

THE LIPSCHITZ FREE BANACH SPACES OF $C(K)$ -SPACES

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ABSTRACT. The aim of this note is to prove that if K is any infinite metric compact space, then the Lipschitz free spaces of $C(K)$ and c_0 are isomorphic. This gives an example of non-Lipschitz-homeomorphic Banach spaces whose free Lipschitz spaces are isomorphic. We also derive some results about Lipschitz homogeneity for Banach spaces, from the results of G. Godefroy and N. J. Kalton on Lipschitz free Banach spaces.

1. THE LIPSCHITZ FREE BANACH SPACE \mathcal{F}_{c_0}

We refer to [5] for the background of this note. Let X be a metric space with a special point 0. We denote by $\text{Lip}_0(X)$ the space of Lipschitz functions from X to \mathbb{R} which vanish at 0. The norm of a function $f \in \text{Lip}_0(X)$ is the Lipschitz constant of f and we denote it by $\|f\|_L$. The space $\text{Lip}_0(X)$ has a canonical predual $\mathcal{F}(X)$ which is the linear span in $\text{Lip}_0(X)^*$ of the evaluation functionals $\delta_x : f \mapsto f(x)$ for $x \in X$. The space $\mathcal{F}(X)$ is called the Lipschitz free space of X . It is clearly invariant under Lipschitz homeomorphisms. So it is natural to ask how much the Lipschitz free space of a Banach space X captures of the Lipschitz structure of X . To the best of the authors' knowledge, it was not even known if two Banach spaces could have the same Lipschitz free space without being Lipschitz homeomorphic. In this note, we prove that, for any infinite compact metric space K , the spaces $\mathcal{F}(C(K))$ and $\mathcal{F}(c_0)$ are isomorphic. In fact we prove that the Banach-Mazur distance between $\mathcal{F}(C(K))$ and $\mathcal{F}(c_0)$ is uniformly bounded. By Theorem 3.1 in [7], we know that $C(K)$ and c_0 are not uniformly homeomorphic (and thus not Lipschitz homeomorphic) if $C(K)$ is not linearly isomorphic to c_0 . Hence this gives an example of a couple of Banach spaces which have isomorphic Lipschitz free spaces but are not uniformly homeomorphic.

Let us begin with a definition. Let X_1, X_2, \dots be Banach spaces and $X = (\bigoplus X_n)_{c_0}$. We denote by \hat{X}_n the subspace of X equal to $\sum_{i=1}^{n-1} \{0\} \oplus X_n \oplus \sum_{i=n+1}^{\infty} \{0\}$. The set $\bigcup \hat{X}_n$ is denoted by $\boxtimes X_n$ and called the *cross* of the X_n 's. It is equipped with the metric d inherited by the norm of X .

Proposition 1. *Let X_1, X_2, \dots be Banach spaces. The Banach spaces $(\bigoplus \mathcal{F}(X_n))_{\ell_1}$ and $\mathcal{F}(\boxtimes X_n)$ are isomorphic.*

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Proof. The dual spaces of $A = (\bigoplus \mathcal{F}(X_n))_{\ell_1}$ and $B = \mathcal{F}(\boxtimes X_n)$ are respectively $A^* = (\bigoplus \text{Lip}_0(X_n))_{\ell_\infty}$ and $B^* = \text{Lip}_0(\boxtimes X_n)$.

We define the map $\Phi : A^* \rightarrow B^*$ by $\Phi(f) = g$ where $f = (f_n)_{n \in \mathbb{N}}$ and $g(x) = f_n(x)$ if $x \in \hat{X}_n$ (here, we identify X_n and \hat{X}_n , which does not carry any confusion).

