S}(Z_1 \oplus Z_2, X) = 4.$$

If Y is H.I., we do a similar proof, passing to a subspace Z of Y which embeds into X_{GM}, and obtain by the same H.I. properties that

$$d \leqslant \dim \mathcal{L}(Z, X)/\mathcal{S}(Z, X) = 2. \qquad \square$$

3. Construction of real H.I. spaces $X(\mathbb{H})$ and $X(\mathbb{C})$

We shall describe the construction of the real H.I. space denoted $X(\mathbb{H})$ in the quaternionic case, assuming familiarity with Gowers–Maurey type constructions as in [13] and mainly [14]. The reader will adapt the construction for the example with complex algebra of operators, denoted $X(\mathbb{C})$. We shall then proceed to give the proofs of the operators properties in each case.

3.1. Preliminaries

Let c_{00} be the vector space of all real sequences which are eventually 0. Let $(e_n)_{n\in\mathbb{N}}$ be the standard basis of c_{00}. Given a family of vectors $\{x_i, i \in I\}$, $[x_i, i \in I]$ denotes the closed linear span of $\{x_i, i \in I\}$. For $k \in \mathbb{N}$, let $F_k = [e_{4k-3}, e_{4k-2}, e_{4k-1}, e_{4k}]$. The sequence F_k will provide a 4-dimensional decomposition of $X(\mathbb{H})$.

We proceed to definitions which are adaptations of the usual Gowers–Maurey definitions to the 4-dimensional decomposition context.

If $E \subset \mathbb{N}$, then we shall also use the letter E for the projection from c_{00} to c_{00} defined by $E(\sum_{i\in\mathbb{N}} x_i) = \sum_{i\in E} x_i$, where $x_i \in F_i, \forall i \in \mathbb{N}$. If $E, F \subset \mathbb{N}$, then we write $E < F$ to mean that $\max E < \min F$, and if $k \in \mathbb{N}$ and $E \subset \mathbb{N}$, then we write $k < E$ to mean $k < \min E$. The support of a vector $x = \sum_i x_i \in c_{00}$, $x_i \in F_i$, is the set of $i \in \mathbb{N}$ such that $x_i \neq 0$. An interval of integers is the intersection of an interval of $\mathbb{R}$ with $\mathbb{N}$. The range of a vector, written $\mathrm{ran}(x)$, is the smallest interval containing its support. We shall write $x < y$ to mean $\mathrm{ran}(x) < \mathrm{ran}(y)$. If $x_1 < \cdots < x_n$, we shall say that $x_1, \ldots, x_n$ are successive.

The class of functions $\mathcal{F}$ is defined as in [13], and the function $f \in \mathcal{F}$ is defined on $[1, +\infty)$ by $f(x) = \log_2(1 + x)$. Let $\mathcal{X}_4$ be the set of normed spaces of the form $X = (c_{00}, \|.\|)$ such that $(F_i)_{i=1}^{\infty}$ is a monotone Schauder decomposition of X where each e_k is normalized. If $f \in \mathcal{F}$, $X \in \mathcal{X}_4$ and every $x \in X$ satisfies the inequality

$$\|x\| \geqslant \sup\left\{ f(N)^{-1} \sum_{i=1}^{N} \|E_i x\| \colon N \in \mathbb{N},\ E_1 < \cdots < E_n \right\},$$

where the E_i's are intervals, then we shall say that X satisfies a lower f-estimate (with respect to $(F_i)_{i=1}^{\infty}$).

Special vectors are considered in [14]. We give their definitions in our context, as well as some lemmas without proof: indeed their proof is essentially the proof from [14] word by word, with the difference that "successive" and "lower f-estimate" mean with respect to (e_i) in their case and to (F_k) in ours. As our definitions are also based on the decomposition (F_k) instead of (e_i), it is easy to check that the proofs are indeed valid. We shall only point out the parts of the proofs which require a non-trivial modification. Alternatively, in the case of the real H.I. space with complex algebra of operators, these lemmas correspond exactly to the lemmas of [14] for the complex space $X_{\mathrm{GM}}^{\mathbb{C}}$, with our 2-dimensional real decomposition interpreted as a Schauder basis on $\mathbb{C}$.

For $X \in \mathcal{X}_4$, $x \in X$, and every integer $N \geqslant 1$, we consider the equivalent norm on X defined by

$$\|x\|_{(N)} = \sup \sum_{i=1}^{N} \|E_i x\|,$$

where the supremum is over all sequences $E_1, E_2, \ldots, E_N$ of successive intervals.

For $0 < \epsilon \leqslant 1$ and $f \in \mathcal{F}$, we say that a sequence $x_1, \ldots, x_N$ of successive vectors satisfies the RIS(ϵ) condition (for f) if there exists a sequence $n_1 < \cdots < n_N$ of integers such that $\|x_i\|_{(n_i)} \leqslant 1$ for each $i = 1, \ldots, N$, $n_1 > (2N/f'(1)) f^{-1}(N^2/\epsilon^2)$, and $\epsilon\sqrt{f(n_i)} > |\mathrm{ran}(\sum_{j=1}^{i-1} x_j)|$, for $i = 2, \ldots, N$. Observe that when $x_1, \ldots, x_N$ satisfies the RIS(ϵ) condition, then $Ex_1, \ldots, Ex_N$ also does for any interval E.

Given $g \in \mathcal{F}$, $M \in \mathbb{N}$ and $X \in \mathcal{X}_4$, an (M, g)-form on X is a functional of norm at most 1 which can be written $\sum_{j=1}^{M} x_j^*$ for a sequence $x_1^* < \cdots < x_M^*$ of successive functionals of norm at most $g(M)^{-1}$. Observe that if x^* is an (M, g)-form then $|x^*(x)| \leqslant g(M)^{-1}\|x\|_{(M)}$ for any x.

Lemma 8. *Let $X \in \mathcal{X}_4$. Let $f, g \in \mathcal{F}$ be such that $\sqrt{f} \leqslant g$. Assume that $x_1, \ldots, x_N \in X$ satisfies the* RIS(ϵ)*-condition for f. If x^* is a (k, g)-form for some integer $k \geqslant 2$ then*

$$\left| x^*\left(\sum_{i=1}^{N} x_i\right)\right| \leqslant \epsilon + 1 + N/\sqrt{f(k)}.$$

In particular, $|x^*(x_1 + \cdots + x_N)| < 1 + 2\epsilon$ when $k > f^{-1}(N^2/\epsilon^2)$.

Proof. Reproduce the proof of [14, Lemma 1], noting that, for $x \in c_{00}$, $\|x\|_{c_0} = \max_{i\in\mathbb{N}} \|x_i\|$ if $x = \sum_{i\in\mathbb{N}} x_i$ with $x_i \in F_i$, $\forall i \in \mathbb{N}$. □

Lemma 9. *Let $X \in \mathcal{X}_4$. Let $f, g \in \mathcal{F}$, $\sqrt{f} \leqslant g$, and let $x_1, \ldots, x_N \in X$ satisfies the* RIS(ϵ) *condition for f. Let $x = \sum_{i=1}^{N} x_i$, and suppose that*

$$\|Ex\| \leqslant 1 \vee \sup\bigl\{\bigl|x^*(Ex)\bigr|:\ x^* \textit{ is a } (k, g)\textit{-form},\ k \geqslant 2\bigr\},$$

for every interval E. Then $\|x\| \leqslant (1 + 2\epsilon)Ng(N)^{-1}$.

Proof. Reproduce the proof of [14, Lemma 3], using [14, Lemma 4] in its 4-dimensional decomposition version. □

Lemma 10. *Let $X \in \mathcal{X}_4$ satisfies a lower f-estimate. Then for every $n \in \mathbb{N}$ and $\epsilon > 0$, every subspace of X generated by a sequence of successive vectors contains a vector x of finite support such that $\|x\| = 1$ and $\|x\|_{(n)} \leqslant 1 + \epsilon$. Hence, for every $N \in \mathbb{N}$, every subspace generated by a sequence of successive vectors contains a sequence $x_1, \dots, x_N$ satisfying the* RIS(ϵ) *condition with $\|x_i\| \geqslant (1+\epsilon)^{-1}$.*

Proof. Given a sequence $(x_n)_{n\in\mathbb{N}}$ of successive vectors in X, (x_n) is basic bimonotone, and for vectors in $[x_n, n \in \mathbb{N}]$, the notions of lower f-estimate, successive vectors, etc... with respect to $(F_k)_{k\in\mathbb{N}}$ correspond to the usual notions of lower f-estimate, successive vectors, etc... with respect to the basis (x_n). Therefore the conclusion holds by Lemma 4 in [14]. □

3.2. Definition of $X(\mathbb{H})$

We now pass to the definition of $X = X(\mathbb{H})$. Let $\mathbf{Q} \subset c_{00}$ be the set of sequences with rational coordinates and modulus at most 1. Let $J \subset \mathbb{N}$ be a set such that if $m < n$ and $m, n \in J$, then $\log\log\log n \geqslant 2m$. We write J in increasing order as $\{j_1, j_2, \dots\}$. We shall also need $f(j_1) > 256$ where $f(x)$ is still the function $\log_2(x+1)$. Let $K, L \subset J$ be the sets $\{j_1, j_3, \dots\}$ and $\{j_2, j_4, \dots\}$.

Let σ be an injection from the collection of finite sequences of successive elements of $\mathbf{Q}$ to L. Given $X \in \mathcal{X}_4$ and $f \in \mathcal{F}$ such that X satisfies a lower f-estimate (with respect to (F_k)), and given an integer $m \in \mathbb{N}$, let $A_m^*(X)$ be the set of functionals of the form $f(m)^{-1} \sum_{i=1}^m x_i^*$ such that $x_1^* < \cdots < x_m^*$ and $\|x_i^*\| \leqslant 1$.

If $k \in \mathbb{N}$, let Γ_k^X be the set of sequences $y_1^* < \cdots < y_k^*$ such that $y_i^* \in \mathbf{Q}$ for each i,

$$y_1^* \in A_{j_{2k}}^*(X) \quad \text{and} \quad y_{i+1}^* \in A_{\sigma(y_1^*,\dots,y_i^*)}^*(X) \quad \text{for each } 1 \leqslant i \leqslant k-1.$$

These sequences are called special sequences. If $(g_i)_{i=1}^k$, $k \in K$, is a special sequence, then the functional $f(k)^{-1/2} \sum_{j=1}^k g_j$ is a special functional on X of size k. The set of such functionals is denoted $B_k^*(X)$. If $f \in \mathcal{F}$ and $g(k) = f(k)^{1/2}$, then a special functional of size k is also a (k, g)-form.