Let $x \in \hat{X}_n$ and $y \in \hat{X}_m$ be two points of the cross. If $n = m$, we have

$$|\Phi(f)(x) - \Phi(f)(y)| \leq \|f_n\|_L \cdot \|x - y\|_{X_n} \leq \|f\|_\infty d(x, y).$$

On the other hand, when $n \neq m$, we have

$$|\Phi(f)(x) - \Phi(f)(y)| \leq \|f_n\|_L \cdot \|x\|_{X_n} + \|f_m\|_L \cdot \|y\|_{X_m} \leq 2 \|f\|_\infty d(x, y).$$

Hence, $\|\Phi(f)\|_L \leq 2 \|f\|_\infty$ for all $f \in A^*$, and thus Φ has norm at most 2. The inverse of Φ is given by $\Phi^{-1}(f) = \left(f|_{X_n}\right)_{n \in \mathbb{N}}$ and it has clearly norm at most 1. Moreover Φ is weak* to weak* continuous so it induces an isomorphism between A and B . □

The next step in our proof will be to show that $(\bigoplus \mathcal{F}(c_0))_{\ell_1}$ is isomorphic to $\mathcal{F}(c_0)$. In order to have that, we need a lemma:

Lemma 2. *Let X_1, X_2, \dots be Banach spaces and $X = (\bigoplus X_n)_{c_0}$. Then there is a 3-Lipschitz retraction from X onto $\boxtimes X_n$ which maps 0 to 0.*

Proof. For any integer q , let p_q denote the canonical projection from X onto $\left(\bigoplus_{n \neq q} X_n\right)_{c_0}$, that is to say, for any vector $x = (x_n)$ in X , we have $p_q(x) = (x_1, \dots, x_{q-1}, 0, x_{q+1}, \dots)$. We can note that $\|p_q(x)\| = \max_{r \neq q} \|x_r\| \leq \|x\|$.

Let us define $L : X \rightarrow \boxtimes X_n$ by $L(0) = 0$ and, if $x = (x_n)$ is different from 0, then

$$L(x) = \left(0, \dots, 0, \left(1 - \frac{\|p_q(x)\|}{\|x_q\|}\right)x_q, 0, \dots\right)$$

where q is such that $\|x_q\| = \|x\|$.

This definition is correct since, if $\|x_q\| = \|x_r\| = \|x\|$ for $q \neq r$, then both calculations give $L(x) = 0$.

If $x = (x_n)$ and $y = (y_n)$ are in X and $\|x_q\| = \|x\|$, $\|y_r\| = \|y\|$ with $q \neq r$, then:

$$\begin{aligned} d(L(x), L(y)) &= (\|x_q\| - \|p_q(x)\|) \vee (\|y_r\| - \|p_r(y)\|) \\ &\leq (\|x_q\| - \|x_r\|) \vee (\|y_r\| - \|y_q\|) \\ &\leq \|x_q\| - \|y_q\| + \|y_r\| - \|x_r\| \leq 2 \|x - y\|. \end{aligned}$$

Note that this includes the case $x = 0$ or $y = 0$. Now, if $\|x_q\| = \|x\|$ and $\|y_r\| = \|y\|$ for $q = r$, with $x \neq 0$ and $y \neq 0$, then

$$\begin{aligned} d(L(x), L(y)) &= \left\| \left(1 - \frac{\|p_q(x)\|}{\|x\|}\right)x_q - \left(1 - \frac{\|p_q(y)\|}{\|y\|}\right)y_q \right\| \\ &\leq \left\| \left(1 - \frac{\|p_q(x)\|}{\|x\|}\right)(x_q - y_q) \right\| + \left\| \left(\frac{\|p_q(y)\|}{\|y\|} - \frac{\|p_q(x)\|}{\|x\|}\right)y_q \right\| \\ &\leq \|x_q - y_q\| + \|y\| \left(\left| \frac{\|p_q(y)\|}{\|y\|} - \frac{\|p_q(x)\|}{\|x\|} \right| + \left| \frac{\|p_q(x)\|}{\|y\|} - \frac{\|p_q(x)\|}{\|x\|} \right| \right) \\ &\leq \|x - y\| + \|p_q(y - x)\| + \|y\| \|p_q(x)\| \left| \frac{1}{\|y\|} - \frac{1}{\|x\|} \right| \\ &\leq 2\|x - y\| + \frac{\|p_q(x)\|}{\|x\|} \left| \|x\| - \|y\| \right| \leq 3\|x - y\|. \end{aligned}$$

Hence L is 3-Lipschitz. Since $x \in \hat{X}_n$ implies $p_n(x) = 0$, it is clear that L is a retraction. \square

Lemma 3. *Let M be a metric space with a special point 0 and let N be a subset of M containing 0. We suppose that there exists a Lipschitz-retraction L from M onto N such that $L(0) = 0$. Then $\mathcal{F}(N)$ is isomorphic to some complemented subspace of $\mathcal{F}(M)$.*

Proof. Using Theorem 2.2.4.b in [10] (where $\mathcal{F}(M)$ is denoted by $\text{AE}(X)$), we see that there exist linear maps $i : \mathcal{F}(N) \rightarrow \mathcal{F}(M)$ and $l : \mathcal{F}(M) \rightarrow \mathcal{F}(N)$ such that for any $x \in N$, $y \in M$, $f \in \text{Lip}_0(M)$ and $g \in \text{Lip}_0(N)$, we have $i(\delta_x)(f) = f(x)$ and $l(\delta_y)(g) = g(L(y))$. One can easily see that i is a linear isomorphism from $\mathcal{F}(N)$ into $\mathcal{F}(M)$ and that il is a linear continuous projection from $\mathcal{F}(M)$ onto $i(\mathcal{F}(N))$. \square

Proposition 4. *The spaces $(\bigoplus \mathcal{F}(c_0))_{\ell_1}$ and $\mathcal{F}(c_0)$ are isomorphic.*

Proof. Let X_1, X_2, \dots be Banach spaces. Then it follows from Lemmas 2 and 3 and Proposition 1 that the space $(\bigoplus \mathcal{F}(X_n))_{\ell_1}$ is isomorphic to a complemented subspace of $\mathcal{F}((\bigoplus X_n)_{c_0})$. Assume now $X_n = c_0$ for each n . We infer that the spaces $(\bigoplus \mathcal{F}(c_0))_{\ell_1}$ and $\mathcal{F}(c_0)$ are isomorphic to complemented subspaces of each other. Since $(\bigoplus \mathcal{F}(c_0))_{\ell_1}$ is clearly isomorphic to its ℓ_1 sum, the standard Pełczyński decomposition method shows that they are isomorphic. \square

Theorem 5. *Let K be any infinite metric compact space. The spaces $\mathcal{F}(C(K))$ and $\mathcal{F}(c_0)$ are isomorphic. Moreover the Banach-Mazur distance is uniformly bounded.*

Proof. The space c_0 is 2-complemented in $C(K)$, so Lemma 3 shows that $\mathcal{F}(c_0)$ is isometric to a 2-complemented subspace of $\mathcal{F}(C(K))$.

By a classical result of Aharoni (see Theorem 7.11, p. 176 in [1]), we know that there is a 3-Lipschitz-homeomorphism between $C(K)$ and some subset of c_0 . Since the spaces $C(K)$ are absolute Lipschitz retracts (with uniform constant), there is a Lipschitz retraction from c_0 onto $C(K)$ which maps 0 to 0 (with uniform constant). Using Lemma 3, we see that $\mathcal{F}(C(K))$ is isomorphic to a complemented subspace of $\mathcal{F}(c_0)$ (with uniform constants). Finally, Proposition 4 ensures that $(\bigoplus \mathcal{F}(c_0))_{\ell_1}$

is isomorphic to $\mathcal{F}(c_0)$. Using the standard Pełczyński decomposition method, we infer that $\mathcal{F}(c_0)$ and $\mathcal{F}(C(K))$ are isomorphic with uniform constants. \square

Note that by the result of Aharoni, the space $\mathcal{F}(c_0)$ is universal for separable Banach spaces. However it is not isomorphic to $C(0, 1)$ which is \mathcal{L}^∞ whereas $L_1(0, 1)$ is complemented in $\mathcal{F}(c_0)$.

An open question is whether a converse to Aharoni's result is true, that is, if $\mathcal{F}(X)$ is universal (equivalently contains $\mathcal{F}(c_0)$), does it follow that c_0 embeds into X ? It is not even known if X must contain the ℓ_∞^n 's uniformly when $\mathcal{F}(X)$ is universal.

2. LIPSCHITZ HOMOGENEOUS BANACH SPACES

We include in this note a consequence of the results of Godefroy and Kalton on Lipschitz free Banach spaces. Define a Banach space to be *Lipschitz homogeneous* if it is Lipschitz homeomorphic to all its closed infinite-dimensional subspaces. The following question was asked in [2].

Question 6. Is every Lipschitz homogeneous Banach space isomorphic to ℓ_2 ?