The quaternionic division algebra may be represented as an algebra of operators on $\mathbb{R}^4$. It is then generated by a family $\{Id_{\mathbb{R}^4}, u, v, w\}$, where u, v, w satisfy the relations $u^2 = v^2 = w^2 = -Id_{\mathbb{R}^4}$, $uv = -vu = w$, $vw = -wv = u$, and $wu = -uw = v$. We may identify u, v, and w with operators u_k, v_k and w_k on each F_k using the identification to $\mathbb{R}^4$ via the canonical basis $e_{4k-3}, \dots, e_{4k}$ of F_k. We then define linear operators U, V and W on c_{00} by, for all $k \in \mathbb{N}$, $U|_{F_k} = u_k$ (respectively $V|_{F_k} = v_k$, $W|_{F_k} = w_k$).

In particular it is clear that $U^2 = V^2 = W^2 = -Id$ and that $UV = -VU = W$, $VW = -WV = U$, $WU = -UW = V$, so that Id, U, V and W generate an algebra which is isomorphic to $\mathbb{H}$.

Our space $X = X(\mathbb{H})$ is then defined inductively as the completion of c_{00} in the smallest norm satisfying the following equation:

$$\|x\| = \|x\|_{c_0} \vee \sup\left\{ f(n)^{-1} \sum_{i=1}^n \|E_i x\|\colon 2 \leqslant n,\ E_1 < \cdots < E_n \right\}$$
$$\vee \sup\left\{ \left|x^*(Ex)\right|\colon k \in K,\ x^* \in B_k^*(X),\ E \subset \mathbb{N} \right\} \vee \|Ux\| \vee \|Vx\| \vee \|Wx\|,$$

where E, and $E_1, \dots, E_n$ are intervals of integers.

We may immediately observe that U, V and W extend to norm 1 operators on X, and even isometries on X, by the quaternionic relations between them. Note also that U, V and W commute with any interval projection, and that whenever $x < y$ and $T \in \{U, V, W\}$, we have $Tx < Ty$. It follows that $\|Tx\|_{(N)} = \|x\|_{(N)}$, whenever $N \geqslant 1$ and $T \in \{U, V, W\}$; when a sequence $x_1, \ldots, x_N$ satisfies the RIS(ϵ) condition, then so does $Tx_1, \ldots, Tx_N$. The adjoints Id_{X^*}, W^*, V^*, U^*, in this order, also satisfy the quaternionic commutation relations, the commutation with interval projections, and the relation with successive vectors. It follows that when $x^* \in A^*_m$ for some $m \in \mathbb{N}$ and $T \in \{U, V, W\}$, we have $T^*x^* \in A^*_m$. However, and this is fundamental, the sequence $T^*x^*_1, \ldots, T^*x^*_k$ is not in general special when $x^*_1, \ldots, x^*_k$ is a special sequence.

The next lemma is taken directly from [14].

Lemma 11. *For every $K_0 \subset K$, there is a function $g \in \mathcal{F}$ such that $f \geqslant g \geqslant f^{1/2}$, $g(k) = f(k)^{1/2}$ whenever $k \in K_0$, and $g(x) = f(x)$ whenever $N \in J \setminus K_0$ and $x \in [\log N, \exp N]$.*

Lemma 12. *Let $0 < \epsilon \leqslant 1$, $0 \leqslant \delta < 1$, $M \in L$ and let n and N be integers such that $N/n \in [\log M, \exp M]$ and $f(N) \leqslant (1+\delta) f(N/n)$. Assume that $x_1, \ldots, x_N$ satisfies the* RIS(ϵ) *condition and let $x = x_1 + \cdots + x_N$. Then $\|(f(N)/N)x\|_{(n)} \leqslant (1+\delta)(1+3\epsilon)$. In particular, if $n = 1$, we have $\|(f(N)/N)x\| \leqslant 1 + 3\epsilon$.*

Proof. We may reproduce the proof of Lemma 7 from [14], provided we show that if a vector Ex is such that $\|Ex\| > 1$, then it is normed by a (k, g)-form, where g is given by Lemma 11 in the case $K_0 = K \setminus \{k\}$. To see this, note that the only new case with respect to the classical Gowers–Maurey's proof is when Ex is normed by some $T^*_1 E^*_1 \ldots T^*_m E^*_m x^*$, with $T_i \in \{U, V, W\}$ and E_i an interval projection, for all $1 \leqslant i \leqslant m$, and x^* a (k, g)-form. By the commutation properties of U, V, W, we may assume Ex is normed by T^*x^* with $T \in \{U, V, W\}$ and x^* a (k, g)-form. But in this case, T^*x^* is also a (k, g)-form, since T^* is an isometry which respects successive vectors. □

To reproduce the proof of Gowers and Maurey, after having added the isometries U, V and W in the definition of the norm, we shall need to distinguish the action of a functional x^* from the actions of U^*x^*, V^*x^* and W^*x^*. This is expressed by the next lemma.

Lemma 13. *Let x be a finitely supported vector in X. Then there exists a functional x^* of norm at most 1 such that $x^*(x) \geqslant 1/2\|x\|$ and such that $x^*(Ux) = x^*(Vx) = x^*(Wx) = 0$.*

Proof. We observe that for any reals α, β, γ, the inverse of the operator $Id - \alpha U - \beta V - \gamma W$ is equal to $(1 + \alpha^2 + \beta^2 + \gamma^2)^{-1}(Id + \alpha U + \beta V + \gamma W)$. It follows that

$$\left\|(Id - \alpha U - \beta V - \gamma W)^{-1}\right\| \leqslant \frac{1 + |\alpha| + |\beta| + |\gamma|}{1 + \alpha^2 + \beta^2 + \gamma^2} \leqslant 3/2$$

by elementary calculus. So for any $x \in c_{00}$,

$$d\big(x, [Ux, Vx, Wx]\big) \geqslant 2/3\|x\|.$$

We conclude using the Hahn–Banach Theorem. □

3.3. The H.I. property

Given $N \in L$, and $\delta > 0$, define a *δ-norming N-pair* to be a pair (x, x^*) defined as follows. Let $y_1, \dots, y_N$ be a sequence satisfying the RIS(1) condition. Let $x = N^{-1} f(N)(y_1 + \cdots + y_N)$. Let, for $1 \leqslant i \leqslant N$, y_i^* be a functional of norm at most 1, such that $\text{ran}(y_i^*) \subset \text{ran}(y_i)$ and $y_i^*(y_i) = \delta$. Let $x^* = f(N)^{-1}(y_1^* + \cdots + y_N^*)$. Note that if (x, x^*) is a δ-norming N-pair, then $x^*(x) = \delta$ and Lemma 12 implies that $\|x\|_{(\sqrt{N})} \leqslant 8$. By Lemma 10 and Hahn–Banach Theorem, δ-norming N-pairs (x, x^*) with arbitrary constant $\delta \leqslant 1/2$ exist with x in an arbitrary block-subspace and N arbitrary.

Proposition 14. *The space $X(\mathbb{H})$ is hereditarily indecomposable.*

Proof. Write $X = X(\mathbb{H})$. Let Y and Z be subspaces of X and $\epsilon > 0$. We may assume that Y and Z are generated by successive vectors in X. Let $k \in K$ be such that $(\epsilon/72) f(k)^{1/2} > 1$. We construct sequences $x_1, \dots, x_k$ and $x_1^*, \dots, x_k^*$ as follows. Let $N_1 = j_{2k}$ and by Lemma 10, let (x_1, x_1^*) be a 1/3-norming N_1-pair such that $x_1 \in Y$, with $|x_1^*(Tx_1)| < k^{-1}$ if $T = U, V$ or W: this is possible by Lemma 13 applied to each of the N_1 vectors forming x_1. Since we allow an error term k^{-1}, x_1 and the functional x_1^* may be perturbed so that x_1^* is in $\mathbf{Q}$ and $\sigma(x_1^*) > f^{-1}(4)$. In general, after the first $i-1$ pairs were constructed, let (x_i, x_i^*) be a 1/3-norming N_i-pair such that x_i and x_i^* are supported after x_{i-1} and x_{i-1}^*, with $x_i \in Y$ if i is odd and $x_i \in Z$ if i is even, such that $|x_i^*(Tx_i)| < k^{-1}$ if $T = U$, V or W, having perturbed x_i^* in such a way that $N_{i+1} = \sigma(x_1^*, \dots, x_i^*)$ satisfies

$$f(N_{i+1}) > 2^{i+1} \quad \text{and} \quad \sqrt{f(N_{i+1})} > 2\left|\text{ran}\left(\sum_{j=1}^{i} x_j\right)\right|.$$

Now let $y = x_1 + x_3 + \cdots + x_{k-1}$, $z = x_2 + x_4 + \cdots + x_k$. Let also $x^* = f(k)^{-1/2}(x_1^* + \cdots + x_k^*)$. Our construction guarantees that x^* is a special functional, and therefore of norm at most 1. It is also clear that $y \in Y$, $z \in Z$, and that

$$\|y + z\| \geqslant x^*(y + z) \geqslant 1/3kf(k)^{-1/2}.$$

Our aim is now to obtain an upper bound for $\|y - z\|$. Let $x = y - z$. Let g be the function given by Lemma 11 in the case $K_0 = K \setminus \{k\}$. By the definition of the norm, all vectors Ex are either normed by (M, g)-forms, by special functionals of length k, by images of such functionals by U^*, V^* or W^*, or they have norm at most 1. In order to apply Lemma 9, it is enough to show that $|T^*z^*(Ex)| = |z^*(TEx)| \leqslant 1$ for any special functional z^* of length k in K, E an interval, T in the set $\{Id, U, V, W\}$. Let $z^* = f(k)^{-1/2}(z_1^* + \cdots + z_k^*)$ be such a functional with $z_l^* \in A_{m_l}^*$ for $1 \leqslant l \leqslant k$.

We evaluate $|z_l^*(ETx_i)|$ for $1 \leqslant l \leqslant k$ and $1 \leqslant i \leqslant k$. Recall that T and E commute.

Let t be the largest integer such that $m_t = N_t$. Then $z_i^* = x_i^*$ for all $i < t$. There are at most two values of $i < t$ such that $x_i \neq Ex_i \neq 0$ or $z_i^* \neq Ez_i^* \neq 0$, and for them $|z_i^*(ETx_i)| \leqslant 1$. The values of $i < t$ for which $x_i = Ex_i$ and $z_i^* = Ez_i^*$ form an interval e and satisfy $z_i^*(Tx_i) = x_i^*(x_i) = 1/3$ if $T = Id$, or $|z_i^*(Tx_i)| = |x_i^*(Tx_i)| \leqslant k^{-1}$, when $T = U, V$ or W. Therefore $|\sum_{i \in e} z_i^*(ET(-1)^i x_i)| \leqslant 1$.

When $i = l = t$, we obtain $|z_t^*(TEx_t)| \leqslant 1$.

If $i = l > t$ or $i \neq l$ then $z_l^*(Tx_i) = (T^* z_l^*)(x_i)$ and we have $T^* z_l^* \in A_{m_i}^*$ for some m_l. Moreover, because σ is injective and by definition of t, $m_l \neq N_i$. If $m_l < N_i$, then by the remark after the definition of N-pairs, $\|x_i\|_{\sqrt{N_i}} \leqslant 8$, so the lower bound of j_{2k} for m_1 tells us that $|T^* z_l^*(x_i)| \leqslant k^{-2}$. If $m_l > N_i$ the same conclusion follows from Lemma 8. There are also at most two pairs (i, l) for which $0 \neq z_l^*(ETx_i) \neq z_l^*(Tx_i)$, in which case $|z_l^*(ETx_i)| \leqslant 1$.

Putting these estimates together we obtain that

$$\left|z^*(TEx)\right| = \left|z^*\left(TE\left(\sum_{i=1}^{k}(-1)^i x_i\right)\right)\right| \leqslant f(k)^{-1/2}\left(2+1+1+2+k^2.k^{-2}\right) \leqslant 1.$$

We also know that $(1/8)(x_1, \dots, x_k)$ satisfies the RIS(1) condition. Hence, by Lemma 9, $\|x\| \leqslant 24kg(k)^{-1} = 24kf(k)^{-1}$. It follows that $\|y - z\| \leqslant 72 f(k)^{-1/2}\|y + z\| \leqslant \epsilon\|y + z\|$. It follows that Y and Z do not form a direct sum and so X is H.I. □

We may also construct a complex version $X(\mathbb{C})$ of our space, with a 2-dimensional decomposition, and a canonical isometry J satisfying $J^2 = -Id$ corresponding to a representation of the complex numbers as operators on each 2-dimensional summand. We leave the reader adapt our definitions and proofs to that case. Alternatively one could use the previous 4-dimensional decomposition setting and put only the operator U, instead of U, V and W, in the definitions and the proofs. We therefore obtain:

Proposition 15. *The space $X(\mathbb{C})$ is hereditarily indecomposable.*

3.4. Properties of operators on $X(\mathbb{H})$ and on $X(\mathbb{C})$

We now turn to the operator properties of our spaces $X(\mathbb{C})$ and $X(\mathbb{H})$. The quaternionic case is immediate from Theorem 5.

Proposition 16. *Let $X = X(\mathbb{H})$. Then the algebra $\mathcal{L}(X)/\mathcal{S}(X)$ is isomorphic to $\mathbb{H}$. Furthermore, for any $Y \subset X$, $\dim \mathcal{L}(Y, X)/\mathcal{S}(Y, X) = 4$, i.e. every operator from Y into X is of the form $ai_{Y,X} + bU|_Y + cV|_Y + dW|_Y + s$, where a, b, c, d are reals and s is strictly singular.*

Proof. The operators Id, U, V, and W generate a quaternionic division algebra, so $\mathcal{L}(X)/\mathcal{S}(X)$ is of dimension at least 4. By Theorem 5, it is isomorphic to $\mathbb{H}$, and furthermore, since $\mathcal{L}(X)/\mathcal{S}(X)$ embeds into $E_Y = \mathcal{L}(Y, X)/\mathcal{S}(Y, X)$, and E_Y is of dimension at most 4 for any $Y \subset X$ by [8], we deduce that $\dim E_Y = 4$ for any $Y \subset X$. □

The complex case requires the following lemma, which is inspired by Lemma 4.14 from [3].

Lemma 17. *Let X be a real H.I. space and J be an operator on X such that $J^2 = -Id$. Let Y be a subspace of X. Let $T \in \mathcal{L}(Y, X)$ be an operator which is not of the form $\lambda i_{Y,X} + \mu J|_Y + s$, with λ, μ scalars and s strictly singular. Then there exists a finite-codimensional subspace Z of Y and some $\alpha > 0$ such that*:

$$\forall z \in Z, \quad d\left(Tz, [z, Jz]\right) \geqslant \alpha\|z\|.$$

Proof. Otherwise we may construct a normalized basic sequence $(y_n) \in Y$, and scalar sequences (λ_n), (μ_n) with for all $n \in \mathbb{N}$,

$$\|Ty_n - \lambda_n y_n - \mu_n J y_n\| \leqslant 2^{-n}.$$

It follows that for $C = 1 + \|T\|$, for all $n \in \mathbb{N}$,

$$\|\lambda_n y_n + \mu_n J y_n\| \leqslant C.$$

We may assume for convenience that J is isometric. We note that

$$(\lambda_n Id + \mu_n J)^{-1} = \frac{\lambda_n Id - \mu_n J}{\lambda_n^2 + \mu_n^2},$$

from which it follows that

$$1 = \|y_n\| \leqslant C \frac{\|\lambda_n Id - \mu_n J\|}{\lambda_n^2 + \mu_n^2},$$

so

$$\lambda_n^2 + \mu_n^2 \leqslant C\big(|\lambda_n| + |\mu_n|\big).$$

It follows immediately that $\max(|\lambda_n|, |\mu_n|) \leqslant 2C$. Thus we may assume that the sequences (λ_n) and (μ_n) converge, and, passing to a subsequence, deduce that for some λ, μ,

$$\|Ty_n - \lambda y_n - \mu J y_n\| \leqslant 3.2^{-n}.$$

From this it follows that the restriction of $T - \lambda i_{Y,X} - \mu J|_Y$ to the space generated by the basic sequence (y_n) is compact, therefore strictly singular. By Lemma 6, we deduce that $T - \lambda i_{Y,X} - \mu J|_Y$ is strictly singular on Y, a contradiction. $\square$

Proposition 18. *Let $X = X(\mathbb{C})$. Then the algebra $\mathcal{L}(X)/\mathcal{S}(X)$ is isomorphic to $\mathbb{C}$. Furthermore, for any $Y \subset X$, $\dim \mathcal{L}(Y, X)/\mathcal{S}(Y, X) = 2$, i.e. any operator from Y into X is of the form $\lambda i_{Y,X} + \mu J|_Y + s$, where λ, μ are reals and s is strictly singular.*

Proof. Let Y be a subspace of X. The operator $J|_Y$ is not of the form $\lambda i_{Y,X} + s$, s strictly singular, otherwise by the H.I. property, Lemma 6, $J - \lambda Id$ would be strictly singular, which would contradict the fact that $J^2 = -Id$. It follows that $\dim \mathcal{L}(Y, X)/\mathcal{S}(Y, X) \geqslant 2$.

We assume $\dim \mathcal{L}(Y, X)/\mathcal{S}(Y, X) > 2$ and look for a contradiction. Let $T \in \mathcal{L}(Y, X)$ which is not of the form $\lambda i_{Y,X} + \mu J|_Y + s$ and assume without loss of generality that $\|T\| \leqslant 1$. Then by Lemma 17 we may find some $\alpha > 0$ and some subspace Z of Y such that for all $z \in Z$,

$$d\big(Tz, [z, Jz]\big) \geqslant \alpha \|z\|.$$

We may assume that Z is generated by successive vectors with respect to the 2-dimensional decomposition of X.

We fix a sequence (y_n) in Z such that for all n, y_{n+1} and Ty_{n+1} are supported after y_n and Ty_n, $\|y_n\|_{(n)} \leqslant 1$ while $\|y_n\| \geqslant 1/2$. Let $k \in K$ and construct sequences $x_1, \ldots, x_k$ and

$x_1^*, \ldots, x_k^*$ as follows. Let $N_1 = j_{2k}$. Let $x_1 = N_1^{-1} f(N_1)(y_{n_1} + \cdots + y_{n_{N_1}})$ where y_{n_i} is a subsequence satisfying the RIS(1) condition. Then $Tx_1 = N_1^{-1} f(N_1)(Ty_{n_1} + \cdots + Ty_{n_{N_1}})$, where the sequence (Ty_{n_i}) satisfies the RIS(1) condition as well. We let $x_1^* = f(N_1)^{-1}(y_{n_1}^* + \cdots + y_{n_{N_1}})$ be associated to Tx_1 so that (x_1^*, Tx_1) is an $\alpha/2$-norming N_1-pair, and such that $|x_1^*(x_1)| < k^{-1}$ and $|x_1^*(Jx_1)| < k^{-1}$ (to find $y_{n_i}^*$'s realizing this, apply the inequality from Lemma 17 to each y_{n_i}, with Hahn–Banach Theorem). Lemma 12 implies that $\|Tx_1\|_{(\sqrt{N_1})} \leqslant 8$. Repeating this, and up to perturbations, we build, for $1 \leqslant i \leqslant k$, $x_i \in Z$ and x_i^* so that (x_i^*, Tx_i) is an $\alpha/2$-norming N_i-pair, and such that $x_1^*, \ldots, x_k^*$ is a special sequence. We let $x = x_1 + \cdots + x_k$ and $x^* = f(k)^{-1/2}(x_1^* + \cdots + x_k^*)$. We therefore have

$$\|x\| \geqslant \|Tx\| \geqslant x^*(Tx) \geqslant (\alpha/2)kf(k)^{-1/2}.$$

We now use Lemma 9 to obtain an upper estimate for $\|x\|$. Let g be the function given by Lemma 11 in the case $K_0 = K \setminus \{k\}$. By the definition of the norm, all vectors Ex are either normed by (M, g)-forms, by special functionals of length k, by images of such functionals by J, or they have norm at most 1. In order to apply Lemma 9, it is enough to show that $|z^*(Ex)| \leqslant 1$ and $|J^* z^*(Ex)| = |z^*(JEx)| \leqslant 1$ for any special functional z^* of length k in K and any interval E. Let $z^* = f(k)^{-1/2}(z_1^* + \cdots + z_k^*)$ be such a functional with $z_j^* \in A_{m_j}^*$.

We evaluate $|z_l^*(Ex_i)|$ and $|z_l^*(EJx_i)|$ for $l \leqslant k$ and $i \leqslant k$.

Let t be the largest integer such that $m_t = N_t$. Then $z_i^* = x_i^*$ for all $i < t$. There are at most two values of $i < t$ such that $x_i \neq Ex_i \neq 0$ or $z_i^* \neq Ez_i^* \neq 0$, and for them $|z_i^*(Ex_i)| \leqslant 1$ and $|z_i^*(JEx_i)| \leqslant 1$. The values of $i < t$ for which $x_i = Ex_i$ and $z_i^* = Ez_i^*$ give $|z_i^*(Ex_i)| = |x_i^*(x_i)| < k^{-1}$ and $|z_i^*(EJx_i)| = |x_i^*(Jx_i)| < k^{-1}$.

When $i = l = t$, we obtain $|z_t^*(Ex_t)| \leqslant 1$ and $|z_t^*(EJx_t)| \leqslant 1$.