That the answer should be yes is a natural conjecture which would generalize the positive answer given to the homogeneous Banach space problem by Gowers [4] and Komorowski, Tomczak-Jaegermann [8]. They proved that a Banach space which is isomorphic to its (closed infinite-dimensional) subspaces must be isomorphic to ℓ_2 . It is also related to the conjecture that for separable Banach spaces, Lipschitz homeomorphism and isomorphism coincide; if the latter is true, then obviously ℓ_2 is the only Lipschitz homogeneous Banach space.

Using the results of Gowers and Komorowski, Tomczak-Jaegermann, partial positive answers to Question 6 were given in [2]: for example, if a dual Banach space is Lipschitz homogeneous, then it must be isomorphic to ℓ_2 .

We derive additional results about this problem using a result of Godefroy and Kalton [5] which states that the Bounded Approximation Property is stable under Lipschitz homeomorphism. We recall a few results about local unconditional structure (or l.u.st.). There are two notions of l.u.st. in the literature. The first, that we shall call l.u.st., was introduced by Figiel, Johnson and Tzafriri [3], on the model of L_p -spaces. The second was introduced by Gordon and Lewis [6], and we shall write it GL-l.u.st. For the definition:

Definition 7. A Banach space X is said to have l.u.st. if there is a constant $C \geq 1$ and an increasing net E_α of finite-dimensional subspaces of X , such that $X = \overline{\bigcup_\alpha E_\alpha}$ and such that for every α , E_α has a C -unconditional basis.

A Banach space X is said to have GL-l.u.st. if there is a constant $C \geq 1$ such that for every finite-dimensional subspace E of X , there exists a finite-dimensional space F with a 1-unconditional basis, and maps $T : E \rightarrow F$, $U : F \rightarrow X$, such that $UT(x) = x$ for all $x \in E$ and such that $\|T\| \|U\| \leq C$.

While it is clear that l.u.st. implies GL-l.u.st., the converse is only known to hold in some special cases. The following proposition is proved in [3] with l.u.st. instead of GL-l.u.st. But we notice that gathering some facts in that article allows us to generalize it to Banach spaces with GL-l.u.st.

Proposition 8. *Let X be a Banach space with GL-l.u.st. which does not contain ℓ_∞^n 's uniformly. Then every subspace of X has a further subspace with an unconditional basis.*

Proof. Assume X has GL-l.u.st. Then by [3], Remark 2.3, X^{**} is complemented in a Banach lattice L . So $X^{(4)}$ (and thus X^{**}) is complemented in L^{**} which is a σ -complete Banach lattice. By local reflexivity, X^{**} does not contain ℓ_∞^n 's uniformly, so by [3], Proposition 2.6, X^{**} is complemented in a Banach lattice M which does not contain ℓ_∞^n 's uniformly. In particular M does not contain a copy of c_0 , so is σ -complete and σ -order continuous, and X is isomorphic to a subspace of M . Thus we may apply [3], Theorem 4.1, to conclude that every subspace of X contains an unconditional basic sequence. \square

The following proposition mixes new results with some results previously obtained in [2].

Proposition 9. *Let X be a Lipschitz homogeneous Banach space which is not isomorphic to ℓ_2 . Then*

- (a) *no subspace of X has the Radon-Nikodým Property,*
- (b) *no subspace of X has GL-l.u.st.,*
- (c) *X has a hereditarily indecomposable subspace,*
- (d) *every subspace of X has the Bounded Approximation Property,*
- (e) *X has cotype $2 + \epsilon$ and type $2 - \epsilon$ for any $\epsilon > 0$.*

Proof. (a) was proved in [2], where it was also proved that X does not contain a subspace with an unconditional basis, and so by Gowers' dichotomy theorem, (c) follows. Every Banach space has a subspace with a Schauder basis, and so (d) follows from the result of Godefroy and Kalton that we recalled. (e) follows from (d) and the results of Szankowski about the Approximation Property (see e.g. [9]). By (e), X does not contain ℓ_∞^n 's uniformly and so (b) follows from the previous proposition. \square

Properties (a)-(e) are a strange combination of positive and negative results, and of course, no example of a Banach space satisfying (a)-(e) is known. The task of finding such a space with the additional property of being Lipschitz homogeneous seems to be out of reach. This suggests that ℓ_2 should be the only Lipschitz homogeneous Banach space.

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