If $i = l > t$ or $i \neq l$ then $z_l^*(Jx_i) = (J^* z_l^*)(x_i)$ and we have as before $J^* z_l^* \in A_{m_l}^*$ for some $m_l \neq N_i$. If $m_l < N_i$, then as we remarked above, $\|x_i\|_{\sqrt{N_i}} \leqslant 8$, so the lower bound of j_{2k} for m_1 tells us that $|J^* z_l^*(x_i)| \leqslant k^{-2}$. If $m_l > N_i$ the same conclusion follows from Lemma 8. There are also at most two pairs (i, l) for which $0 \neq z_l^*(EJx_i) \neq z_l^*(Jx_i)$, in which case $|z_l^*(EJx_i)| \leqslant 1$. The same proof holds for $|z_l^*(Ex_i)|$.

Putting these estimates together we obtain that

$$\left|z^*(Ex)\right| \vee \left|z^*(EJx)\right| \leqslant f(k)^{-1/2}\left(2 + k.k^{-1} + 1 + 2 + k^2.k^{-2}\right) \leqslant 1.$$

We also know that $(1/8)(x_1, \ldots, x_k)$ satisfies the RIS(1) condition. Hence, by Lemma 9, $\|x\| \leqslant 24kg(k)^{-1} = 24kf(k)^{-1}$.

Finally we deduce that $\alpha\sqrt{f(k)} \leqslant 48$, a contradiction, since k was arbitrary in K. We conclude that $\dim \mathcal{L}(Y, X)/\mathcal{S}(Y, X) = 2$ for any $Y \subset X$, and that $\mathcal{L}(X)/\mathcal{S}(X)$ is isomorphic to $\mathbb{C}$. $\square$

As mentioned in the introduction, it is not difficult to show that the techniques used by Gowers and Maurey to study $\mathbb{C}$-linear operators on $X_{\mathrm{GM}}^{\mathbb{C}}$ actually imply that any $\mathbb{R}$-linear operator from an $\mathbb{R}$-linear subspace Y of $X_{\mathrm{GM}}^{\mathbb{C}}$ into itself is an $\mathbb{R}$-linear strictly singular perturbation of a complex multiple of the canonical injection map. Therefore $\mathbb{R}$-linear operators on $X_{\mathrm{GM}}^{\mathbb{C}}$ have the same properties as operators on $X(\mathbb{C})$, and the complex structure properties of $X(\mathbb{C})$ will be shared by $X_{\mathrm{GM}}^{\mathbb{C}}$ seen as a real space.

However, our definition of $X(\mathbb{C})$ uses $\mathbb{R}$-linear special functionals constructed with the real version of Hahn–Banach Theorem, as opposed to $\mathbb{C}$-linear functionals obtained in the complex setting for the definition of $X_{\mathrm{GM}}^{\mathbb{C}}$. In this sense, we feel that our construction is more natural, and it is certainly easier to adapt to produce other examples.

4. General results about complex structure and application to $X(\mathbb{C})$

4.1. Quotienting by strictly singular operators

Given X a real Banach space, let $\mathcal{GL}(X)$ denote the group of invertible operators on X, and let $\mathcal{I}(X)$ denote the subset of operators I on X such that $I^2 = -Id$.

Lemma 19. *Let X be a real Banach space. Let $I \in \mathcal{I}(X)$, and let $S \in \mathcal{S}(X)$ be such that $I + S \in \mathcal{I}(X)$. Then the complex structures X^I and X^{I+S} associated to I and $I + S$ respectively are isomorphic.*

Proof. Write $T = I + S$. From $T^2 = -Id$, we deduce $S^2 = -IS - SI$. Now let $\alpha = Id - (SI/2)$. This is an $\mathbb{R}$-linear map on X. Moreover, it is easy to check, using the relation satisfied by S^2, that

$$\alpha I = I + (S/2) = (I + S)\alpha = T\alpha.$$

This relation ensures that α may be seen as a $\mathbb{C}$-linear operator from X^I into X^T.

Furthermore, by properties of strictly singular operators (see e.g. [16]), the $\mathbb{R}$-linear operator α is Fredholm of index 0 on X as a strictly singular perturbation of Id. This means that $\alpha(X)$ is closed, and that

$$\dim_{\mathbb{R}} \operatorname{Ker} \alpha = \dim_{\mathbb{R}}\bigl(X/\alpha(X)\bigr) < +\infty.$$

This is also true when α is seen as $\mathbb{C}$-linear (note in particular that $\operatorname{Ker}\alpha$ is I-stable and $\alpha(X)$ is T-stable). That is, $\alpha(X^I)$ is closed in X^T, and

$$\dim_{\mathbb{C}} \operatorname{Ker} \alpha = \dim_{\mathbb{C}}\bigl(X^T/\alpha\bigl(X^I\bigr)\bigr) < +\infty.$$

Therefore α is $\mathbb{C}$-Fredholm with index 0, i.e. there exist $\mathbb{C}$-linear decompositions $X^I = X_0 \oplus F_0$ and $X^T = Y_0 \oplus G_0$, with $\dim_{\mathbb{C}} F_0 = \dim_{\mathbb{C}} G_0 < +\infty$, such that the restriction of α to X_0 is a $\mathbb{C}$-linear isomorphism onto Y_0. Since F_0 and G_0 are isomorphic, we deduce that there exists a $\mathbb{C}$-linear isomorphism from X^I onto X^T. □

Let π denote the quotient map from $\mathcal{L}(X)$ onto $\mathcal{L}(X)/\mathcal{S}(X)$. We also let $(\mathcal{L}(X)/\mathcal{S}(X))_0$ denote the group $\pi(\mathcal{GL}(X))$, and $\tilde{\mathcal{I}}(X)$ denote the set of elements of $(\mathcal{L}(X)/\mathcal{S}(X))_0$ whose square is equal to $-\pi(Id)$. In the following we shall identify a complex structure on X with the associated operator $I \in \mathcal{I}(X)$.

Proposition 20. *Let X be a real Banach space. Then the quotient map π induces an injective map $\tilde{\pi}$ from the set of isomorphism classes of complex structures on X into the set of conjugation* *classes of elements of $\tilde{\mathcal{I}}(X)$ for the group $(\mathcal{L}(X)/\mathcal{S}(X))_0$. The image of $\tilde{\pi}$ is the set of conjugation classes of elements of $\tilde{\mathcal{I}}(X)$ which may be lifted to an element of $\mathcal{I}(X)$. If $\mathcal{S}(X)$ admits a supplement in $\mathcal{L}(X)$ which is a subalgebra of $\mathcal{L}(X)$, then $\tilde{\pi}$ is bijective.*

Proof. For any operator T on X, we write $\tilde{T} = \pi(T)$. Let I and T be operators in $\mathcal{I}(X)$. If α is a $\mathbb{C}$-linear isomorphism from X^I onto X^T, then the $\mathbb{C}$-linearity means that $\alpha I = T\alpha$. Therefore $\tilde{\alpha}\tilde{I} = \tilde{T}\tilde{\alpha}$, and $\tilde{I}$ and $\tilde{T}$ satisfy a conjugation relation. Conversely, if $\tilde{I} = \tilde{\alpha}^{-1}\tilde{T}\tilde{\alpha}$ for some $\alpha \in \mathcal{GL}(X)$, then $\alpha^{-1}T\alpha = I + S$, where S is strictly singular. Note that $(I+S)^2 = -Id$, and since $T\alpha = \alpha(I+S)$, α is a $\mathbb{C}$-linear isomorphism from X^{I+S} onto X^T. By Lemma 19, it follows that X^I and X^T are isomorphic. This proves that $\tilde{\pi}$ is well defined and injective.

If $\mathcal{H}(X)$ is a subalgebra of $\mathcal{L}(X)$ which supplements $\mathcal{S}(X)$, then let $T \in \mathcal{L}(X)$ be such that $\tilde{T}^2 = -\tilde{Id}$; we may assume that T (and therefore T^2) belongs to $\mathcal{H}(X)$. Then since $T^2 = -Id + S$, S strictly singular, T^2 must be equal to $-Id$. Any class $\tilde{T} \in \tilde{\mathcal{I}}(X)$ may therefore be lifted to an element of $\mathcal{I}(X)$. □

4.2. Complex structures on $X(\mathbb{C})$

We recall that $X^J(\mathbb{C})$ denotes the complex structure on $X(\mathbb{C})$ associated to the operator J, and that two Banach spaces X and X' are said to be totally incomparable if no subspace of X is isomorphic to a subspace of X'. The first application of Proposition 20 is the complete description of the complex structures on $X(\mathbb{C})$.

Proposition 21. *The space $X^J(\mathbb{C})$ and its conjugate $X^{-J}(\mathbb{C})$ are complex H.I. and totally incomparable. Moreover, any complex structure on $X(\mathbb{C})$ is isomorphic either to $X^J(\mathbb{C})$ or to $X^{-J}(\mathbb{C})$.*

Proof. Any complex structure on $X(\mathbb{C})$ is H.I., since $X(\mathbb{C})$ is real H.I. We have $\mathcal{L}(X(\mathbb{C})) = [Id, J] \oplus \mathcal{S}(X(\mathbb{C}))$, and

$$\big(\mathcal{L}\big(X(\mathbb{C})\big)/\mathcal{S}\big(X(\mathbb{C})\big)\big)_0 \simeq \mathbb{C}^*.$$

The only two elements of $\mathbb{C}$ of square -1 are i and $-i$. By Proposition 20, it follows that $X^J(\mathbb{C})$ and $X^{-J}(\mathbb{C})$ are the only two complex structures on $X(\mathbb{C})$ up to isomorphism.

Assume now α is a $\mathbb{C}$-linear map from a $\mathbb{C}$-linear subspace Y of $X^J(\mathbb{C})$ into $X^{-J}(\mathbb{C})$. This is in particular an $\mathbb{R}$-linear map from Y into $X(\mathbb{C})$. So by Proposition 18, $\alpha = aId|_Y + bJ|_Y + s$, where s is strictly singular. The fact that α is $\mathbb{C}$-linear means that $\alpha J|_Y = -J\alpha$. This implies by ideal properties of strictly singular operators that $aId|_Y + bJ|_Y$ is strictly singular and therefore, $a = b = 0$. It follows that α is $\mathbb{R}$-strictly singular and thus $\mathbb{C}$-strictly singular. Therefore $X^J(\mathbb{C})$ and $X^{-J}(\mathbb{C})$ are totally incomparable. □

4.3. Totally incomparable complex structures

Following [13], we shall say that an operator $W \in \mathcal{L}(Y, Z)$ is *finitely singular* if the restriction of W to some finite-codimensional subspace of Y is an isomorphism into Z. This means that WY is closed and that the Fredholm index $i(W)$ is defined with values in $\mathbb{Z} \cup \{-\infty\}$, where

$$i(W) = \dim\big(\mathrm{Ker}(W)\big) - \dim(Z/WY).$$

The next proposition and corollaries show that the example of $X(\mathbb{C})$ is essentially the only one to ensure the total incomparability property.

The basis of our proof is the fact that, when T and U define complex structures,

$$(T+U)T = U(T+U),$$

which means that $T+U$ is $\mathbb{C}$-linear from X^T into X^U. The similar result holds for $T-U$ between X^T and X^{-U}.

Proposition 22. *Let X be a real Banach space, $T, U \in \mathcal{I}(X)$. Then*

(i) *if Y is a $\mathbb{C}$-linear subspace of X^T, then the map $T+U$ induces an isomorphism from a finite-codimensional subspace of Y into X^U or the map $T-U$ induces an isomorphism from a subspace of Y into X^{-U};*
(ii) *X^T is isomorphic to X^U, or $T-U$ induces an isomorphism from a subspace of X^T into X^{-U}.*

Proof. Let Y be a $\mathbb{C}$-linear subspace of X^T. Assume $T-U$ does not induce a $\mathbb{C}$-linear isomorphism from a subspace of Y onto a subspace of X^{-U}. Therefore the map $(T-U)|_Y$ is strictly singular as a $\mathbb{C}$-linear homeomorphism. To prove (i) it is enough to deduce that $(T+U)|_Y$ is finitely singular from Y into X^U. If this were false, then by [16, Proposition 2.c.4], we could find a ($\mathbb{C}$-linear) infinite-dimensional subspace Z of Y such that $\|(T+U)|_Z\| < \|T\|^{-1}$. Since $T-U$ is strictly singular on Y, we would find a norm 1 vector z in Z with $\|(T-U)z\| < \|T\|^{-1}$. We would then deduce that

$$\|Tz\| \leqslant 2^{-1}\big(\big\|(T+U)z\big\| + \big\|(T-U)z\big\|\big) < \|T\|^{-1}\|z\|,$$

a contradiction.

We now prove (ii). Assume $T-U$ does not induce a $\mathbb{C}$-linear isomorphism from a subspace of X^T onto a subspace of X^{-U}. Then $T-U$ is strictly singular on X^T, and we intend to deduce that $T+U$ is "essentially" an isomorphism from X^T onto X^U.

First we note that by (i), $T+U$ is finitely singular from X^T into X^U, and therefore finitely singular as an $\mathbb{R}$-linear operator on X.

Then we prove that whenever $\lambda \in]0,1[$, the operator $T+\lambda U \in \mathcal{L}(X)$ is finitely singular. Assume on the contrary that $T+\lambda U$ is not finitely singular for some $\lambda \in]0,1[$. Let $c = (1-\lambda^2)(2(1+\|T\|+2\lambda\|U\|))^{-1}$ and take an arbitrary $0<\epsilon<c$. As before there exists some infinite-dimensional ($\mathbb{R}$-linear) subspace Y of X such that $\|(T+\lambda U)|_Y\| < \epsilon$. Therefore, for all $y \in Y$,

$$\|Ty + \lambda Uy\| \leqslant \epsilon\|y\| \leqslant c\|y\|, \tag{1}$$

and by composing by U,

$$\|UTy - \lambda y\| \leqslant \epsilon\|U\|\|y\| \leqslant c\|U\|\|y\|. \tag{2}$$

We deduce from this that Y and TY form a direct sum. Indeed, if there existed norm 1 vectors y and z in Y with $\|z - Ty\| \leqslant c$, then we would have, by (1), $\|z + \lambda Uy\| \leqslant 2c$, therefore

$$\|Uz - \lambda y\| \leqslant 2c\|U\|, \tag{3}$$

but also $\|Tz + y\| \leqslant c\|T\|$, so by (1) again,

$$\|y - \lambda Uz\| \leqslant c\big(1 + \|T\|\big). \tag{4}$$

From (3) and (4), we would get

$$1 - \lambda^2 = \big\|\big(1 - \lambda^2\big)y\big\| \leqslant c\big(1 + \|T\| + 2\lambda\|U\|\big),$$

a contradiction by choice of c. Therefore $Y + TY$ forms a direct sum with projection constants depending only on λ, $\|T\|$ and $\|U\|$.

Now since $T - U$ is $\mathbb{C}$-strictly singular from X^T into X^{-U}, and $Y \oplus TY$ is a $\mathbb{C}$-linear subspace of X^T, there exist $y, z \in Y$ with $\|y + Tz\| = 1$ and $\|(T - U)(y + Tz)\| \leqslant \epsilon$, which means $\|-z - UTz + Ty - Uy\| \leqslant \epsilon$. By (1) and (2), and the fact that $Y \oplus TY$ is direct, we deduce

$$\big\|-z - \lambda z + Ty + \lambda^{-1}Ty\big\| \leqslant \epsilon + \epsilon\|U\|\|z\| + \epsilon\lambda^{-1}\|y\| \leqslant C\epsilon,$$

where C depends only on λ, $\|T\|$ and $\|U\|$. In the same way,

$$\big\|-z - \lambda z + Ty + \lambda^{-1}Ty\big\| \geqslant C'\big((1 + \lambda)\|z\| + \big(1 + \lambda^{-1}\big)\|Ty\|\big) \geqslant C'',$$

where again C'' depends only on λ, $\|T\|$ and $\|U\|$. As ϵ was arbitrary, we obtain a contradiction.

We have therefore proved that $T + \lambda U$ is finitely singular whenever $\lambda \in]0, 1]$, and this is obvious for $\lambda = 0$. In other words, the Fredholm index of $T + \lambda U$ is defined for all $\lambda \in [0, 1]$. By continuity of the Fredholm index, [16, Proposition 2.c.9], we deduce that $\operatorname{ind}(T + U) = \operatorname{ind}(T) = 0$, since T is an isomorphism. Therefore $T + U$ is Fredholm with index 0. It is therefore also Fredholm with index 0 as a $\mathbb{C}$-linear operator from X^T into X^U, and we deduce that X^T and X^U are isomorphic. □

Corollary 23. *Let X be a real Banach space with two totally incomparable complex structures. Then these complex structures are conjugate up to isomorphism and both HI-saturated.*

Proof. By HI-saturated we mean that any subspace has a further subspace which is H.I. Assume X^T is totally incomparable with X^U. By Proposition 22 (applied to U and $-T$), X^U is isomorphic to X^{-T}. To show that X^T is HI-saturated, it is enough by Gowers' dichotomy theorem to prove that X^T does not contain a subspace with an unconditional basis. Indeed, if Y were such a subspace, then by the remark in the introduction, Y would be isomorphic to $\overline{Y}$, $\mathbb{C}$-linear subspace of X^{-T}, which would contradict the total incomparability of X^T with X^{-T}. □

Corollary 24. *There cannot exist more than two mutually totally incomparable complex structures on a Banach space.*

Note that $X^J(\mathbb{C}) \oplus X^J(\mathbb{C})$ is a non-H.I. space which is totally incomparable with its conjugate $X^{-J}(\mathbb{C}) \oplus X^{-J}(\mathbb{C})$. Therefore it is not possible to replace HI-saturated by HI in the conclusion of Corollary 23.

G. Godefroy asked the author whether there also existed a Banach space with exactly three complex structures. The answers turns out to be yes.

Proposition 25. *For any $n \in \mathbb{N}$, the space $X(\mathbb{C})^n$ has exactly $n+1$ complex structures up to isomorphism.*

Proof. We have that $\mathcal{L}(X(\mathbb{C})^n) \simeq \mathcal{M}_n(\mathbb{C}) \oplus \mathcal{S}(X(\mathbb{C})^n)$, and

$$\big(\mathcal{L}\big(X(\mathbb{C})^n\big)/\mathcal{S}\big(X(\mathbb{C})^n\big)\big)_0 \simeq \mathcal{GL}_n(\mathbb{C}).$$

Any complex (n,n)-matrix whose square is equal to $-Id_{\mathbb{C}^n}$ has minimal polynomial X^2+1, and is therefore similar to a diagonal matrix with i or $-i$'s down the diagonal; and there are $n+1$ similarity classes of such matrices according to the number of i's. Therefore by Proposition 20, there are $n+1$ complex structures on $X(\mathbb{C})^n$, up to isomorphism. If we let $X = X^J(\mathbb{C})$, these structures are isomorphic to the spaces $X^k \oplus \overline{X}^{n-k}$, $0 \leqslant k \leqslant n$. □

4.4. Stable properties of complex structures

We now observe that Proposition 22, elementary Fredholm theory, and properties of H.I. spaces imply that a certain number of properties must be preserved among the complex structures of a given Banach space. We recall that a space is said to be HI-saturated if any of its subspaces contains a H.I. subspace, and *unconditionally saturated* if any of its subspaces contains an unconditional basic sequence.

Proposition 26. *Let X be a real Banach space, $T, U \in \mathcal{I}(X)$. Then*

(i) *If X^T contains an unconditional basic sequence, then X^U contains an unconditional basic sequence. In fact any subspace of X^T with an unconditional basis has a subspace which embeds into X^U.*
(ii) *If X^T is unconditionally saturated, then X^U is unconditionally saturated. In fact any subspace of X^U has a further subspace which embeds into X^T.*
(iii) *If X^T has a H.I. subspace then X^U has a H.I. subspace. In fact whenever Y is a H.I. subspace of X^T, it follows that Y or $\overline{Y}$ embeds into X^U.*
(iv) *If X^T is HI-saturated then X^U is HI-saturated.*
(v) *If X^T is H.I. then X^U is H.I. In fact there is either a unique complex structure on X, or exactly two complex structures which are conjugate and such that neither one embeds into the other.*

Proof. Let Y be a subspace of X^T with an unconditional basis. By Proposition 22(i), some subspace Z of Y, which we may assume to have an unconditional basis, embeds into X^U or X^{-U}. Since $Z \simeq \overline{Z}$, we deduce that Z embeds into X^U. We have therefore obtained (i), and (iv) follows by Gowers dichotomy theorem.

Assume X^T is unconditionally saturated. Since $Y \simeq \overline{Y}$ when Y has an unconditional basis, it is clear that X^{-T} is also unconditionally saturated. Let $Y \subset X^U$, then by Proposition 22(i), some subspace Z of Y embeds into X^T or X^{-T}, and by the saturation property, we may assume that Z has an unconditional basis. Therefore $Z \simeq \overline{Z}$ and Z embeds into X^T if and only Z embeds into X^{-T}. We have therefore proved (ii), and the first part of (iii) follows by Gowers' dichotomy theorem.

Assume X^T is H.I. and that X^U is not isomorphic to X^T. By Proposition 22(ii), $T - U$ induces an isomorphism from a subspace of X^T into X^{-U}. Since X^T is H.I., by Lemma 6 and [16, Proposition 2.c.4], it follows that $T - U$ is finitely singular from X^T into X^{-U} and therefore from X into X. If $\mathrm{ind}(T - U) = k \leqslant 0$ when $T - U$ is seen as a real operator on X, then $T - U : X^T \to X^{-U}$ has index $k/2$ (with $-\infty/2 = -\infty$) and therefore X^T is isomorphic to a $-k/2$-codimensional subspace of X^{-U}. The map $T - U$ may also be seen as a $\mathbb{C}$-linear operator from X^{-U} into X^T, and its index is also $k/2$, therefore X^{-U} is also isomorphic to a $-k/2$-codimensional subspace of X^T. Since a H.I. space is never isomorphic to a proper subspace, it follows that $k = 0$, which means that X^T is isomorphic to X^{-U}. If we assume that $k \geqslant 0$, a similar reasoning will give us that X^T is isomorphic to X^{-U}. Therefore X^U is isomorphic to X^{-T}. Since $X^{-T} = \overline{X^T}$ is H.I., the first half of (v) follows, and it only remains to prove the non-embedding part in the second half of (v).

It remains therefore to assume that say X^T embeds into X^{-T}, and to deduce that X^T is isomorphic to X^{-T}. But if α is $\mathbb{C}$-linear from X^T into X^{-T} then it is also $\mathbb{C}$-linear from X^{-T} into X^T, and therefore if X^T embeds into X^{-T}, then X^{-T} embeds into X^T. Since X^T is H.I. this is only possible if X^T and X^{-T} are isomorphic.

Assume finally that Y is a H.I. subspace of X^T, then by Proposition 22(i), the restriction of $T + U$ (or $T - U$) to some subspace Z of Y is an isomorphism into X^U (or X^{-U}). Assume for example that $T - U$ induces an isomorphism from Z into X^{-U}, then Lemma 6 and [16, Proposition 2.c.4] apply to deduce that $(T - U)|_Y$ is finitely singular from Y into X^{-U}.

If $\mathrm{ind}((T - U)|_Y) = -\infty$ then Y embeds into X^{-U}. If $\mathrm{ind}((T - U)|_Y)$ is finite, then X^{-U} is a finite-dimensional perturbation of Y and is therefore H.I., and furthermore some finite-codimensional subspace of X^{-U} is then isomorphic to a subspace of X^T. Applying (v), we deduce that X^T is isomorphic to X^{-U} and therefore Y embeds into X^{-U}.

The second part of (iii) follows. Since $\overline{Y}$ is H.I. whenever Y is H.I., this is also a direct way of proving the first part of (iii). □

Note that (iv) could easily have been proved without Gowers' dichotomy theorem.

Recall that a Banach space is said to be *minimal* if it embeds into any of its infinite-dimensional subspaces. The most important examples of minimal spaces are c_0 and the l_p spaces, $1 \leqslant p < +\infty$. If X is a Banach space, a space Y is said to be *X-saturated* if X embeds into any subspace of Y.

Corollary 27. *Let X be a minimal complex Banach space. Then any other complex structure on X is X-saturated.*

Proof. By Gowers' dichotomy theorem and the fact that H.I. spaces can never be minimal, we have that X is unconditionally saturated. Then by Proposition 26(i), any subspace of a complex structure on X contains a subspace which embeds into X, and therefore a further subspace isomorphic to X. □

Therefore any complex structure on the real space ℓ_p, $1 \leqslant p < +\infty$, must be ℓ_p-saturated. It remains open whether ℓ_p, $p \neq 2$, has unique complex structure.

5. Examples of spaces with unique complex structure

S. Szarek asked whether there exists a Banach space not isomorphic to a Hilbert space, with unique complex structure [20, Pb 7.2]. In this section, we provide various examples to answer this question by the positive.

5.1. Conditional examples

We first look at the complex structures on $X(\mathbb{H})$. Since $U^2 = -Id$, consider the complex structure $X^U(\mathbb{H})$ associated to U. Since $X(\mathbb{H})$ is H.I., the space $X^U(\mathbb{H})$ must be complex H.I.

Any $\mathbb{R}$-linear operator T on X is of the form $aId + bU + cV + dW + S$, where S is strictly singular. Observe that saying that T is $\mathbb{C}$-linear means that T commutes with U, which implies that $c = d = 0$. Therefore we deduce that any $\mathbb{C}$-linear operator on $X^U(\mathbb{H})$ is of the form $aId + bU + S = (a + ib).Id + S$, which was expected since $X^U(\mathbb{H})$ is H.I. (here we used that any $\mathbb{R}$-strictly singular $\mathbb{C}$-linear operator is $\mathbb{C}$-strictly singular).

Proposition 28. *The space $X(\mathbb{H})$ admits a unique complex structure up to isomorphism.*

Proof. We may write $\mathcal{L}(X(\mathbb{H})) = [Id, U, V, W] \oplus \mathcal{S}(X(\mathbb{H}))$, and

$$\big(\mathcal{L}\big(X(\mathbb{H})\big)/\mathcal{S}\big(X(\mathbb{H})\big)\big)_0 \simeq \mathbb{H}^*.$$

Write the generators of $\mathbb{H}$ as $\{1, i, j, k\}$. The elements of $\mathbb{H}$ of square -1 are of the form $r = bi + cj + dk$ with $b^2 + c^2 + d^2 = 1$. Any element r of this form is in the conjugation class of i, since $i(i + r) = (i + r)r$, for $r \neq -i$, and $ij = -ji$, for $r = -i$. Therefore by Proposition 20 all complex structures on $X(\mathbb{H})$ are isomorphic. □

Our next class of examples uses real spaces on which operators are of the form $\lambda Id + S$, S strictly singular (the $\lambda Id + S$-property). Note that non-H.I. examples of real spaces with the $\lambda Id + S$-property may be found e.g. in [4,9].

Proposition 29. *Let $\{X_i, 1 \leqslant i \leqslant N\}$ be a family of pairwise totally incomparable real Banach spaces with the $\lambda Id + S$-property. For $1 \leqslant i \leqslant N$, let $n_i \in \mathbb{N}$. Then $\sum_{1\leqslant i\leqslant N} \oplus X_i^{2n_i}$ has a unique complex structure up to isomorphism.*

Proof. If X has the $\lambda Id + S$-property, and $n \in \mathbb{N}$, let $\mathcal{M}_{2n}(Id_X)$ be the space of $(2n, 2n)$-matrix operators on X^{2n} with homothetic coefficients, which we identify with the space $\mathcal{M}_{2n}$ of real $(2n, 2n)$-matrices. Let $\mathcal{GL}_{2n}$ denote the group of invertible real $(2n, 2n)$-matrices. We have that $\mathcal{L}(X^{2n}) = \mathcal{M}_{2n}(Id_X) \oplus \mathcal{S}(X^{2n})$ and

$$\big(\mathcal{L}\big(X^{2n}\big)/\mathcal{S}\big(X^{2n}\big)\big)_0 \simeq \mathcal{GL}_{2n}.$$

Now any real $(2n, 2n)$-matrix whose square is $-Id_{\mathbb{R}^{2n}}$ is diagonalizable with $\mathbb{C}$-eigenvalues i and $-i$, each with multiplicity n. So any two such matrices are $\mathbb{C}$-similar and therefore $\mathbb{R}$-similar. By Proposition 20, it follows that all complex structures on X^{2n} are isomorphic.

If X is a direct sum $\sum_{1\leqslant i\leqslant N} \oplus X_i^{2n_i}$, where the X_i's are pairwise totally incomparable, then as $\mathcal{L}(X_i, X_j) = \mathcal{S}(X_i, X_j)$ whenever $i \neq j$, we have

$$\mathcal{L}(X) \simeq \left(\sum_{1\leqslant i\leqslant N} \oplus \mathcal{M}_{2n_i}(Id_{X_i}) \right) \oplus \mathcal{S}(X),$$

and

$$\big(\mathcal{L}(X)/\mathcal{S}(X)\big)_0 \simeq \prod_{1\leqslant i\leqslant N} \mathcal{GL}_{2n_i}.$$

It follows immediately from the case $N = 1$ that there is a unique conjugacy class of elements of square -1 for the group $\prod_{1\leqslant i\leqslant N} \mathcal{GL}_{2n_i}$. So all complex structures on X are isomorphic. □

5.2. *An unconditional example*

All the examples considered so far fail to have an unconditional basis. Indeed for each of them the quotient algebra $\mathcal{L}(X)/\mathcal{S}(X)$ is finite-dimensional. We now show how to construct a real Banach space $X(\mathbb{D}_2)$ with an unconditional basis, non-isomorphic to l_2, and with unique complex structure (here $\mathbb{D}_2$ stands for "2-block diagonal"). This is exactly the unconditional version of $X(\mathbb{C})$, precisely in the same way as the space of Gowers [11], on which every operator is the sum of a diagonal and a strictly singular operator, is the unconditional version of Gowers–Maurey's space X_{GM}. In other words, it is a space with "as few" operators as possible to ensure the existence of an unconditional basis and of a complex structure. It is not difficult to show that the unconditionality of a 2-dimensional decomposition, and the existence of a map J such that $J^2 = -Id$ defined on each 2-dimensional summand, already imply that any 2-block diagonal operator associated to bounded $(2,2)$-matrices must be bounded. This will motivate the following definition of $X(\mathbb{D}_2)$. We thank B. Maurey for a discussion which clarified this example.

The basis (e_i) is as before the natural basis of $c_{00}(\mathbb{R})$. For $k \in \mathbb{N}$, let $F_k = [e_{2k-1}, e_{2k}]$. Notions of successivity are taken with respect to the 2-dimensional decomposition associated to the F_k's. Let $\mathcal{D}_2(c_{00})$ denote the space of 2-block diagonal operators on c_{00}, i.e. the space of operators T on c_{00} such that $T(F_k) \subset F_k$ for all $k \in \mathbb{N}$. Any operator in $\mathcal{D}_2(c_{00})$ corresponds to a sequence $(M_n) \in \mathcal{M}_2^{\mathbb{N}}$ of real $(2,2)$-matrices, and will be denoted $D(M_n)$. For $M \in \mathcal{M}_2$, we shall consider the norm $\|M\|$, when M is seen as an operator on l_∞^2, or sometimes $\|M\|_2$ (respectively $\|M\|_1$), the euclidean norm (respectively the l_1-norm) on $\mathcal{M}_2$ identified with $\mathbb{R}^4$.

The space $X(\mathbb{D}_2)$ is defined inductively as the completion of c_{00} in the smallest norm satisfying the following equation:

$$\begin{aligned}\|x\| = \|x\|_{c_0} &\vee \sup\left\{ f(n)^{-1} \sum_{i=1}^{n} \|E_i x\|:\ 2 \leqslant n,\ E_1 < \cdots < E_n \right\} \\ &\vee \sup\big\{ \big|x^*(Ex)\big|:\ k \in K,\ x^* \in B_k^*(X),\ E \subset \mathbb{N} \big\} \\ &\vee \sup\big\{ \big\|D(M_n)x\big\|:\ \forall n \in \mathbb{N},\ \|M_n\| \leqslant 1 \big\},\end{aligned}$$

where E, and $E_1, \ldots, E_n$ are intervals of integers, and (M_n) is a sequence of $(2,2)$-matrices.

From the definition we observe immediately that any 2-block diagonal operator $D(M_n)$, where the sequence (M_n) is bounded, extends to a bounded operator on $X(\mathbb{D}_2)$. The space of such operators will be denoted $\mathcal{D}_2(X(\mathbb{D}_2))$. Furthermore, the norm on each F_k is the l_∞-norm; and whenever $n \in \mathbb{N}$, and y_k, z_k belong to F_k, with $\|z_k\| \leqslant \|y_k\|$ for all $1 \leqslant k \leqslant n$, it follows that

$$\left\| \sum_{k=1}^{n} z_k \right\| \leqslant \left\| \sum_{k=1}^{n} y_k \right\|.$$

This is a strong unconditionality property of $X(\mathbb{D}_2)$ from which we deduce that $(e_i)_{i \in \mathbb{N}}$ is an unconditional basis for $X(\mathbb{D}_2)$.

Proposition 30. *Any operator on $X(\mathbb{D}_2)$ may be written $D + S$, where $D \in \mathcal{D}_2(X(\mathbb{D}_2))$ is 2-block diagonal and $S \in \mathcal{S}(X(\mathbb{D}_2))$ is strictly singular.*

Proof. We sketch how to reproduce a proof from [14]. Let T be an operator on $X(\mathbb{D}_2)$ with 0's down the 2-block diagonal. First we show that if (x_n) is a sequence of successive vectors such that $\|x_n\|_{(n)} \leqslant 1$, $A_n = \mathrm{supp}(x_n)$ and for each n, $B_n \cup C_n$ is a partition of A_n in two subsets, then $C_n T B_n x_n$ converges to 0 (see [14, Lemma 27]). The proof is based on a construction of a special sequence in the "usual" way, similar to our proofs for $X(\mathbb{C})$ and $X(\mathbb{H})$. Arbitrary choices of signs -1 or 1 in the proof of [14] correspond to arbitrary choices of norm 1 $(2,2)$-matrices in our case. If $D = D(M_n) \in \mathcal{D}_2(X)$, with $\|M_n\| \leqslant 1$ for all n, then D preserves successive vectors and $\|D\| \leqslant 1$. Therefore the estimates of the end of [14, Lemma 27] based on properties of RIS vectors and (M, g)-forms are still valid. The other argument based on disjointness of supports of $y_n = B_n x_n$ and $z_n = C_n T B_n x_n$ is immediately seen to be preserved as well. Corollary 28 from [14] may then be reproduced to obtain that $(T x_n)$ converges to 0.

Finally if T is an operator on X, and $\mathrm{diag}(T)$ is its 2-block diagonal part, then $((T - \mathrm{diag}(T))x_n)$ converges to 0 whenever (x_n) is a successive sequence such that $\|x_n\|_{(n)} \leqslant 1$. By Lemma 10 this implies that $T - \mathrm{diag}(T)$ is strictly singular. □

We recall that $(\mathcal{L}(X(\mathbb{D}_2))/\mathcal{S}(X(\mathbb{D}_2)))_0$ is defined as the group of elements of $\mathcal{L}(X(\mathbb{D}_2))/\mathcal{S}(X(\mathbb{D}_2))$ which may be lifted to an invertible operator. Likewise, $(l_\infty(\mathcal{M}_2)/c_0(\mathcal{M}_2))_0$ is the group of elements of $l_\infty(\mathcal{M}_2)/c_0(\mathcal{M}_2)$ which may be lifted to an invertible element of $l_\infty(\mathcal{M}_2)$, that is, to a sequence $(M_n) \in l_\infty(\mathcal{M}_2)$ of real $(2,2)$-matrices such that M_n is invertible for each n and the sequence (M_n^{-1}) is bounded.

Lemma 31. *The algebras $\mathcal{L}(X(\mathbb{D}_2))/\mathcal{S}(X(\mathbb{D}_2))$ and $l_\infty(\mathcal{M}_2)/c_0(\mathcal{M}_2)$ are isomorphic, and the groups $(\mathcal{L}(X(\mathbb{D}_2))/\mathcal{S}(X(\mathbb{D}_2)))_0$ and $(l_\infty(\mathcal{M}_2)/c_0(\mathcal{M}_2))_0$ are isomorphic.*

Proof. Write $X = X(\mathbb{D}_2)$. We note that $\mathcal{D}_2(X) \cap \mathcal{S}(X)$ is equal to the set $\{D(M_n)\colon \lim_{n \to +\infty} M_n = 0\}$. Indeed if (M_n) converges to 0, fix $\epsilon > 0$ and N such that $\|M_n\| \leqslant \epsilon$ for all $n \geqslant N$. Let $Y = [F_n]_{n \geqslant N}$. Any $y \in Y$ may be written $y = \sum_{n \geqslant N} y_n$, $y_n \in F_n$, and therefore

$$\|D(M_n)(y)\| = \left\| \sum_{n \geqslant N} M_n y_n \right\| \leqslant \epsilon \left\| \sum_{n \geqslant N} y_n \right\| = \epsilon \|y\|,$$

by the strong unconditionality properties of the basis. This implies that $D(M_n)$ is compact and therefore strictly singular. Conversely, if $\|M_n\|$ does not converge to 0 then for some $\alpha > 0$ and some infinite set N, $\|M_n\| \geqslant \alpha$ for any $n \in N$, and let $x_n \in F_n$ be a norm 1 vector such that $\|M_n x_n\| \geqslant \alpha$. Let $y_n = M_n x_n$. The map C on $[y_n, n \in N]$ defined by $Cy_n = x_n$ is bounded by the strong unconditionality properties of X. Therefore the restriction of $D(M_n)$ to $[x_n, n \in N]$ is an isomorphism with inverse C and $D(M_n)$ is not strictly singular.

We deduce from this that

$$\mathcal{L}(X)/\mathcal{S}(X) \simeq \mathcal{D}_2(X)\big/\big(\mathcal{D}_2(X) \cap \mathcal{S}(X)\big) \simeq l_\infty(\mathcal{M}_2)/c_0(\mathcal{M}_2).$$

Now if $T \in \mathcal{L}(X)$ is invertible with inverse T', write $T = D + S$ and $T' = D' + S'$, with $D = D(M_n)$, $D' = D(N_n)$ and S, S' strictly singular. From $TT' = Id_X$ we deduce that $DD' = Id_X + s$ where s is strictly singular. Furthermore $s = DD' - Id_X$ is 2-block diagonal, and therefore of the form $D(s_n)$ where s_n converges to 0. Therefore from $M_n N_n = Id_{\mathbb{R}^2} + s_n$, we deduce that for n large enough, M_n is invertible and $M_n^{-1} = N_n(Id + s_n)^{-1}$ is bounded above. Modifying the first terms of the sequences (M_n) and (N_n) (up to modifying S and S'), we may assume that this is true for all $n \in \mathbb{N}$.

Conversely if M_n is invertible for all $n \in \mathbb{N}$, and (M_n^{-1}) is bounded, then $D(M_n)$ is an invertible operator with inverse $D(M_n^{-1})$.

It follows that the elements of $(\mathcal{L}(X)/\mathcal{S}(X))_0$ are those that may be lifted to a diagonal operator $D(M_n)$ where for all n, M_n is invertible, and (M_n^{-1}) is bounded; such an operator $D(M_n)$ corresponds canonically to an invertible element of $l_\infty(\mathcal{M}_2)$, and it follows that

$$\big(\mathcal{L}(X)/\mathcal{S}(X)\big)_0 \simeq \big(l_\infty(\mathcal{M}_2)/c_0(\mathcal{M}_2)\big)_0. \qquad \square$$

Lemma 32. *Let $A \in \mathcal{M}_2$. There exists a map f_A on $\mathcal{M}_2$ such that whenever $A^2 = -Id + r$ with $\|r\| < 1$, it follows that $(A + f_A(r))^2 = -Id$, and such that if $\|r\| < 1$ then $\|f_A(r)\| \leqslant \|A\|((1 - \|r\|)^{-1/2} - 1)$.*

Proof. For $\|r\| < 1$, let $f_A(r) = A(Id - r)^{-1/2} - A$, where $(Id - r)^{-1/2}$ is defined as an infinite series $Id + \sum_{n\geqslant 1} c_n r^n$. Note that when $r = A^2 + Id$, $(Id - r)^{-1/2}$ commutes with A. It follows by an elementary computation that $(A + f_A(r))^2 = -Id$. Furthermore,

$$\big\|f_A(r)\big\| \leqslant \|A\| \sum_{n\geqslant 1} c_n \|r\|^n = \|A\|\big(\big(1 - \|r\|\big)^{-1/2} - 1\big). \qquad \square$$

Lemma 33. *Let $M \in \mathcal{M}_2$ be such that $M^2 = -Id$. Then there exists $P \in \mathcal{GL}_2$ such that $\|P\|_2 = \|P^{-1}\|_2 \leqslant \sqrt{\|M\|_1}$ and $P\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)P^{-1} = M$.*

Proof. If $M^2 = -Id$ then M is of the form $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$ with $a^2 = -1 - bc$. If $c > 0$ put $P = c^{-1/2}\left(\begin{smallmatrix} 1 & a \\ 0 & c \end{smallmatrix}\right)$, then $P^{-1} = c^{-1/2}\left(\begin{smallmatrix} c & -a \\ 0 & 1 \end{smallmatrix}\right)$, and $P\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)P^{-1} = M$. Furthermore, $\|P^{-1}\|_2^2 = \|P\|_2^2 = c^{-1}(1 + a^2 + c^2) = c - b \leqslant \|M\|_1$. If $c \leqslant 0$ then $b > 0$ and a similar proof holds. $\square$

Proposition 34. *The space $X(\mathbb{D}_2)$ has unique complex structure up to isomorphism.*

Proof. Let G be the group $(l_\infty(\mathcal{M}_2)/c_0(\mathcal{M}_2))_0$ and let $I = \{g \in G\colon g^2 = -1\}$. By Lemma 31 and Proposition 20, it is enough to prove that all elements of I are G-conjugate.

Let $g \in I$, so g is the class of some (M_n) which is invertible in $l_\infty(\mathcal{M}_2)$, that is $\|M_n\|_1$ and $\|M_n^{-1}\|_1$ are bounded by some constant C. Since $g \in I$, the sequence $r_n = M_n^2 + Id$ converges to 0. Let $N \in \mathbb{N}$ be such that for all $n \geqslant N$, $\|r_n\| < 1$. Use Lemma 32 to define $N_n = M_n + f_{M_n}(r_n)$ for $n \geqslant N$; we have therefore that $N_n^2 = -Id$ for all $n \geqslant N$ and that $M_n - N_n$ converges to 0. For $n < N$ we just put $N_n = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. It follows that g is also the class of (N_n) modulo $c_0(\mathcal{M}_2)$.

By Lemma 33, there exists $P_n \in \mathcal{GL}_2$ such that $\|P_n\|_2 = \|P_n^{-1}\|_2 \leqslant \sqrt{C}$, and such that

$$N_n = P_n \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P_n^{-1}.$$

Therefore $(P_n)_{n\in\mathbb{N}}$ and $(P_n^{-1})_{n\in\mathbb{N}}$ define inverse elements p and p^{-1} in G. Let j be the element of I associated to the sequence $(j_n) \in l_\infty(\mathcal{M}_2)$ of constant value $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We deduce that

$$g = pjp^{-1},$$

and therefore there is a unique G-conjugacy class of elements of I. □

It remains unknown whether there exists a complex space X with an unconditional basis (and therefore isomorphic to its conjugate), which does not admit a unique complex structure. Note that by Proposition 26(i), any complex structure on such a space X should be saturated with subspaces of X and therefore unconditionally saturated. By Proposition 25, the space $X^J(\mathbb{C}) \oplus X^{-J}(\mathbb{C})$, while isomorphic to its conjugate, does not admit a unique complex structure; this space is isomorphic to the sum of two H.I. spaces and therefore does not have an unconditional basis.

6. Final remarks and questions

The fact that any complex Banach space which is real H.I. is also complex H.I. raises the following question. Does there exist a complex H.I. space which is not H.I. when seen as a real Banach space? The answer turns out to be positive.

To prove this, we consider the canonical complexification $X_{\mathrm{GM}} \oplus_{\mathbb{C}} X_{\mathrm{GM}}$ of the real version of Gowers–Maurey's space, i.e. $X_{\mathrm{GM}} \oplus X_{\mathrm{GM}}$ with the complex structure associated to the operator I defined by $I(x, y) = (-y, x)$. Note that by Proposition 29, any other complex structure on $X_{\mathrm{GM}} \oplus X_{\mathrm{GM}}$ would be isomorphic.

Proposition 35. *The complexification of X_{GM} is complex H.I. but not real H.I.*

Proof. Seen as a real space, $X = X_{\mathrm{GM}} \oplus_{\mathbb{C}} X_{\mathrm{GM}}$ is clearly not H.I. as a direct sum of infinite-dimensional spaces. Let now Y be a $\mathbb{C}$-linear subspace of X. We denote by p_1 and p_2 the projections on the first and the second summand of $X = X_{\mathrm{GM}} \oplus X_{\mathrm{GM}}$ respectively. Either $p_{1|Y}$ or $p_{2|Y}$ is not strictly singular, and without loss of generality this is true of $p_{1|Y}$; so $p_{1|Y_1}$ is an isomorphism for some $\mathbb{R}$-linear subspace Y_1 of Y. We may therefore find a subspace Z of X_{GM} and a map $\alpha : Z \to X_{\mathrm{GM}}$ such that $Y_1 = \{(z, \alpha z), z \in Z\} \subset Y$ (take $Z = p_1(Y_1)$ and $\alpha = p_2(p_{1|Y_1})^{-1}$). By $\mathbb{C}$-linearity, $Y_2 := iY_1 = \{(-\alpha z, z), z \in Z\}$ is also an $\mathbb{R}$-linear subspace of Y.

By the properties of X_{GM}, α is of the form $\lambda i_{Z,X_{\mathrm{GM}}} + s$, where s is strictly singular. For our computation we may assume that the norm $|||\cdot|||$ on $X \oplus X$ is the l_1-sum norm. Let $\epsilon > 0$ be such that $2(1 + |\lambda| + \epsilon)(1 + |\lambda|)\epsilon < 1$. Passing to a subspace, we may assume that $\|s\| \leqslant \epsilon$. We prove that whenever $y_1 \in Y_1$, $y_2 \in Y_2$ are norm 1 vectors, we have $|||y_1 - y_2||| > \epsilon$. Indeed, otherwise let $y_1 = (z_1, \alpha z_1)$ be of norm 1 with $z_1 \in Z$, and $y_2 = (-\alpha z_2, z_2)$ be of norm 1 with $z_2 \in Z$, with

$$\epsilon \geqslant \big|\big|\big|(z_1 + \alpha z_2, \alpha z_1 - z_2)\big|\big|\big|,$$

therefore

$$\epsilon \geqslant \|z_1 + \lambda z_2 + s z_2\|,$$

and since $\|s\| \leqslant \epsilon$,

$$\|z_1 + \lambda z_2\| \leqslant 2\epsilon.$$

Likewise,

$$\|\lambda z_1 - z_2\| \leqslant 2\epsilon.$$

Combining these two inequalities, we obtain

$$\|z_2\| \leqslant \big\|\big(1 + \lambda^2\big) z_2\big\| \leqslant \big(1 + |\lambda|\big) 2\epsilon,$$

which implies

$$1 = |||y_2||| = \|z_2\| + \|\alpha z_2\| \leqslant \big(1 + |\lambda| + \epsilon\big)\big(1 + |\lambda|\big) 2\epsilon,$$

a contradiction.

From this we deduce that Y_1 and Y_2 form a direct sum in Y. Therefore Y is not $\mathbb{R}$-HI. As X is HD_2 as a real space [8, Corollary 2] and $Y \subset X$, it follows that Y is HD_2 as a real space. It follows that Y is $\mathbb{R}$-quasi-maximal in X, and therefore $\mathbb{C}$-quasi-maximal in X (recall that Y is quasi-maximal in X if $Y + Z$ is never a direct sum for Z an infinite-dimensional subspace of X). As every $\mathbb{C}$-linear subspace Y of X is $\mathbb{C}$-quasi-maximal in X, it follows according to the remark before Lemma 6 that X is H.I. as a complex space. □

This remark and the previous examples illustrate how various the relations can be between real and complex structure in a complex H.I. space. It remains unknown whether these structures may differ widely, for example, does there exist a complex H.I. space which contains an unconditional basic sequence when seen as a real space?

Question 36. By Theorem 5, when X is real H.I., there exists a division algebra E isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, such that for any $Y \subset X$, $\mathcal{L}(X)/\mathcal{S}(X)$ embeds into $\mathcal{L}(Y, X)/\mathcal{S}(Y, X)$ which embeds into E. It follows that for any $Y \subset X$,

$$\dim \mathcal{L}(X)/\mathcal{S}(X) \leqslant \mathcal{L}(Y, X)/\mathcal{S}(Y, X) \leqslant \dim E.$$

If $X = X_{\mathrm{GM}}$ all these dimensions are equal to 1. We provided examples $X(\mathbb{C})$ and $X(\mathbb{H})$ for which all these dimensions are equal to 2 or to 4 respectively. It remains open whether these dimensions may differ. For example, does there exist a real H.I. Banach space X such that every operator on X is of the form $\lambda Id_X + S$, but such that there exists an operator T on some subspace

Y of X which is not of the form $\lambda i_{Y,X} + s$? In this case, $\mathcal{L}(X)/\mathcal{S}(X)$ would be isomorphic to $\mathbb{R}$, while E would be complex or quaternionic.

This question is related to whether non-embedding may be replaced by total incomparability in the last part of Proposition 26(v). Assume indeed that there exists a real H.I. space X such that

$$\mathcal{L}(X) \simeq [Id_X, J] \oplus \mathcal{S}(X),$$

where $J^2 = -Id$, while there exists a subspace Y of X such that

$$\mathcal{L}(Y) \simeq [Id_Y, u, v, w] \oplus \mathcal{S}(Y),$$

where $u = J|_Y$ and Id_Y, u, v and w satisfy the quaternionic relations. Then by Proposition 20, X admits exactly two complex structures X^J and X^{-J}, do not embed into each other by Proposition 26(v); and Y^u and Y^{-u} are subspaces of X^J and X^{-J} respectively, which are isomorphic, by Proposition 20 and the proof of Proposition 28, so X^J and X^{-J} are not totally incomparable.

References

[1] R. Anisca, Subspaces of L_p with more than one complex structure, Proc. Amer. Math. Soc. 131 (9) (2003) 2819–2829.

[2] S. Argyros, A universal property of reflexive hereditarily indecomposable Banach spaces, Proc. Amer. Math. Soc. 129 (2001) 3231–3239.

[3] S. Argyros, Ramsey Methods in Analysis (Saturated and Conditional Structures in Banach Spaces), Adv. Courses Math. CRM Barcelona, Birkhäuser Verlag, Basel, 2005.

[4] S. Argyros, A. Manoussakis, An indecomposable and unconditionally saturated Banach space, Studia Math. 159 (1) (2003) 1–32.

[5] S. Argyros, A. Tolias, Methods in the theory of hereditarily indecomposable Banach spaces, Mem. Amer. Math. Soc. 170 (2004) 806.

[6] J. Bourgain, Real isomorphic complex Banach spaces need not be complex isomorphic, Proc. Amer. Math. Soc. 96 (2) (1986) 221–226.

[7] V. Ferenczi, Operators on subspaces of hereditarily indecomposable Banach spaces, Bull. London Math. Soc. 29 (1997) 338–344.

[8] V. Ferenczi, Hereditarily finitely decomposable Banach spaces, Studia Math. 123 (2) (1997) 135–149.

[9] V. Ferenczi, Quotient hereditarily indecomposable Banach spaces, Canad. J. Math. 51 (3) (1999) 566–584.

[10] G. Godefroy, N.J. Kalton, Lipschitz-free Banach spaces, Studia Math. 159 (1) (2003) 121–141.

[11] W.T. Gowers, A solution to Banach's hyperplane problem, Bull. London Math. Soc. 26 (6) (1994) 523–530.

[12] W.T. Gowers, An infinite Ramsey theorem and some Banach-space dichotomies, Ann. of Math. (2) 156 (3) (2002) 797–833.

[13] W.T. Gowers, B. Maurey, The unconditional basic sequence problem, J. Amer. Math. Soc. 6 (4) (1993) 851–874.

[14] W.T. Gowers, B. Maurey, Banach spaces with small spaces of operators, Math. Ann. 307 (1997) 543–568.

[15] N. Kalton, An elementary example of a Banach space not isomorphic to its complex conjugate, Canad. Math. Bull. 38 (2) (1995) 218–222.

[16] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces, Springer-Verlag, New York, Heidelberg, Berlin, 1979.

[17] B. Maurey, Banach spaces with few operators, in: W.B. Johnson, J. Lindenstrauss (Eds.), Handbook of the Geometry of Banach Spaces, vol. 2, Elsevier, Amsterdam, 2002.

[18] S. Mazur, S. Ulam, Sur les transformations isométriques d'espaces vectoriels normés, C. R. Acad. Sci. Paris 194 (1932) 946–948.

[19] S. Szarek, On the existence and uniqueness of complex structure and spaces with "few" operators, Trans. Amer. Math. Soc. 293 (1) (1986) 339–353.

[20] S. Szarek, A superreflexive Banach space which does not admit complex structure, Proc. Amer. Math. Soc. 97 (3) (1986) 437–444.